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SOME RESULTS ON THE SUPREMUM AND ON THE FIRST PASSAGE TIME OF THE GENERALIZED TELEGRAPH PROCESS

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Abstract

We analyze the process $M(t)$ representing the maximum of the one-dimensional telegraph process $X(t)$ with exponentially distributed upward random times and generally distributed downward random times. The evolution of $M(t)$ is governed by an alternating renewal of two phases: a rising phase R and a constant phase C. During a rising phase, $X(t)$ moves upward, whereas, during a constant phase, it moves upward and downward, continuing to move until it attains the maximal level previously reached. Under some choices of the distribution of the downward times, we are able to determine the distribution of C, which allows us to obtain some bounds for the survival function of $M(t)$. In the particular case of exponential downward random times, we derive an explicit expression for the survival function of $M(t)$. Finally, the moments of the first passage time Θ_w of the process $X(t)$ through a fixed boundary $w > 0$ are analyzed.

Keywords: Finite-velocity random motion; telegraph process; first passage time; supremum

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Secondary 60J25

1. Introduction

The one-dimensional (integrated) telegraph process is a suitable mathematical model to describe the random motion of a particle along the real line. The particle moves with finite velocity, and the direction of the motion is reversed according to the arrival epochs of a homogeneous Poisson process. This process was first studied by Goldstein [18] and Kac [23], while some properties of the solution of the Goldstein–Kac telegraph equation were analyzed by Bartlett [3].

Starting from these papers, the telegraph process and its generalizations have drawn the attention of many scientists, especially since they represent an alternative to diffusion processes, which are often unsuitable for describing natural phenomena in the life sciences. For instance, in Beghin *et al.* [4] and in Lopez and Ratanov [27], the authors analyze the asymmetric telegraph process, whereas Di Crescenzo and Martinucci [12] and Martinucci and Meoli

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[28] treat the case of a general distribution for the random times between consecutive reversals of direction. The telegraph process with an arbitrary number of velocities is analyzed in Kolesnik [24], whereas Stadje and Zacks [37] and De Gregorio [10] consider the case of random velocities. The telegraph process perturbed by jumps is studied in Ratanov [35], and the large deviation principle applied to the telegraph process can be found in De Gregorio and Macci [11]. See also Garra *et al.* [31] and Cinque and Orsingher [7] for some multidimensional extensions of the telegraph process, and Di Crescenzo *et al.* ([13, 14]) for the telegraph process confined by boundaries.

The rising interest in the telegraph process concerns not only the theoretical description of the model but also its application in several fields, such as biology (see Hillen and Othmer [21] for the use of the telegraph equation to model chemotactic behavior), physics (see, for instance, Weiss [40] for applications in electromagnetic theory), ecology (see Holmes *et al.* [22] for a model of the dispersal of wild animals) and financial market modeling (see Kolesnik and Ratanov [25] and Pogorui *et al.* [32]).

Recently, Cinque and Orsingher ([5, 6]) have addressed the problem of finding the distribution of the maximum level reached by the particle up to time t in the case of the standard telegraph process. A similar topic has also been studied by Masoliver and Weiss [29], who analyze the process representing the maximum displacement (i.e. the difference between the maximum and the minimum) of a telegraph process, and by Ratanov [36], where some explicit formulae for the distributions of the running maximum/minimum, first passage times, and telegraphic meanders are obtained. In general, there are few processes for which explicit expressions for the distribution of the maximum and that of the first passage time through a fixed boundary are known. However, these distributions lend themselves to important applications. For instance, the motion of some microorganisms is often driven by a telegraph-type equation (see Komin *et al.* [26]), and it may be of interest in finding the maximum displacement reached by such organisms or the first time instant at which the motion reaches a fixed boundary representing a critical threshold. In addition, the study of the maximum may be of interest in finance (for asset pricing models based on the telegraph processes, see Di Crescenzo and Pellerey [15]), and also in geology (for the ground displacements sometimes described as the superposition of an asymmetric telegraph process and a diffusive component, see Travaglino *et al.* [38]).

In the present paper, we aim to analyze the process $M(t)$ which represents the maximum level reached by the particle during the time interval $[0, t]$ in the case of constant velocities with equal absolute value. The evolution of $M(t)$ is represented by an alternating renewal of two phases: a rising phase R and a constant phase C. In particular, the constant phase C represents the duration of an interval in which the particle moves upward and downward, starting with a downward movement, until it reaches the maximal level previously attained. Note that the constant phase C can be regarded as the sojourn time (see, for instance, Ray [34]) of the process $M(t)$ in the previously reached maximal level. The problem of finding the distribution of the sojourn time is of great interest in many research areas (see, for instance, Dębicki *et al.* [9] and Foss and Miyazawa [17]). In our context, the distribution of the constant phase C is related to the survival function of $M(t)$. Moreover, the moments of the first passage time Θ_w of the telegraph process (or equivalently of $M(t)$) can be determined starting from those of C.

The paper is organized as follows. In Section 2, we describe the telegraph process with exponentially distributed upward random times U_i and generally distributed downward random times D_i, and we introduce some basic definitions. In Section 3, we present the general reasoning for finding the distribution of the maximum $M(t)$, together with some general results

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regarding the moments of the first passage time Θ_w. Then, in Sections 4 and 5, we specialize the results obtained for a general distribution of D to the cases when the downward random times have the following distributions: (i) exponential, (ii) Erlang, (iii) weighted exponential, and (iv) mixture of exponentials.

2. The model

Let $\{X(t); t \geq 0\}$ be a one-dimensional integrated telegraph process. Such a process describes the motion of a particle which starts from the origin at the time $t = 0$ and then moves upward and downward alternately with velocity $v = 1$. The first motion of the particle is upward, and the duration of this motion is denoted by U_1. Then the particle spends a random length of time, denoted by D_1, moving downward, before changing its velocity from $v = -1$ to $v = 1$, and so on. More precisely, we denote by U_i, $i \in \mathbb{N}$, the random variables representing the durations of intervals spent by the particle moving upward and by D_i, $i \in \mathbb{N}$, those representing the downward periods. We assume that the sequences of positive, independent and identically distributed (i.i.d.) random times $\{U_1, U_2, \cdots\}$ and $\{D_1, D_2, \cdots\}$ are in turn mutually independent. In particular, the random variables U_i are set to be exponentially distributed with parameter λ, whereas the variables D_i, for any $i \in \mathbb{N}$, follow a general absolutely continuous distribution function G, with probability density g.

Denoting by T_n the nth random instant at which the motion changes velocity, we have

$$T_{2n} = U^{(n)} + D^{(n)}, \qquad T_{2n+1} = T_{2n} + U_{n+1}, \qquad n = 0, 1, \ldots,$$

where $U^{(0)} = D^{(0)} = 0$ and

$$U^{(n)} := U_1 + \ldots + U_n, \qquad D^{(n)} := D_1 + \ldots + D_n, \qquad n = 1, \ldots.$$

Hence the position of the particle at time t can be expressed as

$$X(t) = \int_0^t v\,(-1)^{\Lambda_s}\,\mathrm{d}s,$$

where $v = 1$ and

$$\Lambda_t := \sum_{n=1}^{+\infty} \mathbf{1}_{\{T_n \leq t\}}, \qquad \Lambda_0 = 0,$$

is the alternating counting process characterized by random times $T_1, T_2, \cdots$, which counts the number of the particle velocity changes in $[0, t]$.

Let us consider the process $M(t)$ which represents the maximum level reached by the particle during the time interval $[0, t]$, i.e.

$$M(t) = \sup_{0 \leq s \leq t} X(s).$$

The evolution of $M(t)$ is governed by an alternating renewal of two phases: a rising phase R and a constant phase C, as shown in Figure 1. During a rising phase of the supremum, the particle moves with positive velocity, whereas during a constant phase C of $M(t)$, the particle moves upward and downward, starting with a downward movement and then continuing to move until it attains the maximal level previously reached. Because of the memoryless property, the rising phases R_i, for any $i \in \mathbb{N}$, are independent and exponentially distributed random variables with

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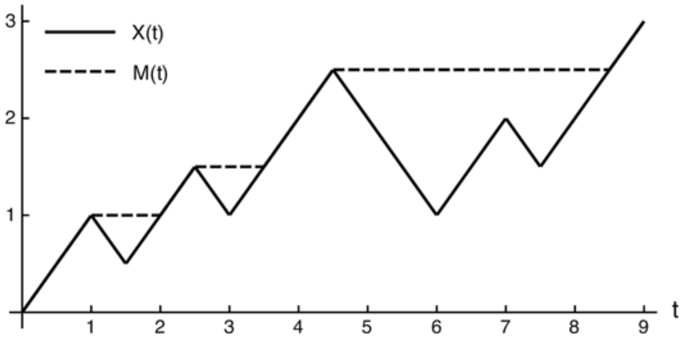

FIGURE 1. The telegraph process $X(t)$ and the corresponding supremum process $M(t)$.

parameter λ. Moreover, the random variables C_i, $i \in \mathbb{N}$, are i.i.d. Obtaining their distributions will be the task of the next sections. Obviously, since during a constant phase the maximal level of the particle does not change, the distribution of $M(t)$, for any fixed time t, is equal to that of the portion of time in $[0, t]$ spent during a rising phase.

3. The distribution of the supremum

Let us introduce the auxiliary compound Poisson process

$$Y(t) = \sum_{n=0}^{N(t)} D_n, \qquad t \geq 0, \tag{1}$$

where $D_0 = 0$ and

$$N(t) := \max \left\{ n \in \mathbb{N}_0 : \sum_{i=1}^{n} U_i \leq t \right\}, \tag{2}$$

which is a Poisson process with intensity λ. From the definition, the condition $Y(t) = s - t$, $0 \leq t \leq s$, means that during the interval $[0, s]$ the particle moves up for t time instants and down for the remaining $s - t$ time instants. From Equation (1), we have that the probability law of $Y(t)$ is characterized by a discrete component

$$\mathbb{P}(Y(t) = 0) = \mathbb{P}(N(t) = 0) = \mathrm{e}^{-\lambda t}$$

and an absolutely continuous component

$$h(u, t) := \frac{\mathrm{d}}{\mathrm{d}u} \mathbb{P}(Y(t) \leq u) = \sum_{n=1}^{+\infty} p(n, \lambda t) g^{(n)}(u), \quad t > 0, \tag{3}$$

where

$$p(n, \lambda t) := \mathbb{P}(N(t) = n) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}, \qquad n \in \mathbb{N}_0, \tag{4}$$

is the probability distribution of the Poisson process $N(t)$ (see Equation (2)), and $g^{(n)}$ is the n-fold convolution of the density g of the random variables D_i.

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Denoting by D_1 the duration of the first downward period, or equivalently the length of the first downward movement, let us set

$$T := T(D_1) := \inf\{t : Y(t) = -D_1 + t\}.$$

The stopping time $T(D_1)$ represents the smallest time such that

$$\sum_{i=1}^{N(T(D_1))} D_i - \sum_{i=1}^{N(T(D_1))+1} U_i + D_1 = 0,$$

so that $T(D_1)$ corresponds to half the time taken by the particle to reach the previous maximal level. Hence, denoting by $C := C(D_1)$ the duration of the constant phase, we have

$$C(D_1) \overset{d}{=} 2 \cdot T(D_1), \tag{5}$$

where $\overset{d}{=}$ denotes equality in distribution. Moreover, from Equation (5), a lower bound for $T(D_1)$ easily follows. Specifically, since the duration of a constant phase is at least equal to $2 \cdot D_1$, we have $T(D_1) \geq D_1$.

Note that, because of the assumption of i.i.d. downward times, we have $T \equiv T(D)$, where the random variable $T(D)$ has finite moments if and only if (see Zacks [39])

$$\mathbb{E}(D) < \mathbb{E}(U). \tag{6}$$

Theorem 1. *The probability density of the stopping time T is given by*

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T \leq t) = e^{-\lambda t} g(t) + \frac{1}{t}\sum_{n=1}^{+\infty} p(n, \lambda t) \int_0^t x g^{(n)}(t-x) g(x)\mathrm{d}x, \qquad t \geq 0, \tag{7}$$

where $p(n, \lambda t)$ is as defined in Equation (4) and $g^{(n)}(x)$ denotes the n-fold convolution of the density g.

Proof. The conditional density of the stopping time $T(D_1)$, given $D_1 = x$, can be expressed as follows:

$$f_T(t|D_1 = x) = \frac{x}{t} h(t-x, t), \qquad t > x > 0, \tag{8}$$

where $h(u, t)$ is as defined in Equation (3).

Moreover, the probability that $T(D_1) = D_1$, conditional on the duration of the first downward movement D_1, can be obtained by means of the compound process $Y(t)$. Indeed, we have

$$\mathbb{P}\left(T(D_1) = D_1 | D_1\right) = \mathbb{P}\left(Y(D_1) = 0\right) = e^{-\lambda D_1}. \tag{9}$$

Hence, by Equations (8) and (9), and recalling Equation (3), we immediately obtain Equation (7). □

Using Equation (5), and recalling Theorem 1, we immediately obtain the following corollary.

Corollary 1. *For $t > 0$, the probability density of the constant phase C can be expressed as*

$$f_C(t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(C \leq t) = \frac{1}{2} e^{-\frac{\lambda t}{2}} g\left(\frac{t}{2}\right) + \frac{1}{t}\sum_{n=1}^{+\infty} p\left(n, \frac{\lambda t}{2}\right) \int_0^{t/2} x\, g^{(n)}\left(\frac{t}{2} - x\right) g(x)\mathrm{d}x. \tag{10}$$

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In the sequel we shall denote by $F_C(t) := \mathbb{P}(C \le t)$, $t > 0$, the distribution function of the random variable C and by $F_C^{(n)}(t)$ its n-fold convolution.

Aiming to formulate the distribution of the supremum process $M(t)$, let us introduce the compound Poisson process

$$Z(t) = \sum_{n=0}^{N(t)} C_n, \tag{11}$$

where $N(t)$ is as defined in Equation (2) and the random variables C_n are i.i.d. random variables with probability density given in Equation (10). By Equation (11), we immediately get the following remark.

Remark 1. The moment generating function of the process $Z(t)$ is given by

$$\psi_Z(t, s) := \mathbb{E}(e^{sZ(t)}) = e^{-\lambda t[1-\psi_C(s)]}, \tag{12}$$

where

$$\psi_C(s) := \mathbb{E}(e^{sC}) \tag{13}$$

is the moment generating function of the random variable C. Moreover, the expected value and the variance of $Z(t)$ can be expressed in terms of the moments of C. In particular, we have

$$\mathbb{E}(Z(t)) = \lambda t \mathbb{E}(C), \qquad Var(Z(t)) = \lambda t \mathbb{E}(C^2). \tag{14}$$

The following theorem provides the expression for the survival distribution of $M(t)$.

Theorem 2. *For $t > 0$, the survival function of $M(t)$ is given by*

$$\mathbb{P}(M(t) > w) = \sum_{n=0}^{+\infty} p(n, \lambda w) F_C^{(n)}(t - w), \tag{15}$$

where $p(n, \lambda w)$ is as defined in Equation (4) and $F_C^{(n)}(t)$ is the n-fold convolution of the distribution function of C.

Proof. By Equation (11), the distribution function of $Z(t)$ can be expressed as

$$H_Z(z, t) := \mathbb{P}(Z(t) \le z) = \sum_{n=0}^{+\infty} p(n, \lambda t) F_C^{(n)}(z).$$

Hence, the proof immediately follows from noting that

$$\mathbb{P}(M(t) > w) = H_Z(t - w, w).$$

□

The relevance of Theorem 2 is its validity whatever the distribution of D. However, the expression given in Equation (15) may lead to complicated calculations because of the presence of the n-fold convolution of the distribution function of C. Therefore it is useful to obtain some bounds for the survival function of $M(t)$, which are provided in the following proposition.

Proposition 1. *For $t \ge w > 0$, the following relationships hold:*

$$L(t, w) \le \mathbb{P}(M(t) > w) \le U(t, w),$$

where

$$L(t, w) := \sum_{k=0}^{+\infty} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \left[F_C\left(\frac{t-w}{k}\right)\right]^k, \qquad U(t, w) := \exp\left[-\lambda w(1 - F_C(t-w))\right], \tag{16}$$

and $F_C(t)$ is the distribution function of the constant phase C.

Proof. The result easily follows from considering $C_{(k)} := \max\{C_1, \ldots, C_k\}$, $k \in \mathbb{N}$, and recalling Equation (11). Indeed, we have

$$\begin{aligned}\mathbb{P}(Z(t) \le w) &= \sum_{k=0}^{+\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \cdot \mathbb{P}\left(\sum_{n=0}^{k} C_n \le w\right) \\ &\le \sum_{k=0}^{+\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \left[F_C(w)\right]^k = \exp\left[-\lambda t(1 - F_C(w))\right],\end{aligned}$$

and, similarly,

$$\mathbb{P}(Z(t) \le w) \ge \sum_{k=0}^{+\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \mathbb{P}\left(C_{(k)} \le \frac{w}{k}\right) = \sum_{k=0}^{+\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \left[F_C\left(\frac{w}{k}\right)\right]^k.$$

□

3.1. First passage time distribution

Let Θ_w be the first passage time of the process $M(t)$ through a fixed boundary $w > 0$, i.e.

$$\Theta_w := \inf\{t \ge 0 : M(t) = w\}. \tag{17}$$

Note that the random variable Θ_w is identified with the first time instant at which the process $X(t)$ crosses a fixed level w. The relationship between the moments of Θ_w and those of the constant phase C is shown in the following proposition.

Proposition 2. *For $w > 0$, the expected value and the variance of the first passage time Θ_w are given by*

$$\mathbb{E}(\Theta_w) = \mathbb{E}(Z(w)) = \lambda w \mathbb{E}(C), \tag{18}$$

$$Var(\Theta_w) = 2w\mathbb{E}(Z(w)) + Var(Z(w)) = 2\lambda w^2 \mathbb{E}(C) + \lambda w \mathbb{E}(C^2), \tag{19}$$

where the compound Poisson process Z(t) is as defined in Equation (11).

Proof. By using integration by parts, we have

$$\begin{aligned}\psi_Z(w, -s) &= \int_0^{+\infty} e^{-sx} h_Z(x, w) \mathrm{d}x = s \int_0^{+\infty} e^{-sx} \mathbb{P}(Z(w) \le x) \mathrm{d}x \\ &= s \int_0^{+\infty} e^{-sx} \mathbb{P}(M(x+w) > w) \mathrm{d}x = s e^{sw} \int_w^{+\infty} e^{-sy} \mathbb{P}(M(y) > w) \mathrm{d}y.\end{aligned}$$

Hence, the equality

$$\frac{e^{-sw}\psi_Z(w, -s)}{s} = \int_w^{+\infty} e^{-sy} \mathbb{P}(M(y) > w) \mathrm{d}y \tag{20}$$

holds, so that

$$\int_w^{+\infty} e^{-sy}(1-\mathbb{P}(M(y)>w))\mathrm{d}y = \frac{e^{-sw}}{s} - \int_w^{+\infty} e^{-sy}\mathbb{P}(M(y)>w)\mathrm{d}y = \frac{e^{-sw}}{s}(1-\psi_Z(w,-s)). \tag{21}$$

Since

$$\lim_{s\to 0}\int_w^{+\infty} e^{-sy}\,(1-\mathbb{P}(M(y)>w))\,\mathrm{d}y = \int_w^{+\infty}\mathbb{P}(M(y)\le w)\mathrm{d}y = \int_w^{+\infty}\mathbb{P}(\Theta_w>y)\mathrm{d}y = \mathbb{E}(\Theta_w),$$

and if we recall that

$$\lim_{s\to 0}\frac{e^{-sw}}{s}(1-\psi_Z(w,-s)) = \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\psi_Z(w,-s)\Big|_{s=0} = \mathbb{E}(Z(w)),$$

the proof follows from Equation (21) and the first of the equations in (14).

Moreover, from Equation (20), by differentiating both members twice with respect to s and evaluating the result for $s=0$, we obtain

$$2w\mathbb{E}(Z(w)) + \mathbb{E}(Z^2(w)) = 2w\frac{\mathrm{d}}{\mathrm{d}s}M_Z(w,s)\Big|_{s=0} + \frac{\mathrm{d}^2}{\mathrm{d}s^2}M_Z(w,0) = \int_w^{+\infty} 2y\mathbb{P}(M(y)\le w)\mathrm{d}y,$$

so that

$$\mathbb{E}(\Theta_w^2) = \mathbb{E}(Z(w))(2w+\mathbb{E}(Z(w))) + Var(Z(w)). \tag{22}$$

Hence, Equation (19) follows from Equations (14) and (22). □

4. The distribution of the maximum

This section is devoted to the analysis of the distribution of the maximum under different choices of the distribution of the downward random times.

4.1. Exponentially distributed downward random times

Let us assume that the random variables D_i, $i\in\mathbb{N}$, are exponentially distributed with parameter $\mu>0$. From now on, we assume $\mu>\lambda$ so that the condition (6) is satisfied. The probability density function of the constant phase C is provided in the following theorem.

Theorem 3. *In the case of exponentially distributed downward times, for $t>0$, the probability density function of the constant phase random variable is given by*

$$f_C(t) = \frac{\mu e^{-(\lambda+\mu)t/2}}{t\sqrt{\lambda\mu}} I_1(t\sqrt{\lambda\mu}), \tag{23}$$

where $I_n(x)$ is the nth-order modified Bessel function of the first kind,

$$I_n(x) := \sum_{m=0}^{+\infty}\frac{1}{\Gamma(m+n+1)m!}\left(\frac{x}{2}\right)^{2m+n}. \tag{24}$$

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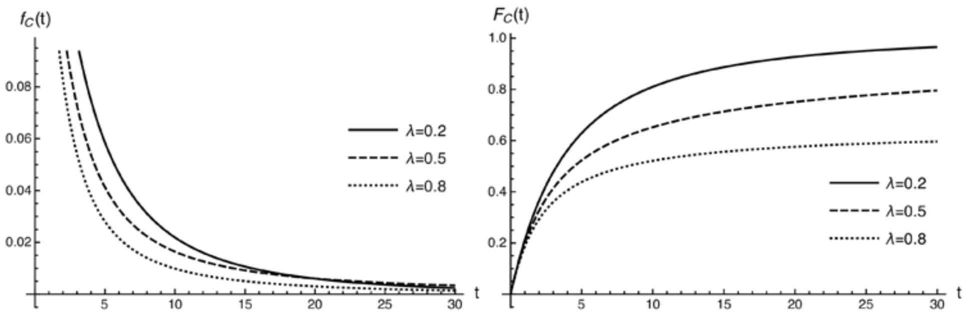

FIGURE 2. The density $f_C(t)$ (left) and the distribution function $F_C(t)$ (right) for $\lambda = 0.2$ (solid), $\lambda = 0.5$ (dashed), $\lambda = 0.8$ (dotted), and $\mu = 0.5$.

Proof. If $D \sim Exp(\mu)$, then $g^{(n)}(x) = \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!}$, $n \in \mathbb{N}$, $x > 0$. Hence, from Equation (7), for $t \geq 0$, it follows that

$$\frac{1}{t} \sum_{n=1}^{+\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \int_0^t \frac{x\mu^n (t-x)^{n-1} e^{-\mu(t-x)}}{(n-1)!} \mu e^{-\mu x} \mathrm{d}x$$

$$= \frac{\mu e^{-(\lambda+\mu)t}}{t} \sum_{n=1}^{+\infty} \frac{(\lambda \mu t)^n}{n!(n-1)!} \int_0^t x(t-x)^{n-1} \mathrm{d}x$$

$$= \mu e^{-(\lambda+\mu)t} \sum_{n=1}^{+\infty} \frac{(\lambda \mu t^2)^n}{n!(n+1)!} = \mu e^{-(\lambda+\mu)t} \left(-1 + \frac{I_1(2t\sqrt{\lambda\mu})}{t\sqrt{\lambda\mu}} \right),$$

so that the proof follows from Equation (10). □

Plots of the density $f_C(t)$ and of the corresponding distribution function $F_C(t)$ are given in Figure 2 for some choices of the parameters.

In the following proposition we provide the expression for the n-fold convolution of the density f_C.

Proposition 3. *In the case of exponentially distributed downward times, for $t > 0$ and $n \in \mathbb{N}$, the n-fold convolution of the density f_C is given by*

$$f_C^{(n)}(t) = \frac{n}{t} \left(\frac{\mu}{\lambda} \right)^{n/2} e^{-(\lambda+\mu)t/2} I_n(t\sqrt{\lambda\mu}). \tag{25}$$

Proof. The proof is by induction on n. The equality holds for $n = 1$. We suppose that Equation (25) holds for $n \in \mathbb{N}$, and we show it is then true for $n + 1$. We have

$$f_C^{(n+1)}(t) = \int_0^t f_C^{(n)}(x) f_C(t-x) \mathrm{d}x$$

$$= \int_0^t \frac{n}{x} \left(\frac{\mu}{\lambda} \right)^{n/2} e^{-(\lambda+\mu)x/2} I_n(x\sqrt{\lambda\mu}) \frac{\mu e^{-(\lambda+\mu)(t-x)/2}}{(t-x)\sqrt{\lambda\mu}} I_1((t-x)\sqrt{\lambda\mu}) \mathrm{d}x$$

$$= \frac{n+1}{t} \left(\frac{\mu}{\lambda} \right)^{(n+1)/2} e^{-(\lambda+\mu)t/2} I_{n+1}(t\sqrt{\lambda\mu}),$$

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following the equation (cf. Equation 11 in Section 2.15.19 of Prudnikov *et al.* [33])

$$\int_0^a \frac{1}{x(a-x)} I_n(ac-cx)I_m(cx)\mathrm{d}x = \frac{n+m}{anm} I_{m+n}(ac), \qquad a, n, m > 0.$$

□

Starting from Proposition 3, we now provide an explicit expression for the density $h_Z(z, t)$ of the compound process $Z(t)$.

Proposition 4. *In the case of exponentially distributed downward times, for* $t > 0$*, the density function of* $Z(t)$ *is given by*

$$h_Z(z, t) := \frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}(Z(t) \le z) = \frac{e^{-\lambda t} e^{-(\lambda+\mu)z/2}}{z} (t\sqrt{\lambda\mu}) \left(2\frac{t}{z} + 1\right)^{-1/2} I_1\left(z\sqrt{\lambda\mu\left(1+\frac{2t}{z}\right)}\right).$$

Moreover, we have $\mathbb{P}(Z(t) = 0) = e^{-\lambda t}$.

Proof. The equality follows from straightforward calculations. Indeed, for any $t > 0$,

$$\begin{aligned} h_Z(z, t) &= \sum_{n=0}^{+\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \frac{\mu^n}{(\lambda\mu)^{n/2}} \frac{n}{z} e^{-(\lambda+\mu)z/2} I_n(z\sqrt{\lambda\mu}) \\ &= \frac{e^{-\lambda t} e^{-(\lambda+\mu)z/2}}{z} \sum_{n=1}^{+\infty} \frac{(t\sqrt{\lambda\mu})^{n/2}}{(n-1)!} I_n(z\sqrt{\lambda\mu}) \\ &= \frac{e^{-\lambda t} e^{-(\lambda+\mu)z/2}}{z} t\sqrt{\lambda\mu} \left(2\frac{t}{z}+1\right)^{-1/2} I_1\left(\sqrt{\lambda\mu z^2 + 2\lambda\mu t z}\right) \\ &= \frac{e^{-\lambda t} e^{-(\lambda+\mu)z/2}}{z} t\sqrt{\lambda\mu} \left(2\frac{t}{z}+1\right)^{-1/2} I_1\left(z\sqrt{\lambda\mu(1+2t/z)}\right), \end{aligned}$$

where we have used the following relation (cf. Equation 4 in Section 5.8.3 of Prudnikov *et al.* [33]):

$$\sum_{k=0}^{+\infty} \frac{t^k}{k!} I_{n+k}(z) = \left(2\frac{t}{z}+1\right)^{-n/2} I_n\left(\sqrt{z^2+2tz}\right).$$

□

Some plots of the density $h_Z(z, t)$ are provided in Figure 3 for different choices of the parameters.

The following proposition provides an expression for the survival function of the supremum $M(t)$ in terms of a series of products of modified Bessel functions and generalized Laguerre polynomials.

Proposition 5. *In the case of exponentially distributed downward times, for* $t > 0$ *and* $0 \le w < t$*, the survival function of the supremum* $M(t)$ *can be expressed as*

$$\mathbb{P}(M(t) > w) = 1 - \frac{2w\sqrt{\lambda\mu}}{(\lambda+\mu)\sqrt{t^2-w^2}} \mathrm{e}^{-\frac{(\lambda+\mu)t}{2}} \mathrm{e}^{\frac{(\mu-\lambda)w}{2}}$$

$$\times \sum_{j=0}^{+\infty} j! \left(-\frac{2\sqrt{\lambda\mu}}{(\lambda+\mu)^2\sqrt{t^2-w^2}}\right)^j I_{j+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right) L_j^{-2j-1}(t(\lambda+\mu)), \qquad (26)$$

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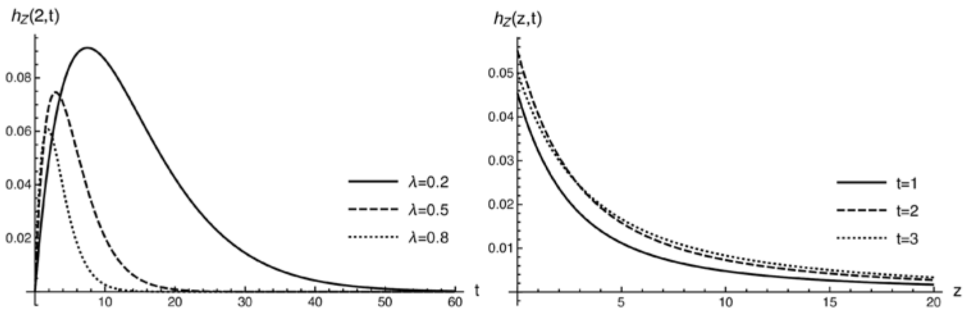

FIGURE 3. Left: the density $h_Z(z, t)$ as a function of t for $z = 2$, $\lambda = 0.2$ (solid), $\lambda = 0.5$ (dashed), $\lambda = 0.8$ (dotted), and $\mu = 0.8$. Right: the density $h_Z(z, t)$ as a function of z for $\mu = 0.3$, $\lambda = 0.5$, $t = 1$ (solid), $t = 2$ (dashed), and $t = 3$ (dotted).

where $L_n^k(x)$, $n \in \mathbb{N}$, *denotes the generalized Laguerre polynomial. Moreover,* $\mathbb{P}(M(t) = t) = e^{-\lambda t}$.

The proof of Proposition 5 is given in Appendix A.

The following proposition provides an asymptotic result concerning the survival probability.

Proposition 6. *For* $w > 0$*, we have*

$$\lim_{t \to +\infty} \mathbb{P}\left(M(t) > w\right) = 1.$$

Proof. This follows immediately from Equation (26), if we recall Equation (9.7.1) of [1] and the definition of generalized Laguerre polynomials (see, for instance, [2, p. 220]). □

In the case $\lambda = \mu$, the following proposition provides an expression for the survival function of the supremum $M(t)$ in terms of the Marcum Q-function, which is a special function occurring as a complementary cumulative distribution function for a non-central chi-squared random variable.

Proposition 7. *In the case of exponentially distributed downward times, for* $t > 0$, $0 \le w < t$, *and* $\lambda = \mu$, *the survival function of the supremum M(t) is given by*

$$\mathbb{P}\left(M(t) > w\right) = e^{-\lambda t} I_0\left(\lambda \sqrt{t-w}\sqrt{t+w}\right) + 2\left[1 - Q_1\left(\sqrt{\lambda(t+w)}, \sqrt{\lambda(t-w)}\right)\right], \qquad (27)$$

where $Q_1(a, b)$ *denotes the Marcum Q-function of order 1, defined as*

$$Q_1(a, b) = \int_b^{+\infty} x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(ax) \mathrm{d}x, \qquad b > 0.$$

Moreover, $\mathbb{P}(M(t) = t) = e^{-\lambda t}$.

Proof. From the definition of the compound Poisson process $Z(t)$ (see Equation (11)), and by Proposition 4, we have that, for $0 \le w \le t$,

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$$\mathbb{P}(M(t)>w)=H_Z(t-w,\,w)=\int_0^{t-w} h_Z(x,\,w)\mathrm{d}x$$
$$=e^{-\lambda w}w\sqrt{\lambda\mu}\int_0^{t-w}\frac{e^{-\frac{x(\lambda+\mu)}{2}}}{x}\frac{1}{\sqrt{2\frac{w}{x}+1}}I_1\left(x\sqrt{\lambda\mu\left(1+2\frac{w}{x}\right)}\right)\mathrm{d}x+e^{-\lambda w}.$$

Making use of integration by parts, we have

$$\begin{aligned}\mathbb{P}\left(M(t)>w\right)&=e^{-\lambda t}I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right)\\&+\int_0^{t-w}e^{-\lambda(x+w)}\left[\lambda I_0\left(I_0\left(\lambda\sqrt{x}\sqrt{x+2w}\right)-\frac{\lambda\sqrt{x}}{\sqrt{x+2w}}I_1\left(\lambda\sqrt{x}\sqrt{x+2w}\right)\right)\right]\mathrm{d}x\\&=e^{-\lambda t}I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right)+\lambda\int_w^t e^{-\lambda y}\left[I_0\left(\lambda\sqrt{y-w}\sqrt{y+w}\right)-\frac{\sqrt{y-w}}{\sqrt{y+w}}I_1\left(\lambda\frac{\sqrt{y-w}}{\sqrt{y+w}}\right)\right]\mathrm{d}y.\end{aligned}\tag{28}$$

Considering the definition of the modified Bessel function of the first kind, we have

$$\begin{aligned}&\int_w^t e^{-\lambda y/2}\left(I_0\left(\lambda\sqrt{t+w}\sqrt{y-w}\right)-\sqrt{\frac{y-w}{t+w}}I_1\left(\lambda\sqrt{t+w}\sqrt{y-w}\right)\right)\mathrm{d}y\\&=\frac{2e^{-\lambda w/2}}{\sqrt{t+w}}\int_0^{\sqrt{t-w}}ze^{-\lambda z^2/2}\left(\sqrt{t+w}I_0\left(\lambda z\sqrt{t+w}\right)-zI_1\left(\lambda z\sqrt{t+w}\right)\right)\mathrm{d}z\\&=\frac{2e^{-\lambda w/2}}{\sqrt{t+w}}\int_0^{\sqrt{t-w}}ze^{-\lambda/2z^2}\left(\sum_{n=0}^{+\infty}\frac{\left(\sqrt{t+w}\right)^{2n+1}\left(\frac{\lambda}{2}z\right)^{2n}}{n!n!}-z\sum_{n=0}^{+\infty}\frac{\left(\sqrt{t+w}\right)^{2n+1}\left(\frac{\lambda}{2}z\right)^{2n+1}}{n!(n+1)!}\right)\mathrm{d}z\\&=\frac{2e^{-\lambda w/2}}{\lambda}\left[\sum_{n=0}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^n}{n!n!}\gamma\left(n+1,\,\frac{\lambda}{2}(t-w)\right)-\sum_{n=0}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^n}{n!(n+1)!}\gamma\left(n+2,\,\frac{\lambda}{2}(t-w)\right)\right],\end{aligned}\tag{29}$$

where $\gamma(s,x)$ denotes the lower incomplete gamma function. Hence, recalling the recurrence relation (see, for instance, Equation (6.5.22) of [1])

$$\gamma(s+1,\,x)=s\gamma(s,\,x)-x^s e^{-x},\tag{30}$$

we find that Equation (29) becomes

$$\begin{aligned}&\frac{2e^{-\lambda w/2}}{\lambda}\left[\sum_{n=0}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^n}{n!n!}\gamma\left(n+1,\,\frac{\lambda(t-w)}{2}\right)\right.\\&\left.-\sum_{n=0}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^n}{n!(n+1)!}\left((n+1)\gamma\left(n+1,\,\frac{\lambda(t-w)}{2}\right)-e^{-\lambda/2(t-w)}\left(\frac{\lambda}{2}(t-w)\right)^{n+1}\right)\right]\\&=\frac{2e^{-\lambda w/2}}{\lambda}\left[\sum_{n=0}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^n\left(\frac{\lambda}{2}(t-w)\right)^{n+1}}{n!(n+1)!}e^{-\lambda(t-w)/2}\right]\\&=\frac{2e^{-\lambda t}}{\lambda}\frac{\sqrt{t-w}}{\sqrt{t+w}}I_1\left(\lambda\sqrt{t-w}\sqrt{t+w}\right).\end{aligned}$$

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Hence

$$\int_w^t e^{-\lambda t/2} e^{-\lambda y/2} I_0\left(\lambda\sqrt{y-w}\sqrt{t+w}\right) \mathrm{d}y$$
$$= \int_w^t e^{-\lambda y}\left[I_0\left(\lambda\sqrt{y-w}\sqrt{y+w} - \frac{\sqrt{y-w}}{\sqrt{y+w}} I_1\left(\lambda\sqrt{y-w}\sqrt{y+w}\right)\right)\right] \mathrm{d}y.$$

Finally, from Equation (28) it follows that

$$\mathbb{P}\left(M(t) > w\right) = e^{-\lambda t} I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right) + \lambda\int_w^t e^{-\lambda t/2} e^{-\lambda y/2} I_0\left(\lambda\sqrt{y-w}\sqrt{t+w}\right) \mathrm{d}y$$
$$= e^{-\lambda t} I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right) + \int_0^{\lambda(t-w)} e^{-\lambda/2(t+w)-x/2} I_0\left(\sqrt{\lambda x}\sqrt{t+w}\right) \mathrm{d}x,$$

and the desired result immediately follows from recalling the definition of the Marcum Q-function. □

The result obtained in Proposition 7 is in agreement with that given in Cinque and Orsingher [6] (cf. Equation (2.26)), as proven in the following remark.

Remark 2. The survival function of $M(t)$, obtained by considering the distribution function given in Equation (2.26) of [6] for $c_1 = c_2 = 1$, can be identified with Equation (27). Indeed,

$$\mathbb{P}\left(M(t) > w\right) = 1 - e^{-\lambda t}\sum_{j=0}^{+\infty} I_{j+1}\left(\lambda\sqrt{(t-w)(t+w)}\right)\left[\left(\sqrt{\frac{t+w}{t-w}}\right)^{j+1} - \left(\sqrt{\frac{t-w}{t+w}}\right)^{j+1}\right]$$

$$= 1 - e^{-\lambda t}\sum_{m=1}^{+\infty}\frac{\left(\frac{\lambda}{2}\right)^m}{m!}(t+w)^m\sum_{j=0}^{m-1}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}(t-w)^j + e^{-\lambda t}\sum_{j=0}^{+\infty}\frac{\left(\frac{\lambda}{2}\right)^j}{j!}(t+w)^j\sum_{h=j+1}^{+\infty}\frac{\left(\frac{\lambda}{2}\right)^h}{h!}(t-w)^h$$

$$= 1 + e^{-\lambda t}\left(-1 + e^{\lambda/2(t-w)}\right) + e^{\lambda/2(t-w)} e^{-\lambda t}\sum_{m=1}^{+\infty}\frac{\left(\frac{\lambda}{2}\right)^m}{m!}(t+w)^m$$

$$\times\left[\frac{\gamma\left(m+1, \frac{\lambda}{2}(t-w)\right)}{\Gamma\left(m+1\right)} + \frac{\gamma\left(m, \frac{\lambda}{2}(t-w)\right)}{\Gamma\left(m\right)} - 1\right].$$

Hence, by Equation (30), the survival function can be expressed as

$$\mathbb{P}\left(M(t) > w\right) = -2e^{-\lambda t} + 2e^{-\frac{\lambda}{2}(t+w)} + e^{-\lambda t} I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right)$$
$$+ 2e^{-\frac{\lambda}{2}(t+w)}\sum_{m=1}^{+\infty}\frac{\left(\frac{\lambda}{2}(t+w)\right)^m}{m!}\gamma\left(m+1, \frac{\lambda}{2}(t-w)\right)$$
$$= e^{-\lambda t} I_0\left(\lambda\sqrt{t-w}\sqrt{t+w}\right) + 2\left[1 - Q_1\left(\sqrt{\lambda(t+w)}, \sqrt{\lambda(t-w)}\right)\right],$$

where the last equality holds by virtue of Equation (61.2.5) of Hansen [20].

Figure 4 shows the behavior of the distribution function $\mathbb{P}(M(t) \le w)$ for some choices of the parameters, obtained from (26) as the complementary function.

In the following proposition, we provide the expression for the expected value of $M(t)$.

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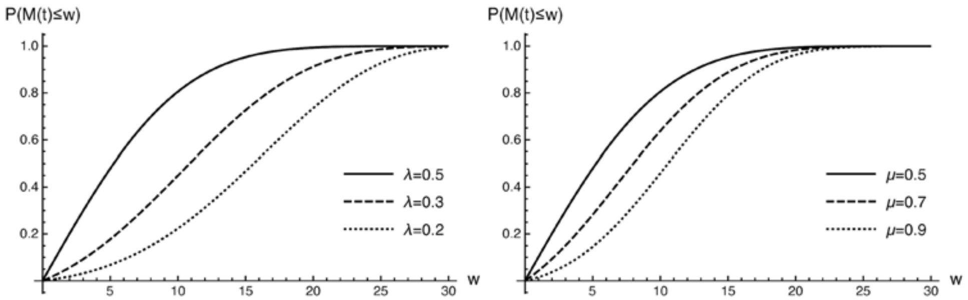

FIGURE 4. The distribution function $\mathbb{P}(M(t) \leq w)$, in the case of exponentially distributed downward times, when $t = 30$. Left: $\mu = 0.5$, $\lambda = 0.5$ (solid), $\lambda = 0.3$ (dashed), and $\lambda = 0.2$ (dotted). Right: $\lambda = 0.5$, $\mu = 0.5$ (solid), $\mu = 0.7$ (dashed), and $\mu = 0.9$ (dotted).

Proposition 8. *In the case of exponentially distributed downward times, the expected value of the supremum M(t) is given by*

$$\mathbb{E}\left(M(t)\right) = \frac{1 - e^{-\lambda t}}{\lambda} + \frac{\mu}{2\lambda} \sum_{h=0}^{+\infty} \frac{\left(\frac{\lambda\mu}{4}\right)^h}{h!(h+1)!} \sum_{r=0}^{h} \binom{h}{r} \left(\frac{2}{\lambda}\right)^r (r+1)!$$
$$\times \left[\left(\frac{2}{\lambda+\mu}\right)^{1+2h-r} \gamma\left(1+2h-r, \frac{(\lambda+\mu)t}{2}\right) - e^{-\lambda t} t^{1+2h-r}\right.$$
$$\left.\times \sum_{k=0}^{r+1} \frac{(\lambda t)^k}{k!} \beta\left(2h-r+1, k+1\right) {}_1F_1\left(1+2h-r; 2+2h+k-r; \frac{(\lambda-\mu)t}{2}\right)\right], \tag{31}$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function

$${}_1F_1(a, b; z) := \sum_{k=0}^{+\infty} \frac{(a)_k z^k}{(b)_k k!}, \tag{32}$$

$\gamma(s, x)$ is the lower incomplete gamma function, and $\beta(x, y)$ denotes the beta function.

The proof of Proposition 8 is given in Appendix B.

Remark 3. In the case $\lambda = \mu$, the expected value of $M(t)$ is given by

$$\mathbb{E}\left(M(t)\right) = e^{-\lambda t} t \left(I_0(\lambda t) + I_1(\lambda t)\right), \qquad t \geq 0. \tag{33}$$

The result follows from Equation (31) if we note that for $\mu = \lambda$,

$$\sum_{k=0}^{r+1} \frac{(\lambda t)^k}{k!} \beta\left(2h-r+1, k+1\right) {}_1F_1\left(1+2h-r; 2+2h+k-r; 0\right)$$
$$= \frac{(\lambda t)^{1-2(h+1)+r}\left[(2h-r)!\Gamma\left(3+h, \lambda t\right) + (2h+2)!\Gamma(1+2h-r, \lambda t)\right]}{(2h+2)!}.$$

We remark that Equation (33) coincides with Equation (5.10) of Cinque and Orsingher [5].

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4.2. Erlang distributed downward random times

Let us now consider the case $D \sim Erlang(\mu, k)$ with $k \in \mathbb{N}$, $\mu > 0$. In the sequel we shall assume $k/\mu < 1/\lambda$, which ensures the validity of the condition (6). The density of D is given by

$$g(t) = \frac{\mu^k t^{k-1} e^{-\mu t}}{(k-1)!}, \quad t \geq 0.$$

In the following proposition, we provide the expression for the density of the random variable C.

Proposition 9. *In the case of Erlang distributed downward random times, for any $t > 0$, the density of the constant phase C is given by*

$$f_C(t) = \frac{1}{2}\mu^k k \left(\frac{t}{2}\right)^{k-1} e^{-(\lambda+\mu)t/2} W_{k,k+1}\left[\lambda\mu^k \left(\frac{t}{2}\right)^{k+1}\right], \tag{34}$$

where

$$W_{\rho,\beta}(z) := \sum_{j=0}^{+\infty} \frac{z^j}{j!\Gamma(\rho j + \beta)}, \quad \rho > -1, \beta \in \mathbb{C}, \tag{35}$$

is the Wright function.

Proof. From Equation (3), the density of $Y(t)$ is

$$h(u, t) = \frac{e^{-\lambda t - \mu u}}{u} \sum_{n=1}^{+\infty} \frac{(\lambda t \mu^k u^k)^n}{n!(nk-1)!}, \quad 0 < u < t.$$

Hence, taking into account Equation (7), the probability density of T, for any $t \geq 0$, is

$$\begin{aligned}
f_T(t) &= \mathrm{e}^{-\lambda t} g(t) + \int_0^t \frac{x}{t} h(t-x, t) g(x) \mathrm{d}x \\
&= \mathrm{e}^{-\lambda t} g(t) + \int_0^t \frac{x}{t} \left(\frac{e^{-\lambda t - \mu(t-x)}}{t-x} \sum_{n=1}^{+\infty} \frac{(\lambda t \mu^k (t-x)^k)^n}{n!(nk-1)!} \right) \frac{\mu^k x^{k-1} e^{-\mu x}}{(k-1)!} \mathrm{d}x \\
&= \mathrm{e}^{-\lambda t} g(t) + \frac{\mu^k}{t(k-1)!} e^{-\lambda t - \mu t} \sum_{n=1}^{+\infty} \frac{(\lambda t \mu^k)^n}{n!(nk-1)!} \int_0^t x^k (t-x)^{kn-1} \mathrm{d}x \\
&= \mathrm{e}^{-\lambda t} g(t) + \frac{\mu^k}{t(k-1)!} e^{-\lambda t - \mu t} \sum_{n=1}^{+\infty} \frac{(\lambda t \mu^k)^n}{n!(nk-1)!} \cdot \frac{t^{k(n+1)} k!(nk-1)!}{(k+kn)!} \\
&= \mathrm{e}^{-\lambda t} g(t) + \mu^k k t^{k-1} e^{-t(\lambda+\mu)} \sum_{n=1}^{+\infty} \frac{(\lambda \mu^k t^{k+1})^n}{n!(k+kn)!}.
\end{aligned}$$

Finally, by Equation (10), we have for $t \geq 0$

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$$f_C(t) = \frac{1}{2}e^{-(\lambda+\mu)t/2}\frac{\mu^k (t/2)^{k-1}}{(k-1)!} + \frac{1}{2}\mu^k k\left(\frac{t}{2}\right)^{k-1} e^{-(\lambda+\mu)t/2}\sum_{n=1}^{+\infty}\frac{(\lambda\mu^k\,(t/2)^{k+1}\,)^n}{n!(k+kn)!}$$

$$= \frac{1}{2}\mu^k k\left(\frac{t}{2}\right)^{k-1} e^{-(\lambda+\mu)t/2}\sum_{n=0}^{+\infty}\frac{(\lambda\mu^k\,(t/2)^{k+1}\,)^n}{n!(k+kn)!},$$

so that the proof immediately follows from Equation (35). □

Remark 4. Note that for $k = 1$ Equation (34) becomes

$$f_C(t)\big|_{k=1} = \frac{1}{2}\mu e^{-(\lambda+\mu)t/2}\, W_{1,2}\left(\lambda\mu\left(\frac{t}{2}\right)^2\right) = \frac{1}{2}\mu e^{-(\lambda+\mu)t/2}\sum_{m=0}^{+\infty}\frac{\left(\lambda\mu\left(\frac{t}{2}\right)^2\right)^m}{m!(m+1)!}$$

$$= \mu e^{-(\lambda+\mu)t/2}\frac{I_1\left(\sqrt{\lambda\mu}t\right)}{\sqrt{\lambda\mu}t}, \qquad t \geq 0,$$

which is the density of C in the case of exponentially distributed downward random times.

Proposition 10. *In the case of Erlang distributed downward times, the upper bound U(t, w) provided in Proposition 1 has the following expression: for any $t > w$,*

$$U(t, w) = \exp\left\{-\lambda w\left[1 - k\left(\frac{\mu}{\lambda+\mu}\right)^k \sum_{j=0}^{+\infty}\frac{(\lambda\mu^k)^j\gamma\left(j+jk+k,\, \frac{(\lambda+\mu)(t-w)}{2}\right)}{j!(\lambda+\mu)^{j+jk}\Gamma\,(jk+k+1)}\right]\right\}, \tag{36}$$

where $\gamma(s, x)$ is the lower incomplete gamma function.

Proof. From Equation (34), the distribution function of C is given by

$$F_C(t) = \int_0^t f_C(s)\mathrm{d}s = k\left(\frac{\mu}{\lambda+\mu}\right)^k \sum_{j=0}^{+\infty}\frac{(\lambda\mu^k)^j\gamma\left(j+jk+k,\, \frac{(\lambda+\mu)t}{2}\right)}{j!(\lambda+\mu)^{j+jk}\Gamma\,(jk+k+1)}. \tag{37}$$

Hence, the result immediately follows from Proposition 1. □

In Figure 5 we provide plots of the lower bound $L(t, w)$ defined in Equation (16) with $F_C(w)$ as given in Equation (37) and U(t, w) as given in Equation (36), for different choices of parameters.

4.3. Weighted exponentially distributed downward random times

Let us assume that $D \sim WE(\alpha, \mu)$, with $\alpha, \mu > 0$, so that the density of D is given by

$$g(t) = \frac{\alpha+1}{\alpha}\mu e^{-\mu t}(1 - e^{-\alpha\mu t}), \qquad t > 0. \tag{38}$$

A random variable having such a density is said to have a weighted exponential distribution (see Gupta and Kundu [19] and Das and Kundu [8]). Note that the exponential distribution can be obtained from Equation (38) by letting $\alpha \to +\infty$. We assume $0 < \lambda \leq \frac{\mu}{2}$, $\alpha > 0$ (or

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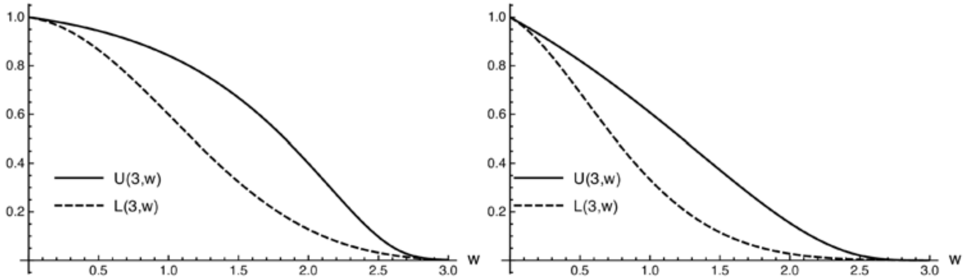

FIGURE 5. The lower bound $L(t, w)$ and the upper bound $U(t, w)$ of the survival function of $M(t)$, in the case of Erlang distributed downward times, for $\mu = 9$, $k = 2$, $t = 3$, $\lambda = 2$ (left), and $\lambda = 3$ (right).

$\frac{\mu}{2} < \lambda < \mu$, $\alpha > \frac{\mu - 2\lambda}{\lambda - \mu}$), so that the condition (6) is fulfilled. In the following proposition we obtain the expression for the density of the constant phase C.

Theorem 4. *In the case of weighted exponentially distributed downward times, the probability density function of the random variable C is given by*

$$f_C(t) = \frac{(1+\alpha)\mu}{2\alpha} e^{-\frac{(\lambda+\mu)t}{2} - \frac{\alpha\mu t}{4}} \sqrt{\pi\alpha\mu\frac{t}{2}} \sum_{n=0}^{+\infty} \frac{\left(\lambda\frac{t^2}{4}\right)^n}{n!(n+1)!} \left(\frac{(1+\alpha)\mu}{\alpha}\right)^n I_{n+1/2}\left(\frac{\alpha\mu t}{4}\right), \qquad t \geq 0. \tag{39}$$

Proof. From Condition 4 of [19], by considering two independent exponentially distributed random variables, i.e. $U \sim Exp(\mu)$ and $V \sim Exp(\mu(1+\alpha))$, we can easily see that

$$U + V \overset{d}{=} D \sim WE(\alpha, \mu).$$

Hence, if $\{D_1, \ldots, D_n\}$ is a collection of n independent random variables such that $D_i \sim WE(\alpha, \mu)$ for any $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} D_i \overset{d}{=} \widetilde{U} + \widetilde{V},$$

where $\widetilde{U} \sim Erlang(\mu(1+\alpha), n)$ and $\widetilde{V} \sim Erlang(\mu, n)$. Moreover, denoting by $f_{\widetilde{U}}(x)$ and $f_{\widetilde{V}}(x)$ the densities of $\widetilde{U}$ and $\widetilde{V}$ respectively, we have

$$\begin{aligned} g^{(n)}(t) &= \int_{-\infty}^{+\infty} f_{\widetilde{U}}(x) f_{\widetilde{V}}(t-x)\mathrm{d}x = \int_0^t \frac{\mu^n(1+\alpha)^n x^{n-1} e^{-\mu(1+\alpha)x}}{(n-1)!} \cdot \frac{\mu^n(t-x)^{n-1}e^{-\mu(t-x)}}{(n-1)!}\mathrm{d}x \\ &= \frac{\mu^{2n}(1+\alpha)^n}{[(n-1)!]^2} \int_0^t x^{n-1} e^{-\mu(1+\alpha)x}(t-x)^{n-1}e^{-\mu(t-x)}\mathrm{d}x \\ &= \left[\frac{\mu(1+\alpha)}{\alpha}\right]^n \frac{e^{-\frac{1}{2}(2+\alpha)\mu t}}{(n-1)!} \sqrt{\frac{\alpha\mu\pi}{t}} t^n I_{n-1/2}\left(\frac{\alpha\mu t}{2}\right). \end{aligned}$$

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Hence, from Equation (7), it follows that

$$f_T(t) = \mu e^{-(\lambda+\mu)t}\frac{\alpha+1}{\alpha}(1-e^{-\alpha\mu t}) - \frac{1}{t}\sum_{n=1}^{+\infty}\frac{(\lambda t)^n e^{-\lambda t}}{n!}\int_0^t e^{-\mu t-\alpha\mu t-1/2\alpha\mu x}\left(-1+e^{\alpha\mu(t-x)}\right)$$

$$\times\sqrt{\frac{\alpha\mu\pi}{x}}\left(\frac{(1+\alpha)\mu x}{\alpha}\right)^{n+1}(x-t)I_{n-1/2}\left(\frac{\alpha\mu x}{2}\right)\frac{1}{n!\,x}\mathrm{d}x$$

$$= \mu e^{-(\lambda+\mu)t}\frac{\alpha+1}{\alpha}(1-e^{-\alpha\mu t}) - \frac{e^{-\mu t-\lambda t-\alpha\mu t}}{t}\sqrt{\alpha\mu\pi}$$

$$\times\sum_{n=1}^{+\infty}\frac{(\lambda t)^n}{n!(n-1)!}\left(\frac{(1+\alpha)\mu}{\alpha}\right)^{n+1}\int_0^t e^{\frac{\alpha\mu x}{2}}x^{n-1/2}(t-x)I_{n-1/2}\left(\frac{\alpha\mu x}{2}\right)\mathrm{d}x$$

$$+\frac{e^{-\mu t-\lambda t}}{t}\sqrt{\alpha\mu\pi}\sum_{n=1}^{+\infty}\frac{(\lambda t)^n}{n!(n-1)!}\left(\frac{(1+\alpha)\mu}{\alpha}\right)^{n+1}\int_0^t e^{-\frac{\alpha\mu x}{2}}x^{n-1/2}(t-x)I_{n-1/2}\left(\frac{\alpha\mu x}{2}\right)\mathrm{d}x$$

$$= \mu e^{-(\lambda+\mu)t}\frac{\alpha+1}{\alpha}(1-e^{-\alpha\mu t}) + e^{-\mu t-\lambda t-\frac{\alpha\mu t}{2}}\sqrt{\alpha\mu\pi t}\sum_{n=1}^{+\infty}\frac{(\lambda t^2)^n}{n!(n+1)!}\left(\frac{(1+\alpha)\mu}{\alpha}\right)^{n+1}$$

$$\times I_{n+1/2}\left(\frac{\alpha\mu t}{2}\right) = e^{-\mu t-\lambda t-\frac{\alpha\mu t}{2}}\sqrt{\alpha\mu\pi t}\sum_{n=0}^{+\infty}\frac{(\lambda t^2)^n}{n!(n+1)!}\left(\frac{(1+\alpha)\mu}{\alpha}\right)^{n+1}I_{n+1/2}\left(\frac{\alpha\mu t}{2}\right).$$

Finally, the result follows from taking into account Equation (10). □

Remark 5. For $\alpha \to +\infty$, since (see, for instance, Equation (9.7.1) of [1])

$$I_{n+1/2}\left(\frac{\alpha\mu t}{4}\right) \propto \frac{e^{\alpha\mu t/4}}{\sqrt{\frac{\pi\alpha\mu t}{2}}},$$

we have that

$$\lim_{\alpha\to+\infty} f_C(t) = \frac{1}{2}\mu e^{-(\lambda+\mu)t/2}\frac{I_1\left(t\sqrt{\lambda\mu}\right)}{\sqrt{\lambda\mu}t}, \qquad t \geq 0,$$

which is identical to the density function of C in the case of exponentially distributed downward random times.

The following proposition provides the expression for the upper bound for the survival function of $M(t)$.

Proposition 11. *In the case of weighted exponentially distributed downward times, the upper bound U(t, w) (see Proposition 1) has the following expression:*

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$$U(t,w) = \exp\left\{-\lambda w\left[1 - \frac{(1+\alpha)\mu}{2\alpha}\frac{\sqrt{2\pi}\alpha\mu}{(\lambda+\mu+\alpha\mu)^2}\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\left(\frac{(1+\alpha)\mu\lambda}{\alpha}\right)^n\right.\right.$$
$$\times\frac{1}{(\lambda+\mu+\alpha\mu)^{3n}}\left(\frac{\alpha\mu}{2}\right)^n\sum_{k=0}^{+\infty}\left(\frac{\alpha\mu/2}{\lambda+\mu+\alpha\mu}\right)^{2k}\frac{1}{\Gamma\,(k+n+3/2)\,k!} \tag{40}$$
$$\left.\left.\times\gamma\left(2+2k+3n,\,(\lambda+\alpha\mu+\mu)\frac{t-w}{2}\right)\right]\right\},\qquad t>w>0,$$

where $\gamma(s,x)$ is the lower incomplete gamma function.

Proof. Recalling Equation (24), from Equation (39) we have that the distribution of C, for $t>w>0$, is given by

$$F_C(w) = \frac{(1+\alpha)\mu}{2\alpha}\sqrt{\pi\alpha\mu}\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\left(\frac{(1+\alpha)\mu\lambda}{\alpha}\right)^n$$
$$\times\int_0^w\exp\left(-\frac{(\lambda+\mu)t}{2}-\frac{\alpha\mu t}{4}\right)\left(\frac{t}{2}\right)^{2n+1/2}I_{n+1/2}\left(\frac{\alpha\mu t}{2}\right)\mathrm{d}t$$
$$=\frac{(1+\alpha)\mu}{2\alpha}\sqrt{\pi\alpha\mu}\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\left(\frac{(1+\alpha)\mu\lambda}{\alpha}\right)^n$$
$$\times\sum_{k=0}^{+\infty}\frac{1}{\Gamma(k+n+3/2)k!}\int_0^w\exp\left(-\frac{(\lambda+\mu)t}{2}-\frac{\alpha\mu t}{4}\right)\left(\frac{t}{2}\right)^{2n+1/2}\left(\frac{\alpha\mu t}{4}\right)^{2k+n+1/2}\mathrm{d}t$$
$$=\frac{(1+\alpha)\mu}{2\alpha}\sqrt{\pi\alpha\mu}\sum_{n=0}^{+\infty}\frac{1}{n!(n+1)!}\left(\frac{(1+\alpha)\mu\lambda}{\alpha}\right)^n$$
$$\times\sum_{k=0}^{+\infty}\frac{2^{1/2-2k-n}}{\Gamma\,(k+n+3/2)\,k!}\cdot\frac{(\alpha\mu)^{1/2+2k+n}}{(\lambda+\mu+\alpha\mu)^{2+2k+3n}}\,\gamma\left(2+2k+3n,\,(\lambda+\alpha\mu+\mu)\frac{w}{2}\right). \tag{41}$$

Hence, the result follows from Proposition 1. □

Figure 6 shows plots of the lower bound $L(t,w)$ defined in Equation (16) with $F_C(w)$ as given in Equation (41) and the upper bound $U(t,w)$ given in Equation (40), in the case of weighted exponentially distributed downward times, for various choices of the parameters.

4.4. Mixed exponential downward random times

This section is devoted to the case in which the downward time D is distributed as a mixture of two exponential distributions, so that the density of D is given by

$$f_D(t) = b_1\mu_1 e^{-\mu_1 t} + b_2\mu_2 e^{-\mu_2 t},\qquad t\geq 0, \tag{42}$$

with $b_i\geq 0$, for $i=1,2$ and $b_1+b_2=1$. Throughout the section we assume that

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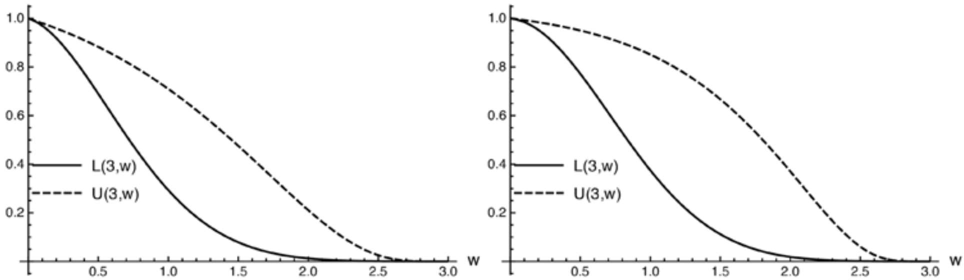

FIGURE 6. The lower bound $L(t, w)$ and the upper bound $U(t, w)$ of the survival function of $M(t)$, in the case of weighted exponentially distributed downward times, for $\alpha = 1$, $\lambda = 4$, $t = 3$, $\mu = 10$ (left), and $\mu = 12$ (right).

$$\frac{b1}{\mu_1} + \frac{b_2}{\mu_2} < \frac{1}{\lambda},$$

so that the condition (6) is fulfilled.

The following theorem provides the expression for the density of the constant phase C.

Theorem 5. *In the case of mixed exponential downward times (see Equation (42)), for $t \geq 0$, the probability density function of the constant phase C is given by*

$$f_C(t) = \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{\lambda t^2}{4}\right)^n \frac{e^{-\lambda t/2}}{(n+1)!}\sum_{k=0}^{n}\frac{\binom{n}{k}}{n!}b_2^n\mu_2^n\left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k$$
$$\times\left[b_1\mu_1 e^{-\mu_1 t/2}{}_1F_1\left(n-k, n+2, (\mu_1-\mu_2)\frac{t}{2}\right) + b_2\mu_2 e^{-\mu_2 t/2}{}_1F_1\left(k, n+2, (\mu_2-\mu_1)\frac{t}{2}\right)\right], \tag{43}$$

where ${}_1F_1(a, b; z)$ is as defined in Equation (32).

Proof. Let us start with the computation of the n-fold convolution of f_D. Setting $f_i(t) := \mu_i e^{-\mu_i t}$ for any $i = 1, 2$, we have

$$f_D^{(n)}(t) = \sum_{k=0}^{n}\binom{n}{k}b_1^k b_2^{n-k} f_1^{(k)}(t) \star f_2^{(n-k)}(t), \qquad t \geq 0,$$

where $f \star g$ denotes the convolution of f and g. In particular, we have that

$$f_1^{(k)}(t)\star f_2^{(n-k)}(t) = \int_{-\infty}^{+\infty} f_1^{(k)}(x)f_2^{(n-k)}(t-x)\mathrm{d}x = \frac{\mu_1^k\mu_2^{n-k}e^{-\mu_2 t}}{(k-1)!(n-k-1)!}$$
$$\times\int_0^t x^{k-1}(t-x)^{n-k-1}e^{-\mu_1 x}e^{\mu_2 x}\mathrm{d}x = \frac{\mu_1^k\mu_2^{n-k}e^{-\mu_2 t}t^{n-1}{}_1F_1(k, n, (\mu_2-\mu_1)t)}{(n-1)!}.$$

Hence, $f_D^{(n)}(t)$ can be expressed as follows:

$$f_D^{(n)}(t) = \sum_{k=0}^{n}\binom{n}{k}b_1^k b_2^{n-k}\mu_1^k\mu_2^{n-k}e^{-\mu_2 t}t^{n-1}\frac{{}_1F_1(k, n, (\mu_2-\mu_1)t)}{(n-1)!}.$$

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From Equation (7), we have

$$f_T(t) = \frac{1}{t}\sum_{n=0}^{+\infty} p(n, \lambda t)\int_0^t x f_D^{(n)}(t-x) f_D(x)\mathrm{d}x = \frac{1}{t}\sum_{n=0}^{+\infty} p(n, \lambda t)\sum_{k=0}^{n}\frac{\binom{n}{k}}{(n-1)!} b_2^n \mu_2^n \left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k$$
$$\times \left[b_1\mu_1 \int_0^t x e^{-\mu_2(t-x)}(t-x)^{n-1} e^{-\mu_1 x} {}_1F_1(k, n, (\mu_2-\mu_1)(t-x))\mathrm{d}x \right.$$
$$\left. + \, b_2\mu_2 \int_0^t x e^{-\mu_2(t-x)}(t-x)^{n-1} e^{-\mu_2 x} {}_1F_1(k, n, (\mu_2-\mu_1)(t-x))\mathrm{d}x \right].$$

It is not hard to show that

$$\int_0^t x e^{-\mu_2(t-x)}(t-x)^{n-1} e^{-\mu_1 x} {}_1F_1(k, n, (\mu_2-\mu_1)(t-x))\mathrm{d}x$$
$$= e^{-\mu_1 t}\int_0^t x(t-x)^{n-1} {}_1F_1(n-k, n, (\mu_1-\mu_2)(t-x))\mathrm{d}x$$
$$= e^{-\mu_1 t}\frac{t^{n+1}}{n(n+1)} {}_1F_1(n-k, n+2, (\mu_1-\mu_2)t),$$

and

$$\int_0^t x e^{-\mu_2(t-x)}(t-x)^{n-1} e^{-\mu_2 x} {}_1F_1(k, n, (\mu_2-\mu_1)(t-x))\mathrm{d}x$$
$$= e^{-\mu_2 t}\int_0^t x(t-x)^{n-1} {}_1F_1(k, n, (\mu_2-\mu_1)(t-x))\mathrm{d}x$$
$$= e^{-\mu_2 t}\frac{t^{n+1}}{n(n+1)} {}_1F_1(k, n+2, (\mu_2-\mu_1)t).$$

Hence, Equation (43) follows from Equation (5). □

Remark 6. For $b_1 = b_2 = \frac{1}{2}$ and $\mu_1 = \mu_2 = \mu$, since ${}_1F_1(n-k, n+2, 0) = 1$, we have that

$$f_C(t)\Big|_{b_1=b_2=\frac{1}{2},\, \mu_1=\mu_2=\mu} = \frac{\mu e^{-(\lambda+\mu)t/2}}{2}\sum_{n=0}^{+\infty}\left(\frac{\lambda t^2}{4}\right)^n \frac{\left(\frac{\mu}{2}\right)^n}{n!(n+1)!}\sum_{k=0}^{n}\binom{n}{k}$$
$$= \frac{\mu e^{-(\lambda+\mu)t/2} I_1\left(\sqrt{\lambda\mu}\, t\right)}{\sqrt{\lambda\mu}\, t}, \qquad t \geq 0,$$

which is the same as the corresponding result in the case of exponentially distributed downward random times.

Proposition 12. *In the case of mixed exponential downward times (Equation (42)), the upper bound U(t, w) given in Proposition 1 has the following expression:*

$$U(t, w) = \exp\left[-\lambda w\left(1 - F_C(t-w)\right)\right], \qquad t > w > 0, \tag{44}$$

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where

$$F_C(w)=b_1\mu_1\sum_{n=0}^{+\infty}\frac{(b_2\mu_2\lambda)^n}{n!(n+1)!}\left[\sum_{m=0}^{+\infty}\frac{(m+n-1)!\,{}_2F_1\left(1-n,\,-n;1-m-n;-\frac{b_1\mu_1}{b_2\mu_2}\right)}{(n-1)!(n+2)_m m!(\lambda+\mu_1)^{m+2n+1}}\right.$$
$$\times(\mu_1-\mu_2)^m\,\gamma\left(1+m+2n,\,\frac{(\lambda+\mu_1)w}{2}\right)+n\sum_{m=0}^{+\infty}\frac{m!{}_2F_1\left(m+1,\,1-n;\,2;\,-\frac{b_1\mu_1}{b_2\mu_2}\right)}{(n+2)_m m!(\lambda+\mu_2)^{m+2n+1}}$$
$$\left.\times(\mu_1-\mu_2)^m\,\gamma\left(1+m+2n,\,\frac{(\lambda+\mu_2)w}{2}\right)\right], \tag{45}$$

$\gamma(s,x)$ is the lower incomplete gamma function, and ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric function, defined as

$${}_2F_1(a,\,b,\,c;z):=\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k z^k}{(c)_k k!}. \tag{46}$$

Proof. From Equation (43), the distribution of C is given by

$$F_C(w)=\frac{1}{2}\sum_{n=0}^{+\infty}\frac{b_2^n\mu_2^n}{n!(n+1)!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k$$
$$\times\left[b_1\mu_1\int_0^w e^{-(\lambda+\mu_1)t/2}{}_1F_1\left(n-k,\,n+2,\,\frac{(\mu_1-\mu_2)t}{2}\right)\left(\frac{\lambda t^2}{4}\right)^n\mathrm{d}t\right.$$
$$\left.+\,b_2\mu_2\int_0^w e^{-(\lambda+\mu_2)t/2}{}_1F_1\left(k,\,n+2,\,\frac{(\mu_1-\mu_2)t}{2}\right)\left(\frac{\lambda t^2}{4}\right)^n\mathrm{d}t\right].$$

Recalling Equation (32), one has

$$\int_0^w e^{-(\lambda+\mu_1)t/2}{}_1F_1\left(n-k,\,n+2,\,\frac{(\mu_1-\mu_2)t}{2}\right)\left(\frac{\lambda t^2}{4}\right)^n\mathrm{d}t$$
$$=\sum_{m=0}^{+\infty}\frac{(n-k)_m}{(n+2)_m m!}\int_0^w e^{-(\lambda+\mu_1)t/2}\left(\frac{(\mu_1-\mu_2)t}{2}\right)^m\left(\frac{\lambda t^2}{4}\right)^n\mathrm{d}t$$
$$=\sum_{m=0}^{+\infty}\frac{(n-k)_m}{(n+2)_m m!}2\lambda^n\frac{(\mu_1-\mu_2)^m}{(\lambda+\mu_1)^{1+m+2n}}\gamma\left(1+m+2n,\,\frac{(\lambda+\mu_1)w}{2}\right).$$

Similarly, we have

$$\int_0^w e^{-(\lambda+\mu_2)t/2}{}_1F_1\left(k,\,n+2,\,\frac{(\mu_1-\mu_2)t}{2}\right)\left(\frac{\lambda t^2}{4}\right)^n\mathrm{d}t$$
$$=\sum_{m=0}^{+\infty}\frac{(k)_m}{(n+2)_m m!}2\lambda^n\frac{(\mu_1-\mu_2)^m}{(\lambda+\mu_2)^{1+m+2n}}\gamma\left(1+m+2n,\,\frac{(\lambda+\mu_2)w}{2}\right).$$

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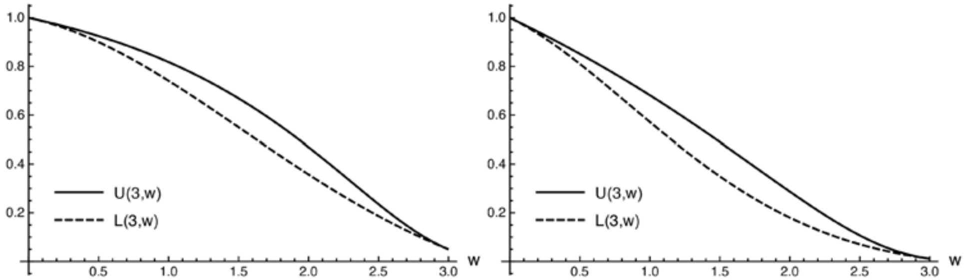

FIGURE 7. The lower bound $L(t, w)$ and the upper bound $U(t, w)$ of the survival function of $M(t)$, in the case of mixed exponential downward times, for $b_1 = b_2 = 0.5$, $t = 3$, $\mu_1 = 2$, $\mu_2 = 3$, $\lambda = 1$ (left), and $\lambda = 1.5$ (right).

The result immediately follows from Proposition 1 and Theorem 5 if we consider that

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k (n-k)_m = \frac{(m+n-1)!\, {}_2F_1\left(1-n, -n; 1-m-n; -\frac{b_1\mu_1}{b_2\mu_2}\right)}{(n-1)!},$$

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k (k)_m = \frac{nb_1\mu_1\, m!\, {}_2F_1\left(1+m, 1-n; 2; -\frac{b_1\mu_1}{b_2\mu_2}\right)}{b_2\mu_2}.$$

□

In Figure 7, we provide plots of the lower bound $L(t, w)$ defined in Equation (16) with $F_C(w)$ as given in Equation (45) and the upper bound $U(t, w)$ given in Equation (44), for different choices of the parameters.

5. Some results on the moments of the first passage time

This section presents some results concerning the moments of the first passage time, defined in Equation (17), under different choices for the distribution of the downward random times.

5.1. Exponentially distributed downward random times

Let us assume that the random variables D_i, $i \in \mathbb{N}$, are exponentially distributed with parameter $\mu > \lambda > 0$. We collect some results concerning the first passage time of $X(t)$ through a fixed level $w > 0$.

Theorem 6. *In the case of exponentially distributed downward times, the expected value of the first passage time Θ_w, defined in Equation (17), is given by*

$$\mathbb{E}(\Theta_w) = \frac{2\lambda}{\mu - \lambda} w, \qquad w \geq 0, \tag{47}$$

for $\mu > \lambda > 0$.

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Proof. Recalling Equation (23), we have

$$\psi_C(-s) = \frac{1}{\lambda}\left[\frac{\lambda+\mu}{2} + s - \sqrt{-\lambda\mu + \frac{(\lambda+\mu+2s)^2}{4}}\right],$$

and

$$\psi_Z(w, -s) = \exp\left[-\frac{w}{2}\left(\lambda - \mu - 2s + \sqrt{(\lambda-\mu)^2 + 4(\lambda+\mu)s + 4s^2}\right)\right]. \tag{48}$$

Hence, for $\lambda < \mu$, we have

$$\mathbb{E}(Z(w)) = -\left.\frac{\mathrm{d}}{\mathrm{d}s}\psi_Z(w, -s)\right|_{s=0} = \frac{2\lambda}{\mu - \lambda} w,$$

so that the proof follows from recalling Equation (18). □

Remark 7. Note that for $\lambda \to \mu^-$,

$$\lim_{\lambda \to \mu^-} \mathbb{E}(\Theta_w) = +\infty.$$

This result is in agreement with the one obtained by Orsingher [30] in the case of a standard symmetric telegraph process.

Theorem 7. *In the case of exponentially distributed downward times, the second-order moment of the first passage time Θ_w, defined in Equation (17), is given by*

$$\mathbb{E}(\Theta_w^2) = \frac{4\lambda\mu w\,[2 + (\mu-\lambda)w]}{(\mu-\lambda)^3}, \qquad w \geq 0,$$

whereas for the variance of Θ_w we have

$$Var(\Theta_w) = \frac{4\lambda w\left[2\mu + (\mu^2 - \lambda^2)w\right]}{(\mu-\lambda)^3}, \qquad w \geq 0. \tag{49}$$

Proof. Recalling Equations (22) and (48), we have

$$\mathbb{E}(\Theta_w^2) = 2w\left.\frac{\mathrm{d}}{\mathrm{d}s}\psi_Z(w, s)\right|_{s=0} + \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\psi_Z(w, s)\right|_{s=0} = \frac{4\lambda\mu w\,[2 + (\mu-\lambda)w]}{(\mu-\lambda)^3}.$$

Hence, Equation (49) follows from recalling Equation (47). □

In Figure 8, we provide plots of the expected value (47) and the variance (49) of the first passage time Θ_w for suitable choices of the parameters.

5.2. Erlang distributed downward random times

Let us now consider the case $D \sim Erlang(\mu, k)$ with $k \in \mathbb{N}$, $\mu > 0$, and $k/\mu < 1/\lambda$. In the following propositions we provide explicit expressions for the moment generating function of the constant phase C and of the compound process $Z(w)$.

Proposition 13. *In the case of $Erlang(\mu, k)$ distributed downward random times, the moment generating function of C is given by*

$$\psi_C(s) = \frac{\mu^k k}{(\lambda+\mu-2s)^k}\, {}_1\Psi_1\left[\begin{matrix}(k, k+1) \\ (k+1, k)\end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right], \tag{50}$$

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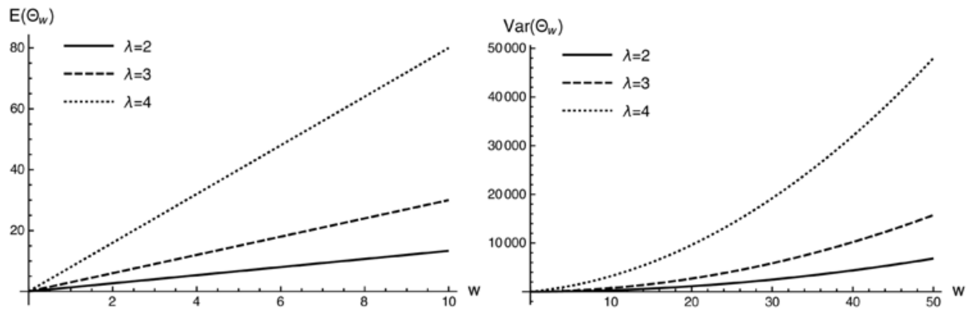

FIGURE 8. The expected value (47) (left) and the variance (49) (right) of the first passage time Θ_w for $\mu = 5$.

where

$$
{}_1\Psi_1\left[\begin{matrix}(a_1, A_1)\\(b_1, B_1)\end{matrix}; z\right] := \sum_{n=0}^{+\infty} \frac{\Gamma(a_1 + A_1 n)}{\Gamma(b_1 + B_1 n)} \cdot \frac{z^n}{n!}
$$

denotes the Fox–Wright function.

Proof. We have

$$
\psi_C(s) = \frac{1}{2}\mu^k k \sum_{n=0}^{+\infty} \frac{(\lambda\mu^k)^n}{n!(k+kn)!} \int_0^{+\infty} e^{st-(\lambda+\mu)t/2} \left(\frac{t}{2}\right)^{(k+1)n+k-1} \mathrm{d}t. \tag{51}
$$

Taking into account that

$$
\int_0^{+\infty} e^{st-(\lambda+\mu)t/2} \left(\frac{t}{2}\right)^{(k+1)n+k-1} \mathrm{d}t = \frac{2(k+n+kn-1)!}{(\lambda+\mu-2s)^{n+k(n+1)}},
$$

from Equation (51) we have

$$
\begin{aligned}
\psi_C(s) &= \frac{\mu^k k}{(\lambda+\mu-2s)^k} \sum_{n=0}^{+\infty} \left(\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right)^n \frac{(k+n+kn-1)!}{n!(k+kn)!} \\
&= \frac{\mu^k k}{(\lambda+\mu-2s)^k} {}_1\Psi_1\left[\begin{matrix}(k, k+1)\\(k+1, k)\end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right].
\end{aligned}
$$

□

Proposition 14. *In the case of Erlang(μ, k) distributed downward random times, the moment generating function of Z(t) is given by*

$$
\psi_Z(t, s) = \exp\left[-\lambda t\left(1 - \frac{\mu^k k}{(\lambda+\mu-2s)^k} {}_1\Psi_1\left[\begin{matrix}(k, k+1)\\(k+1, k)\end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\right)\right]. \tag{52}
$$

Proof. The result follows from Equations (12) and (50), taking into account that the random variables C_j are i.i.d. with distribution as given in Equation (34). □

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The expected value and the variance of the process $Z(t)$ are given in the following proposition.

Proposition 15. *In the case of Erlang*(μ, k) *distributed downward random times, for* $t > 0$, *the expected value of the process* $Z(t)$ *is given by*

$$\mathbb{E}(Z(t)) = \frac{\lambda t \mu^k}{(\lambda+\mu)^{k+1}} \left\{ 2k \left[k\, {}_1\Psi_1 \left[\begin{matrix} (k, k+1) \\ (k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right. \right.$$
$$\left. \left. + \lambda(k+1)\, {}_1\Psi_1 \left[\begin{matrix} (2k+1, k+1) \\ (2k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right] \right\}, \tag{53}$$

and the variance is given by

$$Var(Z(t)) = \lambda t \frac{4k\mu^k(k+1)}{(\lambda+\mu)^{k+2}} \left\{ k\, {}_1\Psi_1 \left[\begin{matrix} (k, k+1) \\ (k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right.$$
$$+ \frac{\lambda\mu^k(3k+2)}{(\lambda+\mu)^{k+1}}\, {}_1\Psi_1 \left[\begin{matrix} (2k+1, k+1) \\ (2k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right]$$
$$\left. + \frac{\lambda^2\mu^{2k}(k+1)}{(\lambda+\mu)^{2k+2}}\, {}_1\Psi_1 \left[\begin{matrix} (3k+2, k+1) \\ (3k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right\}. \tag{54}$$

The proof of Proposition 15 is given in Appendix C.

Finally, we provide some results concerning the expected value of the first passage time.

Proposition 16. *In the case of Erlang*(μ, k) *distributed downward times, the expected value of the first passage time* Θ_w *is given by*

$$\mathbb{E}(\Theta_w) = \lambda w \left\{ \frac{2\mu^k k}{(\lambda+\mu)^{k+1}} \left[k\, {}_1\Psi_1 \left[\begin{matrix} (k, k+1) \\ (k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right. \right.$$
$$\left. \left. + \frac{\lambda\mu^k(k+1)}{(\lambda+\mu)^{k+1}}\, {}_1\Psi_1 \left[\begin{matrix} (2k+1, k+1) \\ (2k+1, k) \end{matrix}; \frac{\lambda\mu^k}{(\lambda+\mu)^{k+1}} \right] \right] \right\}.$$

Proof. The result follows immediately from Proposition 2 and Equation (53). □

Remark 8. When $k = 1$, we have

$$\mathbb{E}(\Theta_w)\Big|_{k=1} = \lambda w \left\{ \frac{2\mu}{(\lambda+\mu)^2} \left[{}_1\Psi_1 \left[\begin{matrix} (1, 2) \\ (2, 1) \end{matrix}; \frac{\lambda\mu}{(\lambda+\mu)^2} \right] + \frac{2\lambda\mu}{(\lambda+\mu)^2}\, {}_1\Psi_1 \left[\begin{matrix} (3, 2) \\ (3, 1) \end{matrix}; \frac{\lambda\mu}{(\lambda+\mu)^2} \right] \right] \right\}$$
$$= \lambda w \left[\frac{2\mu}{(\lambda+\mu)^2} \left(\frac{\lambda+\mu}{\mu} + \frac{2\lambda(\lambda+\mu)}{\mu(\mu-\lambda)} \right) \right] = \frac{2\lambda}{\mu-\lambda} w,$$

which is the same as the expected value in the case of exponential downward random times.

Remark 9. From Equation (53), we can see that the mean of Θ_w depends linearly on w. Moreover, when $\lambda \to \mu^-$, as in the exponential case, one has

$$\mathbb{E}(\Theta_w) \to +\infty.$$

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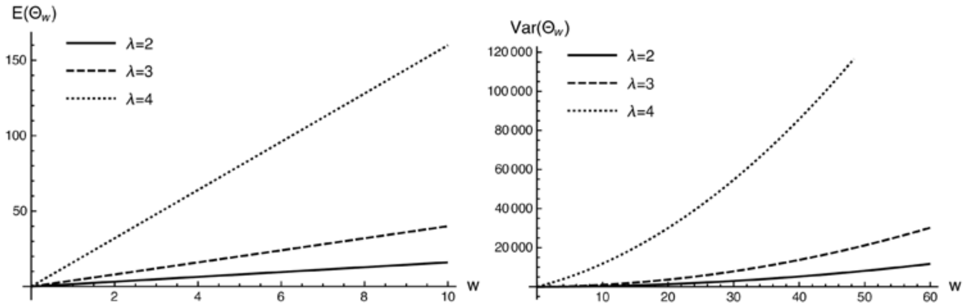

FIGURE 9. The expected value obtained in Proposition 16 (left) and the variance (right) of the first passage time Θ_w for $\mu = 9$ and $k = 2$.

From Equations (53)–(54) and Equation (22), it is possible to get an explicit formula for the second-order moment of the first passage time Θ_w. The resulting expression is cumbersome, so for brevity we omit it.

In Figure 9, we provide plots of the expected value and the variance of the first passage time Θ_w for suitable choices of the parameters.

5.3. Weighted exponentially distributed downward random times

Let us assume that $D \sim WE(\alpha, \mu)$ (see Equation (38)), with $0 < \lambda \le \frac{\mu}{2}$, $\alpha > 0$ (or $\frac{\mu}{2} < \lambda < \mu$, $\alpha > \frac{\mu - 2\lambda}{\lambda - \mu}$).

Proposition 17. *In the case of weighted exponentially distributed downward times, the moment generating function of C is given by*

$$\psi_C(s) = \frac{4(1+\alpha)\mu^2}{(2\lambda + (2+\alpha)\mu - 4s)^2} \sum_{n=0}^{+\infty} \left(\frac{8(1+\alpha)\lambda\mu^2}{(2\lambda + (2+\alpha)\mu - 4s)^3} \right)^n \frac{(3n+1)!}{(n+1)!(2n+1)!} \times {}_2F_1\left(\frac{3(1+n)}{2}, 1 + \frac{3n}{2}, \frac{3}{2} + n, \frac{\alpha^2\mu^2}{(2\lambda + (2+\alpha)\mu - 4s)^2} \right), \tag{55}$$

where ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function (see Equation (46)).

Proof. Since

$$\int_0^{+\infty} e^{-\left(\frac{\lambda+\mu}{2} + \frac{\alpha\mu}{4} - s\right)t} t^{1/2+2n} I_{n+1/2}\left(\frac{\alpha\mu t}{2}\right) \mathrm{d}t = \frac{2^{5/2+3n}(\alpha\mu)^{n+1/2}}{(2\lambda + (2+\alpha)\mu - 4s)^{2+3n}} \frac{\Gamma(3n+2)}{\Gamma\left(\frac{3}{2}+n\right)} \times {}_2F_1\left(\frac{3(1+n)}{2}, 1 + \frac{3}{2}n, \frac{3}{2} + n, \frac{\alpha^2\mu^2}{(2\lambda + (2+\alpha)\mu - 4s)^2} \right),$$

the proof immediately follows from Equation (13). □

In the following proposition, we provide the expected value of the first passage time Θ_w.

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Proposition 18. *In the case of weighted exponentially distributed downward times, the expected value of the first passage time* Θ_w *is given by*

$$\mathbb{E}(\Theta_w) = \lambda w \Bigg\{ \frac{16(1+\alpha)\mu^2}{(2\lambda+(2+\alpha)\mu)^5} \sum_{n=0}^{+\infty} \left(\frac{8(1+\alpha)\lambda\mu^2}{(2\lambda+(2+\alpha)\mu)^3} \right)^n \frac{(2+3n)!}{n!(n+1)!}$$
$$\Bigg[{}_2F_1 \left(\frac{3(1+n)}{2}, 1+\frac{3}{2}n, \frac{3}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right)$$
$$+ \frac{3\alpha^2\mu^2(1+n)}{(3+2n)(2\lambda+(2+\alpha)\mu)^2} {}_2F_1 \left(\frac{3}{2}n+2, \frac{5+3n}{2}, \frac{5}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right) \Bigg] \Bigg\}, \tag{56}$$

where ${}_2F_1(a_1, a_2; b; z)$ *is as defined in Equation (46).*

Proof. Recalling Equation (55), by evaluating the first derivative with respect to s of $\psi_C(s)$ at $s = 0$, we get the expected value of C, which is given by

$$\mathbb{E}(C) = \frac{\mathrm{d}}{\mathrm{d}s} \psi_C(s) \bigg|_{s=0} = \frac{16(1+\alpha)\mu^2}{(2\lambda+(2+\alpha)\mu)^5} \sum_{n=0}^{+\infty} \left(\frac{8(1+\alpha)\lambda\mu^2}{(2\lambda+(2+\alpha)\mu)^3} \right)^n \frac{(2+3n)!}{n!(n+1)!}$$
$$\Bigg[{}_2F_1 \left(\frac{3(1+n)}{2}, 1+\frac{3}{2}n, \frac{3}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right)$$
$$+ \frac{3\alpha^2\mu^2(1+n)}{(3+2n)(2\lambda+(2+\alpha)\mu)^2} {}_2F_1 \left(\frac{3}{2}n+2, \frac{5+3n}{2}, \frac{5}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right) \Bigg].$$

Hence, the result follows easily from Equation (18). □

Remark 10. For $\alpha \to +\infty$, since

$${}_2F_1 \left(\frac{3}{2}+\frac{3}{2}n, 1+\frac{3}{2}n, \frac{3}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right) \propto \frac{(2n)!\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}n\right)\Gamma\left(1+\frac{3}{2}n\right)}$$
$$\times \left(\frac{(2\lambda+(2+\alpha)\mu)^2-\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2} \right)^{-2n-1}$$

(see, for instance, Equation (9) of [16]), we have

$$\lim_{\alpha\to+\infty} \psi_C(s) = -\frac{\lambda+\mu-2s}{2\lambda} \left(-1 + \sqrt{1 - \frac{4\lambda\mu}{(\lambda+\mu-2s)^2}} \right).$$

Hence

$$\lim_{\alpha\to+\infty} \mathbb{E}(\Theta_w) = \frac{2}{\mu-\lambda}\lambda,$$

which corresponds to the expected value of Θ_w in the case of exponentially distributed downward random times.

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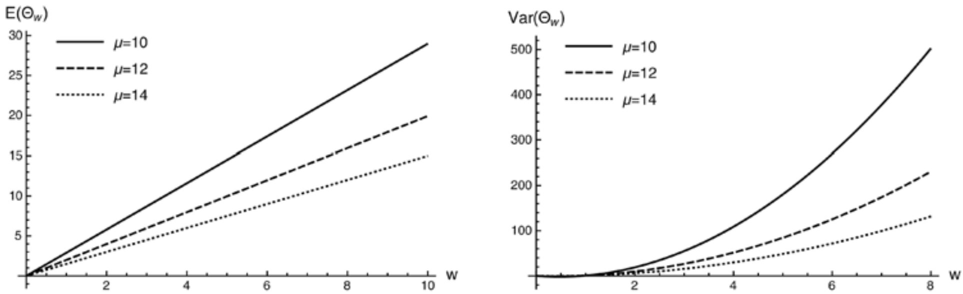

FIGURE 10. The expected value (56) (left) and the variance (right) of Θ_w (obtained from Remark 11) for $\alpha = 1$ and $\lambda = 4$.

Remark 11. From Equation (55) it is possible to get an expression for the second-order moment of Θ_w. In particular, after some calculations we have that

$$\mathbb{E}(C^2) = \frac{d^2}{ds^2}\psi_C(s)\Big|_{s=0} = \frac{8(1+\alpha)\mu^2\sqrt{\pi}}{(2\lambda+(2+\alpha)\mu)^8}\sum_{n=0}^{+\infty}\left(\frac{(1+\alpha)\lambda\mu^2}{2(2\lambda+(2+\alpha)\mu)^3}\right)^n \frac{(3n+3)!}{n!(n+1)!}$$

$$\times\left[\frac{4(2\lambda+(2+\alpha)\mu)^4}{\Gamma\left(\frac{3}{2}+n\right)}{}_2F_1\left(\frac{3(1+n)}{2}, 1+\frac{3}{2}n, \frac{3}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2}\right)\right.$$

$$+\frac{\alpha^4\mu^4(4+3n)(5+3n)}{\Gamma\left(\frac{7}{2}+n\right)}{}_2F_1\left(\frac{3(2+n)}{2}, \frac{7}{2}+\frac{3}{2}n, \frac{7}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2}\right)$$

$$\left.+\frac{2\alpha^2\mu^2(2\lambda+(2+\alpha)\mu)^2(7+6n)}{\Gamma\left(\frac{5}{2}+n\right)}{}_2F_1\left(2+\frac{3}{2}n, \frac{5}{2}+\frac{3}{2}n, \frac{5}{2}+n, \frac{\alpha^2\mu^2}{(2\lambda+(2+\alpha)\mu)^2}\right)\right],$$

where ${}_2F_1(a_1, a_2; b; z)$ is as defined in Equation (46).

The resulting expression for $\mathbb{E}(\Theta_w^2)$ is omitted for brevity. In Figure 10, we provide plots of the expected value and the variance of Θ_w for some choices of the parameters.

5.4. Mixed exponential downward random times

This section is devoted to the case in which the downward time D is distributed as a mixture of two exponential distributions (42), with $b_i \geq 0$, for $i = 1, 2$, $b_1 + b_2 = 1$, and $\frac{b1}{\mu_1} + \frac{b_2}{\mu_2} < \frac{1}{\lambda}$.

Proposition 19. *In the case of mixed exponential distributed downward times, the moment generating function of C is given by*

$$\psi_C(s) = \mathcal{C}(b_1, b_2, \mu_1, \mu_2, \lambda, s) + \mathcal{D}(b_1, b_2, \mu_1, \mu_2, \lambda, s), \tag{57}$$

where

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$$\mathcal{C}(b_1, b_2, \mu_1, \mu_2, \lambda, s) := \frac{b_1\mu_1}{(\lambda+\mu_1-2s)}\sum_{n=0}^{+\infty}\frac{(2n)!}{(n+1)!n!}\left(\frac{b_2\mu_2\lambda}{(\lambda+\mu_1-2s)^2}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k$$
$$\times\, {}_2F_1\left(n-k,\, 2n+1,\, 2+n,\, \frac{\mu_1-\mu_2}{\lambda+\mu_1-2s}\right),$$

with ${}_2F_1(a_1, a_2; b; z)$ *as defined in Equation (46) and*

$$\mathcal{D}(b_1, b_2, \mu_1, \mu_2, \lambda, s) := \frac{b_2\mu_2}{(\lambda+\mu_2-2s)}\sum_{n=0}^{+\infty}\frac{(2n)!}{(n+1)!n!}\left(\frac{b_2\mu_2\lambda}{(\lambda+\mu_2-2s)^2}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k$$
$$\times\, {}_2F_1\left(k,\, 2n+1,\, 2+n,\, \frac{\mu_2-\mu_1}{\lambda+\mu_2-2s}\right).$$

Proof. The proof immediately follows if we recall that

$$\int_0^{+\infty} e^{st}\left(\frac{\lambda t^2}{4}\right)^n e^{-\lambda t/2-\mu_1 t/2}{}_1F_1\left(n-k,\, n+2,\, \frac{(\mu_1-\mu_2)t}{2}\right)\mathrm{d}t$$
$$= \frac{2(2n)!\lambda^n}{(\lambda+\mu_1-2s)^{2n+1}}{}_2F_1\left(n-k,\, 2n+1,\, 2+n,\, \frac{\mu_1-\mu_2}{\lambda+\mu_1-2s}\right),$$

and

$$\int_0^{+\infty} e^{st}\left(\frac{\lambda t^2}{4}\right)^n e^{-\lambda t/2-\mu_2 t/2}{}_1F_1\left(k,\, n+2,\, \frac{(\mu_2-\mu_1)t}{2}\right)\mathrm{d}t$$
$$= \frac{2(2n)!\lambda^n}{(\lambda+\mu_2-2s)^{2n+1}}{}_2F_1\left(k,\, 2n+1,\, 2+n,\, \frac{\mu_2-\mu_1}{\lambda+\mu_2-2s}\right).$$

□

The expected value of the first passage time Θ_w is obtained in the following proposition.

Proposition 20. *In the case of downward times distributed as in Equation (42), the expected value of the first passage time* Θ_w *is given by*

$$\mathbb{E}(\Theta_w) = \lambda w\left(\mathcal{A}(b_1, b_2, \mu_1, \mu_2, \lambda) + \mathcal{B}(b_1, b_2, \mu_1, \mu_2, \lambda)\right),$$

where

$$\mathcal{A}(b_1, b_2, \mu_1, \mu_2, \lambda) := b_1\mu_1\sum_{n=0}^{+\infty}\frac{(2n)!}{(n+1)!n!}(b_2\mu_2\lambda)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k\left(\frac{2(2n+1)}{(\lambda+\mu_1)^{2n+2}}\right.$$
$$\times\, {}_2F_1\left(n-k,\, 2n+1,\, 2+n,\, \frac{\mu_1-\mu_2}{\lambda+\mu_1}\right) + \frac{2(\mu_1-\mu_2)(n-k)(2n+1)}{(\lambda+\mu_1)^{2n+3}(2+n)}$$
$$\left.\times\, {}_2F_1\left(n-k+1,\, 2n+2,\, 3+n,\, \frac{\mu_1-\mu_2}{\lambda+\mu_1}\right)\right),$$

and

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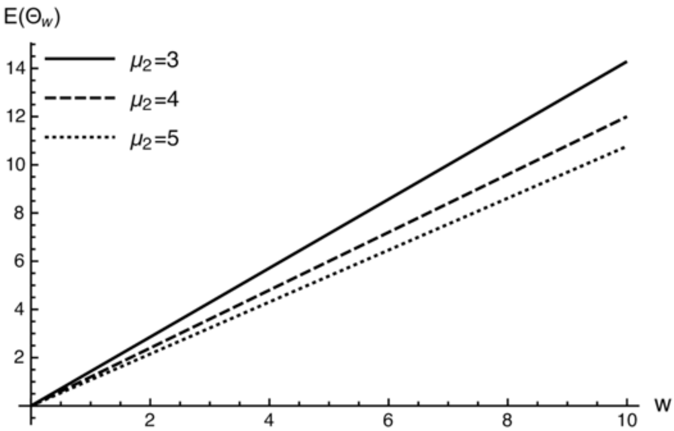

FIGURE 11. The expected value of Θ_w obtained in Proposition 20 for $b_1 = b_2 = 0.5$, $\mu_1 = 2$, and $\lambda = 1$.

$$\mathcal{B}(b_1, b_2, \mu_1, \mu_2, \lambda) := b_2\mu_2 \sum_{n=0}^{+\infty} \frac{(2n)!}{(n+1)!n!}(b_2\mu_2\lambda)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{b_1\mu_1}{b_2\mu_2}\right)^k \left(\frac{2(2n+1)}{(\lambda+\mu_2)^{2n+2}}\right.$$

$$\times\, {}_2F_1\left(k, 2n+1, 2+n, \frac{\mu_2-\mu_1}{\lambda+\mu_2}\right)$$

$$\left.+\frac{2k(\mu_2-\mu_1)(2n+1)}{(\lambda+\mu_2)^{2n+3}(2+n)}{}_2F_1\left(k+1, 2n+2, 3+n, \frac{\mu_2-\mu_1}{\lambda+\mu_2}\right)\right).$$

Proof. The proof follows from Equation (18), if we recall Equation (57) and note that

$$\mathbb{E}(C) = \frac{\mathrm{d}}{\mathrm{d}s}\psi_C(s)\bigg|_{s=0} = \mathcal{A}(b_1, b_2, \mu_1, \mu_2, \lambda) + \mathcal{B}(b_1, b_2, \mu_1, \mu_2, \lambda).$$

□

Some plots of $\mathbb{E}(\Theta_w)$ are provided in Figure 11.

Remark 12. Note that when $b_1 = b_2 = \frac{1}{2}$ and $\mu_1 = \mu_2 = \mu$, we recover the case of exponentially distributed downward random times. Indeed,

$$\mathcal{A}\left(\frac{1}{2}, \frac{1}{2}, \mu, \mu, \lambda\right) = \mathcal{B}\left(\frac{1}{2}, \frac{1}{2}, \mu, \mu, \lambda\right) = \frac{1}{\mu-\lambda}.$$

Hence,

$$\mathbb{E}(\Theta_w)\Big|_{b_1=b_2=\frac{1}{2},\, \mu_1=\mu_2=\mu} = \frac{2\lambda}{\mu-\lambda}w,$$

which corresponds to the expected value of the first passage time Θ_w in the exponential case.

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Appendix A. Proof of Proposition 5

We note that

$$\int_0^{t-w} \frac{e^{-(\lambda+\mu)x/2}}{x} \frac{1}{\sqrt{2\frac{w}{x}+1}} I_1\left(x\sqrt{\lambda\mu\left(1+2\frac{w}{x}\right)}\right) \mathrm{d}x$$
$$= \frac{\sqrt{\lambda\mu}}{2} \sum_{j=0}^{+\infty} \left(\frac{\lambda\mu}{4}\right)^j \frac{1}{j!(j+1)!} \int_0^{t-w} e^{-(\lambda+\mu)x/2} \left((x+w)^w - w^2\right)^j \mathrm{d}x, \tag{58}$$

since

$$I_1\left(x\sqrt{\lambda\mu\left(1+2\frac{w}{x}\right)}\right) = \sum_{j=0}^{+\infty} \frac{\left[\frac{x^2}{4}\left(\lambda\mu(1+2\frac{w}{x})\right)\right]^j}{j!(j+1)!} \left(\frac{x}{2}\sqrt{\lambda\mu\left(1+2\frac{w}{x}\right)}\right)$$
$$= \frac{x}{2}\sqrt{\lambda\mu\left(1+\frac{2w}{x}\right)} \sum_{j=0}^{+\infty} \left(\frac{\lambda\mu}{4}\right)^j \frac{(x^2+2wx)^j}{j!(j+1)!}.$$

In order to evaluate the integral appearing in the right-hand-side of Equation (58), we determine the following integral:

$$\int_{t-w}^{+\infty} e^{-(\lambda+\mu)x/2} \left((x+w)^2 - w^2\right)^j \mathrm{d}x = \int_t^{+\infty} e^{-(\lambda+\mu)(y-w)/2} (y^2-w^2)^j \mathrm{d}y$$
$$= e^{(\lambda+\mu)w/2} \sum_{r=0}^{j} \binom{j}{r} (t^2-w^2)^{j-r} \int_t^{+\infty} e^{-(\lambda+\mu)y/2} (y^2-t^2)^r \mathrm{d}y \tag{59}$$
$$= e^{(\lambda+\mu)w/2} \sum_{r=0}^{j} \binom{j}{r} (t^2-w^2)^{j-r} 2^{2r+1} \left(\frac{t}{\lambda+\mu}\right)^{1/2+r} \frac{r!}{\sqrt{\pi}} K_{1/2+r}\left(\frac{t(\lambda+\mu)}{2}\right),$$

where $K_n(x)$ denotes the nth-order modified Bessel function of the second kind. Note that

$$\frac{e^{-\lambda w} w\lambda\mu}{2} \sum_{j=0}^{+\infty} \left(\frac{\lambda\mu}{4}\right)^j \frac{e^{w/2(\lambda+\mu)}}{j!(j+1)!} \sum_{r=0}^{j} \binom{j}{r} (t^2-w^2)^{j-r} 2^{2r+1} \frac{r!}{\sqrt{\pi}} \left(\frac{t}{\lambda+\mu}\right)^{1/2+r} K_{1/2+r}\left(\frac{t(\lambda+\mu)}{2}\right)$$
$$= \frac{e^{-\lambda w} w\lambda\mu}{2} \sum_{j=0}^{+\infty} \frac{(\lambda\mu)^j}{2^{2j}} \frac{e^{w/2(\lambda+\mu)}}{(j+1)!} \sum_{s=0}^{j} \frac{(t^2-w^2)^s}{s!} 2^{2j-2s+1} \frac{t^{1/2+j-s}}{\sqrt{\pi}(\lambda+\mu)^{1/2+j-s}} K_{1/2+j-s}\left(\frac{t(\lambda+\mu)}{2}\right)$$
$$= \frac{\lambda\mu w\sqrt{t} e^{w(\mu-\lambda)/2}}{\sqrt{\pi(\lambda+\mu)}} \sum_{j=0}^{+\infty} \left(\frac{\lambda\mu t}{\lambda+\mu}\right)^j \frac{1}{(j+1)!} \sum_{s=0}^{j} \frac{1}{s!} \left(\frac{(t^2-w^2)(\lambda+\mu)}{4t}\right)^s K_{1/2+j-s}\left(\frac{t(\lambda+\mu)}{2}\right).$$

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Hence, since

$$\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\left(\frac{\lambda\mu t}{\lambda+\mu}\right)^j\sum_{s=0}^{j}\frac{1}{s!}\left(\frac{(t^2-w^2)(\lambda+\mu)}{4t}\right)^s K_{1/2+j-s}\left(\frac{t(\lambda+\mu)}{2}\right)$$

$$=\sum_{s=0}^{+\infty}\frac{1}{s!}\left(\frac{(t^2-w^2)(\lambda+\mu)}{4t}\right)^s\sum_{j=s}^{+\infty}\frac{1}{(j+1)!}\left(\frac{\lambda\mu t}{\lambda+\mu}\right)^j K_{1/2+j-s}\left(\frac{t(\lambda+\mu)}{2}\right)$$

$$=\sum_{s=0}^{+\infty}\frac{1}{s!}\left(\frac{(t^2-w^2)(\lambda+\mu)}{4t}\right)^s\sum_{h=0}^{+\infty}\frac{1}{(h+s+1)!}\left(\frac{\lambda\mu t}{\lambda+\mu}\right)^{h+s} K_{1/2+h}\left(\frac{t(\lambda+\mu)}{2}\right)$$

$$=\sum_{h=0}^{+\infty}\left(\frac{\lambda\mu t}{\lambda+\mu}\right)^h K_{1/2+h}\left(\frac{t(\lambda+\mu)}{2}\right)\sum_{s=0}^{+\infty}\frac{1}{s!(h+s+1)!}\left(\frac{(t^2-w^2)(\lambda+\mu)\lambda\mu t}{4t(\lambda+\mu)}\right)^s$$

$$=\sum_{h=0}^{+\infty}\left(\frac{\lambda\mu t}{\lambda+\mu}\right)^h\frac{2}{\sqrt{\lambda\mu(t^2-w^2)}}\left(\frac{2}{\sqrt{\lambda\mu(t^2-w^2)}}\right)^h I_{h+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right)K_{1/2+h}\left(\frac{t(\lambda+\mu)}{2}\right)$$

$$=\frac{2}{\sqrt{\lambda\mu(t^2-w^2)}}\sum_{h=0}^{+\infty}\left(\frac{2t\sqrt{\lambda\mu}}{(\lambda+\mu)\sqrt{t^2-w^2}}\right)^h I_{h+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right)K_{1/2+h}\left(\frac{t(\lambda+\mu)}{2}\right),$$

we have

$$e^{-\lambda w}w\sqrt{\lambda\mu}\int_{t-w}^{+\infty}\frac{e^{-(\lambda+\mu)x/2}}{x}\frac{1}{\sqrt{2\frac{w}{x}+1}}I_1\left(x\sqrt{\lambda\mu\left(1+2\frac{w}{x}\right)}\right)\mathrm{d}x$$

$$=\frac{2e^{w/2(\mu-\lambda)}w\sqrt{\lambda\mu t}}{\sqrt{\pi(\lambda+\mu)(t^2-w^2)}}\sum_{h=0}^{+\infty}\left(\frac{2t\sqrt{\lambda\mu}}{(\lambda+\mu)\sqrt{t^2-w^2}}\right)^h I_{h+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right)K_{1/2+h}\left(\frac{t(\lambda+\mu)}{2}\right). \tag{60}$$

Similarly, it can be proved that

$$\int_0^{+\infty}\frac{e^{-(\lambda+\mu)x/2}}{x}\left((x+w)^2-w^2\right)^j\mathrm{d}x=e^{w/2(\lambda+\mu)}\frac{j!}{\sqrt{\pi}}\left(\frac{4w}{\lambda+\mu}\right)^{j+1/2}K_{1/2+j}\left(\frac{w(\lambda+\mu)}{2}\right),$$

so that

$$e^{-\lambda w}w\frac{\lambda\mu}{2}\sum_{j=0}^{+\infty}\left(\frac{\lambda\mu}{4}\right)^j\frac{1}{j!(j+1)!}\int_0^{+\infty}\frac{e^{-(\lambda+\mu)x/2}}{x}\left((x+w)^2-w^2\right)^j\mathrm{d}x$$
$$=e^{w/2(\mu-\lambda)}w\lambda\mu\sqrt{\frac{w}{\lambda+\mu}}\sum_{j=0}^{+\infty}\left(\frac{\lambda\mu w}{\lambda+\mu}\right)^j\frac{K_{1/2+j}\left(\frac{w}{2}(\lambda+\mu)\right)}{(j+1)!\sqrt{\pi}}. \tag{61}$$

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Finally, from Equations (60) and (61), for $0 \le w < t$, we have

$$\begin{aligned}
&\mathbb{P}(M(t) > w) \\
&= \frac{e^{w/2(\mu-\lambda)}w}{\sqrt{(\lambda+\mu)\pi}} \left[\lambda\mu\sqrt{w} \sum_{j=0}^{+\infty} \frac{\left(\frac{\lambda\mu w}{\lambda+\mu}\right)^j}{(j+1)!} K_{1/2+j}\left(\frac{w(\lambda+\mu)}{2}\right) \right. \\
&\left. - \frac{2\sqrt{\lambda\mu t}}{\sqrt{t^2-w^2}} \sum_{j=0}^{+\infty} \left(\frac{2t\sqrt{\lambda\mu}}{(\lambda+\mu)\sqrt{t^2-w^2}}\right)^j I_{j+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right) K_{1/2+j}\left(\frac{t(\lambda+\mu)}{2}\right)\right] + e^{-\lambda w}.
\end{aligned} \tag{62}$$

Since

$$K_{1/2+j}\left(\frac{t(\lambda+\mu)}{2}\right) = \sqrt{\frac{\pi}{t(\lambda+\mu)}}\mathrm{e}^{-\frac{t(\lambda+\mu)}{2}} \frac{j!(-1)^j}{[t(\lambda+\mu)]^j} L_j^{-2j-1}(t(\lambda+\mu)), \tag{63}$$

where $L_n^k(x)$, $n \in \mathbb{N}$, denotes the generalized Laguerre polynomial (see, for instance, [20, p. 411]), and recalling Equation (5.11.4.10) of Prudnikov [33], we have

$$\begin{aligned}
&\frac{e^{w/2(\mu-\lambda)}w}{\sqrt{(\lambda+\mu)\pi}} \lambda\mu\sqrt{w} \sum_{j=0}^{+\infty} \frac{\left(\frac{\lambda\mu w}{\lambda+\mu}\right)^j}{(j+1)!} K_{1/2+j}\left(\frac{w(\lambda+\mu)}{2}\right) \\
&= \frac{e^{-\lambda w}w\lambda\mu}{\lambda+\mu} \sum_{j=0}^{+\infty} \left(-\frac{\lambda\mu}{(\lambda+\mu)^2}\right)^j \frac{L_j^{-2j-1}(w(\lambda+\mu))}{j+1} = 1 - e^{-\lambda w}.
\end{aligned} \tag{64}$$

Hence, substituting Equation (64) in Equation (62), for $0 \le w < t$ we obtain

$$\begin{aligned}
&\mathbb{P}(M(t) > w) \\
&= 1 - \frac{2\sqrt{\lambda\mu t}}{\sqrt{t^2-w^2}} \sum_{j=0}^{+\infty} \left(\frac{2t\sqrt{\lambda\mu}}{(\lambda+\mu)\sqrt{t^2-w^2}}\right)^j I_{j+1}\left(\sqrt{\lambda\mu(t^2-w^2)}\right) K_{1/2+j}\left(\frac{t(\lambda+\mu)}{2}\right),
\end{aligned}$$

so that the statement follows from Equation (63).

Note that, by Equation (5.11.4.10) of [33], the right-hand side of (26) tends to $\mathrm{e}^{-\lambda t}$ as w approaches t.

Appendix B. Proof of Proposition 8

The expected value of $M(t)$ can be computed by considering the integral from 0 to t of the survival function $\mathbb{P}(M(t) > w)$, since $0 \le M(t) \le t$. Considering the series expansion of the Bessel function $I_1(\sqrt{x\lambda\mu(x+2w)})$, we have

$$I_1(\sqrt{x\lambda\mu(x+2w)}) = \sum_{h=0}^{+\infty} \left(\frac{\sqrt{x\lambda\mu(x+2w)}}{2}\right)^{2h+1} \cdot \frac{1}{h!(h+1)!},$$

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so that, after some algebra,

$$\mathbb{E}\,(M(t)) = \int_0^t \mathbb{P}(M(t) > w)\mathrm{d}w = \int_0^t \mathrm{d}w \int_0^{t-w} h_Z(x, w)\mathrm{d}x = \int_0^t \mathrm{d}x \int_0^{t-x} h_Z(x, w)\mathrm{d}w$$

$$= \frac{1 - \mathrm{e}^{-\lambda t}}{\lambda} + \sqrt{\lambda\mu} \sum_{h=0}^{+\infty} \frac{\left(\frac{\sqrt{\lambda\mu}}{2}\right)^{2h+1}}{h!(h+1)!} \int_0^t \mathrm{e}^{-\lambda w} w\,\mathrm{d}w \int_0^{t-w} \mathrm{e}^{-\frac{(\lambda+\mu)x}{2}} x^h (x + 2w)^h \mathrm{d}x. \tag{65}$$

If we expand the term $(x + 2w)^h$ into a finite sum, Equation (65) becomes

$$\frac{1 - \mathrm{e}^{-\lambda t}}{\lambda} + \frac{\mu}{2\lambda} \sum_{h=0}^{+\infty} \frac{\left(\frac{\lambda\mu}{4}\right)^h}{h!(h+1)!} \sum_{j=0}^{h} \binom{h}{j} \left(\frac{2}{\lambda}\right)^j \int_0^t \mathrm{e}^{-\frac{(\lambda+\mu)x}{2}} x^{2h-j} \gamma(j + 2, \lambda(t - x))\mathrm{d}x, \tag{66}$$

where $\gamma(a, x)$ is the lower incomplete gamma function. Since the lower incomplete gamma function can be expanded into a finite sum

$$\gamma(r + 2, \lambda y) = (r + 1)! \left[1 - e^{-\lambda y} \sum_{k=0}^{r+1} \frac{(\lambda y)^k}{k!}\right],$$

we have that

$$\int_0^t \mathrm{e}^{-\frac{(\lambda+\mu)x}{2}} x^{2h-j} \gamma(j + 2, \lambda(t - x))\mathrm{d}x$$

$$= e^{-(\lambda+\mu)t/2}(r + 1)! \left(\int_0^t e^{(\lambda+\mu)y/2}(t - y)^{2h-r}\mathrm{d}y - \sum_{k=0}^{r+1} \frac{\lambda^k}{k!} \int_0^t e^{(\mu-\lambda)y/2} y^k (t - y)^{2h-r}\mathrm{d}y\right)$$

$$= e^{-(\lambda+\mu)t/2}(r + 1)! \left(e^{(\lambda+\mu)t/2} \left(\frac{2}{\lambda + \mu}\right)^{2h-r+1} \gamma\left(2h - r + 1, \frac{(\lambda + \mu)t}{2}\right)\right.$$

$$\left. - \sum_{k=0}^{r+1} \lambda^k t^{2h+1-r+k} (2h - r)! \frac{{}_1F_1\left(k + 1, 2 + 2h - r + k, \frac{(\mu-\lambda)t}{2}\right)}{\Gamma\left(2 + 2h - r + k\right)}\right)$$

$$= (r + 1)! \left(\frac{2}{\lambda + \mu}\right)^{2h-r+1} \gamma\left(2h - r + 1, \frac{(\lambda + \mu)t}{2}\right) - t^{2h-r+1}(r + 1)! e^{-\lambda t}$$

$$\times \sum_{k=0}^{r+1} \frac{\lambda^k t^k}{k!} \beta\left(2h - r + 1, k + 1\right) {}_1F_1\left(2h - r + 1, 2h - r + k + 2, -\frac{(\mu - \lambda)t}{2}\right), \tag{67}$$

where ${}_1F_1(a; b; z)$ is as defined in Equation (32). Hence, the result immediately follows from substituting Equation (67) in Equation (66).

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Appendix C. Proof of Proposition 15

Considering the logarithm of Equation (52) and differentiating the resulting relation with respect to s, one has

$$\frac{1}{\psi_Z(t,s)}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\psi_Z(t,s)=\lambda t\left\{\frac{2k^2\mu^k}{(\lambda+\mu-2s)^{k+1}}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\right.$$
$$\left.+\frac{\mu^k k}{(\lambda+\mu-2s)^k}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\right\}. \tag{68}$$

By the chain rule, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]$$
$$=\frac{\mathrm{d}}{\mathrm{d}z}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};z\right]\Bigg|_{z=\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]$$
$$={}_1\Psi_1\left[\begin{matrix}(2k+1,k+1)\\(2k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\cdot\frac{2\lambda\mu^k(k+1)}{(\lambda+\mu-2s)^{k+2}}, \tag{69}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}z}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};z\right]=\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{+\infty}\frac{\Gamma(k+(k+1)n)}{\Gamma(k+1+kn)}\frac{z^n}{n!}=\sum_{n=1}^{+\infty}\frac{\Gamma(k+(k+1)n)}{\Gamma(k+1+kn)}\frac{z^{n-1}}{(n-1)!}$$
$$=\sum_{m=0}^{+\infty}\frac{\Gamma(2k+1+(k+1)m)}{\Gamma(2k+1+km)}\frac{z^m}{m!}=\frac{\mathrm{d}}{\mathrm{d}z}\,{}_1\Psi_1\left[\begin{matrix}(2k+1,k+1)\\(2k+1,k)\end{matrix};z\right].$$

Hence, Equation (53) follows from substituting Equation (69) into Equation (68) and evaluating the resulting relation at $s=0$.

By differentiating Equation (68) with respect to s, one gets

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\psi_Z(t,s)}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\psi_Z(t,s)\right)=\lambda t\left[\frac{4k^2\mu^k(k+1)}{(\lambda+\mu-2s)^{k+2}}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]+\right.$$
$$+\frac{4k^2\mu^k}{(\lambda+\mu-2s)^{k+1}}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]$$
$$\left.+\frac{\mu^k k}{(\lambda+\mu-2s)^k}\cdot\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,{}_1\Psi_1\left[\begin{matrix}(k,k+1)\\(k+1,k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\right]. \tag{70}$$

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By the chain rule, we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}\,{}_1\Psi_1\left[\begin{matrix}(k, k+1)\\(k+1, k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]$$
$$={}_1\Psi_1\left[\begin{matrix}(2k+1, k+1)\\(2k+1, k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\cdot\frac{2\lambda\mu^k(k+1)}{(\lambda+\mu-2s)^{k+2}},\tag{71}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,{}_1\Psi_1\left[\begin{matrix}(k, k+1)\\(k+1, k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]$$
$$={}_1\Psi_1\left[\begin{matrix}(3k+2, k+1)\\(3k+1, k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\cdot\frac{4\lambda^2\mu^{2k}(k+1)^2}{(\lambda+\mu-2s)^{2k+4}}$$
$$+\,{}_1\Psi_1\left[\begin{matrix}(2k+1, k+1)\\(2k+1, k)\end{matrix};\frac{\lambda\mu^k}{(\lambda+\mu-2s)^{k+1}}\right]\cdot\frac{4\lambda\mu^k(k+1)(k+2)}{(\lambda+\mu-2s)^{k+3}}.\tag{72}$$

Finally, by substituting Equations (71) and (72) into Equation (70) and evaluating the resulting relation at $s=0$, we obtain Equation (54).

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