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THERE ARE NO PROPER TOPOLOGICAL HYPERBOLIC HOMOCLINIC CLASSES FOR AREA-PRESERVING MAPS

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Abstract We begin by defining a homoclinic class for homeomorphisms. Then we prove that if a topological homoclinic class Λ associated with an area-preserving homeomorphism f on a surface M is topologically hyperbolic (i.e. has the shadowing and expansiveness properties), then $\Lambda = M$ and f is an Anosov homeomorphism.

Keywords: shadowing; expansiveness; topological dynamics; homoclinic classes

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1. Introduction

Let $f\colon M \to M$ be a diffeomorphism defined on a closed and connected Riemannian manifold M. A compact and f-invariant set $\Lambda \subset M$ is called a *hyperbolic set* if the tangent bundle over Λ can be decomposed as a direct sum of two sub-bundles, both invariant by the tangent map Df, that are uniformly contracted under forward and backward iterations, respectively (see, e.g. [**9**] for full details). When such a hyperbolic set Λ is given by the orbit of a periodic point, the point is called a *hyperbolic periodic point*. The concept of hyperbolicity played a fundamental part in the development of the stability theory of dynamical systems (see, e.g. [**9**, **11**]).

We recall that a *non-wandering point* $x \in M$ is a point such that any neighbourhood $\mathcal{U}$ of x contains points for which some forward iterate is in $\mathcal{U}$. If the set of non-wandering points is the closure of the set of hyperbolic periodic points and furthermore it is a hyperbolic set, then Smale's spectral decomposition theorem [**11**] ensures that the set of non-wandering points can be partitioned into a finite number of compact blocks exhibiting a dense orbit, called *basic blocks*. When there is a single piece in this decomposition, f is called an *Anosov map*.

In an attempt to generalize the hyperbolic basic sets which had a central role in Smale's spectral decomposition, Newhouse introduced, in the early 1970s, the concept of a homoclinic class (see [**5**]). A *homoclinic class* is defined as the closure of the set of transversal intersections of the stable and unstable invariant manifolds of a hyperbolic periodic saddle of a diffeomorphism f. Homoclinic classes are f-invariant and display a dense orbit of f in the homoclinic class. Yet the hyperbolicity of the periodic saddle, which is the origin of the homoclinic class, is not enough to spread hyperbolicity to the whole homoclinic class. Indeed, it is well known that homoclinic classes may fail to be uniformly hyperbolic.

In the particular case when M is a surface and f is area preserving, Newhouse proved in the mid-1980s [**6**] a simple but elegant result: that f cannot support proper uniformly hyperbolic homoclinic classes. In other words, uniformly hyperbolic homoclinic classes must be the whole manifold, leading to the conclusion that f is an Anosov map.

In the present paper, we intend to reconfigure Newhouse's theorem by considering maps that are not diffeomorphisms. Two problems arise: the pointwise hyperbolicity (a homoclinic class foreshadows the need for a hyperbolic periodic saddle point) and set hyperbolicity (the hypothesis that a homoclinic class is hyperbolic).

Clearly, we first need to seek vestiges of hyperbolicity in non-differentiable contexts and only then try to conjecture what could be the topological counterpart of Newhouse's theorem. Our proposal is to replace the hyperbolicity by two properties with topological flavour—shadowing and expansiveness. In § 2 we fully describe these 'topological hyperbolic sets' in terms of dynamically defined invariant manifolds, canonical coordinates and local product structure (LPS), which are of utmost importance when we realign Newhouse's strategy. In § 3 we define topological homoclinic classes readapted to a type of topological transversality and obtain the Birkhoff–Smale theorem in this non-differentiable setting.

For diffeomorphisms it is well known that periodic points are dense in the whole homoclinic class. However, we do not know how to prove this property for homeomorphisms (see Remark 3.1). Fortunately, we are considering *hyperbolic* topological homoclinic classes, which is enough to obtain that periodic points are dense in the whole homoclinic class (cf. Proposition 2.6).

Like Newhouse, who was inspired by a uniformly hyperbolic basic set to consider the closure of transversal intersections of stable/unstable manifolds of a hyperbolic periodic saddle, we were inspired by a set displaying shadowing and expansiveness (topological set hyperbolicity) to consider the closure of topological transversal intersections of stable/unstable manifolds of a periodic point displaying shadowing and expansiveness (pointwise hyperbolicity).

We hope that our definition of a topological homoclinic class will be useful in several aspects of topological dynamics. Finally, in § 4 we prove the following theorem.

Theorem 1. *Let M be a surface, let $f\colon M \to M$ be an area-preserving homeomorphism and let $\Lambda \subseteq M$ be a topological homoclinic class of f. If f has the shadowing property on Λ and is expansive on Λ, then f is an Anosov homeomorphism (i.e., f has the shadowing property on M and is expansive on M).*

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Notably, in [**3**, **4**] it was proved that there are no expansive homeomorphisms on the two-dimensional sphere $\mathbb{S}^2$. This result, along with Theorem 1, allows us to conclude that $\mathbb{S}^2$ does not support topological homoclinic classes with shadowing and expansiveness associated with an area-preserving homeomorphism.

Throughout the article we assume that M is a closed, connected Riemannian manifold, d is the distance on M induced by the Riemannian structure and λ is the Lebesgue measure on M associated with a volume form on M. Although our main result is about surfaces, some results are stated and proved for manifolds on dimension ≥ 2 for eventual future use.

2. Topological hyperbolicity

2.1. Hyperbolic homeomorphisms

Let $f\colon M \to M$ be a homeomorphism. Given $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let $\Lambda \subseteq M$ be a closed f-invariant set (i.e. $f(\Lambda) = \Lambda$). We say that f has the *shadowing property on* Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f there is a point $z \in \Lambda$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. When $\Lambda = M$, f is said to have the *shadowing property*.

A homeomorphism f is called *expansive on* Λ (see [**9**, § 7]) if there is $e > 0$, called an *expansive constant*, such that for all $x \in \Lambda$ and $y \in M$, if we have $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$ then $x = y$. When $\Lambda = M$, f is simply said to be expansive.

A homeomorphism f is called an *Anosov homeomorphism* if it has the shadowing property and is expansive (see [**1**, § 11.3]). We shall say that f is a *hyperbolic homeomorphism on* Λ if it has the shadowing property on Λ and is expansive on Λ.

2.2. Invariant sets, shadowing and expansiveness

In this section we obtain some useful local results to be used in the sequel. Let $f\colon M \to M$ be a homeomorphism and let $\Lambda \subseteq M$ be a closed and f-invariant set. Given $x \in M$ and $\epsilon > 0$, the *local stable* and *local unstable set* of x are defined, respectively, by

$$W^s_\epsilon(x) = \{y \in M\colon d(f^n(x), f^n(y)) \leq \epsilon \text{ for } n \geq 0\}$$

and

$$W^u_\epsilon(x) = \{y \in M\colon d(f^{-n}(x), f^{-n}(y)) \leq \epsilon \text{ for } n \geq 0\}.$$

Under the expansiveness hypothesis on Λ, stable and unstable sets of points $x \in \Lambda$ are *dynamically defined*, i.e. the following result holds.

Proposition 2.1. *Let $e > 0$. The homeomorphism f is expansive on Λ with expansive constant e if and only if for all $\tilde{e} > 0$ there exists $N > 0$ such that for all $x \in \Lambda$ and all $n \geq N$ we have*

$$f^n(W^s_e(x)) \subset W^s_{\tilde{e}}(f^n(x)) \quad \text{and} \quad f^{-n}(W^u_e(x)) \subset W^u_{\tilde{e}}(f^{-n}(x)). \tag{2.1}$$

Proof. Suppose that f is expansive on Λ with expansive constant e, and suppose, for a contradiction, that there exist sequences $x_n \in \Lambda$ and $y_n \in M$, $n \in \mathbb{N}$, such

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that $y_n \in W^s_e(x_n)$ and $d(f^n(x_n), f^n(y_n)) > \tilde{e}$. Since $y_n \in W^s_e(x_n)$, we have that $d(f^m \circ f^n(x_n), f^m \circ f^n(y_n)) \leq e$ for all $m \geq -n$. Taking subsequences, we may assume that there exists $x \in \Lambda$ and $y \in M$ such that $\lim_n f^n(x_n) = x$ and $\lim_n f^n(y_n) = y$. Hence, $d(f^m(x), f^m(y)) \leq e$ for all $m \in \mathbb{Z}$. Moreover, $d(x, y) = \lim_n d(f^n(x_n), f^n(y_n)) \geq \tilde{e}$. This contradicts the expansiveness on Λ.

Conversely, let $x \in \Lambda$ and $y \in M$ be such that $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$. For all $n \geq 0$ we have $f^{-n}(y) \in W^s_e(f^{-n}(x))$. Then, for any $\tilde{e} > 0$, we have $y \in W^s_{\tilde{e}}(x)$ by the first inclusion of (2.1) and hence $x = y$. □

The *stable* and *unstable sets* of $x \in M$ are defined, respectively, by

$$W^s(x) = \left\{ y \in M \colon \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0 \right\}$$

and

$$W^u(x) = \left\{ y \in M \colon \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0 \right\}.$$

Proposition 2.2. *If f is expansive on Λ with expansive constant e and $\epsilon \in (0, e)$, then for all $x \in \Lambda$:*

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x))) \quad \text{and} \quad W^u(x) = \bigcup_{n \geq 0} f^n(W^u_\epsilon(f^{-n}(x))). \tag{2.2}$$

Proof. Let $y \in \cup_{n \geq 0} f^{-n} W^s_\epsilon(f^n(x))$. There exists $n \geq 0$ such that $f^n(y) \in W^s_\epsilon(f^n(x))$. By Proposition 2.1, for all $\tilde{e} > 0$ there exists $N > 0$ such that for all $m \geq N$,

$$f^{m+n}(y) \in f^m W^s_\epsilon(f^n(x)) \subset W^s_{\tilde{e}}(f^{m+n}(x)).$$

Hence, $d(f^{m+n}(y), f^{m+n}(x)) \leq \tilde{e}$ for all $m \geq N$ and, consequently, $y \in W^s(x)$.

Conversely, let $y \in W^s(x)$. Given $\epsilon > 0$ there exists $N \geq 0$ such that $d(f^n(x), f^n(y)) \leq \epsilon$ for $n \geq N$. Consequently, $d(f^i \circ f^N(x), f^i \circ f^N(y)) \leq \epsilon$ for all $i \geq 0$, i.e. $f^N(y) \in W^s_\epsilon(f^N(x))$. Hence,

$$y \in f^{-N} W^s_\epsilon(f^N(x)) \subset \cup_{n \geq 0} f^{-n} W^s_\epsilon(f^n(x)).$$

The second equality in (2.2) can be proved analogously. □

We say that f has *canonical coordinates* on Λ if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \cap \Lambda \neq \emptyset$.

Lemma 2.3. *If f has the shadowing property on Λ, then f has canonical coordinates on Λ.*

Proof. Given $\epsilon > 0$, let $\delta > 0$ be given by the shadowing property on Λ. Let us be given any $x, y \in \Lambda$ such that $d(x, y) < \delta$. Take $x_i = f^i(x)$ for $i \geq 0$ and $x_i = f^i(y)$ for $i < 0$. Clearly, $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ is a δ-pseudo orbit of f. By the shadowing property on Λ, there exists $z \in \Lambda$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Thus, $d(f^i(z), f^i(x)) < \epsilon$ for all $i \geq 0$ and $d(f^i(z), f^i(y)) < \epsilon$ for all $i < 0$. Hence, $z \in W^s_\epsilon(x)$ and $z \in W^u_\epsilon(y)$. □

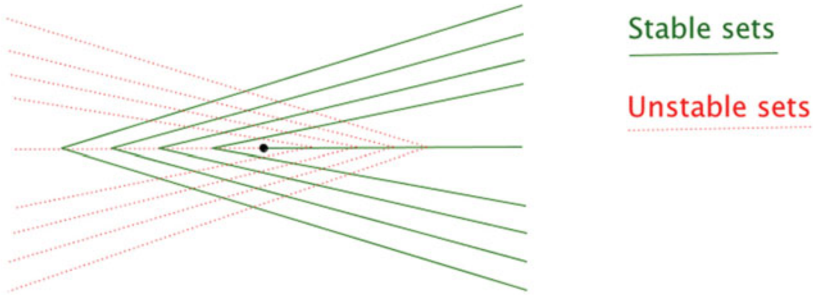

Figure 1. Example of a 1-prong structure. In this case, local stable and local unstable sets intersect in more than one point, contradicting expansivity (cf. Lemma 2.4).

Lemma 2.4. *If f is an expanding homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \leq e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then there is at most one point of intersection of $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$.*

Proof. Let $z \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$. Suppose, for a contradiction, that there exists $w \in M$, $w \neq z$, such that $w \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$. Then, for all $n \geq 0$, $d(f^n(x), f^n(z)) \leq \epsilon$, $d(f^n(x), f^n(w)) \leq \epsilon$, $d(f^{-n}(y), f^{-n}(z)) \leq \epsilon$ and $d(f^{-n}(y), f^{-n}(w)) \leq \epsilon$. This implies that

$$d(f^n(z), f^n(w)) \leq 2\epsilon,$$

for all $n \in \mathbb{Z}$, which contradicts the expansiveness on Λ. □

As a consequence of Lemmas 2.3 and 2.4 we have the following.

Corollary 2.5. *If f is a hyperbolic homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \leq e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then*

$$W^s_\epsilon(x) \cap W^u_\epsilon(y) = \{\text{one point}\} \subset \Lambda.$$

Remark 2.1. We say that a one-dimensional set $W \subset M$ is a *topological manifold* if every $x \in W$ has a neighbourhood homeomorphic to $\mathbb{R}$. It should be enlightening to say that even with expansiveness we can have several 'prongs' as invariant sets, and thus stable (and unstable) sets are not topological manifolds. In Figure 2 we can have expansiveness; nevertheless, a set Λ including a ball around p cannot have canonical coordinates. Actually, there exist points $x, y \in \Lambda$ arbitrarily close but still without any intersection between the local stable set of x and the local unstable set of y. Shadowing implies no n-prong structure for $n > 2$. Expansivity implies no 1-prong structure (cf. Figure 1). Therefore, when we put together shadowing and expansiveness, there can only be a 2-prong structure (Figure 3).

The next result shows that hyperbolic homeomorphisms on Λ behave similarly to the basic pieces of classical hyperbolic dynamics theory (cf. [**9**]).

Proposition 2.6. *Let $f\colon M \to M$ be a volume-preserving homeomorphism which is hyperbolic on Λ. Then, $\overline{Per(f) \cap \Lambda} = \Lambda$.*

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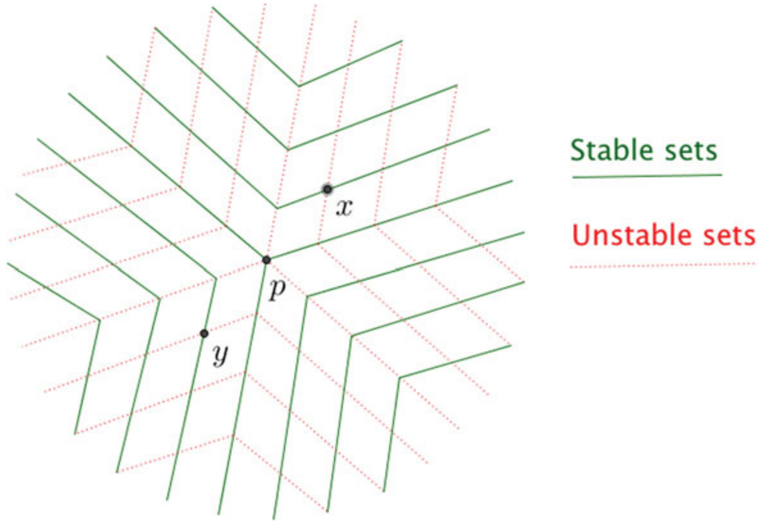

Figure 2. Example of a 3-prong structure. In this case local stable and local unstable sets do not intersect, contradicting the shadowing property (cf. Lemma 2.3).

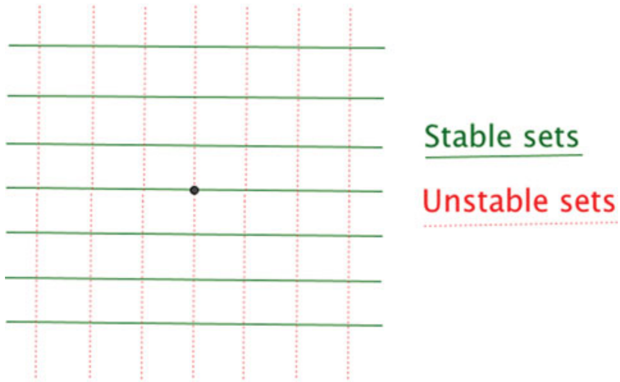

Figure 3. Example of 2-prong structure which is compatible with being a C^0 manifold.

Proof. Let $x \in \Lambda$. Since f is volume preserving, by Poincaré recurrence theorem, almost every point in Λ is recurrent and, therefore, x is a non-wandering point. Hence, it is easy to deduce that for every $\delta > 0$, there exists a δ-chain from x to itself, i.e. there exists a finite δ-pseudo-orbit $\{x_i\}_{i=0}^{n} \subset \Lambda$ such that $x = x_0 = x_n$, $n > 0$.

Let e be the expansive constant of f, let $0 < \epsilon \leq e/2$ be fixed and let $\delta > 0$ be given by the shadowing property on Λ. Let $\{x_i\}_{i=0}^{N} \subset \Lambda$ be a δ-chain from x to itself. The sequence $\{\tilde{x}_i\}_{i\in\mathbb{Z}}$ defined by $\tilde{x}_i = x_i$ if $n \equiv i \,(\mathrm{mod\ N})$ is a δ-pseudo orbit. Therefore, it is ϵ-shadowed by some point $z \in \Lambda$ and, clearly, also by $f^N(z) \in \Lambda$. Applying the triangle inequality, we have that $d(f^n(z), f^{n+N}(z)) \leq 2\epsilon \leq e$ for all $n \in \mathbb{Z}$. Hence, by the expansive property on Λ, we obtain that $z = f^N(z)$. Thus, $z \in \Lambda$ is a periodic point such that $d(z, x) = d(z, x_0) \leq \epsilon$. □

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2.3. Local product structure

Given $\epsilon > 0$, let $\Delta(\epsilon) := \{(x, y) : x, y \in \Lambda \text{ and } d(x, y) \leq \epsilon\}$. We say that a homeomorphism f has an *LPS* on Λ if the following hold.

(A) There are $\delta_0 > 0$ and a continuous function

$$[\cdot,\cdot]\colon \quad \Delta(\delta_0) \longrightarrow \Lambda$$
$$(x, y) \longmapsto [x, y]$$

such that for all $x, y, z \in \Lambda$ we have

$$[x, x] = x, \quad [[x, y], z] = [x, z], \quad [x, [y, z]] = [x, z] \quad \text{and} \quad f([x, y]) = [f(x), f(y)],$$

whenever defined.

(B) There exist $\delta_1 \in (0, \delta_0/2)$ and $\rho \in (0, \delta_1)$ such that for all $x \in \Lambda$ the following three conditions hold:
   (i) denoting $V^\sigma_{\delta_1}(x) := \{y \in W^\sigma_{\delta_0}(x) \cap \Lambda \colon d(x, y) < \delta_1\}$ ($\sigma \in \{s, u\}$) we have that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is an open set of Λ with diameter less than δ_0;
   (ii) $[\cdot,\cdot]\colon V^u_{\delta_1}(x) \times V^s_{\delta_1}(x) \to [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is a homeomorphism; and
   (iii) $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)] \supset \{y \in \Lambda \colon d(x, y) \leq \rho\}$.

When f is a hyperbolic homeomorphism on Λ we shall set $[x, y] := W^s_\eta(x) \cap W^u_\eta(y)$, where $\eta = e/4$ and e is an expansive constant for f (and x is near y).

Lemma 2.7. *If f is a hyperbolic homeomorphism on Λ, then f has an LPS on Λ.*

Proof. We borrow ideas from [**2**, Theorem 5.6] and adapt it to the local case. Let $e > 0$ be an expansive constant for f and fix $\eta = e/4$. By Corollary 2.5, there exists $0 < \delta_0 < \eta$ such that $W^s_\eta(x) \cap W^u_\eta(y) = \{\text{one point}\} \subset \Lambda$ for $x, y \in \Lambda$ with $d(x, y) \leq \delta_0$. We define the map $[\cdot,\cdot] : \Delta(\delta_0) \to \Lambda$ by $[x, y] := W^s_\eta(x) \cap W^u_\eta(y)$, for $(x, y) \in \Delta(\delta_0)$.

The map $[\cdot,\cdot]$ is continuous. Indeed, let $\{(x_n, y_n)\} \subset \Delta(\delta_0)$ be a sequence that converges to $(x, y) \in \Delta(\delta_0)$. Let $z_n = [x_n, y_n]$. Since Λ is compact, taking subsequences, we may assume that $(z_n)_n$ converges to $z \in \Lambda$. Since $z_n \in W^s_\eta(x_n)$, we have that $d(f^i(x_n), f^i(z_n)) \leq \eta$ ($i \geq 0$). Hence, $d(f^i(x), f^i(z)) \leq \eta$ and, therefore, $z \in W^s_\eta(x)$. Analogously, $z \in W^u_\eta(y)$. Thus $z = [x, y]$, which shows that $(z_n)_n = ([x_n, y_n])_n$ converges to $[x, y]$.

Clearly, $[x, x] = x$ for all $x \in \Lambda$. Since $[x, y] \in W^s_\eta(x)$, we have that $[[x, y], z] \in W^s_{2\eta}(x) \cap W^u_\eta(z)$. Hence, by expansivity on Λ, $[[x, y], z] = [x, z]$. Analogously, $[x, [y, z]] = [x, z]$. By uniform continuity of f on Λ, it is easy to conclude that $f[x, y] = [f(x), f(y)]$. Altogether, this proves (A) in the definition of LPS on Λ.

We shall now prove (B) (i)–(iii) in the definition of LPS on Λ. We define a map $g_1\colon \Lambda \times \Delta(\delta_0) \to \mathbb{R}$ by $g_1(x, (y, z)) := d(x, [y, z])$, for $x \in \Lambda$ and $(y, z) \in \Delta(\delta_0)$. Clearly, g_1 is continuous and $g_1(x, (x, x)) = 0$. Since g_1 is uniformly continuous, there exists $\delta_1 \in (0, \delta_0/2)$ such that $\operatorname{diam}\{x, y, z\} < 2\,\delta_1$ implies $d(x, [y, z]) < \delta_0/3$. Therefore, given $(y, z) \in V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$, we have that $d(x, [y, z]) < \delta_0/3$. Take $w_1, w_2 \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$.

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Then there exists $y_1, y_2 \in V^u_{\delta_1}(x)$ and $z_1, z_2 \in V^s_{\delta_1}(x)$ such that $w_1 = [y_1, z_1]$ and $w_2 = [y_2, z_2]$. We have that $d(w_1, w_2) = d([y_1, z_1], [y_2, z_2]) \leq d(x, [y_1, z_1]) + d(x, [y_2, z_2]) \leq \delta_0/3 + \delta_0/3 < \delta_0$. Hence, the diameter of $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is smaller than δ_0. To show that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is open in Λ, let $w \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$. Then there exists $y \in V^u_{\delta_1}(x)$ and $z \in V^s_{\delta_1}(x)$ such that $w = [y, z]$. Since $\operatorname{diam}\{x, y, z\} < 2\,\delta_1$, we have that $d(x, w) < \delta_0/3$. Thus, we can define the maps $P_u : \mathcal{B}_{\delta_0/3}(w) \cap \Lambda \to W^u_\eta(x) \cap \Lambda$ and $P_s : \mathcal{B}_{\delta_0/3}(w) \cap \Lambda \to W^s_\eta(x) \cap \Lambda$ by $P_u(v) := [v, x]$ and $P_s(v) := [x, v]$, for $v \in \mathcal{B}_{\delta_0/3}(w)$. These maps are clearly continuous. Observe that, given $v \in \mathcal{B}_{\delta_0/3}(w) \cap \Lambda$, we have that $d(x, v) < d(x, w) + d(w, v) < \delta_0/3 + \delta_0/3 < \delta_0$. Given that $w = [y, z]$, we have $P_u(w) = y$ and $P_s(w) = z$, by expansivity on Λ. Hence, there is a neighbourhood $U \subset \mathcal{B}_{\delta_0/3}(w) \cap \Lambda$ of w in Λ such that $P_u(U) \subset V^u_{\delta_1}(x)$ and $P_s(U) \subset V^s_{\delta_1}(x)$. Take $v \in U$. By expansivity on Λ, we have that $v = [[v, x], [x, v]]$. Therefore, $v \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$, which proves that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is an open set of Λ. Thus, B(i) is proved.

To prove B(ii), define a map $h : [V^u_{\delta_1}(x), V^s_{\delta_1}(x)] \to V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$ by $h(w) := ([w, x], [x, w])$, for $w \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$. Clearly, h is continuous and h itself is the inverse map of $[\cdot, \cdot]$.

Finally, we shall prove B(iii). Define the map $g_2 : \Delta(\delta_0) \to \mathbb{R}$ by $g_2(x, y) := \operatorname{diam}\{x, [y, x], [x, y]\}$, for $(x, y) \in \Delta(\delta_0)$. Then g_2 is (uniformly) continuous and, therefore, there exists $\rho \in (0, \delta_1)$ such that $d(x, y) < \rho$ implies $g_2(x, y) < \delta_1$. Hence, $[y, x] \in V^u_{\delta_1}(x)$ and $[x, y] \in V^s_{\delta_1}(x)$. Therefore, given $y \in \Lambda$ with $d(x, y) < \rho$, we have that $y = [[y, x], [x, y]] \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$, which proves B(iii). □

3. Topological homoclinic classes

3.1. Topological transversality

The intersection between two curves in the plane can be quite odd. Indeed, given any compact set $K \subset \mathbb{R}$, let us define the continuous function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \min\{|x - k| : k \in K\}$. The intersection between the one-dimensional manifolds defined by the graph of f and the x-axis is the set $(k, 0) \in \mathbb{R}^2$, where $k \in K$. Let $B(0, r) \subset \mathbb{R}^2$ denote the open ball centred in $(0, 0)$ and with radius r. Given one-dimensional manifolds $N_1, N_2 \subset \mathbb{R}^2$, we say that $q \in \mathbb{R}^2$ is a *transversal intersection* between N_1 and N_2 if there exist an open ball B containing q and a homeomorphism $h : B(0, r) \to B$, where $r > 0$, such that $h(x, 0) = \hat{N}_1$ and $h(0, y) = \hat{N}_2$, where $\hat{N}_i$ are the connected components of $N_i \cap B$ containing q.

In the sequel we apply the previous concept of transversality to stable and unstable topological manifolds. Clearly, if $q \in W^s(p) \cap W^u(p)$ is a transversal intersection between the stable and unstable manifolds of p, then this intersection is persistent in the sense that the intersection between these two one-dimensional manifolds cannot disappear under an arbitrarily small C^0-perturbation of the map. We indicate that $q \in W^s(p) \cap W^u(p)$ is a transversal intersection by writing $q \in W^s(p) \pitchfork W^u(p)$.

3.2. Definition of topological homoclinic classes

Let $f : M \to M$ be an area-preserving homeomorphism and $p \in M$ a periodic point of period n.

We say that p is *topologically hyperbolic* if f is a hyperbolic homeomorphism on $\Lambda = \mathcal{O}(p)$, the orbit of p.

By Proposition 2.1, under the expansiveness hypothesis on Λ, stable and unstable sets of points $x \in \Lambda$ are dynamically defined. Clearly, by Remark 2.1, any stable/unstable structure at topological hyperbolic periodic points must be a 2-prong structure.

We define the *topological homoclinic class* of a topological hyperbolic periodic point p by the closure of the transversal intersections of the stable and unstable manifolds of $f^i(p)$, for $i = 0, \ldots, n-1$, that is, by $\overline{W^s(f^i(p)) \pitchfork W^u(f^i(p))}$, for $i = 0, \ldots, n-1$. Clearly, a homoclinic class is compact and f-invariant. The compactness follows from the fact that it is a closed subset of a compact manifold. To check that $f(\Lambda) = \Lambda$, observe that if we have $x \in \Lambda$, then $x_n \to x$ for a sequence $\{x_n\}_n \subset W^s(p) \cap W^u(p)$. By the f-invariance of the sets $W^s(p)$ and $W^u(p)$ we get that $\{f(x_n)\}_n \subset W^s(p) \cap W^u(p)$. Finally, the continuity of f ensures that $f(x) \in \Lambda$.

3.3. The Birkhoff–Smale theorem

In Theorem 3.2 we will obtain a slightly different version of the well-known Birkhoff–Smale theorem, with a topological flavour. The fixed point index will play a crucial role part in the proof since, in rough terms, the existence of a non-null index on a set ensures a fixed point in that set. Let us recall the definition of a fixed point index: take an open ball $B \subset M$ such that $\mathrm{Fix}(f, \partial B) = \emptyset$ (meaning that f has no fixed points in the boundary of B denoted by ∂B) and $\overline{B} \cap \overline{f(B)} \neq \emptyset$, and ∂B, $\overline{B} \cup \overline{f(B)}$ are labelled by the same chart. In this case we say that the index of f in B is the degree of the map defined in chart coordinates by the vector field

$$X_f\colon \quad \partial B \simeq \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$
$$x \longmapsto \frac{f(x) - x}{\|f(x) - x\|}.$$

If the index is $\neq 0$, then f has a fixed point in B. When $\overline{B} \cap \overline{f(B)} = \emptyset$, we say that the index is zero.

In the proof of Theorem 3.2 we will need a weak version of the well-known lambda-lemma [**7**]. This lemma tells us that if Σ is a section transversal to the stable manifold, then $f^n(\Sigma)$ $(n > 0)$ becomes arbitrarily C^1 close to some compact subset contained in the unstable manifold. The difficult part in the proof of the classical lambda-lemma is obtaining the C^1 closeness of the two sets. Indeed, we can obtain easily the following.

Lemma 3.1 (Topological lambda-lemma). *Let Σ and $W^s(p)$ be transversal topological manifolds and let $\Gamma \subset W^u(p)$ be a compact set. Then, given any $\epsilon > 0$, there exist $\Sigma' \subset \Sigma$ and $n_0 \in \mathbb{N}$ such that $d_H(f^{n_0}(\Sigma'), \Gamma) < \epsilon$ (where d_H is the Hausdorff distance between sets).*

Now we are ready to state the main result in this section.

Theorem 3.2 (Topological Birkhoff–Smale theorem). *Let M be a surface, $f\colon M \to M$ a homeomorphism, $p \in M$ a periodic topological hyperbolic point of period n and $q \in W^s(p) \pitchfork W^u(p)$. Then f has a periodic point in any neighbourhood of q.*

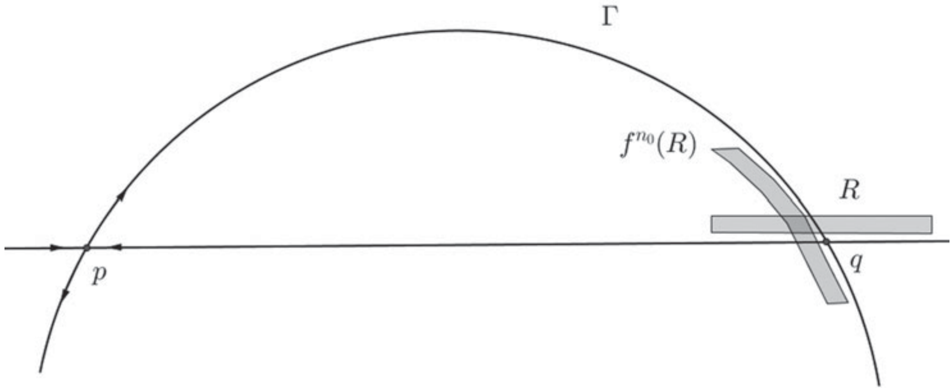

Figure 4. Support for the proof of Theorem 3.2.

Proof. Let $q \in W^s(p) \pitchfork W^u(p)$ and let $\Gamma \subset W^u(p)$ be the compact arc with extremes p and q. By hypothesis there exists a section Σ which is transversal to $W^s(p)$ and containing q. Then, by Lemma 3.1, given any $\epsilon > 0$, there exist $\Sigma' \subset \Sigma$ and $n_0 \in \mathbb{N}$ such that $d_H(f^{n_0}(\Sigma'), \Gamma) < \epsilon$. Take a rectangle R such as that in Figure 4, sufficiently thin and sufficiently close to $W^u(p)$ to contain Σ'. We obtain $R \cap f^{n_0}(R) \neq \emptyset$. Moreover, we can even ensure that $\mathrm{Fix}(f^{n_0}, \partial R) = \emptyset$. Indeed, the top and the bottom of R, in a configuration as in Figure 4, become the bottom and the top of $f^{n_0}(R)$ respectively. Of course, such a configuration can be achieved by allowing a small change in R. Since ∂R is homeomorphic to $\mathbb{S}^1$, we let $X_{f^{n_0}}(x)$ be a vector field defined in ∂R as above. The angular variation of $X_{f^{n_0}}(x)$ as x moves once around ∂R is not zero. Therefore, f^{n_0} has a fixed point in R. Consequently, f has a periodic point in any neighbourhood of q. □

Remark 3.1. We point out that the classic Birkhoff–Smale theorem [**10**] ensures that the periodic points in any neighbourhood of q are homoclinically related to p. This allows us to conclude that the periodic orbits in the homoclinic class are dense in the homoclinic class. However, the arguments used in [**10**] are clearly not valid outside the differentiable context. An alternative approach using tubular family theorems (cf. [**8**]) is also unsuitable under the C^0 hypothesis.

4. Proof of Theorem 1

Let M be a surface, let $f\colon M \to M$ be an area-preserving homeomorphism and let $\Lambda \subseteq M$ be a topological homoclinic class of f. We already know by Corollary 2.5 that if f is a hyperbolic homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \leq e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then

$$W^s_\epsilon(x) \cap W^u_\epsilon(y) = \{\text{one point}\} \subset \Lambda. \tag{4.1}$$

Since Λ is a homoclinic class it is closed by definition. As M is connected, it is sufficient to prove that Λ is open in M.

The next result will be central in the proof of Theorem 1.

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Lemma 4.1. *If $x \in \Lambda$ then $\Lambda \cap W^u(x)$ is dense in $W^u(x)$ and $\Lambda \cap W^s(x)$ is dense in $W^s(x)$.*

Proof. The proof is by contradiction. We assume that $\Lambda \cap W^u(x)$ is not dense in $W^u(x)$. Denote by $I \subset W^u(x)$ a *gap interval* in $W^u(x) \setminus \Lambda$, that is, its extremes are in Λ when I is bounded (from one side, from the other or both) and there are no more points of Λ in I besides the extremes. Clearly, gap intervals can be bounded (when there are two distinct extremes) or unbounded (when there is at most a single extreme).

Case 1. We begin by considering that $I = [y_1, y_2]$ with $y_1, y_2 \in \Lambda$. In § 2.3 we considered small constants such as $\epsilon, \delta_0, \delta_1$ and ρ because the arguments involving the LPS hold only on a microscopic level. Therefore, we iterate backwards until I is sufficiently close to x to be under the hypotheses of § 2.3. Clearly, an iterate of I will still be a gap interval. Using Proposition 2.6, we consider a periodic point $z \in \Lambda$ arbitrarily close to x and such that $W^u(\mathcal{O}(x)) \cap W^u(\mathcal{O}(z)) = \emptyset$. Since $y_1, y_2 \in \Lambda$, by Corollary 2.5, we obtain that $W^s_\epsilon(y_1)$ intersects $W^u_\epsilon(z)$ at a single point $x_1 \in \Lambda$ and $W^s_\epsilon(y_2)$ intersects $W^u_\epsilon(z)$ at a single point $x_2 \in \Lambda$. Let R be the rectangle with sides I, the segment I_1 with extremes y_1 and x_1, the segment I_2 with extremes y_2 and x_2 and the segment I_3 with extremes x_1 and x_2. To ensure that $R \cap \Lambda = \emptyset$, we can pick another periodic point z closer to x, building another rectangle R satisfying $R \cap \Lambda = \emptyset$. Indeed, if such z does not exist, then there would exist a point $w \in R \cap \Lambda$ such that $W^s_\epsilon(w) \pitchfork W^u_\epsilon(x) \in I$, which contradicts the fact that I is a gap interval. From Poincaré's recurrence theorem, λ-almost every point in R is recurrent and so $f^n(R) \cap R \neq \emptyset$ for infinitely many choices of n. Moreover, since f is an area-preserving map, we obtain $\partial R \cap \partial f^n(R) \neq \emptyset$. In fact, preservation of area avoids $R \subset f^n(R)$ or $R \supset f^n(R)$ and so the intersection of boundaries is inevitable. Furthermore, from stable/unstable set arguments we get $f^n(I) \cap I = \emptyset$, $f^n(I) \cap I_3 = \emptyset$, $f^n(I_3) \cap I = \emptyset$, $f^n(I_3) \cap I_3 = \emptyset$, $f^n(I_1) \cap I_1 = \emptyset$, $f^n(I_1) \cap I_2 = \emptyset$, $f^n(I_2) \cap I_2 = \emptyset$ and $f^n(I_2) \cap I_1 = \emptyset$. Therefore, we must have an intersection between f^n iterates of the segments I, I_3 and the segments I_1, I_2, i.e. $f^n(I \cup I_3) \cap (I_1 \cup I_2) \neq \emptyset$, contradicting the fact that I is a gap interval.

Case 2. Finally, we consider the case of having an unbounded interval $I = [y_1, +\infty) \subset W^u(x)$ with $y_1 \in \Lambda$. Once again we pull I near x. Clearly, and since Λ is an f-invariant set, an iterate of I will still be an unbounded interval. We take a periodic point $z \in \Lambda$ arbitrarily close to x. Since $z \in \Lambda$, by Corollary 2.5, we get that $W^u_\epsilon(z) \cap W^s_\epsilon(x) \neq \emptyset$ and $W^u_\epsilon(z) \cap W^s_\epsilon(y_1) \neq \emptyset$ and, moreover, each of these intersections is a single point in the topological homoclinic class Λ.

Therefore, in $W^u(z)$ and since z is periodic, if it exists, a gap interval cannot be unbounded and we can apply Case 1 and conclude that $W^u(z) \cap \Lambda$ is dense in $W^u(z)$. Since, by Proposition 2.6, we can take periodic points $z_n \in \Lambda$ such that $W^u(z_n) \cap \Lambda$ is dense in $W^u(z_n)$, we obtain that $I \cap \Lambda \neq \emptyset$, which is a contradiction. □

Finally, we prove that Λ is open in M. By Lemma 2.7, f has an LPS on Λ. Consider $\delta_0 > 0$ and $\delta_1 \in (0, \delta_0/2)$ given by the LPS on Λ. We have that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is an open set of Λ with diameter less than δ_0, and also that $[\cdot,\cdot]\colon V^u_{\delta_1}(x) \times V^s_{\delta_1}(x) \to [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is a homeomorphism.

By Lemma 4.1, the set $V^u_{\delta_1}(x)$ is dense in $W^u_{\delta_1}(x)$ and the set $V^s_{\delta_1}(x)$ is dense in $W^s_{\delta_1}(x)$. Clearly, the closure of the Cartesian product $V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$ is a topological disk and a neighbourhood of x. Since $[\cdot,\cdot]\colon V^u_{\delta_1}(x) \times V^s_{\delta_1}(x) \to [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is a homeomorphism, $[\cdot,\cdot](\overline{V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)}) = \overline{[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]}$. Hence, x is in the interior of Λ in M. Thus, Λ is open in M.

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