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MOMENTS OF *k*-HOP COUNTS IN THE RANDOM-CONNECTION MODEL

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Abstract

We derive moment identities for the stochastic integrals of multiparameter processes in a random-connection model based on a point process admitting a Papangelou intensity. The identities are written using sums over partitions, and they reduce to sums over non-flat partition diagrams if the multiparameter processes vanish on diagonals. As an application, we obtain general identities for the moments of k-hop counts in the random-connection model, which simplify the derivations available in the literature.

Keywords: Point process; moments; random-connection model; random graph; k-hop

2010 Mathematics Subject Classification: Primary 60G57
Secondary 60G55

1. Introduction

The random-connection model, see, e.g., [12, Chapter 6], is a classical model in continuum percolation. It consists of a random graph built on the vertices of a point process on $\mathbb{R}^d$ by adding edges between two distinct vertices x and y with probability $H(\|x-y\|)$. In the case of the Rayleigh fading $H_\beta(\|x-y\|) = \mathrm{e}^{-\beta\|x-y\|^2}$ with $x, y \in \mathbb{R}^2$, the mean value of the number $N_k^{x,y}$ of k-hop paths connecting $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$ has been computed in [9], together with the variance of 3-hop counts. However, this argument does not extend to $k \geq 3$ as the proof of the variance identity for 3-hop counts in [9] relies on the known Poisson distribution of the 2-hop count. As shown in [9], the knowledge of moments can provide accurate numerical estimates of the probability $\mathrm{P}(N_k^{x,y} > 0)$ of at least one k-hop path by expressing it as a series of factorial moments, and the need for a general theory of such moments was pointed out therein.

On the other hand, moment identities for Poisson stochastic integrals with random integrands have been obtained in [18] based on moment identities for Skorohod's integral on the Poisson space; see [16, 17], and also [19] for a review. These moment identities have been extended to point processes with Papangelou intensities in [5], and to multiparameter processes in [2]. Factorial moments have also been computed in [4] for point processes with Papangelou intensities.

In this paper we derive closed-form expressions for the moments of the number of k-hop paths in the random-connection model. In Proposition 4 the moment of order n of the k-hop count is given as a sum over non-flat partitions of $\{1, \ldots, nk\}$ in a general random-connection model based on a point process admitting a Papangelou intensity. Those results are then specialized to the case of Poisson point processes, with an expression for the variance of the k-hop count given in Corollary 2 using a sum over integer sequences. Finally, in the case of

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Rayleigh fadings we show that some results of [9], such as the computation of variance for 3-hop counts, can be recovered via a shorter argument; see Corollary 4.

We proceed as follows. After presenting some background notation on point processes and Campbell measures, see [8], in Section 2 we review the derivation of moment identities for stochastic integrals using sums over partitions. In the multiparameter case we rewrite those identities for processes vanishing on diagonals, based on non-flat partition diagrams. In Section 3 we apply those results to the computation of the moments of k-hop counts in the random-connection model, and we specialize such computations to the case of Poisson point processes in Section 4. Section 5 is devoted to explicit computations in the case of Rayleigh fadings, which result in simpler derivations than the current literature on moments in the random-connection model.

1.1. Notation on point processes

Let X be a Polish space with Borel σ-algebra $\mathcal{B}(X)$, equipped with a σ-finite non-atomic measure $\lambda(\mathrm{d}x)$. We let

$$\Omega^X = \{\omega = \{x_i\}_{i\in I} \subset X : \#(A \cap \omega) < \infty \text{ for all compact } A \in \mathcal{B}(X)\}$$

denote the space of locally finite configurations on X whose elements $\omega \in \Omega^X$ are identified with the Radon point measures $\omega = \sum_{x\in\omega} \epsilon_x$, where ϵ_x denotes the Dirac measure at $x \in X$. A point process is a probability measure P on Ω^X equipped with the σ-algebra $\mathcal{F}$ generated by the topology of vague convergence.

Point processes can be characterized by their Campbell measure C defined on $\mathcal{B}(X) \otimes \mathcal{F}$ by

$$C(A \times B) := \mathrm{E}\left[\int_A \mathbf{1}_B(\omega \setminus \{x\})\, \omega(\mathrm{d}x)\right], \qquad A \in \mathcal{B}(X),\ B \in \mathcal{F},$$

which satisfies the Georgii–Nguyen–Zessin [14] identity

$$\mathrm{E}\left[\int_X u(x; \omega)\omega(\mathrm{d}x)\right] = \mathrm{E}\left[\int_{\Omega^X}\int_X u(x; \omega \cup x)C(\mathrm{d}x, \mathrm{d}\omega)\right] \tag{1}$$

for all measurable processes $u : X \times \Omega^X \to \mathbb{R}$ such that both sides of (1) make sense.

In the following we deal with point processes whose Campbell measure $C(\mathrm{d}x, \mathrm{d}\omega)$ is absolutely continuous with respect to $\lambda \otimes P$, i.e.

$$C(\mathrm{d}x, \mathrm{d}\omega) = c(x; \omega)\lambda(\mathrm{d}x)P(\mathrm{d}\omega),$$

where the density $c(x; \omega)$ is called the Papangelou density. We will also use the random measure $\hat{\lambda}^n(\mathrm{d}\mathfrak{x}_n)$ defined on X^n by

$$\hat{\lambda}^n(\mathrm{d}\mathfrak{x}_n) = \hat{c}(\mathfrak{x}_n; \omega)\lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_n),$$

where $\hat{c}(\mathfrak{x}_n; \omega)$ is the compound Campbell density $\hat{c} : \Omega_0^X \times \Omega^X \longrightarrow \mathbb{R}_+$ defined inductively on the set Ω_0^X of finite configurations in Ω^X by

$$\hat{c}(\{x_1, \ldots, x_n, y\}; \omega) := c(y; \omega)\hat{c}(\{x_1, \ldots, x_n\}; \omega \cup \{y\}), \qquad n \geq 0; \tag{2}$$

see Relation (1) in [5]. In particular, the Poisson point process with intensity $\lambda(\mathrm{d}x)$ is a point process with Campbell measure $C = \lambda \otimes P$ and $c(x; \omega) = 1$, and in this case the identity (1) becomes the Slivnyak–Mecke formula [20, 11]. Determinantal point processes are examples of point processes with Papangelou intensities, see, e.g., [6, Theorem 2.6], and they can be used for modeling wireless networks with repulsion; see, e.g., [7, 10, 13].

2. Moment identities

The moment of order $n \geq 1$ of a Poisson random variable Z_α with parameter $\alpha > 0$ is given by

$$\mathrm{E}[Z_\alpha^n] = \sum_{k=0}^{n} \alpha^k S(n, k), \qquad n \in \mathbb{N}, \tag{3}$$

where the Stirling number of the second kind $S(n, k)$ is the number of ways to partition a set of n objects into k non-empty subsets; see, e.g., [3, Proposition 3.1]. Regarding Poisson stochastic integrals of deterministic integrands, in [1] the moment formula

$$\mathrm{E}\Big[\Big(\int_X f(x)\omega(\mathrm{d}x)\Big)^n\Big] = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_1,\ldots,r_n \geq 0}} \prod_{k=1}^{n} \Big(\frac{1}{(k!)^{r_k} r_k!}\Big(\int_X f^k(x)\lambda(\mathrm{d}x)\Big)^{r_k}\Big) \tag{4}$$

has been proved for deterministic functions $f \in \bigcap_{p\geq 1} L^p(X, \lambda)$.

The identity (4) has been rewritten in the language of sums over partitions, and extended to Poisson stochastic integrals of random integrands in [18, Proposition 3.1], and further extended to point processes admitting a Panpangelou intensity in [5, Theorem 3.1]; see also [4]. In the following, given $\mathfrak{z}_n = (z_1, \ldots, z_n) \in X^n$, we will use the shorthand notation $\varepsilon^+_{\mathfrak{z}_n}$ for the operator

$$(\varepsilon^+_{\mathfrak{z}_n} F)(\omega) = F(\omega \cup \{z_1, \ldots, z_n\}), \qquad \omega \in \Omega,$$

where F is any random variable on Ω^X. Given $\rho = \{\rho_1, \ldots, \rho_k\} \in \Pi[n]$ a partition of $\{1, \ldots, n\}$ of size $|\rho| = k$, we let $|\rho_i|$ denote the cardinality of each block ρ_i, $i = 1, \ldots, k$.

Proposition 1. *Let $u : X \times \Omega^X \longrightarrow \mathbb{R}$ be a (measurable) process. For all $n \geq 1$ we have*

$$\mathrm{E}\Big[\Big(\int_X u(x;\omega)\omega(\mathrm{d}x)\Big)^n\Big] = \sum_{\rho\in\Pi[n]} \mathrm{E}\Big[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{l=1}^{|\rho|} u^{|\rho_l|}(z_l)\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big],$$

where the sum runs over all partitions ρ of $\{1, \ldots, n\}$ with cardinality $|\rho|$.

Proposition 1 has also been extended, together with joint moment identities, to multiparameter processes $(u_{z_1,\ldots z_r})_{(z_1,\ldots z_r)\in X^r}$; see [2, Theorem 3.1]. For this, let $\Pi[n \times r]$ denote the set of all partitions of the set

$$\Delta_{n\times r} := \{1, \ldots, n\} \times \{1, \ldots, r\} = \{(k, l) : k = 1, \ldots, n, \ l = 1, \ldots, r\},$$

identified to $\{1, \ldots, nr\}$, and let $\pi := (\pi_1, \ldots, \pi_n) \in \Pi[n \times r]$ denote the partition made of the n blocks $\pi_k := \{(k, 1), \ldots, (k, r)\}$ of size r, for $k = 1, \ldots, n$. Given $\rho = \{\rho_1, \ldots, \rho_m\}$ a partition of $\Delta_{n\times r}$, we let $\zeta^\rho : \Delta_{n\times r} \longrightarrow \{1, \ldots, m\}$ denote the mapping defined as

$$\zeta^\rho(k, l) = p \text{ if and only if } (k, l) \in \rho_p, \qquad k = 1, \ldots, n, \ l = 1, \ldots, r, \ p = 1, \ldots, m. \tag{5}$$

In other words, $\zeta^\rho(k, l)$ denotes the index p of the block $\rho_p \subset \Delta_{n\times r}$ to which (k, l) belongs.

Next, we restate Theorem 3.1 of [2] by noting that, in the same way as in Proposition 1, it extends to point processes admitting a Papangelou intensity using the arguments of [4, 5]. When $(u(z_1, \ldots, z_k; \omega))_{z_1,\ldots,z_k\in X}$ is a multiparameter process, we will write

$$\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k; \omega) := u(z_1, \ldots, z_k; \omega \cup \{z_1, \ldots, z_k\}), \quad \mathfrak{z}_n = (z_1, \ldots, z_n) \in X^n,$$

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and in this case we may drop the variable $\omega \in \Omega^X$ by writing $\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k; \omega)$ instead of $\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k; \omega)$.

Proposition 2. *Let $u : X^r \times \Omega^X \longrightarrow \mathbb{R}$ be a (measurable) r-process. We have*

$$\mathrm{E}\left[\left(\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\right)^n\right] = \sum_{\rho\in\Pi[n\times r]} \mathrm{E}\left[\int_{X^{|\rho|}} \varepsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^{n} u(z^{\rho}_{\pi_k})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\right] \quad (6)$$

with $z^{\rho}_{\pi_k} := (z_{\zeta^{\rho}(k,1)}, \ldots, z_{\zeta^{\rho}(k,r)})$, $k = 1, \ldots, n$.

Proof. The main change in the proof argument of [2] is to rewrite the proof of Lemma 2.1 therein by applying (2) recursively, as in the proof of [5, Theorem 3.1], while the main combinatorial argument remains identical. □

When $n = 1$, Proposition 2 yields a multivariate version of the Georgii–Nguyen–Zessin identity (1), i.e.

$$\mathrm{E}\left[\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\right]$$
$$= \sum_{\rho\in\Pi[1\times r]} \mathrm{E}\left[\int_{X^{|\rho|}} \varepsilon^+_{\mathfrak{z}_{|\rho|}} u(z_{\zeta^{\rho}(1,1)}, \ldots, z_{\zeta^{\rho}(1,r)}; \omega)\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\right].$$

2.1. Non-flat partitions

In the following we write $\nu \preceq \sigma$ when a partition $\nu \in \Pi[n \times r]$ is finer than another partition $\sigma \in \Pi[n \times r]$, i.e. when every block of ν is contained in a block of σ, and we let $\hat{0} := \{\{1, 1\}, \ldots, \{n, r\}\}$ denote the partition of $\Delta_{n\times r}$ made of singletons. We write $\rho \wedge \nu = \hat{0}$ when $\mu = \hat{0}$ is the only partition $\mu \in \Pi[n \times r]$ such that $\mu \preceq \nu$ and $\mu \preceq \rho$, i.e. $|\nu_k \cap \rho_l| \leq 1$, $k = 1, \ldots, n$, $l = 1, \ldots, |\rho|$. In this case we say that the partition diagram $\Gamma(\nu, \rho)$ of ν and ρ is *non-flat*; see [15, Chapter 4].

Here, a partition $\rho \in \Pi[n \times r]$ is said to be *non-flat* if the partition diagram $\Gamma(\pi, \rho)$ of ρ and the partition π is *non-flat*, where $\pi := (\pi_1, \ldots, \pi_n) \in \Pi[n \times r]$ with $\pi_k := \{(k, 1), \ldots, (k, r)\}$, $k = 1, \ldots, n$. Figure 1 shows an example of a non-flat partition with $n = 5$, $r = 4$, and

$$\begin{aligned}
\triangle &= \{(1, 2), (2, 1), (2, 2), (3, 3), (4, 2)\},\\
\bigcirc &= \{(1, 1), (3, 1), (4, 4), (5, 3)\},\\
\square &= \{(1, 3), (2, 4), (3, 3), (4, 1), (5, 4)\},\\
\pentagon &= \{(1, 4), (2, 2)\},\\
\times &= \{(2, 3), (3, 4), (4, 2), (5, 1)\}\\
\pi_k &= \{(k, 1), (k, 2), (k, 3), (k, 4), (k, 5)\}, \qquad k = 1, 2, 3, 4, 5.
\end{aligned}$$

2.2. Processes vanishing on diagonals

The next consequence of Proposition 2 shows that when $u(z_1, \ldots, z_r; \omega)$ vanishes on the diagonals in X^r, the moments of

$$\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)$$

reduce to sums over non-flat partition diagrams.

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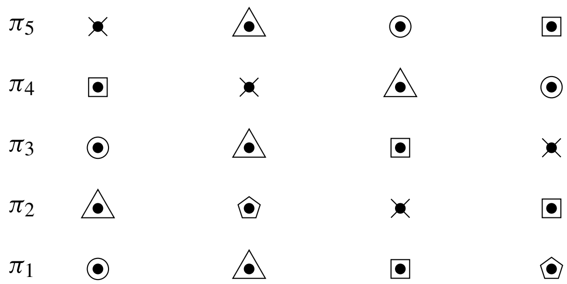

FIGURE 1: Example of a non-flat partition.

Proposition 3. *Assume that* $u(z_1, \ldots, z_r; \omega) = 0$ *whenever* $z_i = z_j$, $1 \le i \ne j \le r$, $\omega \in \Omega^X$. *Then we have*

$$\mathrm{E}\Big[\Big(\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big)^n\Big] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^{n} u(z^{\rho}_{\pi_k})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big].$$

Proof. Assume that $u(z_1, \ldots, z_r; \omega)$ vanishes on diagonals, and let $\rho \in \Pi[n]$. Then, for any $z_1, \ldots, z_r \in X$ we have

$$\prod_{k=1}^{n} u(z^{\rho}_{\pi_k}) = \prod_{k=1}^{n} u(z_{\zeta^{\rho}(k,1)}, \ldots, z_{\zeta^{\rho}(k,r)}) = 0$$

whenever $p := \zeta^{\rho}(k, a) = \zeta^{\rho}(k, b)$ for some $k \in \{1, \ldots, n\}$ and $a \ne b \in \{1, \ldots, r\}$. According to (5) this implies that $(k, a) \in \rho_p$ and $(k, b) \in \rho_p$; therefore ρ is not a non-flat partition, and it should be excluded from the sum over $\Pi[n]$. □

When $n = 1$, the first moment in Proposition 3 yields the Georgii–Nguyen–Zessin identity

$$\begin{aligned}\mathrm{E}\Big[\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big] &= \sum_{\substack{\rho\in\Pi[1\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} u(z^{\rho}_{\pi_1})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big]\\ &= \mathrm{E}\Big[\int_{X^r} \epsilon^+_{\mathfrak{z}_r} u(z_1, \ldots, z_r; \omega)\hat{\lambda}^r(\mathrm{d}\mathfrak{z}_r)\Big]; \qquad (7)\end{aligned}$$

see [9, Lemma IV.1] and [2, Lemma 2.1] for different versions based on the Poisson point process. In the case of second moments, we find that

$$\begin{aligned}&\mathrm{E}\Big[\Big(\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big)^2\Big]\\ &\qquad = \sum_{\substack{\rho\in\Pi[2\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} u(z^{\rho}_{\pi_1})u(z^{\rho}_{\pi_2})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big],\end{aligned}$$

and since the non-flat partitions in $\Pi[2 \times r]$ are made of pairs and singletons, this identity can be rewritten as the following consequence of Proposition 3, in which for simplicity of notation we write $\pi_1 = \{1, \ldots, r\}$ and $\pi_2 = \{r + 1, \ldots, 2r\}$.

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Corollary 1. *Assume that* $u(z_1, \ldots, z_r; \omega) = 0$ *whenever* $z_i = z_j$, $1 \leq i \neq j \leq r$, $\omega \in \Omega^X$. *Then the second moment of the integral of k-processes is given by*

$$\mathrm{E}\Big[\Big(\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big)^2\Big] = \sum_{A\subset\pi_1} \frac{1}{(r-|A|)!}$$
$$\times \sum_{\gamma:\pi_2\to A\cup\{r+1,\ldots,2r-|A|\}} \mathrm{E}\Big[\int_{X^{2r-|A|}} \epsilon^+_{\mathfrak{z}_{2r-|A|}} u(z_{\pi_1}) u(z_{\gamma(r+1)}, \ldots, z_{\gamma(2r)}) \hat{\lambda}^{2r-|A|}(\mathrm{d}\mathfrak{z}_{2r-|A|})\Big],$$

where the above sum is over all bijections $\gamma : \pi_2 \to A \cup \{r+1, \ldots, 2r-|A|\}$.

Proof. We express the partitions $\rho \in \Pi[n\times r]$ with non-flat diagrams $\Gamma(\pi, \rho)$ in Proposition 4 as the collections of pairs and singletons,

$$\rho = \{i, \gamma(i)\}\}_{i\in A} \cup \{\{i\}\}_{i\in\pi_1, i\notin A} \cup \{\{i\}\}_{i\in\pi_2, i\notin\gamma(A)},$$

for all subsets $A \subset \pi_1 = \{1, \ldots, r\}$ and bijections $\gamma : \pi_2 \to A \cup \{r+1, \ldots, 2r-|A|\}$. □

In the case of 2-processes, Corollary 1 shows that

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\int_{X^2} u(z_1, z_2; \omega)\omega(\mathrm{d}z_1)\omega(\mathrm{d}z_2)\Big)^2\Big] \\
&= \sum_{\substack{\rho\in\Pi[n\times 2]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^{n} u(z_{\zeta^\rho(k,1)}, z_{\zeta^\rho(k,2)}) \hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big] \\
&= \sum_{\substack{A\subset\pi_1\\ \gamma:\{3,4\}\to A\cup\{3,\ldots,4-|A|\}}} \frac{1}{(r-|A|)!} \mathrm{E}\Big[\int_{X^{4-|A|}} \epsilon^+_{\mathfrak{z}_{4-|A|}} u(z_1, z_2) u(z_{\gamma(3)}, z_{\gamma(4)}) \hat{\lambda}^{4-|A|}(\mathrm{d}\mathfrak{z}_{4-|A|})\Big] \\
&= \mathrm{E}\Big[\int_{X^4} \epsilon^+_{\mathfrak{z}_4}(u(z_1, z_2)u(z_3, z_4))\hat{\lambda}^4(\mathrm{d}\mathfrak{z}_4)\Big] \\
&\quad + \mathrm{E}\Big[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_1, z_2)u(z_1, z_3))\hat{\lambda}^3(\mathrm{d}\mathfrak{z}_3)\Big] + \mathrm{E}\Big[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_2, z_1)u(z_3, z_1))\hat{\lambda}^3(\mathrm{d}\mathfrak{z}_3)\Big] \\
&\quad + \mathrm{E}\Big[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_1, z_2)u(z_2, z_3))\hat{\lambda}^3(\mathrm{d}\mathfrak{z}_3)\Big] + \mathrm{E}\Big[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_2, z_1)u(z_3, z_2))\hat{\lambda}^3(\mathrm{d}\mathfrak{z}_3)\Big] \\
&\quad + \mathrm{E}\Big[\int_{X^2} \epsilon^+_{\mathfrak{z}_2}(u(z_1, z_2)u(z_1, z_2))\hat{\lambda}^2(\mathrm{d}\mathfrak{z}_2)\Big] + \mathrm{E}\Big[\int_{X^2} \epsilon^+_{\mathfrak{z}_2}(u(z_1, z_2)u(z_2, z_1))\hat{\lambda}^2(\mathrm{d}\mathfrak{z}_2)\Big].
\end{aligned}$$

Similarly, in the case of 3-processes we find

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\int_{X^3} u(z_1, z_2, z_3; \omega)\omega(\mathrm{d}z_1)\omega(\mathrm{d}z_2)\omega(\mathrm{d}z_3)\Big)^2\Big] \\
&= \sum_{\substack{A\subset\{1,2,3\}\\ \gamma:\{4,5,6\}\to A\cup\{4,\ldots,6-|A|\}}} \frac{1}{(3-|A|)!} \mathrm{E}\Big[\int_{X^5} \epsilon^+_{\epsilon^+_{\mathfrak{z}_5}} u(z_1, z_2, z_3) u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)}) \hat{\lambda}^5(\mathrm{d}\mathfrak{z}_5)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\Big[\int_{X^6} \epsilon^+_{\mathfrak{z}_6} u(z_1, z_2, z_3)u(z_4, z_5, z_6)\hat{\lambda}^6(\mathrm{d}\mathfrak{z}_6)\Big] \\
&\quad + \frac{1}{2} \sum_{\gamma:\{4,5,6\}\to\{1,5,6\}} \mathrm{E}\Big[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^5(\mathrm{d}\mathfrak{z}_5)\Big] \\
&\quad + \frac{1}{2} \sum_{\gamma:\{4,5,6\}\to\{2,5,6\}} \mathrm{E}\Big[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^5(\mathrm{d}\mathfrak{z}_5)\Big] \\
&\quad + \frac{1}{2} \sum_{\gamma:\{4,5,6\}\to\{3,5,6\}} \mathrm{E}\Big[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^5(\mathrm{d}\mathfrak{z}_5)\Big] \\
&\quad + \sum_{\gamma:\{4,5,6\}\to\{1,2,6\}} \mathrm{E}\Big[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^4(\mathrm{d}\mathfrak{z}_4)\Big] \\
&\quad + \sum_{\gamma:\{4,5,6\}\to\{1,3,6\}} \mathrm{E}\Big[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^4(\mathrm{d}\mathfrak{z}_4)\Big] \\
&\quad + \sum_{\gamma:\{4,5,6\}\to\{2,3,6\}} \mathrm{E}\Big[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^4(\mathrm{d}\mathfrak{z}_4)\Big] \\
&\quad + \sum_{\gamma:\{4,5,6\}\to\{1,2,3\}} \mathrm{E}\Big[\int_{X^3} \epsilon^+_{\mathfrak{z}_3} z u(z_1, z_2, z_3)u(z_{\gamma(4)}, z_{\gamma(5)}, z_{\gamma(6)})\hat{\lambda}^3(\mathrm{d}\mathfrak{z}_3)\Big].
\end{aligned}
$$

3. Random-connection model

Two point process vertices $x \neq y$ are independently connected in the random-connection graph with the probability $H(x, y)$ given $\omega \in \Omega^X$, where $H : X \times X \longrightarrow [0, 1]$. In particular, the 1-hop count $\mathbf{1}_{\{x\leftrightarrow y\}}$ is a Bernoulli random variable with parameter $H(x, y)$, and we have the relation

$$
\mathrm{E}\bigg[\epsilon^+_{\mathfrak{z}_r} \prod_{i=0}^{r} \mathbf{1}_{\{z_i\leftrightarrow z_{i+1}\}}(\omega) \,\bigg|\, \omega\bigg] = \prod_{i=0}^{r} H(z_i, z_{i+1})
$$

for any subset $\{z_0, \ldots, z_{r+1}\}$ of distinct elements of X, where $\mathfrak{z}_r = \{z_1, \ldots, z_r\}$ and $x \leftrightarrow y$ means that $x \in X$ is connected to $y \in X$.

Given $x, y \in X$, the number of $(r + 1)$-hop sequences $z_1, \ldots, z_r \in \omega$ of vertices connecting x to y in the random graph is given by the multiparameter stochastic integral

$$
N^{x,y}_{r+1} = \int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1) \cdots \omega(\mathrm{d}z_r)
$$

of the $\{0, 1\}$-valued r-process

$$
u(z_1, \ldots, z_r; \omega) := \mathbf{1}_{\{z_i\neq z_j,\ 1\leq i<j\leq r\}}\mathbf{1}_{\{z_1,\ldots,z_r\in\omega\}} \prod_{i=0}^{r} \mathbf{1}_{\{z_i\leftrightarrow z_{i+1}\}}(\omega), \qquad z_1, \ldots, z_r \in X, \tag{8}
$$

which vanishes on the diagonals in X^r, with $z_0 := x$ and $z_{r+1} := y$. In addition, for any distinct $z_1, \ldots, z_r \in X$ and $u(z_1, \ldots, z_r; \omega)$ given by (8) we have

$$
\mathrm{E}[\epsilon^+_{\mathfrak{z}_r} u(z_1, \ldots, z_r; \omega) \mid \omega] = \mathrm{E}\bigg[\epsilon^+_{\mathfrak{z}_r} \prod_{i=0}^{r} \mathbf{1}_{\{z_i\leftrightarrow z_{i+1}\}}(\omega) \,\bigg|\, \omega\bigg] = \prod_{i=0}^{r} H(z_i, z_{i+1}), \tag{9}
$$

and therefore the first-order moment of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given as

$$\mathrm{E}\Big[\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big] = \mathrm{E}\Big[\int_{X^r}\prod_{i=0}^{r} H(z_i, z_{i+1})\hat{\lambda}^r(\mathrm{d}\mathfrak{z}_r)\Big] \tag{10}$$

(see also [9, Theorem II.1]) as a consequence of the Georgii–Nguyen–Zessin identity (7).

In the next proposition we compute the moments of all orders of r-hop counts as sums over non-flat partition diagrams. The role of the powers $1/n^{\rho}_{l,i}$ in (11) is to ensure that all powers of $H(x, y)$ in (11) are equal to one, since all powers of $\mathbf{1}_{\{z\leftrightarrow z'\}}$ in (12) below are equal to $\mathbf{1}_{\{z\leftrightarrow z'\}}$.

Proposition 4. *The moment of order n of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given by*

$$\mathrm{E}[(N^{x,y}_{r+1})^n] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}}\prod_{l=1}^{n}\prod_{i=0}^{r} H^{1/n^{\rho}_{l,i}}(z_{\zeta^{\rho}(l,i)}, z_{\zeta^{\rho}(l,i+1)})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big], \tag{11}$$

where $z_0 = x$, $z_{r+1} = y$, $\zeta^{\rho}(l, 0) = 0$, $\zeta^{\rho}(l, r+1) = r+1$, and

$$n^{\rho}_{l,i} = \#\{(p, j) \in \{1, \ldots, n\} \times \{0, \ldots, r\} : \{\zeta^{\rho}(l, i), \zeta^{\rho}(l, i+1)\} = \{\zeta^{\rho}(p, j), \zeta^{\rho}(p, j+1)\}\},$$

$1 \le l \le n$, $0 \le i \le r$.

Proof. Since $u(z_1, \ldots, z_r; \omega)$ vanishes whenever $z_i = z_j$ for some $1 \le i < j \le r$, by Proposition 3 we have

$$\begin{aligned}
&\mathrm{E}\Big[\Big(\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(\mathrm{d}z_1)\cdots\omega(\mathrm{d}z_r)\Big)^n\Big]\\
&= \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}}\prod_{l=1}^{n}\prod_{i=0}^{r} \mathbf{1}_{\{z_{\zeta^{\rho}(l,i)}\leftrightarrow z_{\zeta^{\rho}(l,i+1)}\}}\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big] \qquad (12)\\
&= \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathrm{E}\Big[\int_{X^{|\rho|}}\prod_{l=1}^{n}\prod_{i=0}^{r} H^{1/n^{\rho}_{l,i}}(z_{\zeta^{\rho}(l,i)}, z_{\zeta^{\rho}(l,i+1)})\hat{\lambda}^{|\rho|}(\mathrm{d}\mathfrak{z}_{|\rho|})\Big],
\end{aligned}$$

where we applied (9). □

As in Corollary 1, we have the following consequence of Proposition 4, which is obtained by expressing the partitions $\rho \in \Pi[n \times r]$ with non-flat diagrams $\Gamma(\pi, \sigma)$ as a collection of pairs and singletons.

Corollary 2. *The second moment of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given by*

$$\begin{aligned}
\mathrm{E}[(N^{x,y}_{r+1})^2] = &\sum_{\substack{A\subset\pi_1\\ \gamma:\{1,\ldots,r\}\to A\cup\{r+1,\ldots,2r-|A|\}}} \frac{1}{(r-|A|)!}\\
&\times \mathrm{E}\Big[\int_{X^{2r-|A|}}\prod_{i=0}^{r} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1})\prod_{j=0}^{r} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{2r-|A|}(\mathrm{d}\mathfrak{z}_{2r-|A|})\Big],
\end{aligned}$$

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where the above sum is over all bijections $\gamma : \{1, \ldots, r\} \to A \cup \{r+1, \ldots, 2r-|A|\}$ *with* $\gamma(0) := 0$, $\gamma(r+1) =: r+1$, $z_0 =: x$, *and* $z_{r+1} := y$, *and*

$$n^{\gamma}_{1,i} = \#\{j \in \{0, \ldots, r\} : \{i, i+1\} = \{\gamma(j), \gamma(j+1)\}\},$$
$$n^{\gamma}_{2,j} = \#\{i \in \{0, \ldots, r\} : (i, i+1) = (\gamma(j), \gamma(j+1))\},$$

for $0 \le i \le r$.

3.1. Variance of 3-hop counts

When $n = 2$ and $r = 2$, Corollary 2 allows us to express the variance of the 3-hop count between $x \in X$ and $y \in X$ as follows:

$$\begin{aligned}
&\operatorname{Var}[N_3^{x,y}] \\
&= \sum_{\substack{\varnothing \neq A \subset \{1,2\} \\ \gamma:\{1,2\}\to A\cup\{3,4-|A|\}}} \frac{1}{(2-|A|)!} \\
&\quad \times \mathrm{E}\bigg[\int_{X^{4-|A|}} \prod_{i=0}^{2} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{4-|A|}(\mathrm{d}\mathfrak{z}_{4-|A|})\bigg] \\
&= \sum_{\gamma:\{1,2\}\to\{1,4\}} \mathrm{E}\bigg[\int_{X^3} \prod_{i=0}^{2} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{3}(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_4)\bigg] \\
&\quad + \sum_{\gamma:\{1,2\}\to\{2,4\}} \mathrm{E}\bigg[\int_{X^3} \prod_{i=0}^{2} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{3}(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_4)\bigg] \\
&\quad + \sum_{\gamma:\{1,2\}\to\{1,2\}} \mathrm{E}\bigg[\int_{X^2} \prod_{i=0}^{2} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{2}(\mathrm{d}z_1, \mathrm{d}z_2)\bigg].
\end{aligned}$$

3.2. Variance of 4-hop counts

When $r = 3$ and $n = 2$, Corollary 2 yields

$$\begin{aligned}
\operatorname{Var}[N_4^{x,y}] &= \sum_{\substack{\varnothing \neq A \subset \pi_1 \\ \gamma:\{1,\ldots,3\}\to A\cup\{4,\ldots,6-|A|\}}} \frac{1}{(3-|A|)!} \mathrm{E}\bigg[\int_{X^{6-|A|}} \prod_{i=0}^{3} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \\
&\qquad \times \prod_{j=0}^{3} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{6-|A|}(d\mathfrak{z}_{6-|A|})\bigg] \\
&= \frac{1}{2} \sum_{\gamma:\{1,\ldots,3\}\to\{1,5,6\}} \mathrm{E}\bigg[\int_{X^5} \prod_{i=0}^{3} H^{1/n^{\gamma}_{1,i}}(z_i, z_{i+1}) \\
&\qquad \times \prod_{j=0}^{3} H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)}) \hat{\lambda}^{5}(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3, \mathrm{d}z_5, \mathrm{d}z_6)\bigg]
\end{aligned}$$

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$$+\frac{1}{2}\sum_{\gamma:\{1,\ldots,3\}\to\{2,5,6\}}\mathrm{E}\bigg[\int_{X^5}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^5(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_5,\mathrm{d}z_6)\bigg]$$
$$+\frac{1}{2}\sum_{\gamma:\{1,\ldots,3\}\to\{3,5,6\}}\mathrm{E}\bigg[\int_{X^5}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^5(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_5,\mathrm{d}z_6)\bigg]$$
$$+\sum_{\gamma:\{1,\ldots,3\}\to\{1,2,6\}}\mathrm{E}\bigg[\int_{X^4}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^4(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_6)\bigg]$$
$$+\sum_{\gamma:\{1,\ldots,3\}\to\{1,3,6\}}\mathrm{E}\bigg[\int_{X^4}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^4(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_6)\bigg]$$
$$+\sum_{\gamma:\{1,\ldots,3\}\to\{2,3,6\}}\mathrm{E}\bigg[\int_{X^4}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^4(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_6)\bigg]$$
$$+\sum_{\gamma:\{1,\ldots,3\}\to\{1,\ldots,3\}}\mathrm{E}\bigg[\int_{X^3}\prod_{i=0}^{3}H^{1/n^{\gamma}_{1,i}}(z_i,z_{i+1})$$
$$\times\prod_{j=0}^{3}H^{1/n^{\gamma}_{2,j}}(z_{\gamma(j)},z_{\gamma(j+1)})\hat{\lambda}^3(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3)\bigg].$$

4. Poisson case

In this section and the next one we will work in the Poisson random-connection model, using a Poisson point process on $X=\mathbb{R}^d$ with intensity $\lambda(\mathrm{d}x)$ on $\mathbb{R}^d$. We let

$$H^{(n)}(x_0,x_n):=\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{i=0}^{n-1}H(x_i,x_{i+1})\lambda(\mathrm{d}x_1)\cdots\lambda(\mathrm{d}x_{n-1}),\qquad x_0,x_n\in\mathbb{R}^d,\ n\geq 1. \tag{13}$$

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The 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is given by the first-order stochastic integral

$$\int_{\mathbb{R}^d} u(z;\omega)\omega(\mathrm{d}z) = \int_{\mathbb{R}^d} \mathbf{1}_{\{x\leftrightarrow z_1\}}\mathbf{1}_{\{z_1\leftrightarrow y\}}(\omega)\omega(\mathrm{d}z_1) = \int_{\mathbb{R}^d} \mathbf{1}_{\{x\leftrightarrow z_1\}}\mathbf{1}_{\{z_1\leftrightarrow y\}}\omega(\mathrm{d}z_1),$$

and its moment of order n is

$$\begin{aligned}
\mathrm{E}\bigg[\bigg(\int_{\mathbb{R}^d} u(z_1;\omega)\omega(\mathrm{d}z_1)\bigg)^n\bigg] &= \mathrm{E}\bigg[\bigg(\int_{\mathbb{R}^d} \mathbf{1}_{\{x\leftrightarrow z_1\}}\mathbf{1}_{\{z_1\leftrightarrow y\}}\omega(\mathrm{d}z_1)\bigg)^n\bigg] \\
&= \sum_{\rho\in\Pi[n\times 1]} \int_{X^{|\rho|}} \prod_{l=1}^{|\rho|} \big(H(x,z_l)H(z_l,y)\big)\lambda^{|\rho|}(\mathrm{d}z_1,\ldots,\mathrm{d}z_{|\rho|}) \\
&= \sum_{k=1}^{n} S(n,k)\bigg(\int_{\mathbb{R}^d} H(x,z)H(z,y)\lambda(\mathrm{d}z)\bigg)^k \\
&= \sum_{k=1}^{n} S(n,k)\big(H^{(2)}(x,y)\big)^k;
\end{aligned}$$

therefore, from (3), the 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is a Poisson random variable with mean

$$H^{(2)}(x,y) = \int_{\mathbb{R}^d} H(x,z)H(z,y)\lambda(\mathrm{d}z).$$

By (10), the first-order moment of the r-hop count is given by

$$H^{(r)}(x,y) = \int_{X^{r-1}} \prod_{i=0}^{r-1} H(z_i,z_{i+1})\lambda^{r-1}(\mathrm{d}z_1,\ldots\mathrm{d}z_{r-1}).$$

Corollary 3. *The variance of the r-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is given by*

$$\begin{aligned}
\mathrm{Var}[N_r^{x,y}] = \sum_{p=1}^{r-1} \sum_{\substack{1\le k_1<\cdots<k_p<r \\ 1\le l_1<\cdots<l_p<r}} \sum_{\sigma\in\Sigma[p]} \int_{X^p} \prod_{0\le i\le p} H^{(k_{i+1}-k_i)}(z_i,z_{i+1}) \\
\times \prod_{\substack{0\le j\le p \\ l_{\sigma(j+1)}-l_{\sigma(j)}+k_{j+1}-k_j>2 \\ \text{or } \{j,j+1\}\neq\{\sigma(j),\sigma(j+1)\}}} H^{(l_{\sigma(j+1)}-l_{\sigma(j)})}(z_{\sigma(j)},z_{\sigma(j+1)})\lambda^p(\mathrm{d}\mathfrak{z}_p),
\end{aligned}$$

with $k_0 = l_0 = 0$, $k_{p+1} = l_{p+1} = r$, $\sigma(0) = 0$, and $\sigma(r) = r$, where the above sum is over all permutations $\sigma \in \Sigma[p]$ of $\{1,\ldots,p\}$.

Proof. We rewrite the result of Corollary 2 by denoting the set $A \subset \pi_1$ as $A = \{k_1,\ldots,k_p\}$, for $1 \le k_1 < \cdots < k_p \le r-1$, and we identify $\gamma(A) \subset A \cup \{r+1,\ldots,2r-|A|\}$ with $\{l_1,\ldots,l_p\}$, which requires a sum over the permutations of $\{1,\ldots,p\}$ since $1 \le l_1 < \cdots < l_p \le r-1$, where $1 \le p \le r-1$. In addition, the multiple integrals over contiguous index sets in A^c are evaluated using (13). □

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4.1. Variance of 3-hop counts

When $n=2$ and $r=2$ Corollary 3 allows us to compute the variance of the 3-hop count between $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$, as follows:

$$\begin{aligned}\mathrm{Var}[N_3^{x,y}] = 2\int_{\mathbb{R}^d} H(x,z_1)H^{(2)}(z_1,y)H^{(2)}(z_1,y)\lambda(\mathrm{d}z_1)\\ +2\int_{\mathbb{R}^d} H(x,z_1)H^{(2)}(x,z_1)H^{(2)}(z_1,y)H(z_1,y)\lambda(\mathrm{d}z_1)\\ +\int_{X^2} H(x,z_1)H(z_1,z_2)H(z_2,y)H(x,z_2)H(z_1,y)\lambda^2(\mathrm{d}z_1,\mathrm{d}z_2)+H^{(3)}(x,y).\end{aligned} \quad (14)$$

By Corollary 3 the variance of 4-hop counts can be similarly computed explicitly as a sum of 33 terms.

5. Rayleigh fading

In this section we consider a Poisson point process on $X=\mathbb{R}^d$ with flat intensity $\lambda(\mathrm{d}x)=\lambda\mathrm{d}x$ on $\mathbb{R}^d$, $\lambda>0$, and a Rayleigh fading function of the form

$$H_\beta(x,y) := \mathrm{e}^{-\beta\|x-y\|^2}, \qquad x,y\in\mathbb{R}^d,\ \beta>0.$$

Lemmas 1 and 2 can be used to evaluate the integrals appearing in Corollary 3 and in the variance (14) of the 3-hop counts.

Lemma 1. *For all $n\ge 1$, $y_1,\dots,y_n\in\mathbb{R}^d$, and $\beta_1,\dots,\beta_n>0$ we have*

$$\begin{aligned}&\int_{\mathbb{R}^d}\prod_{i=1}^n H_{\beta_i}(x,y_i)\mathrm{d}x\\ &\quad=\left(\frac{\pi}{\beta_1+\cdots+\beta_n}\right)^{d/2}\prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1},\frac{\beta_1y_1+\cdots+\beta_iy_i}{\beta_1+\cdots+\beta_i}\right).\end{aligned}$$

Proof. We start by showing that for all $n\ge 1$ we have

$$\begin{aligned}&\prod_{i=1}^n H_{\beta_i}(x,y_i)\\ &=H_{\beta_1+\cdots+\beta_n}\left(x,\frac{\beta_1y_1+\cdots+\beta_ny_n}{\beta_1+\cdots+\beta_n}\right)\prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1},\frac{\beta_1y_1+\cdots+\beta_iy_i}{\beta_1+\cdots+\beta_i}\right).\end{aligned} \quad (15)$$

Clearly, this relation holds for $n=1$. In addition, at the rank $n=2$ we have

$$\begin{aligned}&H_{\beta_1}(x,y_1)H_{\beta_2}(x,y_2)=\mathrm{e}^{-\beta_1\|y_1-x\|^2}\mathrm{e}^{-\beta_2\|x-y_2\|^2}\\ &\quad= \exp\{-\beta_1\|y_1\|^2-\beta_2\|y_2\|^2+2\langle\beta_1y_1+\beta_2y_2,x\rangle-(\beta_1+\beta_2)\|x\|^2\}\\ &\quad= \exp\{-\beta_1\|y_1\|^2-\beta_2\|y_2\|^2-(\beta_1+\beta_2)\|x-(\beta_1y_1+\beta_2y_2)/(\beta_1+\beta_2)\|^2\\ &\qquad\qquad +\|\beta_1y_1+\beta_2y_2\|^2/(\beta_1+\beta_2)\}\\ &\quad= \exp\{-(\beta_1+\beta_2)\|x-(\beta_1y_1+\beta_2y_2)/(\beta_1+\beta_2)\|^2-\beta_1\beta_2\|y_1-y_2\|^2/(\beta_1+\beta_2)\}\\ &\quad= H_{\beta_1+\beta_2}\left(x,\frac{\beta_1y_1+\beta_2y_2}{\beta_1+\beta_2}\right)H_{\frac{\beta_1\beta_2}{\beta_1+\beta_2}}(y_1,y_2).\end{aligned}$$

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Next, assuming that (15) holds at the rank $n \geq 1$, we have

$$
\begin{aligned}
\prod_{i=1}^{n+1} H_{\beta_i}(x, y_i) = H_{\beta_{n+1}}(x, y_{n+1}) H_{\beta_1+\cdots+\beta_n}\left(x, \frac{\beta_1 y_1 + \cdots + \beta_n y_n}{\beta_1 + \cdots + \beta_n}\right) \\
\times \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i}\right) \\
= H_{\beta_1+\cdots+\beta_{n+1}}\left(x, \frac{\beta_1 y_1 + \cdots + \beta_{n+1} y_{n+1}}{\beta_1 + \cdots + \beta_n}\right) \\
\times \prod_{i=1}^{n} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i}\right).
\end{aligned}
$$

As a consequence, we find that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \prod_{i=1}^{n} H_{\beta_i}(x, y_i) \mathrm{d}x = \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i}\right) \\
\times \int_{\mathbb{R}^d} H_{\beta_1+\cdots+\beta_n}\left(x, \frac{\beta_1 y_1 + \cdots + \beta_n y_n}{\beta_1 + \cdots + \beta_n}\right) \mathrm{d}x \\
= \left(\frac{\pi}{\beta_1 + \cdots + \beta_n}\right)^{d/2} \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}}\left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i}\right). \quad \square
\end{aligned}
$$

In particular, applying Lemma 1 for $n = 2$ yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} H_{\beta_1}(y_1, x) H_{\beta_2}(x, y_2) \mathrm{d}x &= \left(\frac{\pi}{\beta_1 + \beta_2}\right)^{d/2} H_{\frac{\beta_1\beta_2}{\beta_1+\beta_2}}(y_1, y_2) \\
&= \left(\frac{\pi}{\beta_1 + \beta_2}\right)^{d/2} \mathrm{e}^{-\beta_1\beta_2\|y_1-y_2\|^2/(\beta_1+\beta_2)}, \quad y_1, y_2 \in \mathbb{R}^d, \qquad (16)
\end{aligned}
$$

and the 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is a Poisson random variable with mean

$$
\begin{aligned}
H_\beta^{(2)}(x, y) &= \lambda \int_{\mathbb{R}^d} H_\beta(x, z) H_\beta(z, y) \mathrm{d}z \\
&= \lambda \left(\frac{\pi}{2\beta}\right)^{d/2} H_{\beta/2}(x, y) \\
&= \lambda \left(\frac{\pi}{2\beta}\right)^{d/2} \mathrm{e}^{-\|x-y\|^2/2}.
\end{aligned}
$$

By an induction argument similar to that of Lemma 1, we obtain the following lemma.

Lemma 2. *For all $n \geq 1$, $x_0, \ldots, x_n \in \mathbb{R}^d$, and $\beta_1, \ldots, \beta_n > 0$ we have*

$$
\begin{aligned}
\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=1}^{n} H_{\beta_i}(x_{i-1}, x_i) \, \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} \\
= \left(\frac{\pi^{n-1}}{\sum_{i=1}^{n} \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n}\right)^{d/2} H_{\frac{\beta_1 \cdots \beta_n}{\sum_{i=1}^{n} \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n}}(x_0, y_n).
\end{aligned}
$$

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Proof. Clearly the relation holds at the rank $n = 1$. Assuming that it holds at the rank $n \geq 1$ and using (16), we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=1}^{n+1} H_{\beta_i}(x_{i-1}, x_i)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \int_{\mathbb{R}^d} H_{\beta_{n+1}}(x_n, x_{n+1}) \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=1}^{n} H_{\beta_i}(x_{i-1}, x_i)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \left(\frac{\pi^{n-1}}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n} \right)^{d/2} \\
&\quad \times \int_{\mathbb{R}^d} H_{\frac{\beta_1 \cdots \beta_n}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n}}(x_0, x_n) H_{\beta_{n+1}}(x_n, x_{n+1})\,\mathrm{d}x_n \\
&= \left(\frac{\pi^{n-1}}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n} \right)^{d/2} \\
&\quad \times \left(\frac{\pi}{\frac{\beta_1 \cdots \beta_n}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_n} + \beta_{n+1}} \right)^{d/2} H_{\frac{\beta_1 \cdots \beta_{n+1}}{\sum_{i=1}^{n+1} \beta_1 \cdots \beta_{i-1}\beta_{i+1} \cdots \beta_{n+1}}}(x_0, x_{n+1}). \qquad \square
\end{aligned}$$

In particular, the first-order moment of the r-hop count between $x_0 \in \mathbb{R}^d$ and $x_r \in \mathbb{R}^d$ is given by

$$\begin{aligned}
H_\beta^{(r)}(x_0, x_r) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=0}^{r-1} H_\beta(x_i, x_{i+1}) \lambda(\mathrm{d}x_1) \cdots \lambda(\mathrm{d}x_{r-1}) \\
&= \lambda^{r-1} \left(\frac{\pi^{r-1}}{r\beta^{r-1}} \right)^{d/2} H_{\beta/r}(x, y) \\
&= \lambda^{r-1} \left(\frac{\pi^{r-1}}{r\beta^{r-1}} \right)^{d/2} \mathrm{e}^{-\beta \|x-y\|^2/r}, \qquad x, y \in \mathbb{R}^d. \qquad (17)
\end{aligned}$$

5.1. Variance of 3-hop counts

Corollary 3 and Lemma 2 allow us to recover Theorem II.3 of [9] for the variance of 3-hop counts by a shorter argument, while extending it from the plane $X = \mathbb{R}^2$ to $X = \mathbb{R}^d$.

Corollary 4. *The variance of the* 3*-hop count between* $x \in \mathbb{R}^d$ *and* $y \in \mathbb{R}^d$ *is given by*

$$\begin{aligned}
\mathrm{Var}[N_3^{x,y}] &= 2\lambda^3 \left(\frac{\pi^3}{8\beta^3} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2/2} + \lambda^2 \left(\frac{\pi^2}{3\beta^2} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2/3} \\
&\quad + 2\lambda^3 \left(\frac{\pi^3}{12\beta^3} \right)^{d/2} \mathrm{e}^{-3\beta\|x-y\|^2/4} + \lambda^2 \left(\frac{\pi^2}{8\beta^2} \right)^{d/2} \mathrm{e}^{-\beta\|x-y\|^2}.
\end{aligned}$$

Proof. By (17) and Lemma 2 we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta^{(2)}(z_1, y) H_\beta^{(2)}(z_1, y) \lambda(\mathrm{d}z_1) \\
&= \lambda^2 \left(\frac{\pi^2}{4\beta^2} \right)^{d/2} \int_{\mathbb{R}^d} H_\beta(x, z_1) H_{\beta/2}^2(z_1, y) \lambda(\mathrm{d}z_1)
\end{aligned}$$

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$$= \lambda^3 \Big(\frac{\pi^2}{4\beta^2}\Big)^{d/2} \int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta(z_1, y) \lambda(\mathrm{d}z_1) = \lambda^3 \Big(\frac{\pi^3}{8\beta^3}\Big)^{d/2} H_{\beta/2}(x, y);$$

$$\int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta^{(2)}(x, z_1) H_\beta^{(2)}(z_1, y) H_\beta(z_1, y) \lambda(\mathrm{d}z_1)$$

$$= \lambda^2 \Big(\frac{\pi^2}{4\beta^2}\Big)^{d/2} \int_{\mathbb{R}^d} H_{3\beta/2}(z_1, y) H_{3\beta/2}(x, z_1) \lambda(\mathrm{d}z_1) = \lambda^3 \Big(\frac{\pi^3}{12\beta^3}\Big)^{d/2} H_{3\beta/4}(x, y);$$

$$\int_{X^2} H_\beta(x, z_1) H_\beta(z_1, z_2) H_\beta(z_2, y) H_\beta(x, z_2) H_\beta(z_1, y) \lambda^2(\mathrm{d}z_1, \mathrm{d}z_2)$$

$$= \lambda \Big(\frac{\pi}{3\beta}\Big)^{d/2} H_\beta(x, y) \int_{\mathbb{R}^d} H_{2\beta/3}(z_2, (x+y)/2) H_{2\beta}(z_2, (x+y)/2) \lambda(\mathrm{d}z_2)$$

$$= \lambda^2 \Big(\frac{\pi^2}{8\beta^2}\Big)^{d/2} H_\beta(x, y);$$

and we conclude by (14). □

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