106.46 On the Gerretsen inequalities in trigonometrical form

The first Gerretsen inequality for a triangle ABC with semi-perimeter s, circumradius R and inradius r is

$$s^2 - 16Rr + 5r^2 \geqslant 0. \qquad (1)$$

This is essential for our proof that the incentre I lies on the symmedicentroidal disc ([1]).

A trigonometrical form will now be found in terms of the sum S and product P of the half-angle tangents $l = \tan\frac{1}{2}A$, $m = \tan\frac{1}{2}B$ and $n = \tan\frac{1}{2}C$.

Key formulae

Note first that l, m, n are positive and that the formula

$$mn + nl + lm = 1 \qquad (2)$$

follows easily from $\cot\frac{1}{2}A = \tan\frac{1}{2}(\pi - A) = \tan\frac{1}{2}(B + C)$. The constraint (2) is used to generate ad hoc results such as

$$l^2 + m^2 + n^2 = (l + m + n)^2 - 2(mn + nl + lm) = S^2 - 2$$

and

$$m^2n^2 + n^2l^2 + l^2m^2 = (mn + nl + lm)^2 - 2(l + m + n)lmn = 1 - 2SP.$$

We next find bounds for $S = l + m + n$ and $P = lmn$.

So from $(m - n)^2 + (n - l)^2 + (l - m)^2 \geqslant 0$ we have

$$2(l + m + n)^2 - 2(mn + nl + lm) = 2\left(S^2 - 2\right) - 2 \geqslant 0,$$

that is $S^2 \geqslant 3$ or $S \geqslant \sqrt{3}$.

Now the geometric and harmonic means of l, m, n are $P^{1/3}$ and $3P$, respectively, so

$$P^{1/3} \geqslant 3P \qquad \text{or} \qquad 0 < P \leqslant \frac{1}{3\sqrt{3}}.$$

Hence

$$S - 9P \geqslant 0. \qquad (3)$$

Next from

$$(1 - 3mn)^2 + (1 - 3nl)^2 + (1 - 3lm)^2 \geqslant 0$$

we have

$$3 - 6(mn + nl + lm) + 9\left(m^2n^2 + n^2l^2 + l^2m^2\right) = 3 - 6 + 9(1 - 2SP) \geqslant 0,$$

that is,

$$1 - 3SP \geqslant 0. \qquad (4)$$

We also have

$$(m + n)(n + l)(l + m) = (S - l)(S - m)(S - n)$$

$$= S^3 - (l + m + n)S^2 + (mn + nl + lm)S - lmn = S - P. \qquad (5)$$

The first Gerretsen inequality

The formulae $r = (s - a)\tan\frac{1}{2}A = (s - b)\tan\frac{1}{2}B = (s - c)\tan\frac{1}{2}C$ lead

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easily to

$$s = \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n}\right)r = \frac{r}{lmn} = \frac{r}{P}. \tag{6}$$

Then by transposing and squaring the formula $r = 4R \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C$ (and using (5)) we have

$$\begin{aligned}\frac{16R^2}{r^2} &= \left(1 + \frac{1}{l^2}\right)\left(1 + \frac{1}{m^2}\right)\left(1 + \frac{1}{n^2}\right) \\ &= \frac{(1 + l^2)(1 + m^2)(1 + n^2)}{P^2} \\ &= \left(\frac{(m + n)(n + l)(l + m)}{P}\right)^2 \\ &= \left(\frac{S - P}{P}\right)^2 \qquad \text{or} \qquad \frac{4R}{r} = \frac{S - P}{P}.\end{aligned} \tag{7}$$

Hence from (1), (6), (7) we have

$$s^2 - 16Rr + 5r^2 = \left(1 - 4(S - P)P + 5P^2\right)\frac{r^2}{P^2} \geqslant 0$$

$$\text{or} \qquad 9P^2 - 4SP + 1 \geqslant 0. \tag{8}$$

Now $\qquad 9P^2 - 4SP + 1 = (1 - 3SP) - P(S - 9P) \geqslant 0$

shows the weighted competition between the two inequalities (3), (4) which leads to enhanced sharpness.

The reader may wish to convert the second Gerretsen inequality $s^2 \leqslant 4R^2 + 4Rr + 3r^2$ to $S^2 + 2SP + 9P^2 \geqslant 4$,

Concluding remarks

Note that (by (7)) the Euler result $R \geqslant 2r$ corresponds to $S - 9P \geqslant 0$. The first Gerretsen inequality (1) is used both as a square and as a linear term in the symmedicentroidal proof [1], whereas just the linear term (8) suffices to complete the original trigonometrical proof with

$$S + P - 4S^2P + 6SP^2 = S\left(9P^2 - 4SP + 1\right) + P(1 - 3SP) \geqslant 0.$$

The reader may also wish to show that $4R + r \geqslant \sqrt{3}s$ and

$$a(s - a) + b(s - b) + c(s - c) \geqslant 4\sqrt{3}\Delta \qquad \text{(triangle area } \Delta\text{)}$$

correspond to $S \geqslant \sqrt{3}$.

Note finally that the trigonometrical form may be the more concise.

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Reference

1. M. Lukarevski and J. A. Scott, Three discs for the incentre, *Math. Gaz.* **106** (July 2022) pp. 332-335.

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