

A STOCHASTIC MATCHING MODEL ON HYPERGRAPHS

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Abstract

Motivated by applications to a wide range of areas, including assemble-to-order systems, operations scheduling, healthcare systems, and the collaborative economy, we study a stochastic matching model on hypergraphs, extending the model of Mairesse and Moyal (*J. Appl. Prob.* **53**, 2016) to the case of hypergraphical (rather than graphical) matching structures. We address a discrete-event system under a random input of single items, simply using the system as an interface to be matched in groups of two or more. We primarily study the stability of this model, for various hypergraph geometries.

Keywords: Markovian queueing theory; stability; matching; hypergraph

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1. Introduction

Matching models have recently received growing interest in the literature on queueing models in which compatibilities between the requests need to be taken into account. These are a natural enrichment of service systems in which the requests must be matched, or put in relation, rather than being served. Among other fields of applications, this is a natural representation of peer-to-peer networks, interfaces of the collaborative economy (such as car- and ride-sharing, dating websites, and so on), assemble-to-order systems, job search applications, and healthcare systems (blood banks and organ transplant networks). All these applications share the same common ground: elements/items/agents enter a system that is just an interface to put them in relation to one another, and relations are possible only if the ‘properties’ (whatever this means) of the elements make them compatible.

A (graphical) stochastic matching model can roughly be defined as follows: items enter a system at random times, and need to be matched into pairs. The possible pairs are given by a compatibility graph whose nodes represent the classes of items, and the classes of incoming items are randomly drawn from a prescribed distribution on the set of nodes. Unmatched items are queued, waiting for a future compatible item, and leave the system in pairs as soon as they are matched. If the items enter the system individually, as in [16] and [19], and more recently in [2] and [18], we say that the system is a *general stochastic matching model* (GM), following the terminology of [16]. If the classes of items are partitioned into two subsets (say the classes

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of ‘customers’ and the classes of ‘servers’) and enter the system pairwise, as in the seminal papers [3, 11] (which viewed such systems as generalizations of skill-based customer/server queueing systems) and then in [1], [2], [9], and [17], we say that the system is a *bipartite stochastic matching model* (BM). Stabilizing policies and fluid/diffusion approximations of two-sided systems are obtained respectively in [7, 8]. Other references address specific models for designated applications: [5] on kidney transplants, [22] on housing allocations systems, and [21] on ride-sharing models. In another line of research, such stochastic matching architectures are addressed from the point of view of stochastic optimization in [10], [13], and [20], among others.

In the seminal papers on general and bipartite matching models, the matching of items is exclusively *pairwise*: a job or a house with an applicant, a kidney with a patient, a cab with a customer, two users of a dating website, etc. However, several of the above applications should naturally incorporate the possibility of matching items in groups of more than two. Let us illustrate this with a concrete example: in organ transplants, (in-)compatibility between givers and receivers may be due to a variety of factors, mainly blood types and immunological factors. In kidney exchange programs, each item represents an intra-incompatible couple (A, B) (e.g. a patient A waiting for a transplant and a relative B of his/hers, who is incompatible with A for a potential organ donation), entering the system to find another intra-incompatible couple (A', B') that is compatible with it, in the sense that A can receive an organ from B' and A' can receive from B . The ability of such a system to accommodate all requests and to maximize the number of successful transplants and avoid congestion is translated into the positive recurrence of a stochastic process representing the stochastic system over time. Then if we view the items as the *couples*, and translate ‘cross-compatibility’ (i.e. A can receive from B' and A' can receive from B) into the existence of an edge between node (A, B) and node (A', B') , such a system is a typical application of the GM introduced in [16].

But let us now consider the case where such exchanges $(A, B) \leftrightarrow (A', B')$ and $(A', B') \leftrightarrow (A'', B'')$ cannot be realized, but A can receive from B' , A' can receive from B'' , and A'' can receive from B . Then it is natural to consider the possibility of executing the three transplants contemporaneously, i.e. to match the triplet $(A, B), (A', B'), (A'', B'')$ all together. In several countries, including the U.S., such ‘exchanges’ in groups of three (or more) are allowed, which raises the issue of maximizing ‘matchings’ that coincide not with sets of edges, but with sets of subsets of nodes of cardinality 3 or more. Hence the need to consider matching models on a compatibility structure that is not a graph but a *hypergraph*, i.e. a set of nodes V equipped with a set of subsets of V of cardinality 3 or more.

Among other fields of applications, the same modeling is suitable for assemble-to-order systems, in which components are produced by independent processes and assembled in groups in a given order. All the same, in operations management, specific operations may be made available at given random times, to be coordinated later into groups of two or more. In all cases, the system controller confronts a random flux of arrivals of items (or operations) and needs to match (or combine/coordinate) them into subgroups of two or more, hence following a hypergraphical structure.

The main purpose of this paper is to study the long-run stability of stochastic matching models, in the sense defined above, on a hypergraphical compatibility structure. Thus, the two closest references to the present work are [13, 20]: in both cases, a general matching model is addressed on a hypergraphical matching structure (notice that [20] also allows matchings including several items of the same class). The first reference addresses continuous-time models; the second considers discrete-time models, but most of the results therein can

easily be extended to the continuous-time setting. In [13] a matching control is introduced that asymptotically minimizes the holding cost of items in an unstable system (we justify the instability of such systems under the assumptions of [13] in Remark 4 below). The paper [20] introduces an algorithm that is a variant of the ‘primal dual algorithm’, allowing stability to be achieved, if this is feasible at all, for a very large class of models. The proposed algorithm furthermore optimizes utility functions that are convex functions of the average matching rates. Both references allow *idling policies*; i.e., scheduling algorithms are allowed to perform no matching at all, even if matchable items are present in the system, in order to wait for more profitable future matches. Allowing idling makes sense in applications such as assemble-to-order systems, advertisement, or operations scheduling, but is much less suitable to kidney transplant networks, in which practitioners always perform a transplant whenever one is possible. In the present work all the matching policies we consider are non-idling, i.e. entering items are always matched right away if this is possible at all. Thus, the model studied in the present paper is a special case of the model studied in [20], for simple arrivals, no same-class matchings, and non-idling matching policies. Our approach is in fact complementary to that in [13] and [20]: generalizing the approach of [16] to hypergraphs instead of graphs, in this paper we are mostly concerned with the structural properties of the underlying hypergraph of the matching model, and determine classes of hypergraph for which there does, or does not, exist a non-idling policy that is able to stabilize the system. In a sense, the present work addresses an upstream problem to that of implementing a performant matching algorithm: we provide simple and comprehensive criteria, *based only on the structural properties of the considered hypergraph*, for the (non-)existence of a stabilizing non-idling policy.

This hypergraphical stochastic matching model addressed in this paper is formally defined as follows: items enter the system by single arrivals, and get matched in groups of two or more, following compatibilities that are represented by a given hypergraph. A matching policy determines the matchings to be executed in the case of a multiple choice, and the unmatched items are stored in a buffer, waiting for a future match.

We address the problem of existence of a steady state for the system: we formally define the stability region of the system as the set of measures on the set of nodes rendering the natural Markov chain of the system positive recurrent, for a given compatibility hypergraph and a given matching policy. We assess the form of the stability region of specific stochastic matching models as a function of the geometry of the underlying hypergraphs. In a nutshell, we show that such systems are not easily stabilizable, by exhibiting wide classes of models having an empty stability region, whatever the non-idling matching policy is. We then provide, or give bounds for, the stability region of particular stabilizable systems.

This paper is organized as follows: we start with some preliminaries in Section 2, in particular introducing the main definitions and properties of hypergraphs. In Section 3 we formally introduce the present model. In Section 4 we provide necessary conditions of stability for the present class of systems: as will be developed therein, and unlike the particular case of the GM on graphs (see [16]), for which a natural necessary condition could be obtained, we introduce various necessary conditions that depend on distinct geometrical properties of the considered hypergraphs. From this we then deduce classes of hypergraphs for which the corresponding matching model cannot be stable; see Section 5. Finally, in Section 6 we provide the precise stability region in the particular case where the compatibility hypergraph is complete 3-uniform, complete 3-uniform k -partite, and then complete up to a partition of its hyperedges (see the precise definitions of these objects below). We conclude this work in Section 7.

2. Preliminaries

2.1. General notation

Let $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$, and \mathbb{N}^+ denote respectively the sets of real numbers, nonnegative real numbers, natural integers, and positive integers, respectively. For a and b in \mathbb{N} , denote by $\llbracket a, b \rrbracket$ the integer interval $[a, b] \cap \mathbb{N}$. We let $a \wedge b$ and $a \vee b$ denote respectively the minimum and the maximum of two numbers $a, b \in \mathbb{R}$.

Given a finite set B , we denote by $\mathcal{M}(B)$ the set of probability measures on B having B as exact support. Denote by \bar{B} the complement of B (within a set of reference that is fixed by the context).

Let $q \in \mathbb{N}^+$. We write any vector $u \in \mathbb{R}^q$ as $u = (u(1), \dots, u(q))$. We denote by \leq the coordinate-wise partial ordering on \mathbb{R}^q ; that is, for any $u, v \in \mathbb{R}^q$ we write $u \leq v$ if and only if $u(i) \leq v(i)$ for any $i \in \llbracket 1, q \rrbracket$. For any $i \in \llbracket 1, q \rrbracket$, let e_i denote the vector in \mathbb{N}^q with components $e_i(j) = \delta_{ij}, j \in \llbracket 1, q \rrbracket$.

The null vector in \mathbb{N}^q is denoted by $\mathbf{0}$. The norm of any vector $u \in \mathbb{N}^q$ is denoted by $\|u\| = \sum_{i=1}^q u(i)$. Let A be a finite set. The cardinality of A is denoted by $|A|$. We let A^* denote the free monoid associated to A , i.e. the set of finite words over the alphabet A . The length of a word $w \in A^*$ is denoted by $|w|$. We write any word $w \in A^*$ as $w = w(1)w(2)\dots w(|w|)$. For any $a \in A$ we denote by $|w|_a$ the number of occurrences of the letter a in the word w . Having set an ordering on A , and denoting by $1, 2, \dots, |A|$ the elements of A in increasing order, we define the commutative image of a word $w \in A^*$ as the $\mathbb{N}^{|A|}$ -valued vector $[w]$ given by $[w] = (|w|_1, \dots, |w|_{|A|})$, i.e. the vector whose i th coordinate is the number of occurrences of the letter i in the word w . Finally, for a word $w \in A^*$ and an ordered list of letters (a, b, c, \dots) appearing in that order in w , we denote by $w \setminus (a, b, c, \dots)$ the word of A^* obtained by just deleting the letters a, b, c, \dots in w .

2.2. Hypergraphs

For easy reference, let us first introduce the basics of hypergraph theory that will be used in this paper. A thorough presentation of the topic can be found e.g. in [4].

Definition 1. A hypergraph \mathbb{H} is defined as a couple (V, \mathcal{H}) for which the following hold:

- The finite set V is the set of nodes of \mathbb{H} . We let $q(\mathbb{H})$ be the cardinality of V , and say that the hypergraph is of order $q(\mathbb{H})$.
- $\mathcal{H} := \{H_1, \dots, H_{m(\mathbb{H})}\}$ is a set of subsets of V , called hyperedges of \mathbb{H} , such that $\bigcup_{i=1}^{m(\mathbb{H})} H_i = V$.

We then say that the hypergraph is simple (or a Sperner family) if $H_i \subset H_j$ implies $i = j$ for all $i, j \in \llbracket 1, m(\mathbb{H}) \rrbracket$, i.e., no hyperedge is included in another one. Whenever no ambiguity is possible, we often write $q := q(\mathbb{H}), m := m(\mathbb{H})$. A sub-hypergraph of \mathbb{H} is a hypergraph $\mathbb{H}' = (V, \mathcal{H}')$ such that $\mathcal{H}' \subset \mathcal{H}$.

Definition 2. Let $\mathbb{H} = (V, \mathcal{H})$ be a hypergraph. The rank of \mathbb{H} is the largest size of a hyperedge, i.e. the integer $r(\mathbb{H}) = \max_{j \in \llbracket 1, m(\mathbb{H}) \rrbracket} |H_j|$. The anti-rank of \mathbb{H} is defined as $a(\mathbb{H}) = \min_{j \in \llbracket 1, m(\mathbb{H}) \rrbracket} |H_j|$, i.e. the smallest size of a hyperedge. If there exists a constant r such that $r(\mathbb{H}) = a(\mathbb{H}) = r$, then \mathbb{H} is said to be r -uniform. The degree of a node $i \in V$ is the number of

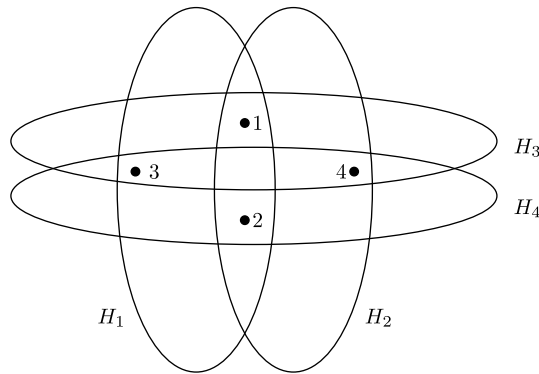


FIGURE 1. Complete 3-uniform hypergraph of order 4.

hyperedges to which i belongs, i.e.

$$d(i) = \sum_{\ell=1}^{m(\mathbb{H})} \mathbb{1}_{H_\ell}(i).$$

If there exists a constant d such that $d(i) = d$ for any i , then \mathbb{H} is said to be d -regular.

Example 1. Figure 1 represents the hypergraph $\mathbb{H} = (V, \mathcal{H})$ with $V = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. The hypergraph \mathbb{H} is of order 4; it is 3-uniform since all hyperedges are of cardinality 3, and 3-regular because all nodes are of degree 3 (they all belong to exactly 3 hyperedges). As all hyperedges of cardinality 3 appear in \mathcal{H} , this hypergraph is said to be *complete 3-uniform of order 4*.

Definition 3. The representative graph of a hypergraph $\mathbb{H} = (V, \mathcal{H})$ is the graph $L(\mathbb{H}) = (\mathcal{H}, \mathcal{E})$ whose nodes are the elements of \mathcal{H} , and such that $(H_i, H_j) \in \mathcal{E}$ (i.e. H_i and H_j share an edge in the graph) if and only if $H_i \cap H_j \neq \emptyset$. The hypergraph \mathbb{H} is said to be connected if $L(\mathbb{H})$ is connected.

As is easily seen, any 2-uniform hypergraph is a graph, whose edges are the elements of \mathcal{H} , and any simple and connected hypergraph contains no isolated node, i.e. has anti-rank at least 2.

Definition 4. A set $T \subset V$ is a transversal of \mathbb{H} if it meets all its hyperedges, that is, $T \cap H \neq \emptyset$, for any $H \in \mathcal{H}$. The set of transversals of \mathbb{H} is denoted by $\mathcal{T}(\mathbb{H})$. A transversal T is said to be minimal if it is of minimal cardinality among all transversals of \mathbb{H} . The transversal number of the hypergraph \mathbb{H} is the cardinality of its minimal transversals. It is denoted $\tau(\mathbb{H})$.

Example 2. (Example 1, continued.) Consider again the hypergraph represented in Figure 1. Then any subset of V of cardinality 2 is a transversal of \mathbb{H} , since it intersects all hyperedges of \mathcal{H} . On the other hand, it is immediate that no singleton of V is a transversal, as all nodes are of degree 3. Thus the transversal number of \mathbb{H} is $\tau(\mathbb{H}) = 2$.

For any set $A \subset V$, we denote by

$$\mathcal{H}(A) = \{H \in \mathcal{H} : H \cap A \neq \emptyset\} \tag{1}$$

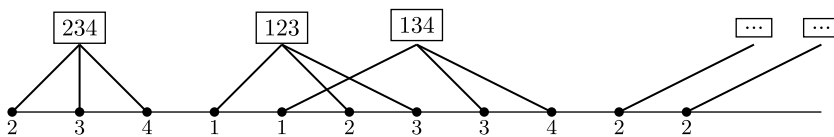


FIGURE 2. The matching model in action, on the matching hypergraph of Figure 1.

the set of hyperedges that intersect A . With some abuse of notation, for any node $i \in V$ we write $\mathcal{H}(i) := \mathcal{H}(\{i\})$.

Throughout this paper, all considered hypergraphs are simple and connected.

3. The model

All the random variables hereafter are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.1. Stochastic matching model on a hypergraph

A (discrete-time, hypergraphical) stochastic matching model is specified by a triple (\mathbb{H}, Φ, μ) , such that

- $\mathbb{H} = (V, \mathcal{H})$ is a simple and connected hypergraph, termed the *matching hypergraph* of the model;
- Φ is a matching policy, precisely defined in Section 3.3 below;
- μ is an element of $\mathcal{M}(V)$.

The matching model (\mathbb{H}, Φ, μ) is then defined as follows. At each time point $n \in \mathbb{N}$, the following occurs:

1. An item enters the system. Its class V_n is drawn from the measure μ on V , independently of everything else. (Thus the sequence of classes of incoming items $\{V_n, n \in \mathbb{N}\}$ is independent and identically distributed (i.i.d.) with common distribution μ .)
2. The incoming item then faces the following alternative:
 - (i) If there exists in the buffer at least one set of items whose respective set of classes forms, together with V_n , a hyperedge of \mathcal{H} , then it is the role of the matching policy Φ to select one of these sets of classes, say $\{i_1, \dots, i_m\}$. Then the $m + 1$ items of respective classes i_1, \dots, i_m, V_n are matched together and leave the system right away. Defining $H_j := \{i_1, \dots, i_m, V_n\} \in \mathcal{H}$, we then say that V_n *completes a matching* of type H_j at time n , and we write $H(n) = H_j$ for the matching performed at n .
 - (ii) Else, the item is stored in the buffer of the system, waiting for a future match, and we write $H(n) = \emptyset$.

Example 3. Consider again the matching hypergraph $\mathbb{H} = (V, \mathcal{H})$ of Figure 1. The dynamic matchings of the realization $\{V_n(\omega), n \in \mathbb{N}\} = 2, 3, 4, 1, 1, 2, 3, 3, 4, 2, 2, \dots$ is represented in Figure 2.

3.2. System dynamics

Fix a hypergraphical matching model on a hypergraph \mathbb{H} of order q . Define for all $n \in \mathbb{N}$ the \mathbb{N}^q -valued random variable $X_n = (X_n(1), \dots, X_n(q))$, where, for each $i \in V$, $X_n(i)$ is the number

of items of class i in the buffer at time n (taking into account the arrival occurring at time n). The vector X_n is then called *class-content* of the system at time n . For any subset B of V , define $X_n(B)$ to be the class-content of elements of B :

$$X_n(B) = \sum_{i \in B} X_n(i),$$

in such a way that the total number of items in the buffer at time n is given by $\|X_n\|$. The *buffer-content* of the system at time n is the word of V^* whose letters are the classes of the items in the buffer, in increasing order of their arrivals. That is,

$$W_n = W_n(1)W_n(2)\dots W_n(|W_n|),$$

where for any ℓ , $W_n(\ell)$ is the class of the ℓ th oldest item in line. Notice that X_n is nothing but the commutative image of W_n , i.e. $X_n = [W_n]$, $n \in \mathbb{N}$.

To simply describe the dynamics of the processes $\{X_n, n \in \mathbb{N}\}$ and $\{W_n, n \in \mathbb{N}\}$, for any $u \in \mathbb{N}^q$ and $H \in \mathcal{H}$ we define the following elements of $\{0, 1\}^q$: p_u is the vector of coordinates $p_u(i) = \mathbb{1}_{\{u(i) > 0\}}$, $i \in [1, q]$, and γ_H is the trace of H , i.e. the vector in $\{0, 1\}^q$ defined by $\gamma_H(i) = \mathbb{1}_H(i)$ for all $i \in [1, q]$. We then set

$$\Gamma(u) = \{H \in \mathcal{H} : p_u \succeq \gamma_H\}.$$

In other words, a hyperedge H belongs to $\Gamma(u)$ if and only if $u(i) > 0$ for any $i \in H$.

3.3. Matching policies

Formally, an *admissible matching policy* is a rule of choice, for any $n \in \mathbb{N}$, of the item(s) matched with the incoming item at time $n + 1$, in case of a multiple choice, that can be made solely on the basis of the knowledge of the buffer-content W_n , on the class V_{n+1} of the incoming item, and possibly on an independent toss. Notice that at all n , $\Gamma(X_n + \mathbf{e}_{V_{n+1}}) = \Gamma([W_n] + \mathbf{e}_{V_{n+1}})$ represents the (possibly empty) set of all hyperedges in \mathcal{H} that can be completed by the arrival of V_{n+1} in a system having buffer-content W_n at time n . Using queueing-related terminology, all admissible matching policies we consider hereafter are furthermore assumed ‘non-idling’, in the sense that at any n , a match is necessarily performed whenever $\Gamma(X_n + \mathbf{e}_{V_{n+1}})$ is non-empty.

3.3.1 Matching policies that depend on the arrival times.

First come, first matched. In the *first come, first matched* (FCFM) policy, the chosen match of the incoming item V_{n+1} at time $n + 1$ is the hyperedge containing the oldest item in line among all hyperedges that can be completed by V_{n+1} . Specifically,

$$W_{n+1} = \begin{cases} W_n V_{n+1} & \text{if } \Gamma([W_n] + \mathbf{e}_{V_{n+1}}) = \emptyset; \\ W_n V_{n+1} \setminus_{(i,j,\dots,k)} & \text{else, for } H(n) = \{i, j, \dots, k\} \in \Gamma([W_n] + \mathbf{e}_{V_{n+1}}), \end{cases}$$

where, in the case where $|\Gamma(X_n + \mathbf{e}_{V_{n+1}})| \geq 2$, i.e. there is more than one possible matching containing V_{n+1} at $n + 1$, $H(n) = \{i, j, \dots, k\}$ is the hyperedge whose first element i appearing in W_n appears first among all elements of $\Gamma([W_n] + \mathbf{e}_{V_{n+1}})$.

Last come, first matched. Likewise, in the *last come, first matched* (LCFM) policy, the newly arrived item at $n + 1$ is matched to form the hyperedge containing the youngest possible

element, i.e.

$$W_{n+1} = \begin{cases} W_n V_{n+1} & \text{if } \Gamma ([W_n] + \mathbf{e}_{V_{n+1}}) = \emptyset; \\ W_n V_{n+1} \setminus (i, j, \dots, k) & \text{else, for } H(n) = \{i, j, \dots, k\} \in \Gamma ([W_n] + \mathbf{e}_{V_{n+1}}), \end{cases}$$

where $H(n) = \{i, j, \dots, k\}$ is the hyperedge whose last element k appearing in W_n appears last among all elements of $\Gamma ([W_n] + \mathbf{e}_{V_{n+1}})$.

3.3.2 Matching policies that depend only on the class-content. A wide class of natural matching policies can be implemented given the sole knowledge of the class-content upon arrival times. In such cases we have, for all n ,

$$X_{n+1} = \begin{cases} X_n + \mathbf{e}_{V_{n+1}} & \text{if } \Gamma (X_n + \mathbf{e}_{V_{n+1}}) = \emptyset; \\ X_n + \mathbf{e}_{V_{n+1}} - \gamma_{H(n)} & \text{else, for some } H(n) \in \Gamma (X_n + \mathbf{e}_{V_{n+1}}), \end{cases} \tag{2}$$

where the choice of the hyperedge $H(n)$ depends on the matching policy. Several examples are provided below.

Match the longest. The matching policy Φ is *match the longest* (denoted by ML) if, for all n , the match realized is that of the hyperedge having the most elements in storage at n . In other words, the chosen hyperedge $H(n)$ in the second case of (2) satisfies

$$H(n) = \operatorname{argmax} \{X_n(H) : H \in \Gamma (X_n + \mathbf{e}_{V_{n+1}})\},$$

ties being broken uniformly at random (and independently of everything else) among hyperedges.

Remark 1. Observe that ml is a particular case of the class of the ‘max weight’ used in [20], constrained to be non-idling (i.e. whenever the entering item can be matched, it is matched right away) and with constant rewards and coefficients in Equation (5) therein.

Match the shortest. Analogously, the *match the shortest* policy (denoted by MS) corresponds to the choice

$$H(n) = \operatorname{argmin} \{X_n(H) : H \in \Gamma (X_n + \mathbf{e}_{V_{n+1}})\},$$

ties being broken uniformly at random, as above.

Fixed priority. In the context of fixed priorities, each vertex $i \in V$ is assigned a full ordering of the hyperedges and chosen to be matched with the first matchable hyperedge following this order. Formally, to each node i is associated a permutation σ_i of the index set $\llbracket 1, d(i) \rrbracket$, and if we write $\mathcal{H}(i) = \{H_{i_1}, H_{i_2}, \dots, H_{i_{d(i)}}\}$, then at any time n ,

$$H(n) = H_{i_{\sigma_i(j)}}, \text{ where } j = \min \{k \in \llbracket 1, d(i) \rrbracket : X_n (H_{i_{\sigma_i(k)}}) > 0\}. \tag{3}$$

Random. For this matching policy, the priority order defined above is not fixed, and is drawn uniformly at random upon each arrival; i.e. for any n , $H(n)$ is defined as in (3), for a permutation $\sigma_i(n)$ that is drawn, independently of everything else, uniformly at random among all permutations of $\llbracket 1, d(i) \rrbracket$.

It is easily seen that under any admissible policy the sequence $\{W_n, n \in \mathbb{N}\}$ is a V^* -valued Markov chain with respect to the filtration generated by the sequence $\{V_n, n \in \mathbb{N}\}$. Additionally, in the cases where $\Phi = \text{ML, MS}$, a fixed priority, or a random policy, the sequence $\{X_n, n \in \mathbb{N}\}$ is an $\mathbb{N}^{|V|}$ -valued Markov chain.

3.4. Stability of the matching model

We say that the matching model (\mathbb{H}, Φ, μ) is stable if the Markov chain $\{W_n, n \in \mathbb{N}\}$ (and thereby $\{X_n, n \in \mathbb{N}\}$) is positive recurrent. For a given hypergraph $\mathbb{H} = (V, \mathcal{H})$ and a given matching policy Φ , we define the *stability region* associated to \mathbb{H} and Φ as the set of probability measures on V rendering the model (\mathbb{H}, Φ, μ) stable, i.e.

$$\text{STAB}(\mathbb{H}, \Phi) = \{\mu \in \mathcal{M}(V) : (\mathbb{H}, \Phi, \mu) \text{ is stable}\}.$$

We then say that a hypergraph \mathbb{H} is stabilizable if $\text{STAB}(\mathbb{H}, \Phi)$ is non-empty for some matching policy Φ . If not, \mathbb{H} is said to be non-stabilizable.

4. Necessary conditions for stability

Fix a matching model (\mathbb{H}, Φ, μ) on a hypergraph $\mathbb{H} = (V, \mathcal{H})$. For any $n, B \subset V$, and $\mathcal{B} \subset \mathcal{H}$, denote by $A_n(B)$ the number of arrivals of elements in B and by $M_n(\mathcal{B})$ the number of matchings of hyperedges in \mathcal{B} realized up to n ; i.e.,

$$A_n(B) = \sum_{k=1}^n \mathbb{1}_{\{V_k \in B\}},$$

$$M_n(\mathcal{B}) = \sum_{k=1}^n \mathbb{1}_{\{H^{(k)} \in \mathcal{B}\}}.$$

With some abuse of notation, write $A_n(i) = A_n(\{i\})$ and $M_n(H) = M_n(\{H\})$ for any $i \in V$ and $H \in \mathcal{H}$. Observe that the following key relation holds for all $B \subset V$:

$$X_n(B) = A_n(B) - \sum_{H \in \mathcal{H}} |H \cap B| M_n(H) \geq 0, \quad n \in \mathbb{N}, \tag{4}$$

since the number of items of classes in B at any time n is precisely the number of arrivals of such items up to time n , minus the number of these items that leave the system upon each matching of a hyperedge that intersects B .

4.1. General conditions

We start by introducing several ‘universal’ stability conditions. Fix a hypergraph $\mathbb{H} = (V, \mathcal{H})$ throughout the section.

Definition 5. We say that $I \subset V$ is an *independent set* of \mathbb{H} if I does not include any hyperedge of \mathbb{H} , i.e, for any $H \in \mathcal{H}, H \cap I \neq \emptyset$. We also let $\mathcal{I}(\mathbb{H})$ be the set of all independent sets of \mathbb{H} .

Let us define, for any $\mu \in \mathcal{M}(V)$ and any $B \subset V$, the set

$$L_\mu(B) = \operatorname{argmin} \{\mu(j) : j \in B\}, \quad B \subset V. \tag{5}$$

To clarify the exposition of the following result, we need to introduce the following notion.

Definition 6. For any $\mu \in \mathcal{M}(V)$ we say that the independent set $I \in \mathcal{I}(\mathbb{H})$ is μ -minimal if the intersection of any hyperedge $H \in \mathcal{H}$ with I is either empty, or reduced to a singleton $\{v_H\}$ such that

- v_H is of degree 1, i.e., H is the only hyperedge to which v_H belongs;
- $L_\mu(H) = \{v_H\}$, i.e., v_H is the only minimum of μ over the set H .

An independent set $I \in \mathcal{I}(\mathbb{H})$ that is not μ -minimal is said to be *non- μ -minimal*. We let $\mathcal{I}_\mu(\mathbb{H})$ be the set of μ -minimal independent sets of \mathbb{H} , and $\mathcal{I}'_\mu(\mathbb{H})$ the set of non- μ -minimal independent sets of \mathbb{H} , that is, the complement of $\mathcal{I}_\mu(\mathbb{H})$ in $\mathcal{I}(\mathbb{H})$.

In other words, a μ -minimal independent sets gathers nodes that are the only minimum of μ over the only hyperedge they belong to. Notice that the collection $\mathcal{I}_\mu(\mathbb{H})$ can be empty. This is the case if and only if all nodes are of degree at least 2, or all nodes of degree 1 are not the only minimum of μ on the single hyperedge they belong to. Observe the following characterization.

Lemma 1. *Let $\mathbb{H} = (V, \mathcal{H})$ be a hypergraph, and let $\mathcal{H} = \{H_1, \dots, H_m\}$. An independent set $I = \{v_1, \dots, v_p\}$ is μ -minimal if and only if for all n and all k_1, \dots, k_m such that $k_j \in L_\mu(H_j)$ for all j ,*

$$A_n(I) = \sum_{j=1}^m |H_j \cap I| A_n(k_j). \tag{6}$$

Proof. First, it is clear that if $I \in \mathcal{I}_\mu(\mathbb{H})$, then $m \geq p$ and the mapping

$$\varphi: \begin{cases} \{j \in \llbracket 1, m \rrbracket : I \cap H_j \neq \emptyset\} & \longrightarrow \llbracket 1, p \rrbracket, \\ j & \longmapsto i : L_\mu(H_j) = H_j \cap I = \{v_i\} \end{cases} \tag{7}$$

is bijective. Thus we have, almost surely for all n ,

$$A_n(I) = \sum_{j=1}^m A_n(v_{\varphi(j)}) = \sum_{\substack{j \in \llbracket 1, m \rrbracket : \\ H_j \cap I \neq \emptyset}} |H_j \cap I| A_n(v_{\varphi(j)}) = \sum_{j=1}^m |H_j \cap I| A_n(k_j).$$

Let us now assume that $I \in \mathcal{I}'_\mu(\mathbb{H})$. Then the following hold:

- If some hyperedge H_j is such that $|H_j \cap I| \geq 2$, then upon each arrival of an element of class k_j , the right-hand side of (6) increases by $|H_j \cap I|$, while the left-hand side increases by 1 if $k_j \in I$, or by 0 otherwise.
- If, for some hyperedge H_j intersecting I , there exists $k_j \in L_\mu(H_j) \cap \bar{I}$, then upon each arrival of a class- k_j item, the right-hand side of (6) increases while the left-hand side does not.
- Finally, if for all $j \in \llbracket 1, m \rrbracket$, $|H_j \cap I| \leq 1$, and for all j such that $|H_j \cap I| = 1$, $L_\mu(H_j) = \{v_{\varphi(j)}\}$ (again defining φ by (7)), then if $\varphi(j) = \varphi(l)$ for some $l \neq j$, upon each arrival of a class- $v_{\varphi(j)}$ item, the right-hand side of (6) increases by 2 while the left-hand side increases by 1.

In all cases, (6) cannot hold for all n , which concludes the proof. □

Now define the following set of measures:

$$\mathcal{N}_1(\mathbb{H}) = \left\{ \mu \in \mathcal{M}(V) : \text{for all } I \in \mathcal{I}'_\mu(\mathbb{H}), \mu(I) < \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k) \right\}.$$

We have the following result.

Proposition 1. For any connected hypergraph \mathbb{H} and any admissible matching policy Φ ,

$$\text{STAB}(\mathbb{H}, \Phi) \subset \mathcal{N}_1(\mathbb{H}).$$

Proof. Fix $\mathbb{H} = (V, \mathcal{H})$ and an admissible policy Φ . Denote by H_1, \dots, H_m the hyperedges of \mathbb{H} . Suppose that $\mu \in \mathcal{M}(V)$ is such that there exists an independent set $I \in \mathcal{I}'_\mu(\mathbb{H})$ such that

$$\mu(I) > \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k). \tag{8}$$

For any $i \in \llbracket 1, m \rrbracket$ and any $k_i \in L_\mu(H_i)$ we have that

$$M_n(H_i) \leq \min_{k \in H_i} A_n(k) \leq A_n(k_i), \quad n \geq 0.$$

Thus, from the equality in (4), for any k_1, \dots, k_m such that $k_i \in L_\mu(H_i)$ for all i , we have that

$$\frac{X_n(I)}{n} \geq \frac{A_n(I)}{n} - \sum_{i=1}^m |H_i \cap I| \frac{A_n(k_i)}{n}, \quad n \geq 1. \tag{9}$$

The strong law of large numbers (SLLN) applied to the right-hand side of (9) implies that for any such k_1, \dots, k_m ,

$$\limsup_n \frac{X_n(I)}{n} \geq \mu(I) - \sum_{i=1}^m |H_i \cap I| \mu(k_i) = \mu(I) - \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k) > 0,$$

so that $X_n(I)$ goes almost surely to infinity, which (as $X_n = [W_n]$ for all n) implies the transience of $\{W_n, n \in \mathbb{N}\}$.

Assume now that μ is such that for some independent set $I \in \mathcal{I}'_\mu(\mathbb{H})$, equality holds in (8). Then, for any k_1, \dots, k_m such that $k_j \in L_\mu(H_j)$ for all j , the Markov chain $\{Y_n, n \in \mathbb{N}\}$ defined as

$$Y_n = A_n(I) - \sum_{j=1}^m |H_j \cap B| A_n(k_j), \quad n \in \mathbb{N},$$

is a random walk with drift 0 that is different from the identically null process, in view of Lemma 1. Hence $\{Y_n, n \in \mathbb{N}\}$ is null recurrent. If the chain $\{W_n, n \in \mathbb{N}\}$ were positive recurrent, the sequence $\{X_n, n \in \mathbb{N}\}$ would visit the state $\mathbf{0}$ infinitely often, with inter-passage time at $\mathbf{0}$ of finite expectation. Thus from (9), the sequence $\{Y_n, n \in \mathbb{N}\}$ would be positive recurrent, an absurdity. This concludes the proof. \square

Define the following sets of measures:

$$\begin{aligned} \mathcal{N}_1^+(\mathbb{H}) &= \left\{ \mu \in \mathcal{M}(V) : \forall I \in \mathcal{I}(\mathbb{H}), \mu(I) < \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H \cap I} \mu(k) \right\}; \\ \mathcal{N}_1^{++}(\mathbb{H}) &= \left\{ \mu \in \mathcal{M}(V) : \forall B \subset V, \mu(B) \leq \sum_{H \in \mathcal{H}} |H \cap B| \min_{k \in H} \mu(k) \right\}. \end{aligned}$$

We have the following result.

Corollary 1. For any connected hypergraph \mathbb{H} and any admissible matching policy Φ ,

$$\text{STAB}(\mathbb{H}, \Phi) \subset \mathcal{N}_1^+(\mathbb{H}) \cap \mathcal{N}_1^{++}(\mathbb{H}).$$

Proof. We simply show that $\mathcal{N}_1(\mathbb{H})$ is included in $\mathcal{N}_1^+(\mathbb{H}) \cap \mathcal{N}_1^{++}(\mathbb{H})$. Again set $\mathcal{H} = \{H_1, \dots, H_m\}$ and fix $\mu \in \mathcal{N}_1(\mathbb{H})$. To show that $\mu \in \mathcal{N}_1^+(\mathbb{H})$, first observe that for any independent set $I = \{v_1, \dots, v_p\} \in \mathcal{I}_\mu(\mathbb{H})$, for any hyperedge H_j intersecting with I , we have that

$$\min_{k \in H_j} \mu(k) = \mu(v_{\varphi(j)}) < \min_{k \in H_j \cap I} \mu(k).$$

Therefore,

$$\mu(I) = \sum_{i=1}^p \mu(v_i) = \sum_{j=1}^m |H_j \cap I| \mu(v_{\varphi(j)}) < \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H \cap I} \mu(k),$$

whereas if $I \in \mathcal{I}'_\mu(\mathbb{H})$, as $\mu \in \mathcal{N}_1(\mathbb{H})$ we have that

$$\mu(I) < \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k) \leq \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H \cap I} \mu(k),$$

and hence $\mu \in \mathcal{N}_1^+(\mathbb{H})$.

It remains to show that $\mu \in \mathcal{N}_1^{++}(\mathbb{H})$, and for this, we first observe the following:

$$\text{For all } I \in \mathcal{I}(\mathbb{H}), \mu(I) \leq \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k). \tag{10}$$

To see this, it suffices to observe that for any independent set $I \in \mathcal{I}_\mu(\mathbb{H})$, recalling (7),

$$\mu(I) = \sum_{j=1}^m \mu(v_{\varphi(j)}) = \sum_{\substack{j \in \llbracket 1, m \rrbracket: \\ H_j \cap I \neq \emptyset}} |H_j \cap I| \mu(v_{\varphi(j)}) = \sum_{j=1}^m |H_j \cap I| \mu(k_j) = \sum_{H \in \mathcal{H}} |H \cap I| \min_{k \in H} \mu(k);$$

hence (10). Now fix B , a subset of V that is not an independent set of \mathbb{H} . Then we construct by induction the family of sets $B := B_0 \supset B_1 \supset B_2 \supset \dots \supset B_r$, where r is properly defined below, as follows: for any $i \geq 0$, if B_i is not an independent set of $\mathcal{I}(\mathbb{H})$, then we take an arbitrary hyperedge $H_{j_i} \in \mathcal{H}$ such that $H_{j_i} \subset B_i$, and set $B_{i+1} = B_i \setminus \{k_i\}$, for an arbitrary $k_i \in L_\mu(H_{j_i})$. Then there exists an integer $r \leq |B| - 1$ such that B_r is an independent set of $\mathcal{I}(\mathbb{H})$, and we stop the construction at this point. Observe that for any $i \in \llbracket 0, r - 1 \rrbracket$,

$$\mu(B_{i+1}) \leq \sum_{H \in \mathcal{H}} |H \cap B_{i+1}| \min_{k \in H} \mu(k) \implies \mu(B_i) \leq \sum_{H \in \mathcal{H}} |H \cap B_i| \min_{k \in H} \mu(k). \tag{11}$$

To see this, fix i and suppose that the left-hand side of the above holds true. Then we have

$$\mu(B_i) = \mu(B_{i+1}) + \mu(k_i), \tag{12}$$

and on the other hand,

$$\begin{aligned} \sum_{H \in \mathcal{H}} |H \cap B_i| \min_{k \in H} \mu(k) &= \sum_{H \in \overline{\mathcal{H}}(k_i)} |H \cap B_i| \min_{k \in H} \mu(k) + \sum_{H \in \mathcal{H}(k_i)} |H \cap B_i| \min_{k \in H} \mu(k) \\ &= \sum_{H \in \overline{\mathcal{H}}(k_i)} |H \cap B_{i+1}| \min_{k \in H} \mu(k) + \sum_{H \in \mathcal{H}(k_i)} (|H \cap B_{i+1}| + 1) \min_{k \in H} \mu(k) \\ &= \sum_{H \in \mathcal{H}} |H \cap B_{i+1}| \min_{k \in H} \mu(k) + \sum_{H \in \mathcal{H}(k_i)} \mu(k_H), \end{aligned} \tag{13}$$

where for any $H \in \mathcal{H}(k_i)$, k_H is an arbitrary element of $L_\mu(H)$. But $\mu(k_i)$ is less than or equal to the second term of the latter sum, because $\mu(k_i) = \mu(k_{H_i})$, and we assumed that B_{i+1} is less than the first term. This completes the proof of (11) in view of (13). To conclude, as $B_r \in \mathcal{I}(\mathbb{H})$ and in view of (10), we have that $\mu(B_r) \leq \sum_{H \in \mathcal{H}} |H \cap B_r| \min_{k \in H} \mu(k)$, which implies, by an immediate induction using (11), that $\mu(B) \leq \sum_{H \in \mathcal{H}} |H \cap B| \min_{k \in H} \mu(k)$. This completes the proof. \square

Remark 2. (Graphical case.) Let us consider the special case where $\mathbb{H} = (V, \mathcal{H})$ is a graph. Then it is shown in Proposition 2 of [16] that the stability region of the model is included in the set

$$\text{NCOND}(\mathbb{H}) = \{ \mu \in \mathcal{M}(V) : \forall I \in \mathcal{I}(\mathbb{H}), \mu(I) < \mu(\mathcal{E}(I)) \},$$

where for any set $B \subset V$, $\mathcal{E}(B) = \{ j \in V : (i, j) \in \mathcal{H} \text{ for some } i \in B \}$. It is then easy to check by hand that $\text{Ncond}(\mathbb{H})$ is included in $\mathcal{N}_1^{++}(\mathbb{H})$. Indeed, if we let $\mu \in \text{NCOND}(\mathbb{H})$ and $I \in \mathcal{I}(\mathbb{H})$ (meaning that I is an independent set of the graph \mathbb{H} , in the usual sense), then, for any edge $H \in \mathcal{H}$, $|H \cap I| = 1$ if I contains a vertex of the edge H , and 0 otherwise, so we get that

$$\sum_{H \in \mathcal{H}} |H \cap I| \min_{j \in H \cap I} \mu(j) = \sum_{(i,j) \in \mathcal{H} : i \in I} \mu(j) \geq \sum_{j \in \mathcal{E}(I)} \mu(j) = \mu(\mathcal{E}(I)),$$

where the inequality above is an equality whenever each element of $\mathcal{E}(I)$ shares an edge with a single element of I , and is otherwise a strict inequality. Thus $\mu \in \mathcal{N}_1^{++}(\mathbb{H})$. In fact, it is necessarily the case that $\text{NCOND}(\mathbb{H}) \subset \mathcal{N}_1(\mathbb{H})$, because if this were not true, there would exist in particular a $\mu \in \overline{\mathcal{N}_1(\mathbb{H})} \cap \text{NCOND}(\mathbb{H})$, making the system $(\mathbb{H}, \text{ML}, \mu)$ unstable (in view of Proposition 4.1) despite the fact that $\mu \in \text{NCOND}(\mathbb{H})$, a contradiction to Theorem 2 in [16]. Observe however that $\text{NCOND}(\mathbb{H}) \neq \mathcal{N}_1(\mathbb{H})$ in general. To see this, consider the case where \mathbb{H} is the cycle of size 5, $1 - 2 - 3 - 4 - 5 - 1$. For a small enough $\epsilon > 0$, set

$$\begin{cases} \mu(1) &= \frac{1}{2} - \frac{3\epsilon}{4}; \\ \mu(2) = \mu(5) &= \frac{1}{4} - \frac{\epsilon}{8}; \\ \mu(3) &= \frac{4\epsilon}{5}; \\ \mu(4) &= \frac{\epsilon}{5}. \end{cases}$$

It is then easily checked that $\mu \in \mathcal{N}_1(\mathbb{H})$. However, $\mu \notin \text{NCOND}(\mathbb{H})$, since the independent set $I = \{1, 3\}$ is such that

$$\mu(I) = \frac{1}{2} + \frac{\epsilon}{20} > \frac{1}{2} - \frac{\epsilon}{20} = \mu(\{2, 4, 5\}) = \mu(\mathcal{E}(I)).$$

In conclusion, if \mathbb{H} is a graph, the necessary condition ‘ $\mu \in \text{NCOND}(\mathbb{H})$ ’ is stronger than the necessary condition ‘ $\mu \in \mathcal{N}_1(\mathbb{H})$ ’.

Let us now define the following set of measures:

$$\mathcal{N}_2(\mathbb{H}) = \left\{ \mu \in \mathcal{M}(V) : \forall T \in \mathcal{T}(\mathbb{H}), \mu(T) > \frac{1}{r(\mathbb{H})} \right\}.$$

We also have the following result.

Proposition 2. For any connected hypergraph \mathbb{H} and any admissible matching policy Φ ,

$$\text{STAB}(\mathbb{H}, \Phi) \subset \mathcal{N}_2(\mathbb{H}).$$

Proof. Suppose that there exists a transversal $T \in \mathcal{T}(\mathbb{H})$ such that $\mu(T) \leq \frac{1}{r(\mathbb{H})}$. As each match contains at least one element whose class is an element of T , at any time the overall number of completed matches cannot exceed the number of arrivals of elements whose class belongs to T ; in other words $M_n(\mathcal{H}) \leq A_n(T)$ for all n . Thus, for all n we have that

$$\frac{X_n(V)}{n} \geq \frac{1}{n} (A_n(V) - r(\mathbb{H})M_n(\mathcal{H})) \geq \frac{1}{n} (A_n(V) - r(\mathbb{H})A_n(T)).$$

Taking n to infinity in the above yields

$$\limsup_n \frac{X_n(V)}{n} \geq 1 - r(\mathbb{H})\mu(T),$$

and we conclude as in the previous proof. □

Remark 3. As an immediate consequence of Proposition 2, if $\mathbb{H} = (V, \mathcal{H})$ is of order q , and such that $\tau(\mathbb{H}) \leq \frac{q}{r(\mathbb{H})}$, then $\text{STAB}(\mathbb{H}, \Phi)$ does not contain the uniform measure $\mu_U = (1/q, \dots, 1/q)$ on V ; in other words the model $(\mathbb{H}, \Phi, \mu_U)$ is unstable for any Φ . Indeed, for any minimal transversal T of \mathbb{H} we have that

$$\mu_U(T) = \frac{\tau(\mathbb{H})}{q} \leq \frac{1}{r(\mathbb{H})}.$$

We now introduce two necessary conditions for stability based on the anti-rank of the considered hypergraph. We first introduce the following sets of measures:

$$\mathcal{N}_3^+(\mathbb{H}) = \left\{ \mu \in \mathcal{M}(V) : \forall i \in V, \mu(i) \leq \frac{1}{a(\mathbb{H})} \right\}; \tag{14}$$

$$\mathcal{N}_3^-(\mathbb{H}) = \left\{ \mu \in \mathcal{M}(V) : \forall i \in V, \mu(i) < \frac{1}{a(\mathbb{H})} \right\}. \tag{15}$$

We have the following.

Proposition 3. For any connected hypergraph $\mathbb{H} = (V, \mathcal{H})$ and any admissible policy Φ ,

$$\text{STAB}(\mathbb{H}, \Phi) \subset \mathcal{N}_3^+(\mathbb{H}). \tag{16}$$

If the hypergraph $\mathbb{H} = (V, \mathcal{H})$ is r -uniform (i.e. $a(\mathbb{H}) = r(\mathbb{H}) = r$) we have that

$$\text{STAB}(\mathbb{H}, \Phi) \subset \mathcal{N}_3^-(\mathbb{H}). \tag{17}$$

In other words, the model (\mathbb{H}, Φ, μ) cannot be stable unless $\mu(i) < 1/r$ for any $i \in V$.

Proof. To prove the first statement, we argue again by contradiction. Suppose that $\mu(i_0) > \frac{1}{a(\mathbb{H})}$ for some node i_0 . As the function

$$\begin{cases} \mathbb{R}^+ & \longrightarrow \mathbb{R}^+ \\ x & \longmapsto \frac{r(\mathbb{H}) - a(\mathbb{H}) + x}{xa(\mathbb{H})} \end{cases}$$

strictly decreases to $\frac{1}{a}$, there exists $x_0 > 0$ such that

$$\mu(i_0) > \frac{r(\mathbb{H}) - a(\mathbb{H}) + x_0}{x_0 a(\mathbb{H})}. \tag{18}$$

Then, applying the inequality in (4) to $B \equiv V \setminus \{i_0\}$, we readily obtain that almost surely for all n ,

$$\begin{aligned} \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} A_n(V \setminus \{i_0\}) &\geq \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \left(\sum_{H \in \mathcal{H}(i_0)} |H - 1| M_n(H) + \sum_{H \in \overline{\mathcal{H}(i_0)}} |H| M_n(H) \right) \\ &\geq \left(r(\mathbb{H}) + x_0 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) M_n(\mathcal{H}(i_0)) + (r(\mathbb{H}) + x_0) M_n(\overline{\mathcal{H}(i_0)}). \end{aligned} \tag{19}$$

Likewise, applying the equality of (4) to $\{i_0\}$ and then $V \setminus \{i_0\}$ also yields

$$\begin{aligned} X_n(V \setminus \{i_0\}) + \left(x_0 + 1 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) X_n(i_0) \\ = A_n(V \setminus \{i_0\}) - \sum_{H \in \mathcal{H}(i_0)} |H - 1| M_n(H) - \sum_{H \in \overline{\mathcal{H}(i_0)}} |H| M_n(H) \\ \quad + \left(x_0 + 1 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) (A_n(i_0) - M_n(\mathcal{H}(i_0))) \\ > A_n(V \setminus \{i_0\}) + \left(x_0 + 1 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) A_n(i_0) \\ \quad - \left(r(\mathbb{H}) + x_0 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) M_n(\mathcal{H}(i_0)) - (r(\mathbb{H}) + x_0) M_n(\overline{\mathcal{H}(i_0)}). \end{aligned}$$

Combining this with (19) implies that almost surely for all n ,

$$X_n(V) + \left(x_0 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) X_n(i_0) > \left(1 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) A_n(V) + x_0 A_n(i_0).$$

Therefore we have that

$$\limsup_n \frac{1}{n} \left(X_n(V) + \left(x_0 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) X_n(i_0) \right) \geq 1 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} + x_0 \mu(i_0); \tag{20}$$

hence the chain $\{W_n, n \in \mathbb{N}\}$ is transient, since the right-hand side of the above is positive from (18).

It remains to check that in the case where the hypergraph is r -uniform, the model cannot be stable whenever

$$\mu(i_0) \geq \frac{1}{a(\mathbb{H})} = \frac{1}{r}$$

for some $i_0 \in V$. For this, notice that, as $r(\mathbb{H}) = a(\mathbb{H}) = r$, a weak inequality holds true in (8) for any $x_0 > 0$. It then readily follows from (20) that for any x_0 ,

$$\limsup_n \frac{1}{n} \left(X_n(V) + \left(x_0 - \frac{r(\mathbb{H}) + x_0}{a(\mathbb{H})} \right) X_n(i_0) \right) \geq 0,$$

and we conclude, as in the proof of Proposition 1, that the chain (W_n) is at best null recurrent. □

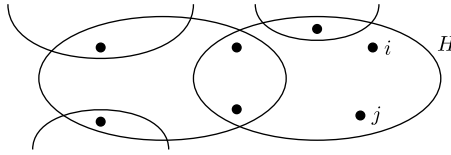


FIGURE 3. Any hypergraph with two isolated nodes is non-stabilizable.

5. Non-stabilizable hypergraphs

Having Corollary 1 and Propositions 1, 2, and 3 in hand, one can identify classes of hypergraphs \mathbb{H} such that (\mathbb{H}, Φ, μ) has an empty stability region for any admissible Φ .

We start with the following elementary observation.

Proposition 4. *If a hyperedge of $\mathbb{H} = (V, \mathcal{H})$ contains two isolated nodes, i.e. there exist $H \in \mathcal{H}$ and $i, j \in H$ such that $d(i) = d(j) = 1$, then the model cannot be stable, i.e. $\text{STAB}(\mathbb{H}, \Phi) = \emptyset$ for any admissible Φ .*

Proof. Let $\mu \in \mathcal{N}_1^+(\mathbb{H})$. Then, considering successively the sets $\{i\}$ and $\{j\}$, as $j \in H \cap \bar{\{i\}}$ and $i \in H \cap \bar{\{j\}}$ we obtain that $\mu(i) < \mu(j)$ and $\mu(i) > \mu(j)$, an absurdity. □

5.1. Stars

First recall that, as for any bipartite graph (see Theorem 2 in [16]), graphical matching models on trees are always unstable. This is true in particular if the matching graph is a ‘star’, i.e. a connected graph in which all vertices but one are of degree 1. The following two results can be seen as generalizations of this fact to hypergraphical models.

Proposition 5. *If an r -uniform hypergraph $\mathbb{H} = (V, \mathcal{H})$ has transversal number $\tau(\mathbb{H}) = 1$, then it is non-stabilizable.*

Proof. Fix Φ and μ in $\text{STAB}(\mathbb{H}, \Phi)$. Let T be a transversal of cardinality 1, i.e. $T = \{i_0\}$, where the vertex i_0 belongs to all hyperedges in \mathcal{H} . Then from Proposition 3, we have that $\mu(i_0) < 1/a(\mathbb{H}) = 1/r$. However, Proposition 2 implies that $\mu(i_0) > 1/r(\mathbb{H}) = 1/r$, an absurdity. □

In other words, any uniform hypergraph whose hyperedges all contain the same node i_0 cannot make the corresponding system stable. Moreover, we have the following.

Proposition 6. *Suppose that there exists a subset $B \subset V$ in the hypergraph $\mathbb{H} = (V, \mathcal{H})$ such that*

- all hyperedges of $\mathcal{H}(B)$ contain at least one node of degree 1;
- at least one of these nodes of degree 1 lies outside of B .

Then \mathbb{H} is non-stabilizable.

Proof. Let $k = |\mathcal{H}(B)|$, i.e. the number of hyperedges intersecting B . Denote by H_1, \dots, H_k these intersecting hyperedges, and for any $l \in \llbracket 1, k \rrbracket$, denote by $i_l \in V$ a node of degree 1 belonging to H_l . Observe that the nodes i_1, \dots, i_k are not necessarily distinct. On the one hand, for any $l \in \llbracket 1, k \rrbracket$ we have that

$$X_n(i_l) = A_n(i_l) - M_n(H_l).$$

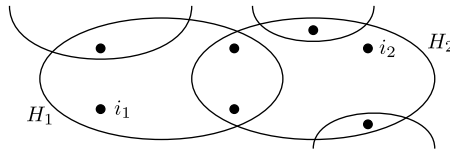


FIGURE 4. Two intersecting hyperedges, each containing an isolated node outside of their intersection, make the system unstable.

Thus, applying again the inequality in (4), we get that for all n ,

$$A_n(B) \geq \sum_{l=1}^k |H_l \cap B| M_n(H_l) = \frac{1}{n} \sum_{l=1}^k |H_l \cap B| (A_n(i_l) - X_n(i_l)).$$

This entails that if $\mu \in \mathcal{N}_1^{++}(\mathbb{H})$,

$$\limsup_n \frac{1}{n} \sum_{l=1}^k |H_l \cap B| X_n(i_l) \geq \sum_{l=1}^k |H_l \cap B| \mu(i_l) - \mu(B) \geq 0.$$

If the above inequality is strict, then the chain $\{W_n\}$ is transient. If the inequality is weak, then as above we can stochastically lower-bound the chain by a zero-drift chain $\{\tilde{Y}_n\}$, defined by

$$\tilde{Y}_n = \left(A_n(B) - \sum_{l=1}^k |H_l \cap B| A_n(i_l) \right), \quad n \in \mathbb{N},$$

which is not identically null, thanks to the assumption that at least one of the nodes i_l , $l = 1, \dots, k$, is not an element of B . This concludes the proof. \square

Example 4. Any hypergraph $\mathbb{H} = (V, \mathcal{H})$ such that there exist two hyperedges H_1 and H_2 with $H_1 \cap H_2 \neq \emptyset$ and two nodes $i_1 \in H_1 \cap \overline{H_2}$, $i_2 \in H_2 \cap \overline{H_1}$ with $d(i_1) = d(i_2) = 1$ is non-stabilizable (see Figure 4). To see this, take $B = H_1 \cap H_2$ in Proposition 6.

Remark 4. (On the DI condition in [13].) Most of the results of [13] hold under Assumption 1 therein, which states that the *dedicated item* (DI) condition is satisfied; namely, each hyperedge contains an isolated node. The above example shows that any matching model (\mathbb{H}, Φ, μ) on a hypergraph \mathbb{H} satisfying the DI condition is unstable for any admissible Φ (the case where \mathbb{H} contains a single hyperedge H is trivial).

5.2. r -partite hypergraphs

We now turn to hypergraphical generalizations of bipartite graphs.

Definition 7. An r -uniform hypergraph $\mathbb{H} = (V, \mathcal{H})$ is said to be r -partite if there exists a partition V_1, V_2, \dots, V_r of V such that every hyperedge in \mathbb{H} meets each V_i at precisely one vertex, i.e. for any $H \in \mathcal{H}$ and any $i \leq r$, $|H \cap V_i| = 1$. With some abuse, we say that an r -uniform hypergraph \mathbb{H} is r -uniform bipartite if there exists a partition V_1, V_2 of V such that for any $H \in \mathcal{H}$, $|H \cap V_1| = 1$ and $|H \cap V_2| = r - 1$.

Remark 5. Notice, first, that in the case $r = 2$, \mathbb{H} being 2-partite means exactly that it is bipartite. Second, a 2-uniform bipartite hypergraph cannot be 2-partite unless it is a bipartite graph.

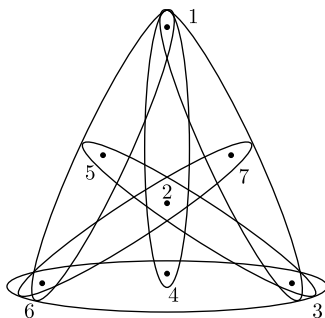


FIGURE 5. The Fano plane minus the hyperedge $\{4, 5, 7\}$.

Proposition 7. Any r -uniform bipartite hypergraph $\mathbb{H} = (V, \mathcal{H})$ is non-stabilizable.

Proof. Applying (4) successively to V_1 and V_2 readily implies that for all n ,

$$X_n(V_1) = A_n(V_1) - M_n(\mathcal{H}) \geq 0 \quad \text{and} \quad X_n(V_2) = A_n(V_2) - (r - 1)M_n(\mathcal{H}) \geq 0,$$

and thus

$$X_n(V_1) \geq A_n(V_1) - \frac{1}{r - 1}A_n(V_2).$$

Then, the usual SLLN-based argument implies that the model cannot be stable unless $\mu(V_2) \geq (r - 1)\mu(V_1)$. But as $\mu(V_1) + \mu(V_2) = 1$ we have that $\mu(V_1) \leq \frac{1}{r}$, and hence $\mu \notin \mathcal{N}_2(\mathbb{H})$ since V_1 is a transversal. \square

Example 5. The so-called Fano plane is a well-known object in discrete geometry. It is the smallest *projective plane*, namely, the smallest set of *points* and *lines* such that any two points share a line, any two lines intersect at a single point, and on every line lie the same number of points. In the setting of hypergraphs (points being nodes and lines being hyperedges), the Fano plane is thus the smallest uniform hypergraph \mathbb{H} in which each pair of nodes belongs to a single hyperedge, and each pair of hyperedges intersects at a single node. It can be checked that $\mathbb{H} = (V, \mathcal{H})$ is of order 7, for $V = \llbracket 1, 7 \rrbracket$ and e.g.

$$\mathcal{H} = \{\{1, 2, 4\}, \{1, 5, 6\}, \{1, 3, 7\}, \{2, 3, 5\}, \{4, 5, 7\}, \{4, 3, 6\}, \{6, 2, 7\}\}.$$

Supported by simulations, we conjecture that Fano planes are stabilizable. However, if $\mathbb{H}' = (V, \mathcal{H}')$ is the sub-hypergraph defined by $\mathcal{H}' = \mathcal{H} \setminus H$, where H is an arbitrary hyperedge of \mathcal{H} , then it is easily seen that \mathbb{H}' is a 3-uniform bipartite hypergraph with $V_1 = H$ and $V_2 = V \setminus H$. So we deduce from Proposition 7 that \mathbb{H}' is non-stabilizable. A Fano plane minus the hyperedge $\{4, 5, 7\}$ is represented in Figure 5.

We know from Theorem 2 in [16] that bipartite graphs are not stabilizable. The next result shows that this can be generalized to r -partite hypergraphs (which generalize bipartite graphs—see Remark 5).

Proposition 8. Any r -partite hypergraph \mathbb{H} is non-stabilizable.

Proof. As in the above proof we get that for any $i \neq j$ and any n ,

$$X_n(V_j) = A_n(V_j) - M_n(\mathcal{H}) \geq 0 \quad \text{and} \quad X_n(V_i) = A_n(V_i) - M_n(\mathcal{H}) \geq 0,$$

implying that $X_n(V_i) \geq A_n(V_i) - A_n(V_j)$ and, in turn, that the model cannot be stable unless $\mu(V_i) \leq \mu(V_j)$. By symmetry, this implies that $\mu(V_i) = \mu(V_j)$. As the V_i are disjoint, we thus have that $\mu(V_i) = 1/r$ for all i . Thus, as any V_i is a transversal of \mathbb{H} , μ is not an element of $\mathcal{N}_2(\mathbb{H})$. \square

Definition 8. A hypergraph $\mathbb{H} = (V, \mathcal{H})$ satisfies Hall’s condition if $|V_2| \geq |V_1|$ for any disjoint subsets V_2 and V_1 of V satisfying $|H \cap V_2| \geq |H \cap V_1|$ for all hyperedges $H \in \mathcal{H}$.

It is well known (see [14] for the particular case of graphs, and the general result in [15]) that Hall’s condition is necessary and sufficient for the existence of a perfect matching on \mathbb{H} , i.e. a spanning sub-hypergraph of \mathbb{H} in which all nodes have degree 1, in the case where the hypergraph is balanced, i.e. it does not contain any odd strong cycle. It is intuitively clear that the construction of stable stochastic matching models on hypergraphs is somewhat reminiscent of that of perfect matchings on a growing hypergraph that replicates the matching hypergraph a large number of times in the long run (in the case of graphs, see the discussion in Section 7 of [19]). This connection has a simple illustration in the next proposition, which provides a family of probability measures, naturally including the uniform measure on V , that cannot stabilize a matching model on the hypergraph \mathbb{H} unless the latter satisfies Hall’s condition. In what follows we define, for any $\mathbb{H} = (V, \mathcal{H})$ and any measure $\mu \in \mathcal{M}(V)$,

$$\mu_{\min} = \min \{ \mu(i) : i \in V \} \quad \text{and} \quad \mu_{\max} = \max \{ \mu(i) : i \in V \}. \tag{21}$$

Proposition 9. For any hypergraph $\mathbb{H} = (V, \mathcal{H})$ that violates Hall’s condition, any matching policy Φ , and any $\mu \in \mathcal{M}(V)$ such that

$$\frac{\mu_{\min}}{\mu_{\max}} > \frac{\lfloor \frac{q(\mathbb{H})+1}{2} \rfloor - 1}{\lfloor \frac{q(\mathbb{H})+1}{2} \rfloor}, \tag{22}$$

the model (\mathbb{H}, Φ, μ) is unstable. In particular, $(\mathbb{H}, \Phi, \mu_U)$ is unstable for μ_U the uniform distribution on V .

Proof. Fix \mathbb{H} , Φ , and a measure μ satisfying (22). We first show that μ is monotonic with respect to the counting measure on V , i.e.

$$\forall E, F \subset V, \quad |E| < |F| \implies \mu(E) < \mu(F). \tag{23}$$

Let E and F be such that $|E| < |F|$, and let $k = |F|$. Also, let α be a bijection from $\llbracket 1, q(\mathbb{H}) \rrbracket$ to V such that

$$\mu_{\min} = \mu(\alpha(1)) \leq \mu(\alpha(2)) \leq \dots \leq \mu(\alpha(q(\mathbb{H}))) = \mu_{\max}; \tag{24}$$

in other words $(\mu(\alpha(1)), \mu(\alpha(2)), \dots, \mu(\alpha(q(\mathbb{H}))))$ is an ordered (in increasing order) version of the family $\{ \mu(i); i \in V \}$. As $|E| \leq k - 1$ we clearly have

$$\mu(F) - \mu(E) \geq \sum_{i=1}^k \mu(\alpha(i)) - \sum_{i=q-k+2}^q \mu(\alpha(i)). \tag{25}$$

First, if

$$k \leq \left\lfloor \frac{q(\mathbb{H}) + 1}{2} \right\rfloor,$$

then (22) entails that $k\mu_{\min} > (k - 1)\mu_{\max}$, whence

$$\sum_{i=1}^k \mu(\alpha(i)) - \sum_{i=q-k+2}^q \mu(\alpha(i)) \geq k\mu_{\min} - (k - 1)\mu_{\max} > 0. \tag{26}$$

If

$$k > \left\lfloor \frac{q(\mathbb{H}) + 1}{2} \right\rfloor,$$

then the index sets $\llbracket 1, k \rrbracket$ and $\llbracket q - k + 2, q \rrbracket$ intersect precisely on $\llbracket q - k + 2, k \rrbracket$. Thus

$$\begin{aligned} \sum_{i=1}^k \mu(\alpha(i)) - \sum_{i=q-k+2}^q \mu(\alpha(i)) &= \sum_{i=1}^{q-k+1} \mu(\alpha(i)) - \sum_{i=k+1}^q \mu(\alpha(i)) \\ &\geq (q - k + 1)\mu_{\min} - (q - k)\mu_{\max} > 0, \end{aligned} \tag{27}$$

where the last inequality follows, as in (26), from the fact that

$$q - k + 1 \leq \left\lfloor \frac{q(\mathbb{H}) + 1}{2} \right\rfloor.$$

Combining (25) with (26)–(27) concludes the proof of (23) in all cases.

Now fix V_2 and V_1 such that $|H \cap V_2| \geq |H \cap V_1|$ for any $H \in \mathcal{H}$, and $|V_2| < |V_1|$, which from (23) implies that $\mu(V_2) < \mu(V_1)$. Then, again applying (4) to V_2 and V_1 , we get that

$$\begin{aligned} X_n(V_2) + X_n(V_1) &\geq A_n(V_2) + A_n(V_1) - 2 \sum_{H \in \mathcal{H}} |H \cap V_2| M_n(H) \\ &\geq A_n(V_2) + A_n(V_1) - 2A_n(V_2). \end{aligned}$$

Thus, by the usual argument, the model cannot be stable unless $\mu(V_2) \geq \mu(V_1)$, a contradiction. □

5.3. Cycles

Definition 9. An r -uniform hypergraph \mathbb{H} ($r \geq 2$) is called an ℓ -(Hamiltonian) cycle ($0 < \ell < r$) if there exists an ordering $V = (v_1, v_2, \dots, v_{q(\mathbb{H})})$ of the nodes of V such that the following hold:

- Every hyperedge of \mathcal{H} consists of r consecutive nodes modulo $q(\mathbb{H})$.
- Any pair of consecutive hyperedges (in an obvious sense) intersects in exactly ℓ vertices.

Proposition 10. Any r -uniform ℓ -cycle of order q such that r divides q is non-stabilizable.

Proof. The partition V_1, V_2, \dots, V_r of V defined by

$$V_i = \{v_{i+(j-1)r} ; j \in \llbracket 1, q/r \rrbracket\}$$

satisfies Proposition 8. □

Figure 6 shows a 3-uniform 2-cycle of order 12.

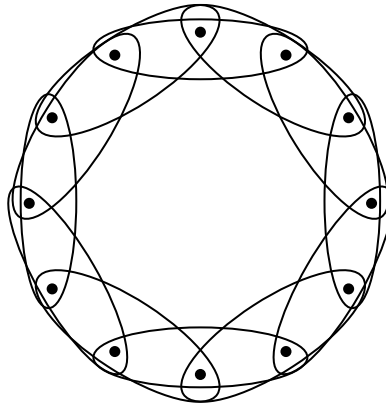


FIGURE 6. A 3-uniform 2-cycle of order 12.

6. Stable systems

We next show that stable matching models on hypergraphs exist. With a view to showing how stability can be established in concrete examples, we provide below two case studies of simple hypergraphs on which a stable stochastic matching model can be defined: complete 3-uniform hypergraphs, and sub-hypergraphs of the latter where several hyperedges are erased.

6.1. Complete 3-uniform hypergraphs

We first consider the case of a complete 3-uniform hypergraph \mathbb{H} , an example of which for $q(\mathbb{H}) = 4$ is represented in Figure 1. We show that, in this case, the necessary condition given in Proposition 3 is also sufficient.

Theorem 1. *Let $\mathbb{H} = (V, \mathcal{H})$ be a complete 3-uniform hypergraph of order $q(\mathbb{H}) \geq 4$. Then, for any admissible policy Φ , we have that*

$$\text{STAB}(\mathbb{H}, \Phi) = \mathcal{N}_3^-(\mathbb{H});$$

that is, the model (\mathbb{H}, Φ, μ) is stable if and only if $\mu(i) < 1/3$ for any $i \in V$.

Proof. The necessity of the condition having been shown in Proposition 3, only the sufficiency remains to be proven. Suppose that $\mu(i) < 1/3$ for any $i \in V$, and fix α such that $\max_{i \in V} \mu(i) < \alpha < 1/3$. Define the planar Markov chain $\{U_n^\alpha, n \in \mathbb{N}\}$ having the following transitions on \mathbb{N}^2 :

$$\left\{ \begin{array}{l} \text{First axis:} \\ \text{Second axis:} \\ \text{Interior:} \end{array} \right. \begin{array}{l} P_{(x,0),(x+1,0)}^\alpha = \alpha, \quad x \in \mathbb{N}^+, \\ P_{(x,0),(x,1)}^\alpha = 1 - \alpha, \quad x \in \mathbb{N}^+, \\ P_{(0,y),(0,y+1)}^\alpha = \alpha, \quad y \in \mathbb{N}^+, \\ P_{(0,y),(1,y)}^\alpha = 1 - \alpha, \quad y \in \mathbb{N}^+, \\ P_{(x,y),(x+1,y)}^\alpha = \alpha, \quad x, y \in \mathbb{N}^+, \\ P_{(x,y),(x,y+1)}^\alpha = \alpha, \quad x, y \in \mathbb{N}^+, \\ P_{(x,y),(x-1,y-1)}^\alpha = 1 - 2\alpha, \quad x, y \in \mathbb{N}^+, \end{array}$$

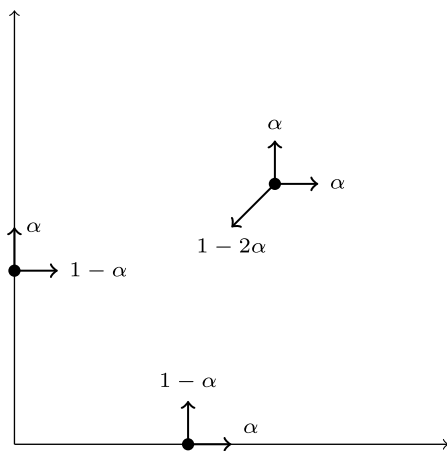


FIGURE 7. Auxiliary Markov chain of the complete 3-uniform hypergraph.

and arbitrary transitions from $(0,0)$ to any element of \mathbb{N}^2 . (These transitions are represented in Figure 7.)

Denote by $\Delta = (\Delta_x, \Delta_y)$, $\Delta' = (\Delta'_x, \Delta'_y)$, and $\Delta'' = (\Delta''_x, \Delta''_y)$ the mean (horizontal and vertical) drifts of the chain $\{U_n^\alpha\}$, respectively on the interior, on the first axis, and on the second axis, so that we have the following:

$$\begin{cases} \text{First axis:} & \Delta'_x = \alpha, & \Delta'_y = 1 - \alpha; \\ \text{Second axis:} & \Delta''_x = 1 - \alpha, & \Delta''_y = \alpha; \\ \text{Interior:} & \Delta_x = 3\alpha - 1, & \Delta_y = 3\alpha - 1. \end{cases}$$

Thus, $\Delta_x < 0$ and $\Delta_y < 0$. Also, we have that

$$\Delta_x \Delta'_y - \Delta_y \Delta'_x = \Delta_x \Delta''_y - \Delta_y \Delta''_x = (3\alpha - 1)(1 - 2\alpha) < 0,$$

so we can apply [12, Theorem 3.3.1(a)] to claim that the Markov chain $\{U_n^\alpha\}$ is positive recurrent. Specifically, it can be checked that, setting

$$u = \frac{1 - 3\alpha}{2} > 0,$$

for any w such that

$$3\alpha - 1 < w < \frac{(3\alpha - 1)\alpha}{1 - \alpha} < 0$$

we have that

$$\begin{cases} 2u\Delta_x + w\Delta_y & < 0, \\ 2u\Delta_y + w\Delta_x & < 0, \\ 2u\Delta'_x + w\Delta'_y & < 0, \\ 2u\Delta''_y + w\Delta_x & < 0. \end{cases} \tag{28}$$

Second, as $4u^2 > w^2$, the quadratic form $Q : (x, y) \mapsto ux^2 + uy^2 + wxy$ is positive definite. Then, in view of Lemma 3.3.3 in [12], it follows from (28) that, defining the mapping

$$L^\alpha : \begin{cases} \mathbb{N}^2 & \longrightarrow \mathbb{R}^+ \\ (x, y) & \longmapsto \sqrt{Q(x, y)} = \sqrt{ux^2 + uy^2 + wxy}, \end{cases}$$

we have that for some compact set $\mathcal{K}^\alpha \subset \mathbb{N}^2$, for any $(x, y) \in \overline{(\mathcal{K}^\alpha)}$,

$$\mathbb{E} [L^\alpha (U_{n+1}^\alpha) - L^\alpha (U_n^\alpha) | U_n^\alpha = (x, y)] < 0. \tag{29}$$

Now, as \mathbb{H} is complete 3-uniform, the states of the Markov chain $\{X_n, n \in \mathbb{N}\}$ have at most two nonzero coordinates; in other words, its state space is

$$\mathcal{E} = \left\{ \mathbf{x} = (x_1, \dots, x_q) \in \mathbb{N}^q : x_i x_j x_k = 0 \text{ for any distinct } i, j, k \in \llbracket 1, q \rrbracket \right\}.$$

Define the mapping

$$L: \begin{cases} \mathcal{E} & \longrightarrow \mathbb{R}^+ \\ \mathbf{x} & \longmapsto \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ L^\alpha((x, 0)) & \text{if } \mathbf{x} = x \cdot \mathbf{e}_i, \text{ for some } x > 0, i \in V, \\ L^\alpha((x, y)) & \text{if } \mathbf{x} = x \cdot \mathbf{e}_i + y \cdot \mathbf{e}_j, \text{ for some } x, y > 0, i \neq j, \end{cases} \end{cases}$$

where the above definition is unambiguous because L^α is a symmetric form on \mathbb{N}^2 . Also define the compact set

$$\mathcal{K} = \{ \mathbf{x} := x \cdot \mathbf{e}_i + y \cdot \mathbf{e}_j \in \mathcal{E} : (x, y) \in \mathcal{K}^\alpha \}.$$

Then, first, if $\mathbf{x} \in \bar{\mathcal{K}} \cap \mathcal{E}$ is such that $\mathbf{x} = x \cdot \mathbf{e}_i + y \cdot \mathbf{e}_j$ for some $x, y > 0$ and $i, j \in V, i \neq j$, we get that

$$\begin{aligned} & \mathbb{E} [L(X_{n+1}) - L(X_n) | X_n = \mathbf{x}] \\ &= (1 - \mu(i) - \mu(j)) (L(\mathbf{x} - \mathbf{e}_i - \mathbf{e}_j) - L(\mathbf{x})) \\ & \quad + \mu(i) (L(\mathbf{x} + \mathbf{e}_i) - L(\mathbf{x})) + \mu(j) (L(\mathbf{x} + \mathbf{e}_j) - L(\mathbf{x})) \\ &= (1 - \mu(i) - \mu(j)) (L^\alpha(x - 1, y - 1) - L^\alpha(x, y)) \\ & \quad + \mu(i) (L^\alpha(x + 1, y) - L^\alpha(x, y)) + \mu(j) (L^\alpha(x, y + 1) - L^\alpha(x, y)) \\ &< (1 - 2\alpha) (L^\alpha(x - 1, y - 1) - L^\alpha(x, y)) \\ & \quad + \alpha (L^\alpha(x + 1, y) - L^\alpha(x, y)) + \alpha (L^\alpha(x, y + 1) - L^\alpha(x, y)) \\ &= \mathbb{E} [L^\alpha(U_{n+1}^\alpha) - L^\alpha(U_n^\alpha) | U_n^\alpha = (x, y)], \end{aligned}$$

where, in the inequality above, we used the facts that L^α is nondecreasing in its first and second variables, and that $L^\alpha(x - 1, y - 1) < L^\alpha(x, y)$. Likewise, if $\mathbf{x} \in \bar{\mathcal{K}} \cap \mathcal{E}$ is such that $\mathbf{x} = x \cdot \mathbf{e}_i$ for some $x > 0$ and $i \in V$, we have that

$$\begin{aligned} & \mathbb{E} [L(X_{n+1}) - L(X_n) | X_n = \mathbf{x}] \\ &= \sum_{j \neq i} \mu(j) (L(\mathbf{x} + \mathbf{e}_j) - L(\mathbf{x})) + \mu(i) (L(\mathbf{x} + \mathbf{e}_i) - L(\mathbf{x})) \\ &= (1 - \mu(i)) (L^\alpha(x, 1) - L^\alpha(x, 0)) + \mu(i) (L^\alpha(x + 1, 0) - L^\alpha(x, 0)) \\ & \quad < (1 - \alpha) (L^\alpha(x, 1) - L^\alpha(x, 0)) + \alpha (L^\alpha(x + 1, 0) - L^\alpha(x, 0)) \\ &= \mathbb{E} [L^\alpha(U_{n+1}^\alpha) - L^\alpha(U_n^\alpha) | U_n^\alpha = (x, 0)], \end{aligned}$$

remarking that $L^\alpha(x, 1) < L^\alpha(x, 0)$. Recalling that $X_n = [W_n]$ for all n , using (29) in both cases, we conclude using the Lyapunov–Foster theorem (see e.g. [6, Section 5.1]) that the chain $\{W_n, n \in \mathbb{N}\}$ is positive recurrent. □

Definition 10. A 3-uniform hypergraph $\mathbb{H} = (V, \mathcal{H})$ is said to be *complete k-partite* if there exists a partition of V into k independent sets I_1, \dots, I_k such that \mathcal{H} contains exactly all subsets

of cardinality 3 of the form $\{v_1, v_2, v_3\}$, where $v_1 \in I_{i_1}, v_2 \in I_{i_2}$, and $v_3 \in I_{i_3}$, for three distinct independent sets I_{i_1}, I_{i_2} , and I_{i_3} .

The complete 3-uniform k -partite hypergraphs generalize the complete k -partite graphs introduced in [17, p. 4], also called *separable* graphs in [16] and [19], or *blow-ups* of the complete graph of order k in some other references. Roughly speaking, a complete 3-uniform k -partite hypergraph is a version of the complete 3-uniform graph of order k , in which the k nodes are replicated several times, each of the k sets of replicas forming an independent set I_i , such that all replicas of the same set do not share any hyperedge with each other, but all share hyperedges of size 3 with all other pairs of replicas belonging to two different other sets of replicas. Observe that in the particular case where all the sets I_1, \dots, I_k are of cardinality 1 (i.e. there are no replicas), the complete 3-uniform k -partite hypergraph is just the complete 3-uniform hypergraph of order k . We can then easily generalize the latter result.

Corollary 2. *For $k \geq 4$, let $\tilde{\mathbb{H}}$ be a complete 3-uniform k -partite hypergraph, and let I_1, \dots, I_k be the corresponding partition into independent sets. Then, for any admissible policy Φ , the model $(\tilde{\mathbb{H}}, \tilde{\Phi}, \tilde{\mu})$ is stable if and only if $\tilde{\mu}(I_i) < 1/3$ for any $i \in \llbracket 1, k \rrbracket$.*

Proof. Macroscopically (i.e., if we do not distinguish between items of classes that belong to the same independent set in the partition I_1, \dots, I_k), the system has the same behavior as the complete 3-uniform hypergraph. Specifically, let q be the order of the hypergraph $\tilde{\mathbb{H}}$, and define the mapping

$$\Psi: \begin{cases} \mathbb{N}^q & \longrightarrow \mathbb{N}^k \\ \tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_q) & \longmapsto \mathbf{x} = (x_1, \dots, x_k): \forall i \in \llbracket 1, k \rrbracket, x_i = \sum_{j \in \llbracket 1, q \rrbracket; j \in I_i} \tilde{x}_j. \end{cases}$$

In words, Ψ maps the detailed class-content of the model onto a class-content where one puts together all the elements of classes belonging to the same independent set of the partition I_1, \dots, I_k . Take L to be the Lyapunov function introduced in the previous proof. Fix an admissible policy $\tilde{\Phi}$ and a probability measure $\tilde{\mu} \in \mathcal{M}(\tilde{V})$, and let $\{\tilde{X}_n, n \in \mathbb{N}\}$ be the class-content process of the model $(\tilde{\mathbb{H}}, \tilde{\Phi}, \tilde{\mu})$. On the other hand, let $\{X_n, n \in \mathbb{N}\}$ be the class-content process of the model (\mathbb{H}, Φ, μ) defined on $\mathbb{H} = (V, \mathcal{H})$, the complete 3-uniform hypergraph of order k , for an arbitrary matching policy Φ and a probability measure $\mu \in \mathcal{M}(V)$ such that $\mu(i) = \tilde{\mu}(I_i)$ for any $i \in \llbracket 1, k \rrbracket$. Then it is easily seen that $\{\tilde{X}_n, n \in \mathbb{N}\}$ and $\{X_n, n \in \mathbb{N}\}$ are connected by the following relation: for all n and all $\tilde{\mathbf{x}} \in \mathbb{N}^q$,

$$\mathbb{E} [L \circ \Psi (\tilde{X}_{n+1}) - L \circ \Psi (\tilde{X}_n) \mid \tilde{X}_n = \tilde{\mathbf{x}}] = \mathbb{E} [L (X_{n+1}) - L (X_n) \mid X_n = \Psi(\tilde{\mathbf{x}})],$$

and the argument in the proof of Theorem 1 shows that the Markov chain $\{\tilde{X}_n, n \in \mathbb{N}\}$ is positive recurrent whenever $\tilde{\mu}(i) < 1/3$, that is, $\mu(i) < 1/3$, for all $i \in \llbracket 1, k \rrbracket$. This concludes the proof. □

6.2. Incomplete 3-uniform hypergraphs

As is shown in Theorem 1, complete 3-uniform hypergraphs are stabilizable for a large class of measures. We show below that incomplete hypergraphs can also be stabilizable.

Theorem 2. *Let $\mathbb{H} = (V, \mathcal{H})$ be a complete 3-uniform hypergraph of order $q \geq 5$, and let $\mathbb{H}' = (V, \mathcal{H}')$ be the (3-uniform) sub-hypergraph of \mathbb{H} obtained by setting $\mathcal{H}' = \mathcal{H} \setminus \mathcal{J}$, where \mathcal{J} is a subset of \mathcal{H} containing disjoint hyperedges. Let J be the union of the elements of \mathcal{J} . Then the*

model (\mathbb{H}', ML, μ) is stable for any μ in

$$\mathcal{S}(\mathbb{H}') = \left\{ \mu \in \mathcal{M}(V) : \left(\max_{i \in J} \lambda_i(\mu) \vee \max_{i \in \bar{J}} v_i(\mu) \right) < 0 \right\} \cap \mathcal{N}_2(\mathbb{H}') \cap \mathcal{N}_3^-(\mathbb{H}'),$$

where the $\lambda_i(\mu) : i \in J$ and $v_i(\mu) : i \in \bar{J}$ are defined respectively by (31) and (32).

Proof. Fix $\Phi = ML$, and let $\mu \in \mathcal{S}(\mathbb{H}')$. For such \mathbb{H}' the study of $\{Y_n, n \in \mathbb{N}\}$ does not boil down to that of a planar Markov chain. Instead, we study the embedded chain $\{Y_n, n \in \mathbb{N}\} = \{X_{4n}, n \in \mathbb{N}\}$, and consider the following quadratic Lyapunov function:

$$Q : \begin{cases} \mathbb{N}^q & \longrightarrow \mathbb{R}^+ \\ \mathbf{x} & \longmapsto \sum_{i=1}^q (x_i)^2. \end{cases}$$

Fix $n \in \mathbb{N}$. We have the following alternatives given the value of the embedded chain $\{Y_n\}$ at time n :

- (i) First, for any $i \in \bar{J}$ and any integer $x_i \geq 2$, the chain $\{Y_n, n \in \mathbb{N}\}$ makes the following transitions from the state $Y_n = x_i \cdot \mathbf{e}_i$:

$$Y_{n+1} = \begin{cases} Y_n + 4\mathbf{e}_i & \text{w.p. } \mu(i)^4; \\ Y_n + 3\mathbf{e}_i + \mathbf{e}_j & \text{w.p. } 4\mu(i)^3\mu(j); \\ Y_n + 2\mathbf{e}_i + 2\mathbf{e}_j & \text{w.p. } 6\mu(i)^2\mu(j)^2; \\ Y_n + \mathbf{e}_i + 3\mathbf{e}_j & \text{w.p. } 4\mu(i)\mu(j)^3; \\ Y_n + 4\mathbf{e}_j & \text{w.p. } \mu(j)^4; \\ Y_n + \mathbf{e}_i & \text{w.p. } 12\mu(i)^2\mu(j)\mu(k); \\ Y_n + \mathbf{e}_j & \text{w.p. } 12\mu(i)\mu(j)^2\mu(k); \\ Y_n - 2\mathbf{e}_i & \text{w.p. } 10\mu(j)^2\mu(k)\mu(\ell) \\ & \text{(the input has } 2j, 1k, \text{ and } 1\ell, \text{ but does not end in } jj); \\ Y_n - \mathbf{e}_i + 2\mathbf{e}_j & \text{w.p. } 2\mu(k)\mu(\ell)\mu(j)^2 \\ & \text{(the input has } 2j, 1k, \text{ and } 1\ell, \text{ and ends in } jj); \\ Y_n - 2\mathbf{e}_i & \text{w.p. } 6\mu(j)^2\mu(k)^2; \\ Y_n - \mathbf{e}_i + 2\mathbf{e}_j & \text{w.p. } 4\mu(j)^3\mu(k); \\ Y_n + \mathbf{e}_j & \text{w.p. } 24\mu(i)\mu(j)\mu(k)\mu(\ell); \\ Y_n - 2\mathbf{e}_i & \text{w.p. } 24\mu(j)\mu(k)\mu(\ell)\mu(m). \end{cases} \tag{30}$$

From this, we deduce using simple algebra that

$$\Delta_i := \mathbb{E} [Q(Y_{n+1}) - Q(Y_n) | Y_n = x_i \cdot \mathbf{e}_i] = \lambda_i(\mu)x_i + \beta_i(\mu),$$

for some bounded $\beta_i(\mu)$ and for

$$\begin{aligned} \lambda_i(\mu) = & 8\mu^4(i) + 24\mu^3(i) \sum_{j \neq i} \mu(j) + 24\mu^2(i) \sum_{j \neq i} \mu^2(j) \\ & + 8\mu(i) \sum_{j \neq i} \mu^3(j) + 24\mu^2(i) \sum_{j, k \neq i} \mu(j)\mu(k) - 44 \sum_{j, k, \ell \neq i} \mu^2(j)\mu(k)\mu(\ell) \\ & - 24 \sum_{j, k \neq i} \mu^2(j)\mu^2(k) - 8 \sum_{j, k \neq i} \mu(j)\mu^3(k) - 96 \sum_{j, k, \ell, m \neq i} \mu(j)\mu(k)\mu(\ell)\mu(m). \end{aligned} \tag{31}$$

Consequently, as the above is negative, there exists a_1^* such that $\Delta_i < 0$ whenever $x_i \geq a_1^*$.

- (ii) For any $i \in J$ and any integer $x_i \geq 2$, the transitions of $\{Y_n, n \in \mathbb{N}\}$ from the state $x_i \cdot \mathbf{e}_i$ can be retrieved in a similar fashion to (30). It follows that

$$\Delta'_i = \mathbb{E} [Q(Y_{n+1}) - Q(Y_n) | Y_n = x_i \cdot \mathbf{e}_i] = v_i(\mu)x_i + \beta'_i(\mu)$$

for some bounded $\beta'_i(\mu)$, and setting $H = \{i, j, k\}$ as the only element of \mathcal{J} such that $i \in H$, we obtain that

$$\begin{aligned} v_i(\mu) &= 8\mu^4(i) + 24\mu^3(i) \sum_{\ell \in V \setminus \{i\}} \mu(\ell) + 24\mu^2(i) \sum_{\ell \in V \setminus \{i\}} \mu^2(\ell) + 8\mu(i) \sum_{\ell \in V \setminus \{i\}} \mu^3(\ell) \\ &\quad - 8 \sum_{\ell \in \bar{H}} \mu(j)\mu^3(\ell) - 4 \sum_{\substack{\ell \in \bar{H}: \\ \text{ends with } kk}} \mu(j)\mu^2(k)\mu(\ell) - 20 \sum_{\substack{\ell \in \bar{H}: \\ \text{otherwise}}} \mu(j)\mu^2(k)\mu(\ell) \\ &\quad - 48\mu(j)\mu(k) \sum_{\ell \in \bar{H}} \mu^2(\ell) - 4 \sum_{\substack{\ell \in \bar{H}: \\ \text{ends with } \ell\ell}} \mu(j)\mu^2(\ell)\mu(m) - 40 \sum_{\substack{\ell \in \bar{H}: \\ \text{otherwise}}} \mu(j)\mu^2(\ell)\mu(m) \\ &\quad + 48\mu^2(i)\mu(j)\mu(k) - 8\mu(j)\mu(k) \sum_{\substack{\ell, m \in \bar{H}: \\ \text{ends with } jk}} \mu(\ell)\mu(m) + 80\mu(j)\mu(k) \sum_{\substack{\ell, m \in \bar{H}: \\ \text{otherwise}}} \mu(\ell)\mu(m) \\ &\quad - 96 \sum_{\ell, m \in \bar{H}} \mu(j)\mu(\ell)\mu(m)\mu(p) + 24\mu^2(i) \sum_{\ell \in \bar{H}} \mu(j)\mu(\ell) + 24\mu^2(i) \sum_{\ell, m \in \bar{H}} \mu(\ell)\mu(m) \\ &\quad - 24 \sum_{\ell \in \bar{H}} \mu^2(j)\mu^2(\ell) - 4 \sum_{\substack{\ell \in \bar{H}: \\ \text{ends with } j\ell}} \mu^2(j)\mu(\ell)\mu(m) - 40 \sum_{\substack{\ell \in \bar{H}: \\ \text{otherwise}}} \mu^2(j)\mu(\ell)\mu(m) \\ &\quad - 8 \sum_{\ell \in \bar{H}} \mu^3(j)\mu(\ell) + 24\mu(i) \sum_{j, k \in H} \mu(j)\mu^2(k) - 8 \sum_{\ell, m \in \bar{H}} \mu^3(\ell)\mu(m) \\ &\quad - 24 \sum_{\ell, m \in \bar{H}} \mu^2(\ell)\mu^2(m) - 4 \sum_{\substack{\ell, m, p \in \bar{H}: \\ \text{ends with } \ell\ell}} \mu^2(\ell)\mu(m)\mu(p) \\ &\quad - 40 \sum_{\substack{\ell, m, p \in \bar{H}: \\ \text{otherwise}}} \mu^2(\ell)\mu(m)\mu(p) - 96 \sum_{\ell, m, p, s \in \bar{H}} \mu(\ell)\mu(m)\mu(p)\mu(s). \end{aligned} \tag{32}$$

Thus, there exists a_2^* such that $\Delta'_i < 0$ whenever $x_i \geq a_2^*$.

- (iii) For any $i \neq j$ such that $\{i, j\}$ is not included in a hyperedge of the family \mathcal{J} , for any integers $x_i, x_j > 0$, we obtain that

$$\Delta_{ij} := \mathbb{E} [Q(X_{n+1}) - Q(X_n) | X_n = x_i \cdot \mathbf{e}_i + x_j \cdot \mathbf{e}_j] = \lambda_{ij}(\mu)x_i + \lambda_{ji}(\mu)x_j + \beta_{ij}(\mu),$$

for a bounded $\beta_{ij}(\mu)$ and for

$$\lambda_{ij}(\mu) = 2\left(\mu(i) - \sum_{\ell \in V \setminus \{i, j\}} \mu(\ell)\right) \text{ and } \lambda_{ji}(\mu) = 2\left(\mu(j) - \sum_{\ell \in V \setminus \{i, j\}} \mu(\ell)\right). \tag{33}$$

Now observe that $V \setminus \{i, j\} \in \mathcal{T}(\mathbb{H})$, so

$$\sum_{\ell \in V \setminus \{i, j\}} \mu(\ell) > \frac{1}{3};$$

then $\lambda_{ij}(\mu) < 0$ and $\lambda_{ji}(\mu) < 0$. Thus, there exists a_3^* such that $\Delta_{ij} < 0$ whenever $x_i \vee x_j \geq a_3^*$.

- (iv) For any i, j such that $i \neq j$ and $\{i, j\} \subset H$ for some $H \in \mathcal{J}$, for any integers $x_i, x_j > 0$, we obtain that

$$\Delta'_{ij} := \mathbb{E} [Q(X_{n+1}) - Q(X_n) | X_n = x_i \cdot \mathbf{e}_i + x_j \cdot \mathbf{e}_j] = v_{ij}(\mu)x_i + v_{ji}(\mu)x_j + \beta'_{ij}(\mu)$$

for a bounded $\beta'_{ij}(\mu)$ and

$$v_{ij}(\mu) = 2\left(\mu(i) - \sum_{\ell \in \bar{H}} \mu(\ell)\right) \text{ and } v_{ji}(\mu) = 2\left(\mu(j) - \sum_{\ell \in \bar{H}} \mu(\ell)\right). \tag{34}$$

As $\bar{H} \in \mathcal{T}(\mathbb{H})$, we have

$$\sum_{\ell \in \bar{H}} \mu(\ell) > \frac{1}{3};$$

then $v_{ij}(\mu) < 0$ and $v_{ji}(\mu) < 0$.

Again, there exists a_4^* such that $\Delta'_{ij} < 0$ whenever $x_i \vee x_j \geq a_4^*$.

- (v) We finally consider the case where $X_n = x_i \cdot \mathbf{e}_i + x_j \cdot \mathbf{e}_j + x_k \cdot \mathbf{e}_k$ for $H = \{i, j, k\}$, for some $H \in \mathcal{J}$ and integers x_i, x_j , and x_k such that $x_i, x_j \geq x_k > 0$. We have

$$\begin{aligned} \Delta_H &:= \mathbb{E} [Q(X_{n+1}) - Q(X_n) | X_n = x_i \cdot \mathbf{e}_i + x_j \cdot \mathbf{e}_j + x_k \cdot \mathbf{e}_k] \\ &= \alpha_i(\mu)x_i + \alpha_j(\mu)x_j + \alpha_k(\mu)x_k + \beta_H(\mu), \end{aligned}$$

for a bounded $\beta_H(\mu)$ and for

$$\alpha_i(\mu) = 2\left(\mu(i) - \sum_{\ell \in \bar{H}} \mu(\ell)\right), \alpha_j(\mu) = 2\left(\mu(j) - \sum_{\ell \in \bar{H}} \mu(\ell)\right) \text{ and } \alpha_k(\mu) = 2\mu(k). \tag{35}$$

As $\bar{H} \in \mathcal{T}(\mathbb{H})$, we have $\alpha_i(\mu) < 0$ and $\alpha_j(\mu) < 0$. From this, we deduce as above the existence of an integer a_5^* such that $\Delta_H < 0$ whenever $x_i \vee x_j \geq a_5^*$.

To conclude, if we let \mathcal{K} be the finite set

$$\mathcal{K} = \left\{ \mathbf{x} \in \mathcal{E} : x_i \leq \max(a_1^*, \dots, a_5^*, 2); i \in V \right\},$$

then it follows from the above arguments that for any $\mathbf{x} \in \mathcal{E} \cap \bar{\mathcal{K}}$ and any $n \in \mathbb{N}$,

$$\mathbb{E} [Q(Y_{n+1}) - Q(Y_n) | Y_n = \mathbf{x}] < 0.$$

We deduce from the Lyapunov–Foster theorem (see [6, Section 5.1]) that the chain $\{Y_n, n \in \mathbb{N}\}$ is positive recurrent. This is the case in turn for the chain $\{X_n, n \in \mathbb{N}\}$. □

Remark 6. Observe that the only incomplete (in the sense of Theorem 2) 3-uniform hypergraph of order 4 would be obtained from the complete one by deleting only one hyperedge. However, as easily seen, the transversal number of the resulting hypergraph is 1, so the latter is non-stabilizable from Proposition 5.

In the following examples, we show how stability can be established for various incomplete 3-uniform hypergraphs using Theorem 2.

Corollary 3. Consider an incomplete 3-uniform hypergraph \mathbb{H}' satisfying the assumptions of Theorem 2. Recall (21), and define the sets

$$\mathcal{A}(\mathbb{H}') := \left\{ \mu \in \mathcal{M}(V) : \frac{\mu_{\max}}{\mu_{\min}} < \left(\frac{2q^4 - 9q^3 + 12q^2 - 13q + 12}{6q^2 + 10q + 24} \right)^{1/4} \right\}$$

and

$$\mathcal{S}_1(\mathbb{H}') := \mathcal{A}(\mathbb{H}') \cap \mathcal{N}_2(\mathbb{H}') \cap \mathcal{N}_3^<(\mathbb{H}').$$

Then the model $(\mathbb{H}', \text{ML}, \mu)$ is stable for any $\mu \in \mathcal{S}_1(\mathbb{H}')$.

Proof. Recalling (31) and (32), simple algebra shows that

$$\mathcal{A}(\mathbb{H}') \subset \left\{ \mu \in \mathcal{M}(V) : \left(\max_{i \in J} \lambda_i(\mu) \vee \max_{i \in \bar{J}} v_i(\mu) \right) < 0 \right\};$$

thus $\mathcal{S}_1(\mathbb{H}') \subset \mathcal{S}(\mathbb{H}')$. □

Example 6. Observe that for any such $\mathbb{H}' = (V, \mathcal{H}')$ satisfying the assumptions of Theorem 2, the model $(\mathbb{H}', \text{ML}, \mu_U)$ is stable for μ_U the uniform distribution on V . Indeed, we have $\mu_U \in \mathcal{S}_1(\mathbb{H}')$. To see this, first observe that

$$\frac{2q^4 - 9q^3 + 12q^2 - 13q + 12}{6q^2 + 10q + 24} > 1.$$

Moreover, it is immediate that $\mu_U \in \mathcal{N}_3^<(\mathbb{H}')$. It remains to show that $\mu_U \in \mathcal{N}_2(\mathbb{H}')$. We proceed in three steps. First, for $q = 5$ the only incomplete 3-uniform hypergraph in the sense of Theorem 2 is the complete hypergraph on $\llbracket 1, 5 \rrbracket$ minus one hyperedge, say $\{1, 2, 3\}$. It is then easily seen that $\{4, 5\}$ is the only minimal transversal of \mathbb{H}' . So $\tau(\mathbb{H}') = 2$, so that for all $T' \in \mathcal{T}(\mathbb{H}')$, $\mu_U(T') \geq 2/5 > 1/3$, showing that $\mu_U \in \mathcal{N}_2(\mathbb{H}')$.

Now, if $q = 6$ there are two incomplete 3-uniform hypergraphs in the sense of Theorem 2: the complete 3-uniform hypergraph on $\llbracket 1, 6 \rrbracket$ minus one hyperedge, say $\{1, 2, 3\}$, and the complete 3-uniform hypergraph on $\llbracket 1, 6 \rrbracket$ minus two disjoint hyperedges, say $\{1, 2, 3\}$ and $\{4, 5, 6\}$. In both cases, $\{4, 5, 6\}$ is a minimal transversal of \mathbb{H}' , so $\tau(\mathbb{H}') = 3$, and thus $\mu_U(T') \geq 3/6 > 1/3$ for all $T' \in \mathcal{T}(\mathbb{H}')$, proving again that $\mu_U \in \mathcal{N}_2(\mathbb{H}')$.

We now address the case where $q > 6$. First observe that

$$q - 2 - \left\lfloor \frac{q}{3} \right\rfloor > \frac{q}{3}. \tag{36}$$

Then let $p = |\mathcal{J}|$ (using the notation of Theorem 2), and let $\mathcal{J} = \{H_1, \dots, H_p\}$. It is easily seen that a transversal of \mathbb{H} can be constructed from any minimal transversal of \mathbb{H}' , by induction, as follows:

- Take a minimal transversal T' of \mathbb{H}' , and set $\mathcal{H}_0 := \mathcal{H}'$ and $T_0 := T'$.
- For any $i = 1, \dots, p$, set $\mathcal{H}_i = \mathcal{H}_{i-1} \cup \{H_i\}$, and let T_i be a transversal of (V, \mathcal{H}_i) of minimal size among those including T_{i-1} . (T_i necessarily exists since $T_{i-1} \cup \{H_i\}$ is a transversal of (V, \mathcal{H}_i) , as easily seen by induction.)

- We obtain $\mathcal{H} = \mathcal{H}_p$ by construction, and $T := T_p$ is a transversal of \mathbb{H} .

We claim that

$$|T| \leq |T'| + p. \tag{37}$$

To see this, observe that for any $i = 1, \dots, p$ we have the following alternative: either $H_i \cap T_{i-1} = \emptyset$, in which case we can take T_i of the form $T_{i-1} \cup \{k\}$ for any $k \in H_i$, or $H_i \cap T_{i-1} \neq \emptyset$, in which case $T_i = T_{i-1}$. In all cases we have that $|T_i| \leq |T_{i-1}| + 1$, and (37) follows by induction. Observing that $|T| \geq \tau(H) = q - 2$ and that, as the H_i are disjoint, $p \leq \lfloor \frac{q}{3} \rfloor$, and using (36) and (37), we finally obtain that

$$\mu_{\cup}(T') = \frac{|T'|}{q} \geq \frac{|T| - p}{q} > \frac{1}{3}.$$

Hence, once again, $\mu_{\cup} \in \mathcal{N}_2(\mathbb{H}')$.

To conclude, μ_{\cup} is in all cases an element of $\mathcal{S}_1(\mathbb{H}')$, implying that the model $(\mathbb{H}', \text{ML}, \mu_{\cup})$ is stable for all such \mathbb{H}' .

7. Conclusion and perspectives

In this paper, we have studied a generalization of stochastic matching models on a graph, by allowing the matching structure to be a hypergraph. This class of models appears to have a wide range of applications in operations management, healthcare, and assemble-to-order systems. After formally introducing the model, we have proposed a simple Markovian representation, under i.i.d. assumptions. We have then addressed the general question of stochastic stability, viewed as the positive recurrence of the underlying Markov chain. For this class of systems, solving this elementary and central question turns out to be an intricate problem. As the results of Sections 4 and 5 demonstrate, stochastic matching models on hypergraphs are in general difficult to stabilize. Unlike the GM on graphs, the non-emptiness of the stability region on matching models on hypergraphs depends on a collection of conditions on the geometry of the compatibility hypergraph: rank, anti-rank, degree, size of the transversals, existence of cycles, and so on.

Nevertheless, we show in Section 6 that the ‘house’ of stable systems is not empty, but shelters models on various uniform hypergraphs that are complete, complete k -partite, or complete up to a partition of their nodes (which is a reasonable assumption for kidney exchange programs with 3-cycles, where, according to the compatibility of blood types and immunological characteristics, most but not all hyperedges of size 3 appear in the compatibility graph). We provide the exact stability region of the system in the first two cases, and a lower bound in the third. For this, we resort to ad-hoc multidimensional Lyapunov techniques.

There is still much to do regarding this class of systems. Providing the precise stability region of a wider class of systems appearing in other applications is a tedious task, and is the subject of our ongoing research on this topic. As the present results tend to demonstrate, such advances are likely to be obtained only on a case-by-case basis. To go beyond the study of stability, crucial questions of interest are, among others, steady-state performance evaluation and a qualitative comparison of systems and matching policies. We believe that the present work thus represents a good starting point for a fruitful avenue of research on such systems.

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