ON SUBSEMIGROUPS OF THE PROJECTIVE GROUP ON THE LINE

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I. Introduction. The subsemigroups of the projective group on the line that are described in this paper are those that can be generated by a pair of infinitesimal transformations. One denotes by G the connected component of the identity of this group; Theorem 1 gives a necessary and sufficient condition for a pair of infinitesimal transformations to generate a subsemigroup which is equal to G (and hence is actually a group). This condition is reformulated in a geometric manner in Theorem 1*. In Theorem 2 a description is given of all possible subsemigroups of G that can be generated by a pair of infinitesimal transformations with no common root. These proper subsemigroups turn out to be, in fact, uniformly finitely generated. Finally, Theorem 3 classifies the subsemigroups generated in the "degenerate" case where the two infinitesimal transformations do have a common root.

The infinitesimal transformations of the projective group on the real line are quadratic functions with real constant coefficients a, $2b$, c. This implies that a one-parameter subsemigroup of this group is given by the solution of the ordinary differential equation:

$$dy/dt = ay^2 + 2by + c \tag{1}$$

with the initial condition $y(0, x) = x$ for all non-negative (or non-positive) values of the parameter t. Thus, for fixed value of t the solution $y = y(t, x)$ of the initial value x represents a projective transformation and the family of these transformations for non-negative t defines the one-parameter semigroup generated by the infinitesimal transformation $\epsilon = ay^2 + 2by + c$.

One defines the subsemigroup generated by a pair of infinitesimal transformations to be the smallest subsemigroup topologically closed with respect to the whole group which contains the one-parameter subsemigroups generated by the two infinitesimal transformations; thus it consists of all finite products of elements of the two one-parameter subsemigroups together with all non-singular limits. It is clear that the generated subsemigroup is contained in or equal to G.

Suppose that ϵ and η are two infinitesimal transformations; since it is well known that the collection of infinitesimal transformations of a closed sub-

Received December 16, and in revised form, July 31, 1967. Theorems 1 and 1* and one case of Theorem 2 are part of the author's doctoral dissertation, written under the guidance of the late Professor Charles Loewner. Additional research was supported by a grant from the University of Oregon.

semigroup of a Lie group forms a convex cone in the Lie algebra, it follows that for all $s \geqq 0$, $t \geqq 0$, the one-parameter semigroup generated by $s\epsilon + t\eta$ is contained in the subsemigroup generated by ϵ and η.

An infinitesimal transformation $\epsilon = ay^2 + 2by + c$ is said to be elliptic, hyperbolic, or parabolic depending upon whether its discriminant $b^2 - ac$ is negative, positive or zero. If an inner automorphism is applied to a one-parameter semigroup, then the corresponding infinitesimal generator is transformed into another generator with the same discriminant. Using an appropriate inner automorphism one can always transform an elliptic generator into the form:

$$\frac{dy}{dt} = y^2 + 1; \quad \text{solution that satisfies } y(0, x) = x \text{ is}^1\text{:}$$
$$\frac{y - i}{y + i} = e^{2it}\frac{x - i}{x + i}, \qquad t \geqq 0; \tag{2}$$

a hyperbolic generator into the form:

$$\frac{dy}{dt} = y^2 - 1; \quad \text{solution that satisfies } y(0, x) = x \text{ is:}$$
$$\frac{y - 1}{y + 1} = e^{2t}\frac{x - 1}{x + 1}, \qquad t \geqq 0; \tag{3}$$

and a parabolic generator into the form:

$$\frac{dy}{dt} = 1; \quad \text{solution that satisfies } y(0, x) = x \text{ is:}$$
$$y = x + t, \qquad t \geqq 0. \tag{4}$$

One observes that a one-parameter semigroup is in fact a one-parameter subgroup if and only if the generator is elliptic. In this case, no point on the projective line remains fixed. In the hyperbolic case there are two fixed points; -1 is called the attractive fixed point as $y(x, t) \to -1$ as $t \to +\infty$ for all $x \neq 1$; similarly, $+1$ is called the repulsive fixed point. In the parabolic case, there is only one fixed point which is both attractive and repulsive; by convention, one considers "$-\infty$" as the repulsive point and "$+\infty$" as the attractive point in this case. For other hyperbolic and parabolic generators, one can make similar definitions and, clearly, under inner automorphisms, attractive fixed points go into attractive fixed points and, similarly, for repulsive fixed points.

II. Using the fact that one can always find a projective transformation that takes any three points on the projective line into any three points, one can establish the following lemma.

[1]In (2), the solution can be put into the form $y = (ax + b)/(cx + d)$, where a, b, c, and d are all real and $ad - bc > 0$.

LEMMA 1. *Any pair ϵ and η of infinitesimal transformations with no common fixed point can always be simultaneously transformed into one and only one of the following normal forms*:

$$
\begin{array}{lll}
(1) & \epsilon = 1, & \eta = \pm y^2, \\
(2) & \epsilon = 1, & \eta = \pm (y^2 - 1), \\
(3) & \epsilon = 1, & \eta = y^2 + 1, \\
(4) & \epsilon = y^2 + 1, & \eta = y^2 + c^2,\ c > 1, \\
(5) & \epsilon = y^2 + 1, & \eta = y^2 - c^2,\ c \geqq 1, \\
(6) & \epsilon = y + \alpha, & \eta = \pm (\alpha y^2 + y),\ 0 < \alpha < 1, \\
(7) & \epsilon = y^2 - 1, & \eta = \pm (y^2 - c^2),\ c > 1.
\end{array}
$$

One denotes by $S(\epsilon, \eta)$ the semigroup generated by ϵ and η and, as mentioned previously, one is interested for which ϵ and η, $S(\epsilon, \eta) = G$. As a preliminary to answering this, one first proves the following lemma.

LEMMA 2. *The subgroup generated by the infinitesimal transformations ϵ and η is equal to G if and only if ϵ and η have no common roots.*

Proof. Necessity is obvious. In order to prove the sufficiency, one shows that if ϵ and η have no common roots, then ϵ, η, and $[\epsilon, \eta]$ ($[\ ,\ ]$ is the usual bracket operation in the Lie algebra, given here by $(d\epsilon/dy)\eta - \epsilon\, d\eta/dy$) are linearly independent over the reals.

Lemma 2 gives a necessary condition for the generated *semigroup* to equal G; if it is satisfied, the next theorem gives a necessary and sufficient condition for it to be true.

THEOREM 1. *Let ϵ, η have no common roots. Then the semigroup $S(\epsilon, \eta)$ generated by ϵ and η is equal to G if and only if for some $s \geqq 0$, $t \geqq 0$, $s\epsilon + t\eta$ is elliptic.*

Proof. Since the semigroup generated by an elliptic transformation is in fact a subgroup, one sees that if ϵ and η are both elliptic with no common roots, then the generated semigroup $S(\epsilon, \eta)$ must indeed be G itself. Now suppose ϵ is elliptic, η arbitrary. Then there are real numbers $s^* \geqq 0$, $t^* > 0$ such that $s^*\epsilon + t^*\eta$ is elliptic, since if one considers $s\epsilon + t\eta$, s, t real, one notes that the discriminant is a continuous function of s and t and for $s = 1$, $t = 0$ it is negative. Clearly, if ϵ and η have no common roots, neither do ϵ and $s^*\epsilon + t^*\eta$. Since the semigroup generated by ϵ and $s^*\epsilon + t^*\eta$ is contained in that generated by ϵ and η, one concludes that if at least one generator is elliptic and there are no common roots, then $S(\epsilon, \eta) = G$.

This proves the sufficiency, for if $\gamma = s\epsilon + t\eta$, $s \geqq 0$, $t \geqq 0$, γ elliptic, then if $s > 0$, γ and η have no common root, and if $t > 0$, γ and ϵ have no common root, provided ϵ and η had no common roots. But the semigroup generated by ϵ, η, and γ is contained in that generated by ϵ and η.

Before proving the necessity of the condition, it is useful to examine which pairs of infinitesimal transformations transformed into the normal form of

Lemma 1 satisfy the condition of the theorem. One easily verifies that in cases (3), (4), and (5) it is always satisfied; in case (1) it is satisfied by $\epsilon = 1, \eta = y^2$; in case (2) by $\epsilon = 1, \eta = y^2 - 1$, and in case (7) by $\epsilon = y^2 - 1$, $\eta = -(y^2 - c^2)$. To prove the necessity, it suffices to show that in all other cases the generated semigroup $S(\epsilon, \eta)$ is a proper subsemigroup of G; a complete description of these proper subsemigroups will be given in Theorem 2 and, in the process, the proof of Theorem 1 will be completed.

First, a geometric reformulation of Theorem 1 will be stated; the proof is left to the reader.

THEOREM 1*. *If at least one of the infinitesimal transformations ϵ, η is elliptic, then $S(\epsilon, \eta)$ equals G. In all other cases G is generated if and only if the attractive fixed points interlace with the repulsive fixed points.*

Remarks. Observe that if neither generator is elliptic, one can consider each generator to have a repulsive and an attractive fixed point. In the case where both generators are hyperbolic and the fixed points interlace, then the attractive fixed points can never interlace with the repulsive ones and thus in this case, $S(\epsilon, \eta)$ is never equal to G.

III. In this section, one is concerned with the case where the attractive fixed points separate the repulsive fixed points; one defines in this case a source and a sink interval of ϵ and η as follows: If z_0, z_1 are the repulsive and attractive fixed points of ϵ, w_0, w_1 those of η, the source interval I consists of all z such that $(z_0, w_0; z_1, z) < 0$ and the sink interval J consists of all z such that $(z_1, w_1; z_0, z) < 0$.

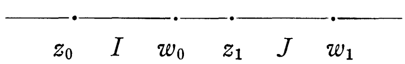

FIGURE 1

The generated semigroup is said to be uniformly finitely generated if there exists an integer n such that any element of the generated semigroup is expressible as a product of at most n transformations of the two one-parameter semigroups. Clearly, such a semigroup is closed.

THEOREM 2. *The subsemigroup $S(\epsilon, \eta)$ generated by a pair of infinitesimal transformations ϵ, η with no common root such that for all $s \geqq 0$, $t \geqq 0$, $s\epsilon + t\eta$ has at least one real root, is uniformly finitely generated; any transformation of the generated subsemigroup can be written as a product of length at most three. The subsemigroup $S(\epsilon, \eta)$ consists precisely of:*

(a) *the two one-parameter semigroups generated by ϵ and η,*

(b) *for each v in the source interval, each w in the sink interval, there exists a function $\lambda(v, w)$ determined by ϵ and η such that $z_1 \leqq \lambda(v, w) < w$ and for* *any point x on the projective line,*[2] $\lambda(v, w) \leqq x < w$, *there is a transformation in the semigroup* $S(\epsilon, \eta)$ *that leaves v and w fixed and takes z_1 into x.*

Remark 1. $\lambda(v, w) = z_1$ if and only if v and w are the roots of some generator $s\epsilon + t\eta$, $s \geqq 0$, $t \geqq 0$.

Remark 2. If $\lambda(v, w) < x$, then the corresponding transformation cannot be obtained as a product of length 2 but can be obtained from 2 different products of length 3, one involving ϵ twice and η once, the other ϵ once and η twice. If $x = \lambda(v, w) > z_1$, the transformation is obtained as a product of length two; if $x = \lambda(v, w) = z_1$, then the transformation is just the identity.

Remark 3. If $s \geqq 0$, $t \geqq 0$, the one-parameter semigroup generated by $s\epsilon + t\eta$ is contained in $S(\epsilon, \eta)$, in agreement with the above remarks; according to the theorem, all the transformations of these one-parameter semigroups can be obtained as products of length 3.

The case $\epsilon = y + \alpha$, $\eta = -\alpha y^2 - y$, $0 < \alpha < 1$, will be considered in detail. Exactly the same technique and proofs work in the other cases. In fact, one can derive the result that $S(\epsilon, \eta)$ is uniformly finitely generated in all the cases from this case. For the case where the roots of two hyperbolic generators separate, and if the attractive and repulsive fixed points separate, the generators may be brought into the form $\epsilon = y + \alpha$, $\eta = -\alpha y^2 - y$, where $\alpha > 1$, or equivalently, $\epsilon = \alpha y + 1$, $\eta = -y^2 - \alpha y$, $0 < \alpha < 1$. Thus, one obtains the transformations of one semigroup from the transformations of the other by replacing α by α^{-1}; so clearly, if any transformation in one semigroup is a product of length at most 3, the same is true for the other semigroup. If $\alpha \to 0$ in the above, one obtains $\epsilon = 1$, $\eta = -y^2$, the case of two parabolic generators and clearly here every transformation must again be a product of length at most 3. Also, one can obtain the semigroup generated by $\epsilon = \gamma y + 1$, $\eta = -y^2 - \beta y$, $-\gamma^{-1} < -\beta < 0$ from an inner automorphism of the one generated by $\epsilon = \alpha y + 1$, $\eta = -y^2 - \alpha y$, for some α, $0 < \alpha < 1$, and thus this is also uniformly finitely generated; letting $\gamma \to 0$ one obtains the result for a hyperbolic and parabolic generator with attractive and repulsive fixed points that separate.

IV. The proof of Theorem 2 in the case where $\epsilon = y + \alpha$, $\eta = -\alpha y^2 - y$, $0 < \alpha < 1$, will be broken up into several lemmas. The main idea of the proof is to first describe the most general element of $S(\epsilon, \eta)$, that is, a product of length 2 in terms of its fixed points, denoted by v and w. Let I denote the interval $(0, \infty)$; J denotes the interval $(-\alpha^{-1}, -\alpha)$. For each $w \in I$, there is a unique $v \in J$ that is the solution of $1 + \alpha(v + w) + vw = 0$; denote this v by $v = F(w)$. One finds that for any $w \in I$ and for any v,

[2]Assume, for definiteness, that the sink interval is just (z_1, w_1) which can be achieved by inner automorphism if necessary.

$F(w) < v < -\alpha$, there is a unique product of length 2, formed by applying first ϵ and then η, that has v and w as its only fixed points; one denotes this transformation by $y = A_v^w(x)$. Further, for any $w \in I$ and for any v, $-\alpha^{-1} < v < F(w)$, there is also a unique product of length 2, formed by applying first η and then ϵ, that has v and w as its only fixed points; one denotes this transformation by $y = B_v^w(x)$. If $v = F(w)$, then there is no product of length 2 that keeps v and w fixed except the identity; it is convenient to adopt the convention that $A_{F(w)}^w(x) = B_{F(w)}^w(x) = x$. The transformations $A_v^w(x)$ and $B_v^w(x)$ turn out to have a crucial role in the proof of the theorem because, by appropriately choosing w, v_1, and v_2, one can express any element of $S(\epsilon, \eta)$ that is a product of length 3 in the form[3] $A_{v_1}^w(B_{v_2}^w(x))$ and also, by appropriately choosing w, v_3, and v_4, one can express this same element in the form $B_{v_3}^w A_{v_4}^w(x)$. But these products are particularly simple to study because the same $w \in I$ is held fixed by both $A_{v_1}^w(x)$ and $B_{v_2}^w(x)$, and, further, the fact that any product of length 3 has the two alternate possible representations above enables one to show that any product of length 4 can in fact be written as a product of length 3. Thus the proof really has two parts, first describing the elements of $S(\epsilon, \eta)$ which can be expressed in the form $A_{v_1}^w B_{v_2}^w(x)$ and then proving $S(\epsilon, \eta)$ is minimal in the sense that all elements of $S(\epsilon, \eta)$ have this form.

LEMMA 1. *If $y = T(x) \in S(\epsilon, \eta)$, $T(x)$ not the identity, then T has two distinct real fixed points v and w, $v \in J$, $w \in I$. For all x, $x \neq v$, $\lim_{n\to\infty} T^n(x) = w$.*

Proof. One observes that T transforms $[0, \infty]$ properly into itself and T^{-1} does the same to $[-\alpha^{-1}, -\alpha]$.

Let $y = T_t(x)$ denote the elements of the one-parameter semigroup generated by ϵ $(t \geqq 0)$, and let $y = S_s(x)$ be the one generated by η $(s \geqq 0)$.

LEMMA 2. *If $y = T(x) = S_sT_t(x)$, $s > 0$, $t > 0$, then the fixed points of T satisfy: $1 + \alpha(v + w) + vw > 0$; conversely, if $v \in J$ and $w \in I$ satisfy this inequality, there exist unique positive numbers $s > 0$, $t > 0$ such that $y = S_sT_t(x)$ has v and w as its fixed points.*

Proof. One finds that

$$T_{t'}(x) = e^{t'}x + (e^{t'} - 1)\alpha, \qquad S_{s'}(x) = x/((e^{s'} - 1)\alpha x + e^{s'}),$$

and one changes parameters to $t = e^{t'}$, $s = e^{s'}$; hence[4] $t \geqq 1$, $s \geqq 1$. If $T = S_sT_t$, $w \in I$ is a fixed point of T, then

$$s = \frac{tw + (t-1)\alpha + \alpha tw^2 + \alpha^2 w(t-1)}{w + \alpha tw^2 + \alpha^2 w(t-1)} \tag{5}$$

[3]Observe that this is clearly a product of length 3. As the order of applying transformations always begins at the right, parentheses will be omitted subsequently.

[4]The parameters t, s will be used from now on.

so that $s = s(t, w)$; clearly, $w > 0$, $t > 1$ imply $s > 1$. If $v \in J$ is also a fixed point of T, then it must be possible to solve for t the equation:

$$s(t, w) = s(t, v), \qquad t > 1. \tag{6}$$

This becomes a quadratic equation for t with solutions

$$t = 1, \qquad t = (\alpha^2 - 1)/(vw + \alpha^2 + \alpha(v + w))$$

and the latter is greater than 1 precisely if the condition of the lemma holds. Conversely, if this condition holds, choose t as above and s by (5); then $S_s T_t(x)$ has v and w as its fixed points.

Remark 1. Let $F(w) = -(1 + \alpha w)/(\alpha + w)$; then the set of v satisfying the condition of the lemma that belong to J is the set of all v: $F(w) < v < -\alpha$. If

$$\tau = \frac{\alpha^2 - 1}{vw + \alpha^2 + \alpha(v + w)}, \tag{7}$$

then

$$\frac{\partial \tau}{\partial v} = \frac{-(\alpha^2 - 1)(w + \alpha)}{[(vw + \alpha^2 + \alpha(v + w)]^2} > 0, \tag{8}$$

thus τ increases monotonically as v increases; as $v \to -\alpha$, $\tau \to \infty$.

Remark 2. Similar results hold for products $T_t S_\sigma(x)$ provided we reverse the inequality in the lemma. Here the set of all v in J that satisfy the inequality for fixed w is the set of all v such that:

$$-\alpha^{-1} < v < F(w) \tag{9}$$

as v decreases, σ increases and as $v \to -\alpha^{-1}$, $\sigma \to \infty$.

Remark 3. If $1 + \alpha(v + w) + vw = 0$, then $v \in J$, $w \in I$ are not the fixed points of any element of $S(\epsilon, \eta)$ which is a product of length 2.

Now, recalling the definitions of $A_v{}^w(x)$ and $B_v{}^w(x)$, one defines

$$\begin{aligned} \lambda(v, w) &= A_v{}^w(0) \quad \text{if } F(w) \leqq v < -\alpha, \\ \lambda(v, w) &= B_v{}^w(0) \quad \text{if } -\alpha^{-1} < v \leqq F(w). \end{aligned} \tag{10}$$

(Observe that $\lambda(F(w), w) = 0$ by either definition.) λ is a continuous function of v and w and, for fixed w,

$$\lim_{v \to -\alpha} \lambda(v, w) = w, \qquad \lim_{v \to -\alpha^{-1}} \lambda(v, w) = w; \tag{11}$$

as $\partial \tau / \partial v > 0$ by (8), one can solve (7) for v as a function of τ; $v(\tau)$ is defined for all $\tau \geqq 1$ and increases continuously and monotonically from $F(w)$ to $-\alpha$ as τ increases from 1 to ∞. Define the function of two variables:

$$G(\gamma, \tau) = A_{v(\tau)}{}^w(\gamma), \qquad \tau \geqq 1, \tag{12}$$

where γ is any point on the projective line. In fact, $G(w, \tau) = w$ and $\lim_{\tau\to\infty} G(\gamma, \tau) = w$ for all $\gamma \neq -\alpha$. The reason for introducing $G(\gamma, \tau)$ and studying its behaviour for fixed γ as a function of τ is that a knowledge of this becomes crucial when we compose an $A_{v_1}{}^w(x)$ with a $B_{v_2}{}^w(x)$ and then let v_1 and v_2 vary.

LEMMA 3. *The equation $\partial G/\partial \tau = 0$ is for fixed τ quadratic in γ, one solution always being $\gamma = w$. If the other solution is denoted by $\gamma = \gamma(\tau)$, this is a continuous function of τ such that $\gamma(1) = F(w)$ and $\gamma(\tau)$ increases monotonically*[5] *to $-\alpha$ as $\tau \to \infty$ and for fixed τ, $v(\tau) < \gamma(\tau)$.*

Proof. That the equation $\partial G/\partial \tau = 0$ is a quadratic in γ follows from a simple calculation and clearly, $\gamma = w$ must be one root for all τ. If $v = v(t^*)$ corresponds to $\tau = t^*$, then $G(v(t^*), t^*) = v(t^*)$ and for $\tau < t^*$, $G(v(t^*), \tau) > v(t^*)$; hence one concludes that $(\partial G/\partial \tau)(v(t^*), t^*) < 0$. But one finds that $(\partial G/\partial \tau)(-\alpha, \tau) > 0$ for all τ as $-\alpha$ is a fixed point of ϵ and hence it follows that $G(-\alpha, \tau) = S_{s(\tau)}(-\alpha)$, and from (5) one sees that $\partial s/\partial \tau > 0$. Thus, one concludes that $v(t^*) < \gamma(t^*) < -\alpha$ and since as $t^* \to \infty$, $v(t^*) \to -\alpha$, the lemma is proved.

Remark 1. Hence if $-\alpha \leqq \gamma < w$, $G(\gamma, \tau)$ is an increasing function of τ; if $\gamma \leqq F(w)$ or if $\gamma > w$, or if $\gamma = \infty$, $G(\gamma, \tau)$ is a decreasing function of τ. All points on $(F(w), -\alpha)$ may be written as $v = v(\tau_0)$ and $G(v, \tau)$ increases for $1 \leqq \tau \leqq \tau^*$ and then decreases for all $\tau \geqq \tau^*$, where $\tau^* < \tau_0$ (i.e., by the time v has returned to its original position, the function is decreasing and remains decreasing).

Remark 2. If one defines $H(\gamma, \sigma) = B_{v(\sigma)}{}^w(\gamma)$, analogous results are obtained.

LEMMA 4. *For each $w \in I$, each v, $F(w) \leqq v < -\alpha$, and each δ, $\lambda(v, w) \leqq \delta < w$, there are v_2, $v \leqq v_2 < -\alpha$, and v_1, $-\alpha^{-1} < v_1 \leqq F(w)$, such that the transformation $B_{v_1}{}^w A_{v_2}{}^w(x)$ leaves v and w fixed and transforms 0 onto δ; hence there exist numbers $t \geqq 1$, $s \geqq 1$, and $u \geqq 1$ such that $T_u S_s T_t(x)$ takes v into v, w into w, and 0 into δ.*

Proof. Suppose that $v = v(\tau_0)$; define $\tau^* = (\alpha - \alpha^{-1})/(v + \alpha)$ so that

$$A_{v(\tau^*)}{}^w(v) = -\alpha^{-1}.$$

Then for τ, $\tau_0 \leqq \tau \leqq \tau^*$, $A_{v(\tau)}{}^w(v)$ decreases monotonically from v to $-\alpha^{-1}$. But for each y, $-\alpha^{-1} < y \leqq v$, there is a unique $v_1(y)$, $-\alpha^{-1} < v_1 \leqq F(w)$, such that $B_{v_1(y)}{}^w(y) = v$. Further, $v_1(y)$ is a continuous function of y, $v_1(v) = F(w)$ and since $v_1(y) < y$,

[5]In fact, $\gamma(\tau) = v(\tau^2)$, $\tau \geqq 1$, but this is not needed. It shows that $\gamma(\tau)$ is monotonically increasing, a fact which subsequently is not used.

(13) $$\lim_{y \to -\alpha^{-1}} v_1(y) = -\alpha^{-1}.$$

Hence for each τ, $\tau_0 \leqq \tau < \tau^*$, there are points $v_1(\tau)$ and $v_2(\tau)$ such that:

(14) $$v = v(\tau_0) \leqq v_2(\tau) < v(\tau^*) < -\alpha, \qquad -\alpha^{-1} < v_1(\tau) \leqq F(w),$$

and

(15) $$B_{v_1}{}^w A_{v_2}{}^w(v) = v.$$

It is obvious that $B_{v_1}{}^w A_{v_2}{}^w(w) = w$ and that $B_{v_1}{}^w A_{v_2}{}^w(0) \geqq \lambda$, since $v_2 \geqq v$ implies $A_{v_2}{}^w(0) \geqq A_v{}^w(0) = \lambda$ and $B_{v_1}{}^w A_{v_2}{}^w(0) \geqq A_{v_2}{}^w(0)$. Clearly, $F(\tau) = B_{v_1(\tau)}{}^w A_{v_2(\tau)}{}^w(0)$ is a monotonically increasing function of τ for $\tau_0 \leqq \tau < \tau^*$ such that $F(\tau_0) = \lambda$ and since as $\tau \to \tau^*$, $v_1(\tau) \to -\alpha^{-1}$, it follows that

(16) $$\lim_{\tau \to \tau^*} F(\tau) = w,$$

which proves the lemma.

Remark 1. Suppose that $B_{v_1}{}^w A_{v_2}{}^w(v) = v$. Then $B_{v_1}{}^w A_{v_2}{}^w(0) \geqq \lambda$; indeed, $v(\tau_0) \leqq v_2 \leqq v(\tau^*)$, since otherwise, $A_{v_2}{}^w(v)$ would either be less than or equal to $-\alpha^{-1}$ or greater than v and in neither case is there a $B_{v_1}{}^w$ which could transform $A_{v_2}{}^w(v)$ into v.

Remark 2. Given any transformation described in Lemma 4, there are v_3, $v(\sigma^*) < v_3 \leqq F(w)$ (where $\sigma^* = v(\alpha^2 - 1)/\alpha(1 + \alpha w)$) and v_4, $v \leqq v_4 < -\alpha$, such that $A_{v_4}{}^w B_{v_3}{}^w$ equals this transformation. Here one remarks that σ^* was chosen so that $B_{v(\sigma^*)}{}^w(v) = -\alpha$. The proof is completely analogous to the proof of Lemma 4.

Remark 3. If $A_{v_2}{}^w B_{v_1}{}^w(v) = v$, then $A_{v_2}{}^w B_{v_1}{}^w(0) \geqq \lambda$.

Remark 4. Similar results hold if $-\alpha^{-1} < v \leqq F(w)$.

The transformations studied in Lemma 4 are, as has already been mentioned, products of length 3, and thus it has been established that, given any $v \in J$, any $w \in I$, and any δ, $\lambda(v, w) \leqq \delta < w$, there are numbers t_1, s_1, and u_1 such that $y = T_{t_1}S_{s_1}T_{u_1}(x)$ transforms 0 into δ and leaves v and w fixed ($t_1 \geqq 1$, $s_1 \geqq 1$, $u_1 \geqq 1$), and, further, there are also numbers t_2, s_2, and u_2 such that $T_{t_1}S_{s_1}T_{u_1}(x) = S_{t_2}T_{s_2}S_{u_2}(x)$ ($t_2 \geqq 1$, $s_2 \geqq 1$, $u_2 \geqq 1$). Next, it will be shown that there are no other elements of $S(\epsilon, \eta)$ that are expressible as a product of length 3.

LEMMA 5. *If $y = T(x) \in S(\epsilon, \eta)$ is expressible as a product of length* 3 *and if w and v are the fixed points of T, $w \in I$, $v \in J$, then $\lambda(v, w) \leqq T(0) < w$.*

Proof. The result is, of course, obvious if $v = F(w)$ since $\lambda(w, F(w)) = 0$. The proof will be given only in the case $v > F(w)$; the other case is similar.

Case 1. $T(x) = T_tS_sT_u(x)$ for some $t \geqq 1$, $s \geqq 1$, $u \geqq 1$. Since $T(w) = w$ and $S_sT_u(w) > 0$, it must be that $1 \leqq t < 1 + w/\alpha$ as $T_{1+w/\alpha}(0) = w$. Now for each τ, $\tau_0 \leqq \tau \leqq \tau^*$, one can find, as before,

$$v_1(\tau), \quad v_2(\tau), \quad -\alpha^{-1} < v_1(\tau) \leqq F(w), \quad v \leqq v_2(\tau) \leqq v(\tau^*)$$

such that $B_{v_1}{}^w A_{v_2}{}^w(x)$ also leaves v and w fixed and 0 is transformed into a value greater than or equal to λ but less than w. $B_{v_1}{}^w(x) = T_{t(\sigma)}S_\sigma(x)$ and as v_1 decreases monotonically from $F(w)$ to $-\alpha^{-1}$, σ increases monotonically from 1 to ∞ and $t(\sigma)$ increases monotonically from 1 to $1 + w/\alpha$. Choose τ so that $t(\sigma)$ equals the t which appears in the product for $T(x)$ above. Then

$$R(x) = T_t S_{s'} T_{u'}(x), \qquad s' \geqq 1, u' \geqq 1, \tag{17}$$

leaves w and v fixed and $\lambda(v, w) \leqq R(0) < w$.

It remains to prove that $s' = s$, $u' = u$.

Now choose $\tilde{w}$ and $\tilde{v}$ so that $T_t(\tilde{w}) = w$ and $T_t(\tilde{v}) = v$. Then

$$\begin{aligned} S_s T_u(v) &= \tilde{v}, \qquad & S_s T_u(w) &= \tilde{w}, \\ S_{s'} T_{u'}(v) &= \tilde{v}, \qquad & S_{s'} T_{u'}(w) &= \tilde{w}, \end{aligned} \tag{18}$$

where $s \geqq 1$, $u \geqq 1$, $s' \geqq 1$, $u' \geqq 1$.

If $V(x) = S_s T_u(x)$, $V(w) = \tilde{w}$, $V(v) = \tilde{v}$, then one finds that

$$\text{(19)} \quad \begin{aligned} &\text{(a)} \qquad \frac{uw + (u-1)\alpha}{\alpha(uw + (u-1)\alpha)(s-1) + s} = \tilde{w}, \\ &\text{(b)} \qquad \frac{uv + (u-1)\alpha}{\alpha(uv + (u-1)\alpha)(s-1) + s} = \tilde{v}. \end{aligned}$$

Solving (a) for s in terms of u and substituting in (b) yields a quadratic equation for u; thus there are *at most two* distinct solutions, s_1, u_1 and s_2, u_2 of (19). But $s = 1$, $u = t^{-1}$ is a solution, for S_1 is the identity and $T_{t^{-1}}(x) = T_t^{-1}(x)$ so that $T_t^{-1}(v) = \tilde{v}$, $T_{t^{-1}}(w) = \tilde{w}$. If $t = 1$, $T(x)$ is a product of length 2 and hence $T(0) = \lambda(v, w)$ so one may assume that $t > 1$ and hence $t^{-1} < 1$. Thus there is *at most one* solution of (19) such that $s \geqq 1$, $u \geqq 1$ and hence $s = s'$, $u = u'$.

Case 2. $T(x) = S_t T_s S_u(x)$. Here it is easily seen that $1 \leqq u < \sigma^*$ since $S_{\sigma^*}(v) = -\alpha$. Now one can choose a transformation $A_{v_4}{}^w B_{v_3}{}^w(x) = S_{t'} T_{s'} S_u(x)$ which also leaves w and v fixed and the remainder of the proof is as in Case 1.

From Lemma 5 and the paragraph preceding it, it follows that, given *any* transformation $T(x)$ of the form $T(x) = T_t S_s T_u(x)$, $t \geqq 1$, $s \geqq 1$, $u \geqq 1$, there are numbers $t' \geqq 1$, $s' \geqq 1$, and $u' \geqq 1$ such that

$$T(x) = T_t S_s T_u(x) = S_{t'} T_{s'} S_{u'}(x).$$

Hence any product of length 4 can be expressed as a product of length 3 and this completes the proof of Theorem 2, at least in the case discussed in detail here.

The other cases have analogous proofs with slightly different calculations. Observe that Lemma 3 did not give precise information on the function

$\gamma(\tau)$, apart from the fact that $v(\tau) < \gamma(\tau) < -\alpha$. The property of the generators that insured the existence of fixed points in the intervals I and J was that the attractive and repulsive fixed points separated each other. In a sense, the fact that $S(\epsilon, \eta)$ contained no elliptic transformations meant that it had to leave aside many other elements of G.

V. There remain the proper subsemigroups generated by pairs of infinitesimal transformations with a common root, which one assumes to be at[6] ∞. If ϵ and η generate different one-parameter semigroups and yet have both roots in common, then $\epsilon = -\eta$ so that the generated subsemigroups is just a one-parameter subgroup.

THEOREM 3. *The subsemigroup $S(\epsilon, \eta)$ generated by a pair of infinitesimal transformations ϵ and η with a common root at ∞ but such that $\epsilon \neq \pm\eta$ is uniformly finitely generated and consists of products of length* 2. *The subsemigroup $S(\epsilon, \eta)$ is precisely the union of the one-parameter subsemigroups generated by $s\epsilon + t\eta$, $s \geqq 0$, $t \geqq 0$ and thus is minimal in the sense that these are always contained in the generated subsemigroup.*

Proof. One need only consider the cases:

$$\text{(20)} \qquad \begin{array}{lll} \text{(a)} & \epsilon = 1, & \eta = y, \\ \text{(b)} & \epsilon = 1, & \eta = -y, \\ \text{(c)} & \epsilon = y, & \eta = y - 1, \\ \text{(d)} & \epsilon = y, & \eta = -(y-1). \end{array}$$

The generated subsemigroups can directly be verified to be

$$\text{(21)} \qquad \begin{array}{lll} \text{(a)} & y = ax + b, & a \geqq 1, b > 0, \\ \text{(b)} & y = ax + b, & 0 < a \leqq 1, b \geqq 0, \\ \text{(c)} & y = ax + b, & a \geqq 1 - b, b \leqq 0, \\ \text{(d)} & y = ax + b, & a \geqq \max\{0, 1 - b\}, b \geqq 0. \end{array}$$

The theorem follows by showing that any generated transformation can be obtained as a product of length two, and that its fixed points are identical to those of all the transformations of the one-parameter subsemigroup generated by $s\epsilon + t\eta$ for an appropriate choice of $s \geqq 0$, $t \geqq 0$, with the same attractive and repulsive fixed points.

[6]If ϵ is elliptic, and ϵ and η have a common root, then they generate the same subgroup.

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