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Geodesic stretch, pressure metric and marked length spectrum rigidity

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Abstract. We refine the recent local rigidity result for the marked length spectrum obtained by the first and third author in [GL19] and give an alternative proof using the geodesic stretch between two Anosov flows and some uniform estimate on the variance appearing in the central limit theorem for Anosov geodesic flows. In turn, we also introduce a new pressure metric on the space of isometry classes, which reduces to the Weil–Petersson metric in the case of Teichmüller space and is related to the works [BCLS15, MM08].

Key words: Anosov geodesic flows, rigidity, marked length spectrum
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1. *Introduction*

Let M be a smooth closed n-dimensional manifold. We denote by $\mathcal{M}$ the Fréchet manifold consisting of smooth metrics on M. We denote by $\mathcal{M}^{k,\alpha}$ the set of metrics with regularity $C^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$. We fix a smooth metric $g_0 \in \mathcal{M}$ with Anosov geodesic flow $\varphi_t^{g_0}$ and define the unit tangent bundle by $S_{g_0}M := \{(x, v) \in TM \mid |v|_{g_0} = 1\}$. Recall that being Anosov means that there exists a flow-invariant continuous splitting

$$T(S_{g_0}M) = \mathbb{R}X \oplus E_s \oplus E_u,$$

such that

$$\|d\varphi_t^{g_0}(w)\| \leq Ce^{-\lambda t}\|w\| \quad \text{for all } w \in E_s, \text{ for all } t \geq 0,$$
$$\|d\varphi_t^{g_0}(w)\| \leq Ce^{-\lambda |t|}\|w\| \quad \text{for all } w \in E_u, \text{ for all } t \leq 0,$$

where the constants $C, \lambda > 0$ are uniform and the norm here is the one induced by the Sasaki metric of g_0. Such a property is satisfied in negative curvature.

1.1. *Geodesic stretch and marked length spectrum rigidity.* The set of primitive free homotopy classes $\mathcal{C}$ of M is in one-to-one correspondence with the primitive conjugacy classes of $\pi_1(M, x_0)$ (where $x_0 \in M$ is arbitrary). When g_0 is Anosov, there exists a unique closed geodesic $\gamma_{g_0}(c)$ in each primitive free homotopy class $c \in \mathcal{C}$ (see [**Kli74**]). This allows us to define the *marked length spectrum* of the metric g_0 by

$$L_{g_0} : \mathcal{C} \to \mathbb{R}_+, \quad L_{g_0}(c) = \ell_{g_0}(\gamma_{g_0}(c)),$$

where $\ell_{g_0}(\gamma)$ denotes the g_0-length of a curve $\gamma \subset M$ computed with respect to g_0. The marked length spectrum can alternatively be defined for the whole set of free homotopy classes, but it is obviously an equivalent definition. Given $c \in \mathcal{C}$, we will write $\delta_{g_0}(c)$ to denote the probability Dirac measure carried by the unique g_0-geodesic $\gamma_{g_0}(c) \in c$.

It was conjectured by Burns and Katok [**BK85**] that the marked length spectrum of negatively curved manifolds determines the metric up to isometry in the sense that two negatively curved metrics g and g_0 with the same marked length spectrum (namely $L_g = L_{g_0}$) should be isometric. Although the conjecture was proved for surfaces by Croke and Otal [**Cro90**, **Ota90**]) and in some particular cases in higher dimension (for conformal metrics by Katok [**Kat88**] and when (M, g_0) is a locally symmetric space by the work of Hamenstädt and of Besson, Courtois and Gallot [**BCG95**, **Ham99**]), it is still open in dimension higher than or equal to 3 and open even in dimension 2 in the more general setting of Riemannian metrics with Anosov geodesic flows. The same type of problems can also be posed for billiards, and we mention recent results on this problem by Avila, De Simoi and Kaloshin [**ADSK16**] and De Simoi, Kaloshin and Wei [**DSKW17**] for convex domains close to ellipses (although the Anosov case would rather correspond to the case of hyperbolic billiards). Recently, the first and last author obtained the following result on the Burns–Katok conjecture.

THEOREM 1.1. (Guillarmou and Lefeuvre [**GL19**]) *Let (M, g_0) be a smooth Riemannian manifold with Anosov geodesic flow, and further assume that its curvature is non-positive if* $\dim M \geq 3$. *Then there exists $k \in \mathbb{N}$ depending only on* $\dim M$ *and $\varepsilon > 0$ small enough depending on g_0 such that the following statement holds: if $g \in \mathcal{M}$ is such that $\|g - g_0\|_{C^k} \leq \varepsilon$ and $L_g = L_{g_0}$, then g is isometric to g_0.*

One of the aims of this paper is to further investigate this result from different perspectives: new stability estimates and a refined characterization of the condition under which the isometry may hold. More precisely, we can relax the assumption that the two marked length spectra of g and g_0 exactly coincide to the weaker assumption that they 'coincide at infinity' and still obtain the isometry. In what follows, we say that $L_g/L_{g_0} \to 1$

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when

$$\lim_{j\to+\infty}\frac{L_g(c_j)}{L_{g_0}(c_j)}=1, \tag{1.1}$$

for any sequence $(c_j)_{j\in\mathbb{N}}$ of primitive free homotopy classes such that $\lim_{j\to\infty} L_{g_0}(c_j)=+\infty$, or equivalently $\lim_{j\to\infty} L_g(c_j)/L_{g_0}(c_j)=1$, if $\mathcal{C}=(c_j)_{j\in\mathbb{N}}$ is ordered by the increasing lengths $L_{g_0}(c_j)$. We prove in Appendix A that $L_g/L_{g_0}\to 1$ is actually equivalent to $L_g=L_{g_0}$. As a consequence, by [GL19], if (1.1) holds and if $\|g-g_0\|_{C^k}<\varepsilon$ for some small enough $\varepsilon>0$, then g is isometric to g_0. If we restrict ourselves to metrics with the same topological entropy, the knowledge of $L_g(c_j)/L_{g_0}(c_j)$ for a subsequence so that the geodesic $\gamma_{g_0}(c_j)$ equidistributes is even sufficient; see Theorem 2.9.

We develop a new proof strategy, different from [GL19], which relies on the introduction of the *geodesic stretch* between two metrics. This quantity was first introduced by Croke and Fathi [CF90] and further studied by the second author [Kni95]. If g is close enough to g_0, then by Anosov structural stability, the geodesic flows φ^{g_0} and φ^g are orbit equivalent via a homeomorphism ψ_g, that is, they are conjugate up to a time reparametrization

$$\varphi^g_{\kappa_g(z,t)}(\psi_g(z))=\psi_g(\varphi^{g_0}_t(z))$$

for some time rescaling $\kappa_g(z,t)$. The *infinitesimal stretch* is the infinitesimal function of time reparametrization $a_g(z)=\partial_t\kappa_g(z,t)|_{t=0}$: it satisfies $d\psi_g(z)X_{g_0}(z)=a_g(z)X_g(\psi_g(z))$ where $z\in S_{g_0}M$ and X_{g_0} (respectively, X_g) denotes the geodesic vector field of g_0 (respectively, g). The *geodesic stretch between g and g_0 with respect to the Liouville (normalized with total mass 1) measure $\mu^{\mathrm{L}}_{g_0}$ of g_0* is then defined by

$$I_{\mu^{\mathrm{L}}_{g_0}}(g_0,g):=\int_{S_{g_0}M} a_g\,d\mu^{\mathrm{L}}_{g_0}.$$

The function a_g is uniquely defined up to a coboundary [dlLMM86] so that the geodesic stretch is well defined. (Although this is only used in §5.2, we also point out that the existence of the conjugacy ψ_g and of the reparametrization a_g is actually *global* and one need not assume that the two metrics are close. This is a very particular feature of the geodesic structure. We refer to Appendix B for a proof of this fact.)

Since obviously $\langle\delta_{g_0}(c_j),a_g\rangle=L_g(c_j)/L_{g_0}(c_j)$, we have

$$I_{\mu^{\mathrm{L}}_{g_0}}(g_0,g)=\lim_{j\to\infty}\frac{L_g(c_j)}{L_{g_0}(c_j)},$$

if $(c_j)_{j\in\mathbb{N}}\subset\mathcal{C}$ is a sequence so that the uniform probability measures $(\delta_{g_0}(c_j))_{j\in\mathbb{N}}$ supported on the closed geodesics of g_0 in the class c_j converge to $\mu^{\mathrm{L}}_{g_0}$ in the weak sense of measures. (The existence of the sequence c_j follows from [Sig72, Theorem 1].) In particular, $L_g=L_{g_0}$ implies that $I_{\mu^{\mathrm{L}}_{g_0}}(g_0,g)=1$ (alternatively, $L_g=L_{g_0}$ implies that a_g is cohomologous to 1 by Livsic's theorem). While being of interest in its own right, it turns out that this method involving the geodesic stretch provides a new estimate which quantifies locally the distance between isometry classes in terms of this geodesic stretch functional (below $H^{-1/2}(M)$ denotes the L^2-based Sobolev space of order $-1/2$ and $\alpha \in (0, 1)$ is any fixed exponent).

THEOREM 1.2. *Let (M, g_0) be a smooth Riemannian n-dimensional manifold with Anosov geodesic flow and further assume that its curvature is non-positive if $n \geq 3$. There exists $k \in \mathbb{N}$ large enough depending only on n, some positive constants C, ε depending on g_0 and $C_n > 0$ depending on n such that for all $\alpha \in (0, 1)$, the following statement holds: for each $g \in \mathcal{M}^{k,\alpha}$ with $\|g - g_0\|_{C^{k,\alpha}(M)} \leq \varepsilon$, there exists a $C^{k+1,\alpha}$-diffeomorphism $\psi : M \to M$ such that*

$$C\|\psi^* g - g_0\|^2_{H^{-1/2}(M)} \leq \boldsymbol{P}\Big(- J^u_{g_0} - a_g + \int_{S_{g_0}M} a_g \, d\mu^L_{g_0}\Big) + C_n (I_{\mu^L_{g_0}}(g_0, g) - 1)^2$$
$$\leq |\mathcal{L}_+(g)| + |\mathcal{L}_-(g)|$$

where $J^u_{g_0}$ is the unstable Jacobian of φ^{g_0}, $\boldsymbol{P}$ denotes the topological pressure for the φ^{g_0} flow defined by (2.11), a_g is the reparametrization coefficient relating φ^{g_0} and φ^g defined above, and

$$\mathcal{L}_+(g) := \limsup_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1, \quad \mathcal{L}_-(g) := \liminf_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1.$$

In particular, if (1.1) holds, then g_0 and g are isometric.

Note that g need not have non-positive curvature in the theorem. We also remark that the curvature condition on g_0 can be replaced by the injectivity of the X-ray transform I_2 on divergence-free symmetric 2-tensors, and similarly for Theorem 1.3 below. From the proof one sees that the exponent k can be taken to be $k = 3n/2 + 17$.

Theorem 1.2 is an improvement over the Hölder stability result [**GL19**, Theorem 3] as it only involves the asymptotic behavior of L_g/L_{g_0} or some natural quantity from thermodynamic formalism. We insist on the fact that the new ingredient here is *the stability estimate in itself* (the rigidity result is not new).

We also emphasize that one of the key facts to prove this theorem still boils down to some elliptic estimate on some variance operator acting on symmetric 2-tensors, denoted by $\Pi_2^{g_0}$ in [**GL19**, **Gui17**]: indeed, we show that the combination of the Hessians of the geodesic stretch at g_0 and of the pressure functional can be expressed in terms of this variance operator, which enjoys uniform lower bounds $C_{g_0}\|\psi^* g - g_0\|_{H^{-1/2}}$ for some $C_{g_0} > 0$, at least once we have factored out the gauge (the diffeomorphism action by pullback on metrics).

We also notice that in Theorem 1.2, although the $H^{-1/2}(M)$ norm is a weak norm, a straightforward interpolation argument using that $\|g\|_{C^{k,\alpha}} \leq \|g_0\|_{C^{k,\alpha}} + \varepsilon$ is uniformly bounded shows that an estimate of the form

$$\|\psi^* g - g_0\|_{C^{k'}} \leq C(|\mathcal{L}_+(g)| + |\mathcal{L}_-(g)|)^\delta$$

holds for any $k' < k - n/2$ and some explicit $\delta \in (0, 1/2)$ depending on k, k' ($C > 0$ depending only on g_0).

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1.2. *Variance and pressure metric.* The variance operator appearing in the proof of Theorem 1.2 can be defined for $h_1, h_2 \in C^\infty(M; S^2T^*M)$ satisfying the condition

$$\int_M \mathrm{Tr}_{g_0}(h_i)\, d\mathrm{vol}_{g_0} = 0, \tag{1.2}$$

for $i = 1, 2$ (see §2.3 for further details on tensor analysis) by

$$\langle \Pi_2^{g_0} h_1, h_2 \rangle := \int_{\mathbb{R}} \int_{SM} \pi_2^* h_1(\varphi_t^{g_0}(z)) \pi_2^* h_2(z)\, d\mu_{g_0}^{\mathrm{L}}(z)\, dt,$$

where $z = (x, v) \in SM$ and, given a symmetric 2-tensor $h \in C^\infty(M; S^2T^*M)$, we define the pullback operator

$$\pi_2^* h(x, v) := h_x(v, v).$$

The quadratic form $\langle \Pi_2^{g_0} h, h \rangle$ corresponds to the variance $\mathrm{Var}_{\mu_L}(\pi_2^* h)$ for $\varphi_t^{g_0}$ with respect to the Liouville measure of the lift $\pi_2^* h$ of the tensor h to SM (see §2.5 and (2.5)). Note that the trace-free condition (1.2) is equivalent to

$$\int_{SM} \pi_2^* h(x, v)\, d\mu_{g_0}^{\mathrm{L}}(x, v) = 0;$$

see §2.3. The integral defining $\Pi_2^{g_0}$ then converges (in the L^1 sense) by the rapid mixing of φ^{g_0} (proved in [**Liv04**]). The operator $\Pi_2^{g_0}$ is a pseudodifferential operator of order -1 that is elliptic on divergence-free tensors (see [**GL**, **GL19**, **Gui17**]). As a consequence, it satisfies elliptic estimates on all Sobolev or Hölder spaces (see Lemma 2.1). More precisely, there is $C_{g_0} > 0$ such that, for all $h \in H^{-1/2}(M; S^2T^*M)$ which is divergence-free (that is, $\mathrm{Tr}_{g_0}(\nabla^{g_0} h) = 0$),

$$\langle \Pi_2^{g_0} h, h \rangle \geq C_{g_0} \|h\|^2_{H^{-1/2}(M)}, \tag{1.3}$$

provided g_0 is Anosov with non-positive curvature (or simply Anosov if $\dim M = 2$). We show in Proposition 4.1 that $g \mapsto \Pi_2^g$ is continuous with values in $\Psi^{-1}(M)$ and this implies that for g_0 a smooth Anosov metric (with non-positive curvature if $\dim M > 2$), (1.3) holds uniformly if we replace g_0 by any metric g in a small C^∞-neighborhood of g_0. This allows us to obtain a more uniform version of Theorem 1.2.

THEOREM 1.3. *Let (M, g_0) be a smooth Riemannian n-dimensional manifold with Anosov geodesic flow and further assume that its curvature is non-positive if $n \geq 3$. Then there exist $k \in \mathbb{N}$, $\varepsilon > 0$ and C_{g_0} depending on g_0 such that for all $g_1, g_2 \in \mathcal{M}$ such that $\|g_1 - g_0\|_{C^k} \leq \varepsilon$, $\|g_2 - g_0\|_{C^k} \leq \varepsilon$, there is a C^k- diffeomorphism $\psi : M \to M$ such that*

$$\|\psi^* g_2 - g_1\|^2_{H^{-1/2}(M)} \leq C_{g_0}(|\mathcal{L}_+(g_1, g_2)| + |\mathcal{L}_+(g_2, g_1)|)$$

with

$$\mathcal{L}_+(g_1, g_2) := \limsup_{j \to \infty} \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} - 1.$$

In particular, if $L_{g_1}/L_{g_2} \to 1$, then g_2 is isometric to g_1.

This result suggests defining a distance on isometry classes of metrics (here we mean isometries homotopic to the identity) from the marked length spectrum by setting, for two $C^{k,\alpha}$ metrics g_1, g_2,

$$d_L(g_1, g_2) := \limsup_{j\to\infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|.$$

We have the following corollary of Theorem 1.3.

COROLLARY 1.4. *The map d_L descends to the space of isometry classes of Anosov non-positively curved metrics and defines a distance near the diagonal.*

We also define the *Thurston asymmetric distance* by

$$d_T(g_1, g_2) := \limsup_{j\to\infty} \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)},$$

and show that this is a distance on isometry classes of metrics with topological entropy equal to 1; see Proposition 5.4. This distance was introduced in Teichmüller theory by Thurston in [**Thu98**].

The elliptic estimate (1.3) also allows us to define a *pressure metric* on the open set consisting of isometry classes of Anosov non-positively curved metrics (contained in $\mathcal{M}/\mathcal{D}_0$ if $\mathcal{D}_0$ is the group of smooth diffeomorphisms isotopic to the identity) by setting, for $h_1, h_2 \in T_{g_0}(\mathcal{M}/\mathcal{D}_0) \subset C^\infty(M; S^2T^*M)$,

$$G_{g_0}(h_1, h_2) := \langle \Pi_2^{g_0} h_1, h_2 \rangle_{L^2(M, d\,\mathrm{vol}_{g_0})}.$$

We show in §3.3.1 that this metric is well defined and restricts to (a multiple of) the Weil–Petersson metric on Teichmüller space if $\dim M = 2$: it is related to the construction of Bridgeman *et al* [**BCLS15**, **BCS18**] and McMullen [**MM08**], but with the difference that we work here in the setting of variable negative curvature and the space of metrics considered here is infinite-dimensional. In a related but different context with infinite dimension, we note that the variance is used to define a metric on the space of Hölder potentials by Giulietti *et al* [**GKLM18**] and its curvature is studied by Lopes and Ruggiero [**LR18**].

We finally notice that, in the study of Katok entropy conjecture near locally symmetric spaces, the variance was an important tool in the work of Pollicott and Flaminio [**Fla95**, **Pol94**]. In that case, one can use representation theory to analyze this operator.

2. *Preliminaries*

2.1. *Notation.* If $H = C^k, H^s, C^{-\infty}$ etc. is a regularity scale and $E \to M$ a smooth bundle over a smooth compact manifold M, we will use the notation $H(M; E)$ for sections of E with regularity H, while if N is a smooth manifold, we use the notation $H(M, N)$ for the space of maps from M to N with regularity H.

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2.2. *Microlocal calculus.* On a closed manifold M, we will denote by $\Psi^m(M; V)$ the space of classical pseudo-differential operators of order $m \in \mathbb{R}$ acting on a vector bundle V over M (see [**GS94**]; the operators could map sections of two distinct vector bundles, but this will not be needed here). We recall that for fixed $m \in \mathbb{R}$, this is a Fréchet space: indeed, using a fixed smooth cutoff function θ supported in a small neighborhood of the diagonal, a fixed system of charts, each $A \in \Psi^m(M; V)$ has Schwartz kernel κ_A that can be decomposed as $\theta\kappa_A + (1-\theta)\kappa_A$. For the $(1-\theta)\kappa_A$ part we can use the $C^\infty(M \times M; V \otimes V^*)$ topology, while for $\chi\kappa_A$ we can use the semi-norms of the full symbols of $\chi\kappa_A$ using the local charts and the left quantization in the charts. We also denote by $H^s(M)$ the L^2-based Sobolev space of order $s \in \mathbb{R}$, with norm given by fixing an arbitrary Riemannian metric g_0 on M. More precisely, denoting by Δ the non-negative Laplacian associated to this metric, we define

$$\|f\|_{H^s(M)} := \|(\mathbb{1} + \Delta)^{s/2} f\|_{L^2(M, d\mathrm{vol})},$$

and $H^s(M)$ is the completion of $C^\infty(M)$ with respect to this norm. This definition is naturally extended to sections of vector bundles. What is important is that the spaces and the norm (up to a scaling factor) do not depend on the choice of metric g_0. For $k \in \mathbb{N}$, $\alpha \in (0, 1)$, the spaces $C^{k,\alpha}(M)$ are the usual Hölder spaces and $\mathcal{D}'(M)$ will denote the space of distributions dual to $C^\infty(M)$. We will denote by $\langle \cdot, \cdot \rangle_{L^2}$ the continuous extension of the pairing

$$C^\infty(M) \times C^\infty(M) \ni (f, f') \mapsto \int_M f \bar{f}' d\mathrm{vol}_{g_0},$$

to the pairing $H^s(M) \times H^{-s}(M) \to \mathbb{C}$ for each $s \in \mathbb{R}$ (and likewise for sections of bundles).

2.3. *Symmetric tensors and X-ray transform.* In this subsection, we assume that the metric g is fixed and that its geodesic flow φ_t^g is Anosov on the unit tangent bundle SM of g. We denote by μ^{L} the Liouville measure, normalized to be a probability measure on SM. For the sake of simplicity, we drop the index g in the notation. Given an integer $m \in \mathbb{N}$, we denote by $\otimes^m T^*M \to M$, $S^m T^*M \to M$ the respective vector bundle of m-tensors and symmetric m-tensors on M. Given $f \in C^\infty(M; S^m T^*M)$, we denote by $\pi_m^* f \in C^\infty(SM)$ the canonical morphism $\pi_m^* f : (x, v) \mapsto f_x(v, \ldots, v)$. We also introduce the trace operator $\mathrm{Tr} : C^\infty(M; S^{m+2} T^*M) \to C^\infty(M; S^m T^*M)$ defined pointwise in $x \in M$ by

$$\mathrm{Tr}(f) = \sum_{i=1}^n f(\mathbf{e}_i, \mathbf{e}_i, \cdot, \ldots, \cdot),$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ denotes an orthonormal basis of TM in a neighborhood of a fixed point $x_0 \in M$. Observe that, for $f = \sum_{i,j=1}^n f_{ij} \mathbf{e}_i^* \otimes \mathbf{e}_j^* \in C^\infty(M; S^2 T^*M)$ defined around x_0,

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we have

$$\begin{aligned}\int_{SM} \pi_2^* f \, d\mu_g^L &= \int_M \bigg(\int_{S_x M} \pi_2^* f(x, v) \, dS_x(v) \bigg) d\mathrm{vol}(x) \\ &= \sum_{i,j=1}^n \int_M f_{ij}(x) \bigg(\int_{S_x M} v_i v_j \, dS_x(v) \bigg) d\mathrm{vol}(x) \\ &= C_n \sum_{i=1}^n \int_M f_{ii}(x) d\mathrm{vol}(x) = C_n \int_M \mathrm{Tr}_g(f) d\mathrm{vol},\end{aligned}$$

for some constant $C_n = \int_{\mathbb{S}^{n-1}} v_1^2 dv$ depending on $n = \dim M$. This justifies the claim that the trace-free condition (1.2) was equivalent to the fact that the pullback of the symmetric tensor to SM was of average 0.

The natural derivation of symmetric tensors is $D := \sigma \circ \nabla$, where ∇ is the Levi-Civita connection and $\sigma : \otimes^m T^*M \to S^m T^*M$ is the operation of symmetrization. This operator satisfies the important identity

$$X\pi_m^* = \pi_{m+1}^* D, \tag{2.1}$$

where X denotes the geodesic vector field on SM. The operator D is elliptic [**GL**, Lemma 2.4] with trivial kernel when m is odd and one-dimensional kernel when m is even, given by the Killing tensors $c\sigma(g^{\otimes m/2})$, $c \in \mathbb{R}$ (this is a simple consequence of (2.1) combined with the fact that the geodesic flow is ergodic in the Anosov setting). We denote by $\langle \cdot, \cdot \rangle$ the scalar product on $C^\infty(M; S^m T^*M)$ induced by the metric g (see [**GL**, §2] for further details). The formal adjoint of D with respect to this scalar product is $D^* = -\operatorname{Tr} \circ \nabla$. We also denote by the same $\langle \cdot, \cdot \rangle$ the natural L^2 scalar product on $C^\infty(SM)$ induced by the Liouville measure μ^{L}. The formal adjoint of π_m^* with respect to these two scalar products is denoted by

$$\pi_{m*} : \mathcal{D}'(SM) \to \mathcal{D}'(M; S^m T^*M),$$

where $\mathcal{D}'$ denotes the space of distributions, dual to C^∞.

We recall that $\mathcal{C}$, the set of free homotopy classes in M, is in one-to-one correspondence with the set of conjugacy classes of $\pi_1(M, x_0)$ for some arbitrary choice of $x_0 \in M$ (see [**Kli74**]) and for each $c \in \mathcal{C}$ there exists a unique closed geodesic $\gamma(c) \in c$. We denote its Riemannian length with respect to g by $L(c) = \ell_g(\gamma(c))$. The *X-ray transform* on SM is the operator defined by

$$I : C^0(SM) \to \ell^\infty(\mathcal{C}), \quad If(c) = \frac{1}{L(c)} \int_0^{L(c)} f(\varphi_t(z)) \, dt,$$

where $z \in \gamma(c)$ is any point. This is a continuous linear operator when $\ell^\infty(\mathcal{C})$ is endowed with the sup norm on the sequences. Then the X-ray transform I_m of symmetric m-tensors is simply defined by $I_m := I \circ \pi_m^*$. Using (2.1), we immediately have

$$\{Dp \mid p \in C^\infty(M; S^{m-1} T^*M)\} \subset \ker I_m \cap C^\infty(M; S^m T^*M). \tag{2.2}$$

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Using the ellipticity of D, any tensor $f \in C^\infty(M; S^m T^*M)$ can be decomposed uniquely as a sum

$$f = Dp + h, \tag{2.3}$$

with $p \in C^\infty(M; S^{m-1}T^*M)$ and $h \in C^\infty(M; S^m T^*M)$ is such that $D^*h = 0$. We call Dp the *potential part* of f and h the *solenoidal part*. The same decomposition holds in Sobolev regularity $H^s(M)$, $s \in \mathbb{R}$, and in $C^{k,\alpha}(M)$ regularity, $k \in \mathbb{N}$, $\alpha \in (0, 1)$. We will write $h = \pi_{\ker D^*} f$ and the solenoidal projection $\pi_{\ker D^*} := \mathbb{1} - D_g \Delta_g^{-1} D_g^*$ is a pseudodifferential operator of order 0 [**GL**, Lemma 2.6] (here $\Delta_g := D_g^* D_g$ is the Laplacian on 1-forms). The X-ray transform is said to be *solenoidal injective* (or s-injective for short) if (2.2) is an equality. It is conjectured that I_m is s-injective as long as the metric is Anosov, but it is only known in the following cases:

- for $m = 0, 1$ [**DS10**];
- for any $m \in \mathbb{N}$ in dimension 2 [**Gui17**, **PSU14**];
- for any $m \in \mathbb{N}$, in any dimension in non-positive curvature [**CS98**].

It is also known that $\ker I_m / \operatorname{ran} D$ is finite-dimensional for general Anosov geodesic flow (see [**DS03**, Theorem 1.5] or [**Gui17**, Remark 3.7]).

The direct study of the analytic properties of I_m is difficult as this operator involves integrals over the set of closed orbits, which is not a manifold. Nevertheless, in [**Gui17**], the second author introduced an operator Π_m that involves a sort of integration of tensors over 'all orbits', and this space is essentially the manifold SM. The construction of $\Pi_m : C^\infty(M; S^m T^*M) \to \mathcal{D}'(M; S^m T^*M)$ relies on microlocal tools coming from [**DZ16**, **FS11**], but a simpler definition that uses the fast mixing of the flow φ_t is given by

$$\begin{gathered}\Pi_m := \pi_{m*}(\Pi + \langle \cdot, 1\rangle)\pi_m^* \quad \text{with} \\ \Pi : C^\infty(SM) \to \mathcal{D}'(SM), \quad \langle \Pi f, f' \rangle := \lim_{T\to\infty} \int_{-T}^{T} \langle e^{tX} f, f' \rangle \, dt\end{gathered} \tag{2.4}$$

if $\langle f, 1\rangle = \int_{SM} f \, d\mu^{\mathrm{L}} = 0$ and $\Pi(1) := 0$. The convergence of the integral as $T \to \infty$ is ensured by the exponential decay of correlations [**Liv04**] (but also follows from the existence of the variance [**KS90**]). We can thus write, for $\langle f, 1\rangle = 0$,

$$\langle \Pi f, f' \rangle = \int_{\mathbb{R}} \langle f \circ \varphi_t, f' \rangle_{L^2(SM)} \, dt.$$

We note the following useful properties of Π, proved in [**Gui17**, Theorem 1.1]:

- $\Pi : H^s(SM) \to H^{-s}(SM)$ is bounded for all $s > 0$;
- if $f \in H^s(SM)$ with $s > 0$, then $X\Pi f = 0$;
- if f and Xf belong to $H^s(SM)$ for $s > 0$, then $\Pi X f = 0$. (In [**Gui17**], f is assumed to be in $H^{s+1}(SM)$, but one can reduce to the case $f \in H^s(SM)$ by using a density argument and [**DZ19**, Lemma E.45].)

As is well known (see; for example; [**KS90**, Proof of Proposition 1.2.]), we can make a link between Π and the variance in the central limit theorem for Anosov geodesic flows. Let us quickly explain this fact by using the fast mixing of the flow. The *variance* of φ_t with respect to the Liouville measure μ^{L} is defined for $u \in C^\alpha(SM)$, $\alpha \in (0, 1)$

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real-valued, by

$$\mathrm{Var}_{\mu^{\mathrm{L}}}(u) := \lim_{T\to\infty} \frac{1}{T} \int_{SM} \bigg(\int_0^T u(\varphi_t(z))\, dt \bigg)^2 d\mu^{\mathrm{L}}(z), \tag{2.5}$$

under the condition that $\int_{SM} u \, d\mu^{\mathrm{L}} = 0$. We observe, since φ_t preserves μ^{L}, that

$$\begin{aligned}\mathrm{Var}_{\mu^{\mathrm{L}}}(u) &= \lim_{T\to\infty} \frac{1}{T} \int_{SM} \int_0^T \int_0^T u(\varphi_{t-s}(z))u(z)\, dt ds d\mu^{\mathrm{L}}(z) \\ &= \lim_{T\to\infty} \int_0^1 \int_{\mathbb{R}} \mathbf{1}_{[(t-1)T,tT]}(r)\langle u\circ\varphi_r, u\rangle_{L^2}\, dr\, dt,\end{aligned}$$

where the L^2 pairing is with respect to μ^{L}. By exponential decay of correlations [**Liv04**], we have, for $|r|$ large,

$$|\langle u\circ\varphi_r, u\rangle_{L^2}| \le Ce^{-\nu|r|}\|u\|^2_{C^\alpha}$$

for some $\alpha > 0$, $\nu > 0$, $C > 0$ independent of u. Thus, by the Lebesgue theorem,

$$\mathrm{Var}_{\mu^{\mathrm{L}}}(u) = \langle \Pi u, u\rangle, \tag{2.6}$$

if $\langle u, \mathbf{1}\rangle = 0$, where $\mathbf{1}$ denotes the constant function equal to 1, showing that the quadratic form associated to our operator Π is nothing more than the variance. For a symmetric 2-tensor h satisfying $\langle h, g\rangle_{L^2} = \int_M \mathrm{Tr}_g(h)\, d\mathrm{vol}_g = 0$, we have $\int_{SM} \pi_2^* h\, d\mu_g^L = 0$ and

$$\langle \Pi_2 h, h\rangle = \langle \Pi\pi_2^* h, \pi_2^* h\rangle = \mathrm{Var}_{\mu^{\mathrm{L}}}(\pi_2^* h).$$

We have the following properties for Π_m.

- Π_m is a positive self-adjoint pseudodifferential operator of order -1, elliptic on solenoidal tensors; see [**Gui17**, Theorem 3.5] and [**GL**, Lemma 4.3].
- $\Pi_m D = 0$ and $D^*\Pi_m = 0$ (by [**Gui17**, Theorem 3.5] and $X\pi^*_{m-1} = \pi^*_m D$).
- If I_m is s-injective, then Π_m is invertible on solenoidal tensors in the sense that there exists a pseudodifferential operator Q of order 1 such that $Q\Pi_m = \pi_{\ker D^*}$; see [**GL**, Theorem 4.7].
- Conversely, if $\Pi_m|_{\ker D^*}$ is injective, then I_m is s-injective. Indeed, by [**Gui17**, Corollary 2.8], if $I_m h = 0$ then $\pi_m^* h = Xu$ for some $u \in C^\infty(SM)$ and thus $\Pi_m h = \pi_{m*}\Pi X u = 0$.

In particular, using the spectral theorem, there is a bounded self-adjoint operator $\sqrt{\Pi_m}$ on L^2 such that $\sqrt{\Pi_m}\sqrt{\Pi_m} = \Pi_m$. We add the following property, the use of which will be crucial in this paper.

LEMMA 2.1. *If (M, g) has Anosov geodesic flow and I_2 is s-injective, there exists a constant $C > 0$ such that, for all tensors $h \in H^{-1/2}(M; S^2T^*M)$,*

$$\langle \Pi_2 h, h\rangle \ge C\|\pi_{\ker D^*} h\|^2_{H^{-1/2}(M)}.$$

Proof. In [**GL**, Theorem 4.4 and Lemma 2.2], the principal symbol of Π_2 was computed and turned out to be

$$\sigma_2 := \sigma(\Pi_2) : (x,\xi) \mapsto |\xi|^{-1}\pi_{\ker i_\xi} A_2^2 \pi_{\ker i_\xi},$$

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for some positive definite diagonal endomorphism A_2 which is constant on both subspaces $S_0^2T^*M := \{h \in S^2T^*M \mid \mathrm{Tr}_g(h) = 0\}$ and $\mathbb{R}g = \{\lambda g \in S^2T^*M \mid \lambda \in \mathbb{R}\}$. Here i_ξ is the interior product with the dual vector $\xi^\sharp \in T_xM$ of ξ with respect to the metric. We introduce the symbol $b \in C^\infty(T^*M)$ of order $-1/2$ defined by $b : (x, \xi) \mapsto \chi(x, \xi)|\xi|^{-1/2}A_2$, where $\chi \in C^\infty(T^*M)$ vanishes near the 0 section in T^*M and is equal to 1 for $|\xi| > 1$, and define $B := \mathrm{Op}(b) \in \Psi^{-1/2}(M; S^2T^*M)$, where Op is a quantization on M. Using that the principal symbol of $\pi_{\ker D^*}$ is $\pi_{\ker i_\xi}$ (see [**GL**, Lemma 2.6]), we observe that $\Pi_2 = \pi_{\ker D^*}B^*B\pi_{\ker D^*} + R$, where $R \in \Psi^{-2}(M; S^2T^*M)$. Thus, given $h \in H^{-1/2}(M, S^2T^*M)$,

$$\langle \Pi_2 h, h\rangle_{L^2} = \|B\pi_{\ker D^*}h\|_{L^2}^2 + \langle Rh, h\rangle_{L^2}. \tag{2.7}$$

By ellipticity of B, there exists a pseudodifferential operator Q of order $1/2$ such that $QB\pi_{\ker D^*} = \pi_{\ker D^*} + R'$, where $R' \in \Psi^{-\infty}(M; S^2T^*M)$ is smoothing. Thus there is $C > 0$ such that, for each $h \in C^\infty(M; S^2T^*M)$,

$$\|\pi_{\ker D^*}h\|_{H^{-1/2}}^2 \le \|QB\pi_{\ker D^*}h\|_{H^{-1/2}}^2 + \|R'h\|_{H^{-1/2}}^2 \le C\|B\pi_{\ker D^*}h\|_{L^2}^2 + \|R'h\|_{H^{-1/2}}^2.$$

Since Lemma 2.1 is trivial on potential tensors, we can already assume that h is solenoidal, that is, $\pi_{\ker D^*}h = h$. Recalling (2.7), we obtain that

$$\begin{aligned} \|h\|_{H^{-1/2}}^2 &\le C\langle \Pi_2 h, h\rangle_{L^2} - C\langle Rh, h\rangle_{L^2} + \|R'h\|_{H^{-1/2}}^2 \\ &\le C\langle \Pi_2 h, h\rangle_{L^2} + C\|Rh\|_{H^{1/2}}\|h\|_{H^{-1/2}} + \|R'h\|_{H^{-1/2}}^2. \end{aligned} \tag{2.8}$$

Now, assume by contradiction that the statement in Lemma 2.1 does not hold, that is, we can find a sequence of tensors $f_n \in C^\infty(M; S^2T^*M)$ such that $\|f_n\|_{H^{-1/2}} = 1$ with $D^*f_n = 0$ and

$$\|\sqrt{\Pi_2}f_n\|_{L^2}^2 = \langle \Pi_2 f_n, f_n\rangle_{L^2} \le \frac{1}{n}\|f_n\|_{H^{-1/2}}^2 = \frac{1}{n} \to 0.$$

Up to a subsequence, and since R is of order -2, we can assume that $Rf_n \to v_1$ in $H^{1/2}$ for some v_1, and $R'f_n \to v_2$ in $H^{-1/2}$. Then, using (2.8), we obtain that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1/2}$ which thus converges to an element $v_3 \in H^{-1/2}$ such that $\|v_3\|_{H^{-1/2}} = 1$ and $D^*v_3 = 0$. By continuity, $\Pi_2 f_n \to \Pi_2 v_3$ in $H^{1/2}$ and thus $\langle \Pi_2 v_3, v_3\rangle = 0$. Since v_3 is solenoidal, we get $\sqrt{\Pi_2}v_3 = 0$, thus $\Pi_2 v_3 = 0$. Since we assumed I_2 s-injective, Π_2 is also injective by [**GL**, Lemma 4.6]. This implies that $v_3 \equiv 0$, thus contradicting $\|v_3\|_{H^{-1/2}} = 1$. □

We note that the same proof also works for tensors of any order $m \in \mathbb{N}$. In fact we can even get a uniform estimate.

LEMMA 2.2. *Let (M, g_0) be a smooth compact Anosov Riemannian manifold with $I_2^{g_0}$ being s-injective. There exist a C^∞ neighborhood $\mathcal{U}_{g_0}$ of g_0 and a constant $C > 0$ such that for all $g \in \mathcal{U}_{g_0}$ and all tensors $h \in H^{-1/2}(M; S^2T^*M)$,*

$$\langle \Pi_2^g h, h\rangle_{L^2} \ge C\|\pi_{\ker D_g^*}h\|_{H^{-1/2}(M)}^2.$$

Proof. First, let g_0 be fixed Anosov metric with $I_2^{g_0}$ s-injective (in particular, it is the case if it has non-positive curvature). Proposition 4.1 (which will be proved later) shows that

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the operator $\Pi_2 = \Pi_2^g$ is a continuous family as a map

$$g \in \mathcal{U}_{g_0} \mapsto \Pi_2^g \in \Psi^{-1}(M; S^2T^*M)$$

where $\mathcal{U}_{g_0} \subset C^\infty(M; S^2T^*M)$ is a C^∞-neighborhood of g_0 and $\Psi^{-1}(M; S^2T^*M)$ is equipped with its Fréchet topology as explained before. Let $h \in \ker D_g^*$ be a solenoidal (with respect to g) symmetric 2-tensor, then $h = \pi_{\ker D_g^*} h$. Let $C_{g_0} > 0$ be the constant provided by Lemma 2.1 applied to the metric g_0. We choose $\mathcal{U}_{g_0}$ small enough so that $\|\Pi_2^g - \Pi_2^{g_0}\|_{H^{-1/2} \to H^{1/2}} \leq C_{g_0}/3$ (this is made possible by the continuity of $g \mapsto \Pi_2^g \in \Psi^{-1}$). Then

$$\langle \Pi_2^g h, h\rangle = \langle (\Pi_2^g - \Pi_2^{g_0})h, h\rangle + \langle \Pi_2^{g_0} h, h\rangle \geq C_{g_0} \|\pi_{\ker D_{g_0}^*} h\|_{H^{-1/2}}^2 - C_{g_0}/3 \times \|h\|_{H^{-1/2}}^2.$$

But the map $\mathcal{U}_{g_0} \ni g \mapsto \pi_{\ker D_g^*} = \mathbb{1} - D_g \Delta_g^{-1} D_g^* \in \Psi^0$ is continuous: this follows from the fact that one can construct a full parametrix $Q_g \in \Psi^{-2}(M)$ of Δ_g modulo smoothing in a continuous way with respect to g (by standard elliptic microlocal analysis), the fact that Δ_g is injective since $\ker D_g = 0$ for g Anosov (as $D_g u = 0$ implies $X\pi_1^* u = 0$, thus $\pi_1^* u$ has to be constant, thus 0 since $\pi_1^* u(x, -v) = -\pi_1^* u(x, v)$) and the continuity of composition of pseudodifferential operators. This implies that for g in a possibly smaller neighborhood $\mathcal{U}_{g_0}$ of g_0, using $h = \pi_{\ker D_g^*} h$,

$$\langle \Pi_2^g h, h\rangle \geq C_{g_0} \|\pi_{\ker D_g^*} h\|_{H^{-1/2}}^2 - \frac{2C_{g_0}}{3} \times \|h\|_{H^{-1/2}}^2 = \frac{C_{g_0}}{3} \|\pi_{\ker D_g^*} h\|_{H^{-1/2}}^2.$$

The proof is complete. □

We also observe that the generalization of the previous lemma to tensors of any order is straightforward. As mentioned earlier, an immediate consequence of the previous lemma is the following proposition.

PROPOSITION 2.3. *Let (M, g_0) be a smooth Riemannian n-dimensional Anosov manifold with $I_m^{g_0}$ s-injective. Then there exists a C^∞-neighborhood $\mathcal{U}_{g_0}$ of g_0 in $\mathcal{M}$ such that, for any $g \in \mathcal{U}_{g_0}$, for any $m \in \mathbb{N}$, I_m^g is s-injective.*

Proof. As mentioned above (before Lemma 2.1), the s-injectivity of I_m^g is equivalent to that of Π_m^g on solenoidal tensors and the previous lemma allows us to conclude. □

2.4. *The space of Riemannian metrics.* We fix a *smooth* metric $g_0 \in \mathcal{M}$ and consider an integer $k \geq 2$ and $\alpha \in (0, 1)$. We recall that the space $\mathcal{M}$ of all smooth metrics is a Fréchet manifold. We denote by $\mathcal{D}_0 := \mathrm{Diff}_0(M)$ the group of smooth diffeomorphisms on M that are isotopic to the identity; this is a Fréchet Lie group in the sense of [**Ham82**, Section 4.6]. The right action

$$\mathcal{M} \times \mathcal{D}_0 \to \mathcal{M}, \quad (g, \psi) \mapsto \psi^* g$$

is smooth and proper [**Ebi68**, **Ebi70**]. Moreover, if g is a metric with Anosov geodesic flow, it is directly seen from ergodicity that there are no Killing vector fields and thus the isotropy subgroup $\{\psi \in \mathcal{D}_0 \mid \psi^* g = g\}$ of g is finite. For negatively curved metrics it is shown in [**Fra66**] that the action is free, that is, the isotropy group is trivial. One

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cannot apply the usual quotient theorem [**Tro92**, pp. 20] in the setting of Banach or Hilbert manifolds but rather smooth Fréchet manifolds instead (using the Nash–Moser theorem). Thus, in the setting of the space of smooth metrics with Anosov geodesic flows (the important fact, to apply Ebin's slice theorem, is that metrics with Anosov geodesic flows do not have Killing vector fields, that is, infinitesimal isometries; this is due to the fact that $\ker D|_{C^\infty(M,T^*M)} = \{0\}$ as mentioned earlier, which itself follows from the ergodicity of the geodesic flow), which is an open set of a Fréchet vector space, the slice theorem says that there exist a neighborhood $\mathcal{U}$ of g_0, a neighborhood $\mathcal{V}$ of Id in $\mathcal{D}_0$ and a Fréchet submanifold $\mathcal{S}$ containing g_0 so that

$$\mathcal{S} \times \mathcal{V} \to \mathcal{U}, \quad (g, \psi) \mapsto \psi^* g \tag{2.9}$$

is a diffeomorphism of Fréchet manifolds and $T_{g_0}\mathcal{S} = \{h \in T_{g_0}\mathcal{M} \mid D^*_{g_0} h = 0\}$; see [**Ebi68**, **Ebi70**]. Moreover, $\mathcal{S}$ parametrizes the set of orbits $g \cdot \mathcal{D}_0$ for g near g_0 and $T_g\mathcal{S} \cap T(g \cdot \mathcal{D}_0) = 0$.

On the other hand, if one considers $\mathcal{M}^{k,\alpha}$, the space of metrics with $C^{k,\alpha}$ regularity and $\mathcal{D}_0^{k+1,\alpha} := \mathrm{Diff}_0^{k+1,\alpha}(M)$, the group of diffeomorphisms isotopic to the identity with $C^{k+1,\alpha}$ regularity, then both spaces are smooth Banach manifolds. However, the action of $\mathcal{D}_0^{k+1,\alpha}$ on $\mathcal{M}^{k,\alpha}$ is no longer smooth but only topological, which also prevents us from applying the quotient theorem.

Nevertheless, recalling that g_0 is smooth, if we consider $\mathcal{O}^{k,\alpha}(g_0) := g_0 \cdot \mathcal{D}_0^{k+1,\alpha} \subset \mathcal{M}^{k,\alpha}$, then this is a smooth submanifold of $\mathcal{M}^{k,\alpha}$ and

$$T_g\mathcal{O}^{k,\alpha}(g_0) = \{D_g p \mid p \in C^{k+1,\alpha}(M; T^*M)\}.$$

Notice that (2.3) in $C^{k,\alpha}$ regularity exactly says that given $g \in \mathcal{O}^{k,\alpha}(g_0)$, we have the decomposition

$$T_g\mathcal{M} = T_g\mathcal{O}^{k,\alpha}(g_0) \oplus \ker D^*_g|_{C^{k,\alpha}(M,S^2T^*M)}. \tag{2.10}$$

Thus, an infinitesimal perturbation of a metric $g \in \mathcal{O}^{k,\alpha}(g_0)$ by a symmetric 2-tensor that is solenoidal with respect to g is actually an infinitesimal displacement *transversally to the orbit* $\mathcal{O}^{k,\alpha}(g_0)$.

We will need a stronger version of the previous decomposition (2.10) which can be understood as a slice theorem. Knowledge of it goes back to [**Ebi68**, **Ebi70**]; see also [**GL19**, Lemma 4.1] for a short proof in the $C^{k,\alpha}$ category.

LEMMA 2.4. *Let k be an integer greater than or equal to 2 and $\alpha \in (0, 1)$, let g_0 be a $C^{k+3,\alpha}$ metric with Anosov geodesic flow. There exists a neighborhood $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$ of g_0 in the $C^{k,\alpha}$-topology such that for any $g \in \mathcal{U}$, there exists a unique $C^{k+1,\alpha}$-diffeomorphism ψ such that $\psi^* g$ is solenoidal with respect to g_0. Moreover, the following map is C^2:*

$$C^{k,\alpha}(M; S^2T^*M) \times C^{k+3,\alpha}(M; S^2T^*M) \to \mathcal{D}_0^{k+1,\alpha}(M), \quad (g, g_0) \mapsto \psi.$$

Remark 2.5. The previous lemma is not stated exactly this way in [**GL19**, Lemma 4.1]. Indeed, the proof assumes that g_0 is smooth and fixed. However, inspecting the proof, it readily applies to $g_0 \in C^{k+3,\alpha}$ and the implicit function theorem used in that proof shows the regularity of ψ with respect to g_0. We do not include the proof of these details in order not to burden the discussion.

We also see that we need to use to $C^{k,\alpha}$ regularity for $\alpha \neq 0, 1$ instead of C^k: this is due to the fact that the pseudodifferential operator inverting the linearization $D_{g_0}^* D_{g_0}$ that arises naturally in the proof of this lemma (see [**GL19**, Lemma 4.1]) acts on these spaces but on C^k, for $k \in \mathbb{N}$. Instead, one would have to resort to Zygmund spaces C_*^k. We refer to [**Tay91**, Appendix A] for further details.

2.5. *Thermodynamic formalism.* Let f be a Hölder-continuous function on $S_{g_0}M$. We recall that its *pressure* [**Wal82**, Theorem 9.10] is defined by

$$\mathbf{P}(f) := \sup_{\mu \in \mathfrak{M}_{\text{inv}}} \left(\mathbf{h}_\mu(\varphi_1^{g_0}) + \int_{S_{g_0}M} f \, d\mu \right), \tag{2.11}$$

where $\mathfrak{M}_{\text{inv}}$ denotes the set of invariant (by the flow φ^{g_0}) Borel probability measures and $\mathbf{h}_\mu(\varphi_1^{g_0})$ is the metric entropy of the flow $\varphi_1^{g_0}$ at time 1. It is actually sufficient to restrict the sup to ergodic measures $\mathfrak{M}_{\text{inv,erg}}$ [**Wal82**, Corollary 9.10.1]. Since the flow is Anosov, the supremum is always achieved for a unique invariant ergodic measure μ_f (by [**BR75**, Theorem 3.3]; see also [**HF19**, Theorem 9.3.4] and the following discussion therein) called the *equilibrium state* of f, and

$$\mu_f = \mu_{f'} \Longrightarrow f - f' = X_{g_0}u + c \quad \text{for some } u \text{ Hölder and } c \text{ constant;} \tag{2.12}$$

see [**HF19**, Theorem 9.3.16]. The measure μ_f is also mixing and positive on open sets, which rules out the possibility of a finite combination of Dirac measures supported on a finite number of closed orbits. Moreover, μ_f can be written as an infinite weighted sum of Dirac masses $\delta_{g_0}(c_j)$ supported over the geodesics $\gamma_{g_0}(c_j)$, where $c_j \in \mathcal{C}$ are the primitive classes (see [**Par88**] for the case $\mathbf{P}(f) \geq 0$ or [**PPS15**, Theorem 9.17] for the general case). For example, when $\mathbf{P}(f) \geq 0$,

$$\int u \, d\mu_f = \lim_{T\to\infty} \frac{1}{N(T, f)} \sum_{\{j | L_{g_0}(c_j) \in [T, T+1]\}} e^{\int_{\gamma_{g_0}(c_j)} f} \int_{\gamma_{g_0}(c_j)} u, \tag{2.13}$$

where $N(T, f) := \sum_{j, L_{g_0}(c_j) \in [T, T+1]} L_{g_0}(c_j) e^{\int_{\gamma_{g_0}(c_j)} f}$. When $f = 0$, this is the measure of maximal entropy, also called the *Bowen–Margulis measure* $\mu_{g_0}^{\text{BM}}$; in that case $\mathbf{P}(0) = \mathbf{h}_{\text{top}}(\varphi_1^{g_0})$ is the topological entropy of the flow. When $f = -J_{g_0}^u$, where $J_{g_0}^u : x \mapsto \partial_t(|\det d\varphi_t^g(x)|_{E_u(x)})|_{t=0}$ is the unstable Jacobian, we obtain the Liouville measure $\mu_{g_0}^{\text{L}}$ induced by the metric g_0; in that case, $\mathbf{P}(-J_{g_0}^u) = 0$. If we fix an exponent of Hölder regularity $\nu > 0$, then the map $C^\nu(S_{g_0}M) \ni f \mapsto \mathbf{P}(f)$ is real analytic (see [**Rue04**, Corollary 7.10] for discrete systems and [**PP90**, Proposition 4.7] for flows).

2.6. *Geodesic stretch.* We refer to [**CF90**, **Kni95**] for the original definition of this notion.

2.6.1. *Structural stability and time reparametrization.* We fix a smooth metric $g_0 \in \mathcal{M}$ with Anosov geodesic flow and we view the geodesic flow and vector fields of any metric

g close to g_0 as living on the unit tangent bundle $S_{g_0}M$ of g_0 by simply pulling them back by the diffeomorphism

$$(x, v) \in S_{g_0}M \to \left(x, \frac{v}{|v|_g}\right) \in S_gM.$$

We fix some constant $k \geq 2$ and $\alpha \in (0, 1)$. There exist a regularity parameter $\nu > 0$ and a neighborhood $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$ of g_0 such that, by the structural stability theorem ([**dlLMM86**, Appendix A] or [**KKPW89**, Proposition 2.2] for the Hölder regularity case), for any $g \in \mathcal{U}$, there exists a C^ν Hölder homeomorphism $\psi_g : S_{g_0}M \to S_{g_0}M$, differentiable in the flow direction, which is an orbit conjugacy that is such that

$$d\psi_g(z)X_{g_0}(z) = a_g(z)X_g(\psi_g(z)) \quad \text{for all } z \in S_{g_0}M, \tag{2.14}$$

where a_g is in $C^\nu(S_{g_0}M)$. Moreover, the map

$$\mathcal{U} \ni g \mapsto (a_g, \psi_g) \in C^\nu(S_{g_0}M) \times C^\nu(S_{g_0}M, S_{g_0}M)$$

is C^{k-2} and ψ_g is homotopic to the identity. For the proof of Theorem 1.3, we will also need the continuity of $a_g = a_{g_0,g}$ and of its g-derivatives of order $\ell \leq k - 2$ as a function of the base metric g_0. This continuity follows essentially from the proof of [**KKPW89**, Proposition 2.2]; we give a proof of this fact in Proposition C.1 in the Appendix.

Note that neither a_g nor ψ_g is unique, but a_g is unique up to a coboundary and in all the following paragraphs; adding a coboundary to a_g will not affect the results. From (2.14), we obtain that for $t \in \mathbb{R}$, $z \in S_{g_0}M$, $\varphi^g_{\kappa_{a_g}(z,t)}(\psi_g(z)) = \psi_g(\varphi^{g_0}_t(z))$ with

$$\kappa_{a_g}(z, t) = \int_0^t a_g(\varphi^{g_0}_s(z))\, ds. \tag{2.15}$$

If $c \in \mathcal{C}$ is a free homotopy class, then

$$L_g(c) = \int_0^{L_{g_0}(c)} a_g(\varphi^{g_0}_s(z))\, ds, \tag{2.16}$$

for any $z \in \gamma_{g_0}(c)$, the unique g_0-closed geodesic in c.

2.6.2. *Definition of the geodesic stretch.* We denote by $\widetilde{M}$ the universal cover of M. Given a metric $g \in \mathcal{M}$ on M, we denote by $\widetilde{g}$ its lift to the universal cover. Given two metrics g_1 and g_2 on M, there exists a constant $c > 0$ such that $c^{-1}g_1 \leq g_2 \leq cg_1$. This implies that any $\widetilde{g_1}$-geodesic is a quasi-geodesic for $\widetilde{g_2}$. We now assume that the two metrics g_1, g_2 are Anosov on M. The *ideal* (or *visual*) boundary $\partial_\infty\widetilde{M}$ is independent of the choice of g and is naturally endowed with the structure of a topological manifold (see Appendix B) whose regularity inherits that of the foliation (that is, it is at least Hölder continuous and is $C^{2-\varepsilon}$ for any $\varepsilon > 0$ on negatively curved surfaces by [**HK90**]). In negative curvature, we refer to [**BH99**, Ch. H.3] and [**Kni02**] for further details. For the general Anosov case, we refer to [**Kni12**] and Appendix B of the present paper.

We denote by $\mathcal{G}_g := S_{\widetilde{g}}\widetilde{M}/\sim$ (where $z \sim z'$ if and only if there exists a time $t \in \mathbb{R}$ such that $\varphi_t(z) = z'$) the set of g-geodesics on $\widetilde{M}$: this is a smooth $2n$-dimensional manifold. Moreover, there exists a Hölder-continuous homeomorphism $\Phi_g : \mathcal{G}_g \to \partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus$

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Δ, where Δ is the diagonal in $\partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M}$. Given a point $z \in S_{\widetilde{g}}\widetilde{M}$, we will denote by $z_+, z_- \in \partial_\infty \widetilde{M}$ the points (in the future and in the past, respectively) on the boundary at infinity of the geodesic generated by z.

We now consider a fixed metric g_0 on M and a metric g in a neighborhood of g_0. If ψ_g denotes an orbit equivalence between the two geodesic flows, then ψ_g induces a homeomorphism $\Psi_g : \mathcal{G}_{g_0} \to \mathcal{G}_g$. The map

$$\Phi_g \circ \Psi_g \circ \Phi_{g_0}^{-1} : \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta \to \partial_\infty \widetilde{M} \times \partial_\infty \widetilde{M} \setminus \Delta$$

is nothing more than the identity.

Given $z = (x, v) \in S_{g_0}M$, we denote by $c_{g_0}(z) : t \mapsto c_{g_0}(z, t) \in M$ the unique geodesic (for the sake of simplicity, we identify the geodesic and its arc-length parametrization) such that $c_{g_0}(z, 0) = x$, $\dot{c}_{g_0}(z, 0) = v$. We consider $\widetilde{c}_{g_0}(z)$, a lift of $c_{g_0}(z)$ to the universal cover $\widetilde{M}$, and introduce the function

$$b : S_{g_0}M \times \mathbb{R} \to \mathbb{R}, \quad b(z, t) := d_{\widetilde{g}}(\widetilde{c}_{g_0}(z, 0), \widetilde{c}_{g_0}(z, t)),$$

which computes the $\widetilde{g}$-distance between the endpoints of the $\widetilde{g}_0$-geodesic joining $\widetilde{c}_{g_0}(z, 0)$ to $\widetilde{c}_{g_0}(z, t)$. It is an immediate consequence of the triangle inequality that $(z, t) \mapsto b(z, t)$ is a subadditive cocycle for the geodesic flow φ^{g_0}, that is,

$$b(z, t + s) \leq b(z, t) + b(\varphi_t^{g_0}(z), s) \quad \text{for all } z \in S_{g_0}M, \text{ for all } t, s \in \mathbb{R}$$

As a consequence, by the subadditive ergodic theorem (see [**Wal82**, Theorem 10.1], for instance), we obtain the following lemma.

LEMMA 2.6. *Let μ be an invariant probability measure for the flow $\varphi_t^{g_0}$. Then the quantity*

$$I_\mu(g_0, g, z) := \lim_{t \to +\infty} b(z, t)/t$$

exists for μ-almost every $z \in S_{g_0}M$, $I_\mu(g_0, g, \cdot) \in L^1(S_{g_0}M, d\mu)$, and this function is invariant by the flow $\varphi_t^{g_0}$.

We define the *geodesic stretch of the metric g, relative to the metric g_0, with respect to the measure μ* by

$$I_\mu(g_0, g) := \int_{S_{g_0}M} I_\mu(g_0, g, z)\, d\mu(z).$$

When the measure μ in the previous definition is ergodic, the function $I_\mu(g_0, g, \cdot)$ is thus (μ-almost everywhere) equal to the constant $I_\mu(g_0, g)$. We recall that $\delta_{g_0}(c)$ is the normalized measure supported on $\gamma_{g_0}(c)$, that is,

$$\delta_{g_0}(c) : u \mapsto \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} u(\varphi_t^{g_0}(z))\, dt.$$

We can actually describe the stretch using the time reparametrization a_g.

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LEMMA 2.7. *Let μ be an ergodic invariant measure with respect to the flow $\varphi_t^{g_0}$. Then,*

$$I_\mu(g_0, g) = \int_{SM_{g_0}} a_g \, d\mu = \lim_{j\to+\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

where $(c_j)_{j\geq 0} \in \mathcal{C}^{\mathbb{N}}$ is such that (the existence of c_j follows from [Sig72, Theorem 1]*) $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu$.*

Proof. We first prove the left-hand equality. Let $\widetilde{M}$ be the universal covering of M and Γ the group of deck transformations. Denote as above by $\widetilde{\psi}_g : S_{\widetilde{g}_0}\widetilde{M} \to S_{\widetilde{g}}\widetilde{M}$ the lift of the conjugacy between the geodesic flow of the metrics $\widetilde{g}$ and $\widetilde{g}_0$. Then, for all $\gamma \in \Gamma$,

$$\widetilde{\varphi}^g_{\kappa_{a_g}(z,t)}(\widetilde{\psi}_g(z)) = \widetilde{\psi}_g(\widetilde{\varphi}_t^{g_0}(z)) \quad \text{and} \quad \widetilde{\psi}_g(\gamma_* z) = \gamma_* \widetilde{\psi}_g(z).$$

If $\pi : T\widetilde{M} \to \widetilde{M}$ is the canonical projection the function $d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(z)), \pi(z))$ is Γ-invariant. This follows since

$$\begin{aligned} d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(\gamma_* z)), \pi(\gamma_* z)) &= d_{\widetilde{g}}(\pi(\gamma_*\widetilde{\psi}_g(z)), \pi(\gamma_* z)) \\ &= d_{\widetilde{g}}(\gamma\pi(\widetilde{\psi}_g(z)), \gamma\pi(z)) = d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(z)), \pi(z)). \end{aligned}$$

Hence, by the compactness of M and the continuity of $d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(z)), \pi(z))$ there is a constant $C > 0$ such that $d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(z)), \pi(z)) \leq C$ for all $z \in S\widetilde{M}$. Using the triangle inequality, we obtain

$$\begin{aligned} |b(z,t) - \kappa_{a_g}(t,z)| &= |d_{\widetilde{g}}(\pi(\widetilde{\varphi}_t^{g_0}(z)), \pi(z)) - d_{\widetilde{g}}(\pi(\widetilde{\varphi}^g_{\kappa_{a_g}(z,t)}(\widetilde{\psi}_g(z))), \pi(\widetilde{\psi}_g(z)))| \\ &= |d_{\widetilde{g}}(\pi(\widetilde{\varphi}_t^{g_0}(z)), \pi(z)) - d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(\widetilde{\varphi}_t^{g_0}(z))), \pi(\widetilde{\psi}_g(z)))| \\ &\leq d_{\widetilde{g}}(\pi(\widetilde{\varphi}_t^{g_0}(z)), \pi(\widetilde{\psi}_g(\widetilde{\varphi}_t^{g_0}(z)))) + d_{\widetilde{g}}(\pi(\widetilde{\psi}_g(z)), \pi(z)) \leq 2C. \end{aligned}$$

This implies, using (2.15), that

$$\lim_{t\to+\infty} b(z,t)/t = \lim_{t\to+\infty} \kappa_{a_g}(z,t)/t = \lim_{t\to+\infty} \frac{1}{t}\int_0^t a_g(\varphi_s^{g_0}(z))\, ds = \int_{S_{g_0}M} a_g \, d\mu,$$

for μ-almost every $z \in S_{g_0}M$, by the Birkhoff ergodic theorem [Wal82, Theorem 1.14]. By (2.16) we also have

$$\int_{S_{g_0}M} a_g \, d\mu_f = \lim_{j\to\infty} \langle \delta_{g_0}(c_j), a_g\rangle = \lim_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

thus the proof is complete. □

As a consequence, we immediately obtain the following corollary.

COROLLARY 2.8. *Let g belong to a fixed neighborhood $\mathcal{U}$ of g_0 in $\mathcal{M}^{k,\alpha}$, and assume that for any sequence of primitive free homotopy classes $(c_j)_{j\geq 0} \in \mathcal{C}^{\mathbb{N}}$ such that $L_{g_0}(c_j) \to \infty$, we have $\lim_{j\to\infty} L_g(c_j)/L_{g_0}(c_j) = 1$. Then, for any equilibrium state μ_f with respect to $\varphi_t^{g_0}$ associated to some Hölder function f, we have $I_{\mu_f}(g_0, g) = 1$.*

Combining this with the results of [GL19, Theorem 1], namely the local rigidity of the marked length spectrum, we also easily obtain the following theorem.

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THEOREM 2.9. *Let (M, g_0) be a smooth Riemannian n-dimensional manifold with Anosov geodesic flow, topological entropy $\boldsymbol{h}_{top}(g_0) = 1$, and assume that its curvature is non-positive if $n \geq 3$. Then there exists $k \in \mathbb{N}$ large enough, depending only on n, $\varepsilon > 0$ small enough such that the following statement holds: there is $C > 0$ depending on g_0 so that, for each $g \in C^k(M; S^2T^*M)$ with $\|g - g_0\|_{C^k} \leq \varepsilon$, if*

$$\boldsymbol{h}_{top}(g) = 1, \quad \lim_{j\to+\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} = 1,$$

for some sequence $(c_j)_{j\in\mathbb{N}}$ of primitive free homotopy classes such that $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu^{BM}_{g_0}$, then g is isometric to g_0.

Proof. Given a metric g, we have by [**Kni95**, Theorem 1.2] (in [**Kni95**] the metric is assumed to be negatively curved, but the argument applies also for Anosov flows, as is shown in [**BCLS15**, Proposition 3.8]: it corresponds to Proposition 3.10 below in the case $f := 1$ and $f' = a_g$) that

$$\mathbf{h}_{\mathrm{top}}(g) \geq \frac{\mathbf{h}_{\mathrm{top}}(g_0)}{I_{\mu^{\mathrm{BM}}_{g_0}}(g_0, g)}, \tag{2.17}$$

with equality if and only if φ^{g_0} and φ^g are, up to a scaling, time-preserving conjugate, that is, there exists a homeomorphism ψ such that $\psi \circ \varphi^{ct}_{g_0} = \varphi^t_g \circ \psi$ with $c := \mathbf{h}_{\mathrm{top}}(g)/\mathbf{h}_{\mathrm{top}}(g_0)$.

In particular, restricting to metrics with entropy 1, we obtain that $I_{\mu^{\mathrm{BM}}_{g_0}}(g_0, g) \geq 1$ with equality if and only if the geodesic flows are conjugate, that is, if and only if $L_g = L_{g_0}$ (by the Livsic theorem). As a consequence, given g_0, g with entropy 1 such that $L_g(c_j)/L_{g_0}(c_j) \to_{j\to+\infty} 1$ for some sequence $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu^{\mathrm{BM}}_{g_0}$, we obtain that $I_{\mu^{\mathrm{BM}}_{g_0}}(g_0, g) = 1$, hence $L_g = L_{g_0}$. If $k \in \mathbb{N}$ was chosen large enough at the beginning, we can then conclude by the local rigidity of the marked length spectrum [**GL19**, Theorem 1]. □

In Theorem 2.9, we assume that g_0 has entropy 1. This is actually a harmless assumption in so far as the same result holds true on metrics of constant topological entropy $\mathbf{h}_{\mathrm{top}}(g) = \lambda > 0$. Recall that by considering $\lambda^2 g_0$ for some constant $\lambda > 0$, the entropy scales as $\mathbf{h}_{\mathrm{top}}(\lambda^2 g_0) = \mathbf{h}_{\mathrm{top}}(g_0)/\lambda$ [**Pat99**, Lemma 3.23] and we can thus always reduce to the previous case $\mathbf{h}_{\mathrm{top}}(g_0) = 1$. We also observe that the previous theorem implies the local rigidity of the marked length spectrum: if $L_g = L_{g_0}$, then $\mathbf{h}_{\mathrm{top}}(g_0) = \mathbf{h}_{\mathrm{top}}(g)$ because the topological entropy $\mathbf{h}_{\mathrm{top}}(g)$ is the first pole of the Ruelle zeta function [**PP90**, Theorem 9.1]

$$\zeta_g(s) := \prod_{c\in\mathcal{C}} (1 - e^{-sL_g(c)}).$$

We can then apply Theorem 2.9 to deduce that g is isometric to g_0. We will provide an alternative proof of this fact in the next section without using the proof of [**GL19**].

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3. *A functional on the space of metrics*

Given a metric g in a $C^{k,\alpha}$-neighborhood $\mathcal{U}$ of g_0, we define the potential

$$V_g := J^u_{g_0} + a_g - 1 \in C^\nu(S_{g_0}M) \tag{3.1}$$

for some $\nu > 0$. We remark that $\mathcal{U} \ni g \mapsto V_g \in C^\nu(S_{g_0}M)$ is C^{k-2} and, for $g = g_0$, $V_{g_0} = J^u_{g_0}$. Consider the map $\psi : \mathcal{M}^{k,\alpha} \to \mathbb{R}$, defined for g_0 a fixed smooth metric with Anosov geodesic flow, by

$$\Psi(g) := \mathbf{P}\bigg(- J^u_{g_0} - a_g + \int_{S_{g_0}M} a_g \, d\mu^L_{g_0}\bigg) = \mathbf{P}(-V_g) + I_{\mu^L_{g_0}}(g_0, g) - 1. \tag{3.2}$$

We also define the maps

$$F : \mathcal{M}^{k,\alpha} \to \mathbb{R}, \quad F(g) := \mathbf{P}(-V_g), \tag{3.3}$$

$$\Phi : \mathcal{M}^{k,\alpha} \to \mathbb{R}, \quad \Phi(g) = I_{\mu^L_{g_0}}(g_0, g) - 1, \tag{3.4}$$

satisfying $\Psi(g) = F(g) + \Phi(g)$. We note that Ψ, Φ, F are C^{k-2} by [**Con92**]. We also make the following observation: since $\mathbf{P}(-J^u_{g_0}) = 0$ and a_{g_0} is cohomologous to 1, we have $\Psi(g_0) = 0$ and

$$\Phi(g) = -\bigg(\mathbf{h}_{\mu^L_{g_0}}(\varphi_1^{g_0}) + \int_{S_{g_0}M} (1 - J^u_{g_0} - a_g)\, d\mu^L_{g_0}\bigg) \geq -\mathbf{P}(1 - J^u_{g_0} - a_g) = -F(g) \tag{3.5}$$

by using the variational definition (2.11) of the pressure. This shows that, for all $g \in \mathcal{M}^{k,\alpha}$,

$$\Psi(g) \geq \Psi(g_0) = 0.$$

Moreover, $\Psi(g) = 0$ if and only if the inequality (3.5) becomes an equality, which means that $\mu^L_{g_0}$ is the equilibrium measure of $-J^u_{g_0} + 1 - a_g$. Since $\mu^L_{g_0}$ is also the equilibrium measure associated to $-J^u_{g_0}$, we conclude by (2.12) that $1 - a_g$ is cohomologous to a constant, or equivalently a_g is cohomologous to a constant. We have thus shown the following lemma.

LEMMA 3.1. *The map Ψ satisfies $\Psi(g) \geq \Psi(g_0) = 0$, and $\Psi(g) = \Psi(g_0) = 0$ if and only if a_g is cohomologous to a constant, or equivalently $L_g = \lambda L_{g_0}$ for some $\lambda > 0$.*

The proof of Theorem 1.2 will be a consequence of the fact that Taylor expansion of Ψ at $g = g_0$ has leading term given by the Hessian, which turns out to be the variance operator Π_2 studied before.

3.1. *The proof of Theorem 1.2.* In what follows, we will compute the derivatives of the map Ψ, Φ, F. As mentioned earlier, they are C^{k-2} by [**Con92**, Theorem C], and explicit computations of their derivatives can be found in [**PP90**, Proposition 4.10] (subshift case) and [**KKPW90**, **KKW91**] (topological entropy case). The first step in the proof is the following proposition.

PROPOSITION 3.2. *The non-negative functional* $\Psi : \mathcal{M}^{k,\alpha} \to \mathbb{R}^+$ *defined in* (3.2) *satisfies the following property: there exist a neighborhood* $\mathcal{U}$ *of* g_0 *in* $C^{5,\alpha}(M, S^2T^*M)$ *and a constant* C_{g_0} *depending on* g_0 *such that, for all* $g \in \mathcal{U}$,

$$\Psi(g) \geq \tfrac{1}{8}(\langle \Pi_2^{g_0}(g-g_0), (g-g_0)\rangle_{L^2} - \langle (g-g_0), g_0\rangle_{L^2}^2) - C_{g_0}\|g-g_0\|_{C^{5,\alpha}}^3.$$

Proof. We shall compute the Taylor expansion of Ψ at $g = g_0$ to second order. By [**PP90**, Proposition 4.10], we have, for $h \in T_g\mathcal{M}^{k,\alpha}$,

$$dF_g.h = -\int_{S_{g_0}M} da_g.h\, dm_g$$

where m_g is the equilibrium measure of $-V_g$. In particular, observe that for $g = g_0$, we have

$$dF_{g_0}.h = -\int_{S_{g_0}M} da_{g_0}.h\, d\mu_{g_0}^{\mathrm{L}}, \tag{3.6}$$

since $m_{g_0} = \mu_{g_0}^{\mathrm{L}}$. Next, we get, for $h \in T_{g_0}\mathcal{M}^{k,\alpha}$,

$$d\Phi_{g_0}.h = \int_{S_{g_0}M} da_{g_0}.h\, d\mu_{g_0}^{\mathrm{L}} = -dF_{g_0}.h, \tag{3.7}$$

thus $d\Psi_{g_0}.h = 0$ for all $h \in T_{g_0}\mathcal{M}^{k,\alpha}$.

Let us next compute the second derivative $d^2\Psi_{g_0}(h,h)$. First, we have

$$d^2\Phi_{g_0} = \int_{S_{g_0}M} d^2a_{g_0}(h,h)\, d\mu_{g_0}^{L}.$$

Then, by [**PP90**, Proposition 4.11] we know that

$$d^2\mathbf{P}_{-V_{g_0}}(dV_{g_0}.h, dV_{g_0}.h) = \mathrm{Var}_{\mu_{g_0}^{\mathrm{L}}}(dV_{g_0}.h - \langle dV_{g_0}.h, 1\rangle) = \langle \Pi^{g_0} dV_{g_0}.h, dV_{g_0}.h\rangle_{L^2},$$
$$d\mathbf{P}_{-V_{g_0}}(dV_{g_0}.h) = \int_{S_{g_0}M} dV_{g_0}.h\, d\mu_{g_0}^{\mathrm{L}}, \tag{3.8}$$

where $\mathrm{Var}_{\mu_{g_0}^{\mathrm{L}}}(h)$ is the variance defined in (2.5), equal to $\langle \Pi^{g_0}h, h\rangle_{L^2}$ by (2.6) and $\Pi^{g_0}1 = 0$. Therefore,

$$\begin{aligned} d^2F_{g_0}(h,h) &= -\, d\mathbf{P}_{-V_{g_0}}.d^2V_{g_0}(h,h) + d^2\,\mathbf{P}_{-V_{g_0}}(dV_{g_0}.h, dV_{g_0}.h) \\ &= -\, d\mathbf{P}_{-V_{g_0}}.d^2a_{g_0}(h,h) + \langle \Pi^{g_0} da_{g_0}.h, da_{g_0}.h\rangle_{L^2}. \end{aligned}$$

All together, we finally get

$$d^2\Psi_{g_0}(h,h) = \langle \Pi^{g_0} da_{g_0}.h, da_{g_0}.h\rangle_{L^2}.$$

To conclude, we claim in Lemma 3.3 below that $da_{g_0}.h - \frac{1}{2}\pi_2^*h$ is a coboundary, so that

$$d^2\Psi_{g_0}(h,h) = \langle \Pi^{g_0}\pi_2^*h, \pi_2^*h\rangle_{L^2} = \tfrac{1}{4}(\langle \Pi_2^{g_0}h, h\rangle_{L^2} - \langle h, g_0\rangle_{L^2}^2).$$

The statement of the proposition is then simply the Taylor expansion of $\Psi(g)$ at $g = g_0$, with $h = g - g_0$. (We need the map to be C^3 for the Taylor expansion, hence the need for the $C^{5,\alpha}$ regularity since we lose two derivatives as mentioned at the beginning of §3.) □

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LEMMA 3.3. *Consider a smooth deformation* $(g_\lambda)_{\lambda\in(-1,1)}$ *of* g_0 *inside* $\mathcal{M}^{k,\alpha}$. *Then there exists a Hölder-continuous function* $f : S_{g_0}M \to \mathbb{R}$ *such that*

$$\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0} f.$$

Proof. Let c be a fixed free homotopy class, and let $\gamma_0 \in c$ be the unique closed g_0-geodesic in the class c, which we parametrize by unit-speed $z_0 : [0, \ell_{g_0}(\gamma_0)] \to S_{g_0}M$. We define $z_\lambda(s) = \psi_\lambda(z_0(s)) = (\alpha_\lambda(s), \dot\alpha_\lambda(s))$ (the dot is the derivative with respect to s), where ψ_λ is the conjugacy between g_λ and g_0: this gives a non-unit-speed parametrization of γ_λ, the unique closed g_λ-geodesic in c. We recall that $\pi : TM \to M$ is the projection. Using (2.14), we obtain

$$\begin{aligned}
&\int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\dot\alpha_\lambda(s), \dot\alpha_\lambda(s))\, ds \\
&\quad= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\partial_s(\pi\circ z_\lambda(s)), \partial_s(\pi\circ z_\lambda(s)))\, ds \\
&\quad= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\partial_s(\pi\circ\psi_\lambda\circ z_0(s)), \partial_s(\pi\circ\psi_\lambda\circ z_0(s)))\, ds \\
&\quad= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s))\, \underbrace{g_\lambda(d\pi(X_{g_\lambda}(z_\lambda(s))), d\pi(X_{g_\lambda}(z_\lambda(s))))}_{=1}\, ds \\
&\quad= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s))\, ds.
\end{aligned}$$

Since $s \mapsto \alpha_0(s)$ is a unit-speed geodesic for g_0, it is a critical point of the energy functional (with respect to g_0). Thus, by differentiating the previous identity with respect to λ and evaluating at $\lambda = 0$, we obtain

$$\int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda g_\lambda|_{\lambda=0}(\dot\alpha_0(s), \dot\alpha_0(s))\, ds = 2\int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda a_\lambda|_{\lambda=0}(z_0(s))\, ds.$$

As a consequence, $\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0}$ is a Hölder-continuous function in the kernel of the X-ray transform: by the usual Livsic theorem, there exists a function f (with the same Hölder regularity), differentiable in the flow direction, such that $\pi_2^*(\partial_\lambda g_\lambda|_{\lambda=0}) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0} f$. □

As a corollary, we obtain the following result.

COROLLARY 3.4. *For* $k \geq 5$, $\alpha \in (0, 1)$, *there exist a neighborhood* $\mathcal{U}$ *of* g_0 *in* $C^{k,\alpha}(M; S^2T^*M)$ *and constants* $C_{g_0}, C'_{g_0} > 0$ *depending on* g_0 *such that, for all* $g \in \mathcal{U}$,

$$C_{g_0}\|\pi_{\ker D^*_{g_0}}(g - g_0)\|^2_{H^{-1/2}(M)} \leq \Psi(g) + \tfrac{1}{4}\langle(g - g_0), g_0\rangle^2_{L^2} + C'_{g_0}\|g - g_0\|^3_{C^{5,\alpha}}.$$

There exist a neighborhood $\mathcal{U}'$ *of* g_0 *in* $C^{k,\alpha}(M; S^2T^*M)$ *and a constant* $C''_{g_0} > 0$ *depending on* g_0 *such that, for all* $g \in \mathcal{U}'$, *there is a diffeomorphism* $\psi \in C^{k+1,\alpha}(M)$ *such that*

$$C_{g_0}\|\psi^*g - g_0\|^2_{H^{-1/2}(M)} \leq \Psi(g) + \tfrac{1}{4}\langle(\psi^*g - g_0), g_0\rangle^2_{L^2} + C''_{g_0}\|\psi^*g - g_0\|^3_{C^{5,\alpha}}$$

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Proof. The first inequality follows from Proposition 3.2 and Lemma 2.2. For the second inequality, we apply the first inequality to $\psi^* g$, where ψ is the diffeomorphism obtained from Lemma 2.4, and we use that $\Psi(\psi^* g) = \Psi(g)$. □

The next step is to control the term $\langle(\psi^* g - g_0), g_0\rangle_{L^2}$ by the geodesic stretch. We will show the following proposition.

PROPOSITION 3.5. *There is $k \in \mathbb{N}$ large enough, depending only on $n = \dim M$, such that if g_0 is smooth with Anosov geodesic flow, and non-positive curvature in the case $n > 2$, there exist $C_{g_0} > 0$ and $C_n > 0$, an open neighborhood $\mathcal{U}$ in $C^{k,\alpha}(M; S^2T^*M)$ of g_0, such that. for each $g \in \mathcal{U}$, there is a diffeomorphism ψ satisfying*

$$\begin{aligned}
C_{g_0}\|\psi^* g - g_0\|^2_{H^{-1/2}(M)} &\\
\leq \boldsymbol{P}\Big(- J^u_{g_0} - a_g + \int_{S_{g_0}M} a_g \, d\mu^L_{g_0}\Big) &+ C_n(I_{\mu^L_{g_0}}(g_0, g) - 1)^2\\
\leq \boldsymbol{P}\Big(- J^u_{g_0} - a_g + \int_{S_{g_0}M} a_g \, d\mu^L_{g_0}\Big) &+ C_n(\boldsymbol{P}(-J^u_{g_0} - a_g + 1))^2\\
\leq \boldsymbol{P}\Big(- J^u_{g_0} - a_g + \int_{S_{g_0}M} a_g \, d\mu^L_{g_0}\Big) &+ (Vol_g(M) - Vol_{g_0}(M))^2.
\end{aligned}$$

Here C_{g_0} depends on g_0 and C_n on n only.

Proof. We write the Taylor expansion of $\Phi(\psi^* g) = \Phi(g) = I_{\mu^L_{g_0}}(g_0, g) - 1$ at $g = g_0$: by Lemma 3.6 and Lemma 3.3,

$$d\Phi_{g_0}.h = \int_{S_{g_0}M} da_{g_0}.h \, d\mu^L_{g_0} = \frac{1}{2}\int_{S_{g_0}M} \pi_2^* h \, d\mu^L_{g_0} = C_n\langle h, g_0\rangle_{L^2}$$

for some $C_n > 0$ depending only on $n = \dim M$. Then

$$\Phi(g) = \Phi(\psi^* g) = C_n\langle\psi^* g - g_0, g_0\rangle_{L^2} + \mathcal{O}(\|\psi^* g - g_0\|^2_{C^{5,\alpha}}).$$

Combining with Corollary 3.4, we obtain

$$C_{g_0}\|\psi^* g - g_0\|^2_{H^{-1/2}(M)} \leq \Psi(g) + 2C_n^{-2}\Phi(g)^2 + C''_{g_0}\|\psi^* g - g_0\|^3_{C^{5,\alpha}} \tag{3.9}$$

if $\|\psi^* g - g_0\|_{C^{5,\alpha}}$ is small enough, which is the case if $\|g - g_0\|_{C^{5,\alpha}}$ is small enough by Lemma 2.4. To obtain the first inequality of Proposition 3.5, we apply Sobolev embedding and interpolation estimates (the interpolation estimate $\|u\|_{H^c} \leq \|u\|^t_{H^a}\|u\|^{1-t}_{H^b}$ for $c = ta + (1-t)b$ is obtained by applying the Hadamard three-line theorem to the holomorphic function $s \mapsto \sum_j (1 + \lambda_j)^s\langle u, e_j\rangle^2_{L^2}$ on $\mathrm{Re}(s) \in [a, b]$, where e_j is an orthonormal basis of eigenfunctions of any positive elliptic self-adjoint differential operator of order 2 on symmetric tensors and λ_j gives the corresponding eigenvalues) [**Tay96**, Ch. 4] and get, for some constants $c_{g_0} > 0$, $c'_{g_0} > 0$ depending on g_0 only,

$$\|\psi^* g - g_0\|^3_{C^{5,\alpha}} \leq c_{g_0}\|\psi^* g - g_0\|^3_{H^{(n/2)+5+\alpha'}} \leq c'_{g_0}\|\psi^* g - g_0\|^2_{H^{-1/2}}\|\psi^* g - g_0\|_{H^k},$$

if $k > (3/2)n + 16 + 3\alpha$ and $\alpha' > \alpha$. This means that if $\|\psi^* g - g_0\|_{H^k}$ is small enough, depending on the constants C_{g_0}, C''_{g_0}, c_{g_0}, c'_{g_0}, one can absorb the $\|\psi^* g - g_0\|^3_{C^{5,\alpha}}$ term of

(3.9) into the left-hand side and get the first inequality of Proposition 3.5. The smallness of $\|\psi^* g - g_0\|_{H^k}$ is implied by the smallness of $\|g - g_0\|_{C^{k,\alpha}}$ by Lemma 2.4. The same exact argument applies by replacing $\Phi(g)$ by $F(g)$ using that $dF_{g_0} = -d\phi_{g_0}$; this proves the second inequality of Proposition 3.5. The last inequality is similar since

$$\mathrm{Vol}_g(M) - \mathrm{Vol}_{g_0}(M) = \frac{1}{2}\int_M \mathrm{Tr}_{g_0}(h)\, d\mathrm{vol}_{g_0} + \mathcal{O}(\|h\|^2_{C^{5,\alpha}}) = \frac{1}{2}\langle h, g_0\rangle_{L^2} + \mathcal{O}(\|h\|^2_{C^{5,\alpha}})$$

for $h := g - g_0$. The proof is complete. □

To conclude the proof of Theorem 1.2, we need to estimate $\Phi(g)$ and $F(g)$ in terms of $\mathcal{L}_\pm(g)$. Recall that (see [PPS15, Corollary 9.17])

$$\begin{aligned}\mathbf{P}(-V_g) &= \lim_{T\to\infty} \frac{1}{T} \log \sum_{c\in\mathcal{C}, L_{g_0}(c)\in[T,T+1]} e^{-\int_{\gamma_{g_0}(c)} V_g} \\ &= \lim_{T\to\infty} \frac{1}{T} \log \sum_{c\in\mathcal{C}, L_{g_0}(c)\in[T,T+1]} e^{-\int_{\gamma_{g_0}(c)} J^u_{g_0}} e^{L_{g_0}(c) - L_g(c)}.\end{aligned}$$

Thus, if we order $\mathcal{C} = (c_j)_{j\in\mathbb{N}}$ by the lengths (that is, $L_{g_0}(c_j) \geq L_{g_0}(c_{j-1})$), and we define

$$\mathcal{L}_+(g) := \limsup_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1, \quad \mathcal{L}_-(g) := \liminf_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1,$$

we see that for all $\delta > 0$ small, there is $T_0 > 0$ large so that, for all j with $L_{g_0}(c_j) \in [T, T+1]$ with $T \geq T_0$,

$$e^{\min((T+1)(-\mathcal{L}_+(g)-\delta), T(-\mathcal{L}_+(g)-\delta))} \leq e^{L_{g_0}(c_j) - L_g(c_j)} \leq e^{\max((T+1)(-\mathcal{L}_-(g)+\delta), T(-\mathcal{L}_-(g)+\delta))}.$$

We deduce, using $\mathbf{P}(-V_{g_0}) = 0$, that

$$-\mathcal{L}_+(g) - \delta \leq \mathbf{P}(-V_g) \leq -\mathcal{L}_-(g) + \delta.$$

Since $\delta > 0$ is arbitrarily small, we obtain $|F(g)| \leq \max(|\mathcal{L}_+(g)|, |\mathcal{L}_-(g)|)$. Similarly, Lemma 2.7 shows that $|\Phi(g)| \leq \max(|\mathcal{L}_+(g)|, |\mathcal{L}_-(g)|)$. So the proof of Theorem 1.2 is complete by combining these bounds with Proposition 3.5 (the right-hand side in the first and second inequalities of Proposition 3.5 being $F(g) + \Phi(g) + C_n\Phi(g)^2$ and $F(g) + \Psi(g) + C_n F(g)^2$).

3.2. *A submanifold of the space of metrics.* It is quite natural to describe the stretch functional Φ on the space

$$\mathcal{N}^{k,\alpha} := \{g \in \mathcal{M}^{k,\alpha} \mid \mathbf{P}(-V_g) = 0\} \tag{3.10}$$

and on $\mathcal{N}^{k,\alpha}_{\mathrm{sol}} := \mathcal{N}^{k,\alpha} \cap \ker D^*_{g_0}$. Indeed, as we shall see, this becomes a strictly convex functional near $g_0 \in \mathcal{N}^{k,\alpha}_{\mathrm{sol}}$ when restricted to $\mathcal{N}^{k,\alpha}_{\mathrm{sol}}$. It is possible that the map is strictly convex globally on $\mathcal{N}^{k,\alpha}_{\mathrm{sol}}$, in which case that would prove the global rigidity of the marked length spectrum.

Given $g \in \mathcal{N}^{k,\alpha}$, we denote by m_g the unique equilibrium state for the potential V_g. We will also write $\mathcal{N}$ for the case where $k = \infty$. First we check that these are (infinite-dimensional) manifolds.

LEMMA 3.6. *There exists a neighborhood $\mathcal{U} \subset \mathcal{M}^{k,\alpha}$ of g_0 such that $\mathcal{N}^{k,\alpha} \cap \mathcal{U}$ is a codimension-one C^{k-2}-submanifold of $\mathcal{U}$ and $\mathcal{N}_{sol}^{k,\alpha} \cap \mathcal{U}$ is a C^{k-2}-submanifold of $\mathcal{U}$. Similarly, there is $\mathcal{U} \subset \mathcal{M}$ an open neighborhood so that $\mathcal{N} \cap \mathcal{U}$ is a Fréchet submanifold of $\mathcal{M}$.*

Proof. To prove this lemma, we will use the notion of differential calculus on Banach manifolds as it is stated in [**Zei88**, Ch. 73]. Note that $\mathcal{M}^{k,\alpha}$ is a smooth Banach manifold and $\mathcal{N}^{k,\alpha} \subset \mathcal{M}^{k,\alpha}$ is defined by the implicit equation $F(g) = 0$ for

$$F : g \mapsto \mathbf{P}(-V_g) \in \mathbb{R}. \tag{3.11}$$

The map F being C^{k-2}, we need only prove that dF_{g_0} does not vanish by [**Zei88**, Theorem 73.C]. This will immediately give that $T_{g_0}\mathcal{N}^{k,\alpha} = \ker dF_{g_0}$. In order to do so, we need a deformation lemma. For the sake of simplicity, we write the objects $\cdot_\lambda$ instead of $\cdot_{g_\lambda}$.

We can now complete the proof of Lemma 3.6. We first prove the first part concerning $\mathcal{N}^{k,\alpha}$. Recall formula (3.6) for dF_{g_0}. Using Lemma 3.3, we obtain

$$dF_{g_0}.h = -\int_{S_{g_0}M} da_{g_0}.h \, d\mu_{g_0}^{\mathrm{L}} = -\frac{1}{2}\int_{S_{g_0}M} \pi_2^* h \, d\mu_{g_0}^{\mathrm{L}} = -C_n \langle h, g_0 \rangle_{L^2}, \tag{3.12}$$

for some constant $C_n > 0$ depending on n. This is obviously surjective and we also obtain

$$T_{g_0}\mathcal{N}^{k,\alpha} = \ker dF_{g_0} = \{h \in C^{k,\alpha}(M; S^2T^*M) \mid \langle h, g_0\rangle_{L^2} = 0\} = (\mathbb{R}g_0)^{\perp},$$

where the orthogonal is understood with respect to the L^2-scalar product.

We now deal with $\mathcal{N}_{\mathrm{sol}}^{k,\alpha}$. First observe that $\ker D_{g_0}^*$ is a closed linear subspace of $\mathcal{M}^{k,\alpha}$ and thus a smooth submanifold of $\mathcal{M}^{k,\alpha}$. By [**Zei88**, Corollary 73.50], it is sufficient to prove that $\ker D_{g_0}^*$ and $\mathcal{N}^{k,\alpha}$ are transverse at g_0. But observe that $g_0 \in \ker D_{g_0}^* \simeq T_{g_0} \ker D_{g_0}^*$ and thus

$$T_{g_0} \ker D_{g_0}^* + T_{g_0}\mathcal{N}^{k,\alpha} = T_{g_0}\mathcal{M}^{k,\alpha},$$

showing transversality.

The case of $\mathcal{N}$ follows directly from the Nash–Moser theorem: F is obviously a smooth tame map from $C^\infty(M; S^2T^*M)$ to $\mathbb{R}$; moreover, dF_g has a right inverse H_g since

$$dF_g.g = -\int_{S_{g_0}M} da_g.g \, dm_g = -\frac{1}{2}\int_{S_{g_0}M} a_g \, dm_g = -\frac{1}{2} I_{m_g}(g_0, g)$$

where we use the fact that $2da_g.g$ is cohomologous to a_g. This can be seen by differentiating $L_g(c) = \int_{\gamma_{g_0}(c)} a_g$ and applying the Livsic theorem. In particular, the right inverse is given by $H_g.1 := -2g/I_{m_g}(g_0, g)$. The family of right inverses $g \mapsto H_g$ is smooth since $g \mapsto a_g$ and $g \mapsto m_g$ are smooth by [**Con92**, Theorem C], and it is clearly also tame, thus we can apply directly [**Ham82**, Theorem 1.1.3, pp. 172] to deduce that F has a smooth tame right inverse, which shows that $\mathcal{N}$ is a Fréchet submanifold. □

We remark that if $L_g = L_{g_0}$, then a_g is cohomologous to 1, so $\mathbf{P}(-V_g) = \mathbf{P}(-V_{g_0}) = 0$ in that case, which means that $g \in \mathcal{N}^{k,\alpha}$. From the second inequality in Proposition 3.5, we obtain the following corollary.

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COROLLARY 3.7. *Let g_0 be a smooth metric with Anosov geodesic flow, with non-positive curvature if $n > 2$. There exist $C_{g_0} > 0$ and a neighborhood $\mathcal{U} \subset \mathcal{N}^{k,\alpha}$ such that, for all $g \in \mathcal{U}$, there is a diffeomorphism $\psi \in \mathcal{D}_0^{k+1,\alpha}$ so that*

$$C_{g_0}\|\psi^* g - g_0\|^2_{H^{-1/2}(M)} \leq I_{\mu^L_{g_0}}(g_0, g) - 1.$$

As suggested by this estimate, the functional Φ turns out to be strictly convex near g_0 when restricted on $\mathcal{N}^{k,\alpha}_{\mathrm{sol}}$. First, we have, for $h \in T_{g_0}\mathcal{N}^{k,\alpha}$,

$$d\Phi_{g_0}.h = -dF_{g_0}.h = 0,$$

so that $\Phi : \mathcal{N}^{k,\alpha} \to \mathbb{R}$ has a critical point at g_0. For the second derivative at g_0, the same computation as in the previous section easily gives the following result.

LEMMA 3.8. *The map $\Phi : \mathcal{N}^{k,\alpha}_{sol} \to \mathbb{R}$ is strictly convex at g_0, and there is $C > 0$ such that*

$$d^2\Phi_{g_0}(h, h) = \tfrac{1}{4}\langle \Pi_2^{g_0} h, h\rangle \geq C\|h\|^2_{H^{-1/2}(M)}$$

for all $h \in T_{g_0}\mathcal{N}^{k,\alpha}_{sol}$.

Proof. The proof follows exactly that of Proposition 3.2, using $T_{g_0}\mathcal{N}^{k,\alpha} = (\mathbb{R}g_0)^{\perp}$. □

3.3. *The pressure metric on the space of negatively curved metrics.* The results of this paragraph are stated in negative curvature, but it is very likely that one could relax the assumption to the Anosov case. Again, the only obstruction for the moment is that it is still not known whether the X-ray transform I_2 (hence the operator Π_2) is injective on solenoidal tensors in the Anosov case when $\dim(M) \geq 3$.

3.3.1. *Definition of the pressure metric using the variance.* On $\mathcal{M}^-$, the cone of smooth negatively curved metrics, we introduce the non-negative symmetric bilinear form

$$G_g(h_1, h_2) := \langle \Pi_2^g h_1, h_2\rangle_{L^2(M, d\,\mathrm{vol}_g)}, \tag{3.13}$$

defined for $g \in \mathcal{M}$, $h_j \in T_g\mathcal{M} \simeq C^\infty(M; S^2T^*M)$. It is non-degenerate on $T_g\mathcal{M} \cap \ker D_g^*$, namely, $G_g(h, h) \geq C_g\|h\|^2_{H^{-1/2}}$ by Lemma 2.2, and the constant C_g turns out to be locally uniform for g near a given metric g_0. Combining these facts, we obtain the following proposition.

PROPOSITION 3.9. *Let $g_0 \in \mathcal{M}^-$. Then the bilinear form G defined in (3.13) produces a Riemannian metric on the quotient space $\mathcal{M}^-/\mathcal{D}_0$ near the class $[g_0]$, where $\mathcal{M}^-/\mathcal{D}_0$ is identified with the slice $\mathcal{S}$ passing through g_0 as in (2.9).*

Proof. It suffices to show that G is non-degenerate on $T\mathcal{S}$. Let $h \in T_g\mathcal{S}$ and assume that $G_g(h, h) = 0$. We can write $h = \mathcal{L}_V g + h'$ where $D_g^* h' = 0$ and V is a smooth vector field and $\mathcal{L}_V$ the Lie derivative with respect to V. By Lemma 2.1 we obtain $0 = G_g(h, h) \geq C\|h'\|_{H^{-1/2}}$. Thus $h = \mathcal{L}_V g$, but we also know that $T_g\mathcal{S} \cap \{\mathcal{L}_V g \mid V \in C^\infty(M; T^*M)\} = \{0\}$ since $\mathcal{S}$ is a slice. Therefore $h = 0$. □

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3.3.2. *Definition using the intersection number.* We now want to relate the pressure metric previously introduced to some renormalized intersection numbers involving some well-chosen potentials. This will be needed to show that the pressure metric coincides with (a multiple of) the Weil–Petersson metric in the case where M is a surface and one restricts to hyperbolic metrics. This also makes a relation with recent work of [**BCLS15**].

Let us assume that g is in a fixed C^2-neighborhood of g_0. Since $J^u_{g_0} > 0$, we obtain that $V_g = J^u_{g_0} + a_g - 1 > 0$ if g is close enough to g_0. By [**Sam14**, Lemma 2.4], there exists a unique constant $\mathbf{h}_{V_g} \in \mathbb{R}$ such that $\mathbf{P}(-\mathbf{h}_{V_g} V_g) = 0$. In particular, $\mathcal{N}$ coincides in a neighborhood of g_0 with the set $\{g \in \mathcal{M} \mid \mathbf{h}_{V_g} = 1\}$. One can express the constant $\mathbf{h}_{V_g}$ as $\mathbf{h}_{V_g} = \mathbf{h}_{\mathrm{top}}(\varphi_t^{g_0, V_g})$, where $\varphi_t^{g_0, V_g}$ is a time reparametrization of the geodesic flow of g_0 (see [**BCLS15**, §3.1.1]). More precisely, given a Hölder-continuous positive function $f \in C^\nu(S_{g_0}M)$ on $S_{g_0}M$, we introduce the unique real number $\mathbf{h}_f$ such that $\mathbf{P}(-\mathbf{h}_f f) = 0$ and we set

$$S_{g_0}M \times \mathbb{R} \ni (z, t) \mapsto \kappa_f(z, t) := \int_0^t f(\varphi_s^{g_0}(z))\, ds.$$

For a fixed $z \in S_{g_0}M$, this is a homeomorphism on $\mathbb{R}$ and thus allows us to define

$$\varphi^{g_0, f}_{\kappa_f(z,t)}(z) := \varphi_t^{g_0}(z). \tag{3.14}$$

We now follow the approach of [**BCLS15**, §3.4.1]. Given two Hölder-continuous functions $f, f' \in C^\nu(S_{g_0}M)$ such that $f > 0$, one can define an *intersection number* [**BCLS15**, Eq. (13)]

$$\mathbf{I}_{g_0}(f, f') := \frac{\int_{S_{g_0}M} f'\, d\mu_{-\mathbf{h}_f f}}{\int_{S_{g_0}M} f\, d\mu_{-\mathbf{h}_f f}}$$

where $d\mu_{-\mathbf{h}_f f}$ is the equilibrium measure for the potential $-\mathbf{h}_f f$. We have the following result, which follows from [**BCLS15**, Proposition 3.8] stated for Anosov flows on compact metric spaces:

PROPOSITION 3.10. (Bridgeman, Canary, Labourie and Sambarino [**BCLS15**]) *Let $f, f' : S_{g_0}M \to \mathbb{R}_+$ be two Hölder-continuous positive functions. Then*

$$\boldsymbol{J}_{g_0}(f, f') := \frac{\boldsymbol{h}_{f'}}{\boldsymbol{h}_f} \boldsymbol{I}_{g_0}(f, f') \geq 1,$$

with equality if and only if $\boldsymbol{h}_f f$ and $\boldsymbol{h}_{f'} f'$ are cohomologous for the geodesic flow $\varphi_t^{g_0}$ of g_0. The quantity $\boldsymbol{J}_{g_0}(f, f')$ is called the renormalized intersection number.

We apply the previous proposition with $f := J^u_{g_0}$ (then $\mathbf{h}_{J^u_{g_0}} = 1$) and $f' := V_g$. Without assuming that $g \in \mathcal{N}$ (that is, we do not necessarily assume that $\mathbf{h}_{V_g} = 1$), we

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have

$$\mathbf{J}_{g_0}(J^u_{g_0}, V_g) = \mathbf{h}_{V_g}\mathbf{I}_{g_0}(J^u_{g_0}, V_g) = \mathbf{h}_{V_g}\frac{\int_{S_{g_0}M}(J^u_{g_0} + a_g - \mathbf{1})\, d\mu^{\mathrm{L}}_{g_0}}{\int_{S_{g_0}M} J^u_{g_0} d\mu^{\mathrm{L}}_{g_0}}$$
$$= \mathbf{h}_{V_g}\frac{\mathbf{h}_L(g_0) + I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) - 1}{\mathbf{h}_L(g_0)} \geq 1,$$

where $\mathbf{h}_L(g_0)$ is the entropy of the Liouville measure for g_0. In the specific case where $g \in \mathcal{N}$, $\mathbf{h}_{V_g} = 1$ and we find that $I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) \geq 1$ with equality if and only if a_g is cohomologous to 1, that is, if and only if $L_g = L_{g_0}$, or alternatively if and only if φ^g and φ^{g_0} are time-preserving conjugate. This computation holds as long as $J^u_{g_0} + a_g - 1 > 0$ (which is true in a C^2-neighborhood of g_0).

In particular, on $\mathcal{N}$, we have the linear relation

$$\mathbf{J}_{g_0}(J^u_{g_0}, V_g) = 1 + \frac{I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) - 1}{\mathbf{h}_L(g_0)}.$$

In the notations of [BCLS15, Proposition 3.11], the second derivative computed for the family $(g_\lambda)_{\lambda\in(-1,1)} \in \mathcal{N}$ is

$$\partial^2_\lambda \mathbf{J}_{g_0}(J^u_{g_0}, V_{g_\lambda})|_{\lambda=0} = \frac{1}{\mathbf{h}_L(g_0)}\partial^2_\lambda I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g_\lambda)|_{\lambda=0} = \frac{\langle \Pi^{g_0}_2 \dot{g}_0, \dot{g}_0\rangle}{4\,\mathbf{h}_L(g_0)} \tag{3.15}$$

and is called the *pressure form*. When considering a slice transverse to the $\mathcal{D}_0$ action on $\mathcal{N}$, it induces a metric called the *pressure metric* by Lemma 2.1. To summarize, we have the following lemma.

LEMMA 3.11. *Given a smooth metric g_0, the metric G_{g_0} restricted to $\mathcal{N}$ can be obtained from the renormalized intersection number by*

$$G_{g_0}(h, h) = 4\boldsymbol{h}_L(g_0)\partial^2_\lambda \boldsymbol{J}_{g_0}(J^u_{g_0}, V_{g_\lambda})|_{\lambda=0}$$

where $(g_\lambda)_{\lambda\in(-1,1)}$ is any family of metrics such that $g_\lambda \in \mathcal{N}$ and $\dot{g}_0 = h \in T_{g_0}\mathcal{N}$.

3.3.3. *Link with the Weil–Petersson metric.* We now assume that $M = S$ is an orientable surface of genus at least 2. Let $\mathcal{T}(S)$ be the Teichmüller space of S. We show that the pressure metric coincides with (a multiple of) the Weil–Petersson metric in restriction to $\mathcal{T}(S)$. We fix a hyperbolic metric g_0. Given $\eta, \rho \in \mathcal{T}(S)$ and g_η, g_ρ the associated hyperbolic metrics, since $\mathcal{T}(S)$ is connected (indeed, a ball in $\mathbb{C}^{3(\mathrm{genus}(M)-1)}$) there is topological conjugacy between g_η, g_ρ and g_0 and one can defined the time rescaling a_{g_η} and a_{g_ρ} by using a path of hyperbolic metrics relating g_0 to g_η or to g_ρ. The intersection number is defined as

$$\mathbf{I}(\eta, \rho) := \mathbf{I}_{g_0}(a_{g_\eta}, a_{g_\rho}) = \frac{\int_{S_{g_0}M} a_{g_\rho} d\mu_\eta}{\int_{S_{g_0}M} a_{g_\eta} d\mu_\eta},$$

where $[g_\eta] = \eta$, $[g_\rho] = \rho$ and μ_η is the equilibrium state of $-\mathbf{h}_{a_{g_\eta}} a_{g_\eta}$. Note that $\mathbf{h}_{a_{g_\eta}} = \mathbf{h}_{\mathrm{top}}(\varphi^{g_0,a_\eta}_t) = 1$ since φ^{g_0,a_η} is conjugate to the geodesic flow of g_η, which in turn has constant curvature, and by [**Sam14**, Lemma 2.4], $a_{g_\eta} d\mu_\eta / \int_{S_{g_0}M} a_{g_\eta} d\mu_\eta$ is the measure of maximal entropy of the flow $\varphi_t^{g_0,a_\eta}$, thus also the normalized Liouville measure of g_η (viewed on $S_{g_0}M$). This number $\mathbf{I}(\eta,\rho)$ is in fact *independent of* g_0 as it can alternatively be written

$$\mathbf{I}(\eta,\rho) = \lim_{T\to\infty} \frac{1}{N_T(\eta)} \sum_{c\in\mathcal{C}, L_{g_\eta}(c)\le T} \frac{L_{g_\rho}(c)}{L_{g_\eta}(c)},$$

where $N_T = \sharp\{c \in \mathcal{C} \mid L_{g_\eta}(c) \le T\}$ (see [**BCS18**, Proof of Theorem 4.3]). In particular, taking $g_0 = g_\eta$, we have

$$\mathbf{I}(\eta,\rho) = I_{\mu^{\mathrm{L}}_{g_\eta}}(g_\eta, g_\rho).$$

As explained in [**BCS18**, Theorem 4.3], up to a normalization constant c_0 depending on the genus only, the Weil–Petersson metric on $\mathcal{T}(S)$ is equal to

$$\|h\|^2_{\mathrm{WP}} = c_0 \partial^2_\lambda \mathbf{I}(\eta,\eta_\lambda)|_{\lambda=0} = c_0 \partial^2_\lambda I_{\mu^{\mathrm{L}}_{g_\eta}}(g_\eta, g_{\eta_\lambda})|_{\lambda=0}, \tag{3.16}$$

where $\dot\eta_0 = h$ and $(g_{\eta_\lambda})_{\lambda\in(-1,1)}$ is a family of hyperbolic metrics such that $[g_{\eta_\lambda}] = \eta_\lambda$, $\eta = \eta_0 = [g_0]$. This fact follows from combined works of Thurston, Wolpert [**Wol86**] and McMullen [**MM08**]: the length of a random geodesic γ on (S, g_0) with respect to g_{η_λ} has a local minimum at $\lambda = 0$ and the Hessian is positive definite (Thurston), is equal to the Weil–Petersson norm squared of $\dot g$ (Wolpert [**FF93**, **Wol86**]) and is given by a variance (McMullen [**MM08**]); here random means equidistributed with respect to the Liouville measure of g_0. We can check that the metric G also corresponds to this metric.

PROPOSITION 3.12. *The metric G on $\mathcal{T}(S)$ is a multiple of the Weil–Petersson metric.*

Proof. This follows directly from (3.15), (3.16) and the fact that $\mathbf{h}_L(g_\eta) = 1$ if g_η has curvature -1. □

Remark 3.13. We notice that the positivity of the metric in the case of Teichmüller space follows only from some convexity argument in finite dimension. In the case of general metrics with negative curvature, the elliptic estimate of Lemma 2.1 on the variance is much less obvious due to the infinite dimensionality of the space. As it turns out, this is the key for the local rigidity in our results.

4. *Uniform elliptic estimates on* Π_2

In this section we prove that the operator $\Pi_2^g \in \Psi^{-1}(M; S^2T^*M)$ depends continuously on g. Let $\mathcal{M}^{\mathrm{An}}$ be the space of smooth Riemannian metrics with Anosov geodesic flow.

PROPOSITION 4.1. *The map $\mathcal{M}^{An} \ni g \mapsto \Pi_2^g \in \Psi^{-1}(M; S^2T^*M)$ is continuous when $\Psi^{-1}(M; S^2T^*M)$ is equipped with its topology of Fréchet spaces.*

Recall that the Fréchet topology was introduced at the beginning of §2.2. We fix a metric g_0 and we work in a neighborhood $\mathcal{U}$ of g_0 in the C^∞ topology. In particular, we will always assume that this neighborhood $\mathcal{U}$ is small enough that any $g \in \mathcal{U}$ has an Anosov geodesic flow that is orbit-conjugated to that of g_0 by structural stability. We will also see

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the geodesic flows $(\varphi_t^g)_{t\in\mathbb{R}}$ as acting on the unit bundle $SM := S_{g_0}M$ for g_0 by using the natural identification $S_gM \to S_{g_0}M$ obtained by scaling in the fibers. The operator π_2^* associated to g becomes: for $(x, v) \in S_{g_0}M$,

$$(\pi_2^* h)(x, v) = h_x(v, v)|v|_g^{-2}.$$

4.1. *The resolvents of X_g and anisotropic spaces.* We first recall the construction of resolvents of X_g from Faure and Sjöstrand [FS11] (see also [DZ16]) and, in particular, the version used in Dang *et al* [DGRS20] that deals with the continuity with respect to the flow X_g. Let $E^*_{u/s}(g) \subset T^*(SM)$ be the annihilators of $E_{u/s}(g) \oplus E_0(g)$, that is,

$$E_u^*(g)(E_u(g) \oplus E_0(g)) = 0, \quad E_s^*(g)(E_s(g) \oplus E_0(g)) = 0.$$

There are two resolvents bounded on L^2 for X_g defined for $\mathrm{Re}(\lambda) > 0$ by

$$R_g^{\pm}(\lambda) := \pm \int_0^\infty e^{-\lambda t} e^{\pm t X_g} f \, dt$$

for $f \in L^2(SM, d\mu_g^{\mathrm{L}})$. They solve $(-X_g \pm \lambda) R_g^{\pm}(\lambda) = \mathrm{Id}$ on L^2. The following results are proved in [FS11], and we use here the presentation of [DGRS20, Sections 3.2 and 3.3] due to the need for uniformity with respect to g: there is $c_0 > 0$ depending only on g, locally uniform with respect to g (c_0 depends only on the Anosov exponents of contraction/dilation of $d\varphi_1^g$), such that for each $N_0 > 0$, $N_1 > 16N_0$, $R_g^{\pm}(\lambda)$ admits a meromorphic extension in $\mathrm{Re}(\lambda) > -c_0 N_0$ as a bounded operator

$$R_g^-(\lambda) : \mathcal{H}^{m_g^{N_0,N_1}} \to \mathcal{H}^{m_g^{N_0,N_1}}, \quad R_g^+(\lambda) : \mathcal{H}^{-m_g^{N_0,N_1}} \to \mathcal{H}^{-m_g^{N_0,N_1}} \tag{4.1}$$

where $\mathcal{H}^{\pm m_g^{N_0,N_1}}$ are Hilbert spaces depending on $N_0 > 0$, $N_1 > 0$ satisfying the properties

$$H^{2N_1}(SM) \subset \mathcal{H}^{m_g^{N_0,N_1}} \subset H^{-2N_0}(SM), \quad H^{2N_0}(SM) \subset \mathcal{H}^{-m_g^{N_0,N_1}} \subset H^{-2N_1}(SM)$$

and defined by

$$\mathcal{H}^{\pm m_g^{N_0,N_1}} = (A_{m_g^{N_0,N_1}})^{\mp 1} L^2(SM), \quad A_{m_g^{N_0,N_1}} := \mathrm{Op}(e^{m_g^{N_0,N_1} \log f}),$$

and $A_{m_g^{N_0,N_1}}$ is an invertible pseudo-differential operator with inverse having principal symbol $e^{-m_g^{N_0,N_1} \log f}$. Here Op denotes a quantization (with a fixed small semi-classical parameter to ensure that $\mathrm{Op}(e^{m_g^{N_0,N_1} \log f})$ is invertible), while $m_g^{N_0,N_1} \in S^0(T^*(SM))$, $f \in S^1(T^*(SM), [1, \infty))$ (the usual classes of symbols) are homogeneous of respective degree 0 and 1 in $|\xi| > R$, for some $R > 1$ independent of g, and constructed from the lifted flow $\Phi_t^g = ((d\varphi_t^g)^{-1})^T$ acting on $T^*(SM)$. The function f can be taken depending only on g_0 for g in a small enough C^∞ neighborhood $\mathcal{U}$ of g_0. Moreover, there are small conic neighborhoods $C_u(g_0)$ and $C_s(g_0)$ of $E_u^*(g_0)$ and $E_s^*(g_0)$ such that, for any smaller open conic neighborhood $C'_u(g_0) \subset C_u(g_0)$ of $E_u^*(g_0)$ and $C'_s(g_0) \subset C_s(g_0)$ of $E_s^*(g_0)$,

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$m_g^{N_1,N_1}$ satisfies

$$\begin{cases} m_g^{N_0,N_1}(z,\xi) \geq N_1, & (z,\xi) \in C_s'(g_0), \\ m_g^{N_0,N_1}(z,\xi) \geq N_1/8, & (z,\xi) \notin C_u(g_0), \\ m_g^{N_0,N_1}(z,\xi) \leq -N_0, & (z,\xi) \in C_u'(g_0), \end{cases} \tag{4.2}$$

and $m_g(x,\xi) \in [-2N_0, 2N_1]$ for all $(z,\xi) \in T^*(SM)$. We note that [**DGRS20**, Lemma 3.3] shows that $m_g^{N_0,N_1}$ is smooth with respect to the metric g and that f can be taken to be independent of g for g close enough to g_0. The spaces $\mathcal{H}^{m_g^{N_0,N_1}}$ are called *anisotropic Sobolev spaces*. The pseudodifferential operators $A_{m_g^{N_0,N_1}}$ belong to the class $\Psi^{2N_1}(SM)$ but also to some anisotropic subclass denoted $\Psi^{m_g^{N_0,N_1}}(SM)$ admitting composition formulas; we refer to [**FRS08**, **FS11**] for details.

Eventually, [**DGRS20**, Proposition 6.1] shows that there is a small open neighborhood W_δ of the circle $\{\lambda \in \mathbb{C} \mid |\lambda| = \delta\}$ for some small $\delta > 0$ so that

$$\mathcal{U} \times W_\delta \ni (g,\lambda) \mapsto A_{m_g^{N_0,N_1}} R_g^-(\lambda) (A_{m_g^{N_0,N_1}})^{-1} \in \mathcal{L}(H^1(SM), L^2(SM)) \tag{4.3}$$

is continuous. (In [**DGRS20**, Proposition 6.1], a small semi-classical parameter $h > 0$ appears: we can just fix this parameter small enough. It does not play any role here except in the quantization procedure Op. We also add that in [**DGRS20**, Proposition 6.1], N_1 is chosen to be equal to $20N_0$ for notational convenience, but the proof does not use that fact.)

4.2. *The operator Π_2^g in terms of resolvents.* Following [**Gui17**], the link between Π^g and the resolvent is given by the Laurent expansion

$$\Pi^g = R_g^+(0) - R_g^-(0),$$

where $R_g^+(\lambda)$ has a pole of order 1, $R_g^\pm(0)$ is defined by

$$R_g^\pm(\lambda) = \pm\lambda^{-1}\langle \cdot, \mathbf{1}\rangle + R_g^\pm(0) + \mathcal{O}(\lambda),$$

and $R_g^-(0) = -(R_g^+(0))^*$ where the adjoint is with respect to the Liouville measure.

LEMMA 4.2. *Let $\chi \in C_c^\infty(\mathbb{R})$ be even and equal to* 1 *in $[-T,T]$ and supported in the interval $(-T-1, T+1)$. Then*

$$\begin{aligned} \Pi^g = \int_{\mathbb{R}} \chi(t) e^{tX_g}\, dt - R_g^+(0) \int_0^{+\infty} \chi'(t) e^{tX_g}\, dt \\ + R_g^-(0) \int_0^{+\infty} \chi'(t) e^{-tX_g}\, dt - \langle \cdot, \mathbf{1}\rangle \int_{\mathbb{R}} \chi. \end{aligned} \tag{4.4}$$

Proof. For $\mathrm{Re}(\lambda) > 0$, we can write, by integration by parts,

$$\begin{aligned} R_g^\pm(\lambda) &= \pm \int_0^\infty \chi(t) e^{-t(\lambda \mp X_g)} dt \pm \int_0^\infty (1-\chi(t)) e^{-t(\lambda\mp X_g)} dt \\ &= \pm \int_0^\infty \chi(t) e^{-t(\lambda\mp X_g)} dt - R_g^\pm(\lambda) \int_0^\infty \chi'(t) e^{t(\pm X_g - \lambda)} dt. \end{aligned}$$

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Then taking the limit as $\lambda \to 0$, we obtain

$$R_g^{\pm}(0) = \pm \int_0^{\infty} \chi(t)e^{\pm tX_g}dt - R_g^{\pm}(0)\int_0^{\infty} \chi'(t)e^{\pm tX_g}dt \mp \int_0^{\infty} \chi(t)dt\langle \cdot, \mathbf{1}\rangle,$$

and summing gives the result. □

Next, we remark that, using that $\varphi_t^g(x, -v) = -\varphi_{-t}^g(x, v)$ (where multiplication by -1 is the symmetry in the fibers of SM), it is straightforward to check that, for all $t \in \mathbb{R}$,

$$\pi_{2*}e^{tX_g}\pi_2^* = \pi_{2*}e^{-tX_g}\pi_2^*,$$

which also implies that $\pi_{2*}R_g^+(0)e^{tX_g}\pi_2^* = -\pi_{2*}R_g^-(0)e^{-tX_g}\pi_2^*$ and thus

$$\Pi_2^g = 2\pi_{2*}\int_0^{\infty} \chi(t)e^{-tX_g}\,dt\pi_2^* \\ + 2\pi_{2*}R_g^-(0)\int_0^{+\infty} \chi'(t)e^{-tX_g}\,dt\pi_2^* + \left(1 - \int_{\mathbb{R}} \chi\right)\langle \cdot, \mathbf{1}\rangle. \tag{4.5}$$

We will prove that these three terms depend continuously on g. Note that

$$\left(1 - \int_{\mathbb{R}} \chi\right)\langle f, \mathbf{1}\rangle = \left(1 - \int_{\mathbb{R}} \chi\right)\int_{SM} f(z)\,d\mu_g^{\mathrm{L}}(z)$$

and thus the g-continuity of this term is immediate. Now, we claim the following result.

LEMMA 4.3. *There exist $T > 0$ large enough and a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of g_0 in $\mathcal{M}^{An}$ so that for all $x \in M$ and all $g \in \mathcal{U}'$ the exponential map of g in the universal cover $\widetilde{M}$,*

$$\exp_x^{\tilde{g}} : \{v \in T_x\widetilde{M};\, |v|_g \le T\} \to \widetilde{M},$$

is a diffeomorphism onto its image and $\Phi_t^g(V^* \cap \ker \iota_{X_g}) \subset C_u'(g_0)$ for all $t \ge T$, if $\Phi_t^g := ((d\varphi_t^g)^{-1})^T$ is the symplectic lift of φ_t^g, $V^* \subset T^*(SM)$ is the annihilator of the vertical bundle $V = \ker d\pi_0 \subset T(SM)$ and $\iota_X : T^*(SM) \to \mathbb{R}$ is the contraction $\iota_{X_g}(\xi) = \xi(X_g)$.

We also mention here as it is used in the following proof that, as a consequence of hyperbolicity,

$$V^* \cap E_s^* = V^* \cap E_u^* = \{0\}.$$

This can be found in [**Pat99**, Theorem 2.50], for instance (formulated for the tangent bundle $T(SM)$ but the adaptation to $T^*(SM)$ is straightforward). The T in Lemma 4.2 will be chosen accordingly so that Lemma 4.3 is satisfied.

Proof. By [**DGRS20**, Lemma 3.1], the cone $C_u'(g_0)$ can be chosen so that there exist $T > 0$ and $\mathcal{U}'$ such that, for all $t \ge T$ and all $g \in \mathcal{U}'$, $\Phi_t^g(C_u'(g_0)) \subset C_u'(g_0)$. We also know that $\Phi_{T_0}^{g_0}(V^* \cap \ker \iota_{X_g}) \subset C_u'(g_0)$ for some $T_0 > T$ by hyperbolicity of g_0 (that is, the stable bundle E_s^* only intersects trivially the vertical bundle $V^* \cap \ker \iota_{X_g}$), but by continuity of $g \mapsto \Phi_{T_0}^g$, the same holds for all g in some possibly smaller neighborhood $\mathcal{U}'' \subset \mathcal{U}'$, thus for all $t \ge T_0$ and all $g \in \mathcal{U}''$, $\Phi_t^g(V^* \cap \ker \iota_{X_g}) \subset C_u'(g_0)$. Now, we claim that, up to

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choosing $\mathcal{U}''$ even smaller, the exponential map is a diffeomorphism on $\{|v|_g \leq T\}$ in the universal cover: indeed, Anosov geodesic flows have no pair of conjugate points. $\square$

4.3. *Proof or Proposition 4.1.* Let us define

$$\Omega_1^g := \pi_{2*} \int_0^\infty \chi(t) e^{-tX_g}\, dt \pi_2^*, \quad \Omega_2^g := \pi_{2*} R_g^-(0) \int_0^{+\infty} \chi'(t) e^{-tX_g}\, dt \pi_2^*.$$

Proposition 4.1 is a consequence of the following two lemmas.

LEMMA 4.4. *For each $g \in \mathcal{U}'$, $\Omega_1^g \in \Psi^{-1}(M; S^2T^*M)$ with principal symbol*

$$\sigma(\Omega_1^g)(x, \xi) = c_n |\xi|^{-1} \pi_{\ker i_\xi} A_2^2 \pi_{\ker i_\xi}$$

for some $c_n > 0$ depending only on $n = \dim M$ and A_2 some positive definite endomorphism defined in Lemma 2.1, and the map $g \mapsto \Omega_1^g$ is continuous with respect to the smooth topology on $\mathcal{U}'$ and the usual Fréchet topology on $\Psi^{-1}(M; S^2T^*M)$.

Proof. The fact that, for each $g \in \mathcal{M}^{\mathrm{An}}$, the operator $\Omega_1^g \in \Psi^{-1}(M; S^2T^*M)$ is proved in [**Gui17**, Theorem 3.5]. The computation of the principal symbol follows from the computation [**SSU05**, **SU04**] and is done in detail in our setting in [**GL**, Theorem 4.4]. We need to check the continuity with respect to g in the $\Psi^{-1}(M; S^2T^*M)$ topology and we can proceed as in [**SU04**, Propositions 1 and 2]. For $h \in C^\infty(M; S^2T^*M)$, we can write explicitly in $(x_i)_i$ coordinates in the universal cover $\widetilde{M}$ near a point $p \in \widetilde{M}$,

$$(\Omega_1^g h(x))_{ij} = \int_{S_x\widetilde{M}} \int_0^\infty \chi(t) \widetilde{h}_{\exp_x^{\widetilde{g}}(tv)}(\partial_t \exp_x^{\widetilde{g}}(tv), \partial_t \exp_x^{\widetilde{g}}(tv)) p_{ij}(x, v)\, dt dS_x(v),$$

where $p_{ij}(x, v)$ are homogeneous polynomials of order 2 in the v variable, $\widetilde{h} \in C^\infty(\widetilde{M}; S^2T^*M)$ is the lift of h to the universal cover $\widetilde{M}$, and dS_x is the natural measure on the sphere $S_x\widetilde{M}$. Using Lemma 4.3, we can perform the change of coordinates $(t, v) \in (0, T) \times S_x\widetilde{M} \mapsto y := \exp_x^{\widetilde{g}}(tv) \in \widetilde{M}$, and we get the distance $t = d_{\widetilde{g}}(x, y)$ in $\widetilde{M}$, and

$$dt dv = \frac{J_x^g(y)}{(d_{\widetilde{g}}(x, y))^{n-1}} d\mathrm{vol}_g(y), \quad v = -\nabla_y^{\widetilde{g}} d_{\widetilde{g}}(x, y), \quad \partial_t \exp_x^{\widetilde{g}}(tv) = (\nabla_x^{\widetilde{g}} d_{\widetilde{g}}(x, y)),$$

for some $J_x^g(y)$ smooth in x, y, g. We claim that this implies that

$$\Omega_1^g h(x) = \int_M K_g(x, y) h(y)\, d\mathrm{vol}_g(y)$$

for some $K_g(x, y)$ which is smooth in (g, x, y) outside the diagonal $x = y$ and, near the diagonal, has the form (for some $L < \infty$)

$$K_g(x, y) = \sum_{\ell=1}^{L} c_\ell(g, x, y) \omega_{\ell, g, x}(x - y)$$

with c_ℓ a matrix valued function, smooth in all its variables and $\omega_{\ell,g,x}(v)$ a vector-valued function smooth in g, x, homogeneous of degree $-(n-1)$ in $v \in \mathbb{R}^n$. Indeed, one can work in the universal cover $\widetilde{M}$ where x_i are globally defined coordinates, so that, writing

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$h(x) = \sum_{i,j} h_{ij}(x)dx_i dx_j$ and $p = \sum_{ij} p_{ij}(x)dx_i dx_j$, we get that $K_g(x, y)$ is a matrix with coefficients

$$(K_g(x, y))_{iji'j'} = \chi(d_{\tilde{g}}(x, y))p_{ij}(x)F_i^g(x, y)F_j^g(x, y)G_{i'}^g(x, y)G_{j'}^g(x, y)\frac{J_x^g(y)}{d_{\tilde{g}}(x, y)^{n-1}}$$

where $F_i^g(x, y) = -dx_i(\nabla_y^{\tilde{g}} d_{\tilde{g}}(x, y))$ and $G_i^g(x, y) = dx_i(\nabla_x^{\tilde{g}} d_{\tilde{g}}(x, y))$. Now we can use the standard fact (see, for example, [SU04, Lemma 1]) that

$$d_{\tilde{g}}^2(x, y) = \sum_{ij} H_{ij}^1(g, x, y)(x - y)_i(x - y)_j,$$

$$dx_i(\nabla_x^g d_{\tilde{g}}(x, y)) = \frac{\sum_{ij} H_{ij}^2(g, x, y)(x - y)_j}{d_{\tilde{g}}(x, y)}$$

(and likewise for $dx_i(\nabla_y^g d_{\tilde{g}}(x, y))$ by symmetry) where $H_{ij}^k(g, x, y)$ are smooth in all variables and positive definite for $x = y$. The kernel K_g is thus smooth outside the diagonal (as a function of g, x, y), and can be written near the diagonal as a sum of terms of the form $c(g, x, y)\omega_{g,x}(x - y)$ where c is smooth in all its variables and $\omega_{g,x}(v)$ is a homogeneous distribution of degree $-(n - 1)$ in the variable v, smooth in g, x. The off-diagonal term for the Fréchet topology is then clearly smooth in g, while the near-diagonal term has full local symbols that are Fourier transforms of $c(g, x, x - v)\omega_{g,x}(v)$:

$$\sigma(g; x, \xi) = \int_{\mathbb{R}} e^{iv\xi} c(g, x, x - v)\omega_{g,x}(v)\, dv.$$

It is then a standard and easy exercise to check that this provides uniform bounds on semi-norms of the symbol. (Alternatively, the semi-norms on the full symbol are equivalent to semi-norms in the space of distributions on $M \times M$ that are conormal to the diagonal, defined through differentiations of $K_g(x, y)$ with respect to smooth fields tangent to $\operatorname{diag}(M \times M)$; see [Mel, Ch. 5, Proposition 6.1.1 and its proof]. Such norms for K_g are clearly uniformly bounded in terms of g.) We deduce the continuity (and indeed, smoothness) of Ω_1^g as an element of $\Psi^{-1}(M; S^2T^*M)$ with respect to the metric g. □

LEMMA 4.5. *The operator Ω_2^g has a smooth Schwartz kernel for each $g \in \mathcal{U}'$, and the map*

$$g \in \mathcal{U}' \mapsto \Omega_2^g \in C^\infty(M \times M; S^2T^*M \otimes (S^2T^*M)^*)$$

is continuous if we identify Ω_g^2 with its Schwartz kernel.

Proof. First we observe that if $B \in \Psi^0(SM)$ is chosen, independently of g, so that $B^* = B$ and B is microsupported in a small conic neighborhood of V^* not intersecting $\mathcal{C}_u(g_0)$ and equal microlocally to the identity in a slightly smaller conic neighborhood of V^*, then

$$\pi_2^* = B\pi_2^* + S_g, \quad \pi_{2*} = \pi_{2*}B + S_g^*,$$

with S_g a continuous family of smoothing operators. This decomposition is a consequence of the fact that π_2^* maps $C^{-\infty}(M; S^2T^*M)$ to the space $C_{V^*}^{-\infty}(SM)$ of distributions with wavefront set contained in V^* (π_2^* being essentially a pullback, this follows, for instance, from [Hör03, Theorem 8.2.4]). We will show that the operator

$$\Omega_3^g := \pi_{2*} B R_g^-(0) \int_T^{T+1} \chi'(t) e^{-tX_g} B \pi_2^* \, dt$$

is a continuous family (with respect to g) of smoothing operators. We need to show that for each $N > 0$, $\Omega_3^g : H^{-N}(SM) \to H^N(SM)$ is a continuous family with respect to g of bounded operators. To study $R_g^-(0)$, it suffices to write it in the form

$$R_g^-(0) = \frac{1}{2\pi i} \int_{|\lambda|=\delta} \frac{R_g^-(\lambda)}{\lambda} d\lambda \tag{4.6}$$

with δ small enough so that the only pole of $R_g^-(\lambda)$ in $|\lambda| \leq \delta$ is $\lambda = 0$ (this is possible for g close enough to g_0 by continuity of $g \mapsto R_g^-(\lambda)$ as proved in [DGRS20]; note that the spectrum (the Pollicott–Ruelle resonances) depends continuously on the metric, as was shown by [Bon20]), so that this amounts to analyzing $R_g^-(\lambda)$ on $\{|\lambda| = \delta\}$. We decompose $B = B^1 + B^2$ with $B^i \in \Psi^0(SM)$, where $\mathrm{WF}(B^1)$ is contained in a conic neighborhood of $\ker \iota_{X_{g_0}}$ not intersecting the annihilator $E_0(g_0)^*$ of $E_u(g_0) \oplus E_s(g_0)$ (the neutral direction) and $\mathrm{WF}(B^2) \cap \ker \iota_{X_{g_0}} = \emptyset$ (B^2 is microsupported in the elliptic region). For $i = 1, 2$ we let $B_T^i \in \Psi^0(SM)$ be microsupported in a conic neighborhood of $\bigcup_{t\in[T,T+1]} \Phi_t^g(\mathrm{WF}(B^i))$, so that by Egorov (or simply the formula of composition of $\Psi^0(SM)$ with diffeomorphisms of SM),

$$\text{for all } t \in [T, T+1], \quad e^{-tX_g} B^i = B_T^i e^{-tX_g} B^i + S'_{g,i}(t),$$

for some continuous family $(g, t) \mapsto S'_{g,i}(t)$ of smoothing operators (for g close enough to g_0). We note that by taking $\mathcal{U}'$ small enough and $\mathrm{WF}(B^1)$ close enough to $V^* \cap \ker \iota_{X_{g_0}}$, Lemma 4.3 ensures that we can choose B_T^1 depending only on T (thus uniform in $g \in \mathcal{U}'$) so that $\mathrm{WF}(B_T^1) \subset C'_u(g_0)$. Thus

$$\int_T^{T+1} \chi'(t) e^{-tX_g} B^1 \, dt = B_T^1 \int_T^{T+1} \chi'(t) e^{-tX_g} B^1 \, dt + S''_{g,1}$$

for some continuous family $g \mapsto S''_{g,1}$ of smoothing operators. Next we use (4.1) with the choice $N_0 = N + 1$ and $N_1/16 = N + 2$. Since, by (4.2),

$$m_g(z, \xi) \leq -N - 1 \quad \text{for all } (z, \xi) \in \mathrm{WF}(B_T^1),$$

we obtain, using the composition properties in [FRS08, Theorem 8] that $A_{m_g^{N_0,N_1}} B_T^1 \in \Psi^{-N-1}(SM)$ is uniformly bounded with respect to g and continuous as a map $g \in \mathcal{U}' \mapsto A_{m_g^{N_0,N_1}} B_T^1 \in \mathcal{L}(H^{-N}(SM), H^1(SM))$. In particular,

$$\mathcal{U}' \ni g \mapsto A_{m_g^{N_0,N_1}} \int_T^{T+1} \chi'(t) e^{-tX_g} B^1 \, dt \in \mathcal{L}(H^{-N}(SM), H^1(SM)) \tag{4.7}$$

is continuous. Next, we deal with the 'elliptic region' term, that is, the term B^2. The idea is to show it is smoothing, since it is a Schwartz function of X_g microlocalized in the elliptic region of X_g. First, $\mathrm{WF}(B_T^2)$ does not intersect $\ker \iota_{X_g}$ for $g \in \mathcal{U}'$ after possibly reducing

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$\mathcal{U}'$ since it does not intersect $\ker \iota_{X_{g_0}}$. Moreover, we have

$$X_g^{2N}\int_T^{T+1}\chi'(t)e^{-tX_g}B^2dt=\int_T^{T+1}\chi^{(1+2N)}(t)e^{-tX_g}B^2dt,$$

and since $\mathrm{WF}(B_T^2)$ does not intersect $\ker \iota_{X_g}$ for $g\in\mathcal{U}'$, there exist by microlocal ellipticity [**DZ19**, Proposition E.32] a family $Q_g\in\Psi^{-2N}(SM)$ and $Z_g\in\Psi^{-\infty}(SM)$, both continuous with respect to g, so that

$$Q_gX_g^{2N}=B_T^2+Z_g.$$

We write

$$\begin{aligned}B_T^2\int_T^{T+1}\chi'(t)e^{-tX_g}B^2\,dt&=Q_gX_g^{2N}\int_T^{T+1}\chi'(t)e^{-tX_g}B^2\,dt+Z_g'\\&=Q_g\int_T^{T+1}\chi^{(1+2N)}(t)e^{-tX_g}B^2dt+Z_g',\end{aligned}$$

where $Z_g'\in\mathcal{L}(H^{-N}(SM),H^N(SM))$ continuously in g. Since $\int_T^{T+1}\chi'(t)e^{-tX_g}B^2\,dt$ is continuous in g as a bounded map $\mathcal{L}(H^{-N}(SM))$ and Q_g is continuous in g as a bounded map $\mathcal{L}(H^{-N}(SM),H^N(SM))$, we get

$$B_T^2\int_T^{T+1}\chi'(t)e^{-tX_g}B^2\,dt\in\mathcal{L}(H^{-N}(SM),H^N(SM))$$

continuously in $g\in\mathcal{U}'$. Combine these facts with (4.7), (4.3) and (4.6), we deduce that

$$\mathcal{U}'\ni g\mapsto A_{m_g^{N_0,N_1}}R_g^-(0)(A_{m_g^{N_0,N_1}})^{-1}A_{m_g^{N_0,N_1}}\int_T^{T+1}\chi'(t)e^{-tX_g}B\,dt$$

is continuous as a map with values in $\mathcal{L}(H^{-N}(SM),L^2(SM))$. Finally, using that $\mathrm{WF}(B)\cap C_u(g_0)=\emptyset$ and $-m_g^{N_0,N_1}\le -2N-4$ outside $C_u(g_0)$ by (4.2), we have that $B(A_{m_g^{N_0,N_1}})^{-1}\in\Psi^{-2N-4}(SM)$ uniformly in g (again using [**FRS08**, Theorem 8] and [**DGRS20**, Lemma 3.2]) and the following map is continuous:

$$\mathcal{U}'\ni g\mapsto B(A_{m_g^{N_0,N_1}})^{-1}\in\mathcal{L}(L^2(SM),H^N(SM)).$$

This shows that $\mathcal{U}'\ni g\mapsto\Omega_3^g\in\mathcal{L}(H^{-N}(M;S^2T^*M),H^N(M;S^2T^*M))$ is continuous. The terms involving the smoothing remainders S_g appearing in the difference between Ω_2^g and Ω_3^g can be dealt with using the same argument, and indeed are even simpler to consider. The proof is then complete. □

The proof of Proposition 4.1 is simply the combination of Lemmas 4.4 and 4.5.

As a corollary we prove Theorem 1.3.

4.4. *Proof of Theorem 1.3.* Let $g_0\in\mathcal{M}^{\mathrm{An}}$ and assume g_0 has non-positive curvature if $n\ge 3$. Using Lemma 2.4, for $g_1,g_2\in\mathcal{M}$ close enough to g_0 in $C^{k+3,\alpha}$ norm, we can find $\psi\in\mathcal{D}_0^{k+1,\alpha}$ (with $k\ge 5$ to be chosen later), depending in a C^2 fashion on (g_1,g_2) such that $D^*_{g_1}(\psi^* g_2) = 0$. Moreover, $g'_2 = \psi^* g_2$ satisfies

$$\|g'_2 - g_1\|_{C^{k,\alpha}} \leq C(\|g_1 - g_0\|_{C^{k,\alpha}} + \|g_2 - g_0\|_{C^{k,\alpha}})$$

for some C depending only on g_0. We can then rewrite the proof of Theorem 1.2, replacing g_0 by g_1. Let $\Psi_{g_1}(g_2) = \mathbf{P}(-J^u_{g_1} - a_{g_1,g_2} + \int_{S_{g_0}M} a_{g_1,g_2}\, d\mu^L_{g_1})$ be the map (3.2) with (g_1, g_2) replacing (g_0, g), and $\Phi_{g_1}(g_2) = I_{\mu^{\mathrm{L}}_{g_1}}(g_1, g_2)$, where a_{g_1,g_2} is the time reparametrization coefficient (2.14) in the conjugacy between the flows φ^{g_1} and φ^{g_2}, and the pressure and the stretch are taken with respect to the flow φ^{g_1}. Combining [**Con92**, Theorem C] and Proposition C.1, the maps $(g_1, g_2) \mapsto \Psi_{g_2}(g_1)$ and $(g_1, g_2) \mapsto \Phi_{g_1}(g_2)$ are C^3 in g_2 if k is chosen large enough, and each g_2-derivative of order $\ell \leq 3$ is continuous with respect to $(g_1, g_2) \in C^{k+3} \times C^{k+3}$ (again k is fixed large enough). Following the proof of Proposition 3.5, this gives that for g_1, g_2 smooth but close enough to g_0 in $\mathcal{M}^{k+3,\alpha}$

$$C_n(\Phi_{g_1}(g_2) - 1)^2 + \Psi_{g_1}(g_2) \geq \tfrac{1}{8}\langle \Pi^{g_1}_2(g'_2 - g_1), (g'_2 - g_1)\rangle - C'_{g_1}\|g_2 - g_1\|^3_{C^{k_0,\alpha}},$$

where C_n depends only on $n = \dim M$, and C'_{g_1} depends on $\|g_1\|_{C^{k_0,\alpha}}$ for some fixed k_0.

Combining Proposition 4.1 and Lemma 2.2, we deduce that there exist $C_{g_0}, C'_{g_0} > 0$ depending only on g_0 so that for $g_1, g_2 \in \mathcal{M}$ in a small enough neighborhood of g_0 in the $C^{k+3,\alpha}$ topology (for $k \geq k_0$),

$$C_n(\Phi_{g_1}(g_2) - 1)^2 + \Psi_{g_1}(g_2) \geq C_{g_0}\|g'_2 - g_1\|_{H^{-1/2}(M)} - C'_{g_0}\|g_2 - g_1\|^3_{C^{5,\alpha}}.$$

This means that there exists $\varepsilon > 0$ depending on g_0 and k large enough so that, for all $g_1, g_2 \in \mathcal{M}$ smooth satisfying $\|g_j - g_0\|_{C^{k+3,\alpha}(M)} \leq \varepsilon$, the estimates of Proposition 3.5 with (g_1, g_2) replacing (g_0, g) hold uniformly with respect to (g_1, g_2). This proves the theorem.

5. *Distances from the marked length spectrum*

In this section we discuss different notions of distances involving the marked length spectrum on the space of isometry classes of negatively curved metrics. Again, if the X-ray transform I_2 were known to be injective, it is likely that one could only assume the Anosov property for the metrics in this paragraph.

5.1. *Length distance.*

We define the following map.

Definition 5.1 Let k be as in Theorem 1.3. We define the marked length distance map $d_L : \mathcal{M}^{k,\alpha} \times \mathcal{M}^{k,\alpha} \to \mathbb{R}^+$ by

$$d_L(g_1, g_2) := \limsup_{j\to\infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|.$$

This is indeed well defined. If g_1, g_2 are two such metrics, then there exists a constant $C = C(g_1, g_2) \geq 1$ such that for all $(x, v) \in TM$, $(1/C) \times |v|_{g_1(x)} \leq |v|_{g_2(x)} \leq C \times |v|_{g_1(x)}$. As a consequence, using that a geodesic is a minimizer of the length among

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a free homotopy class, we obtain

$$\frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} = \frac{\ell_{g_1}(\gamma_{g_1}(c_j))}{\ell_{g_2}(\gamma_{g_2}(c_j))} \leq \frac{\ell_{g_1}(\gamma_{g_2}(c_j))}{\ell_{g_2}(\gamma_{g_2}(c_j))} \leq C^{1/2}\frac{\ell_{g_2}(\gamma_{g_2}(c_j))}{\ell_{g_2}(\gamma_{g_2}(c_j))} = C^{1/2},$$

and the lower bound follows from a similar computation. We obtain the following corollary of Theorem 1.3.

COROLLARY 5.2. *The map d_L descends to the set of isometry classes near g_0 and defines a distance in a small $C^{k,\alpha}$-neighborhood of the isometry class of g_0.*

Proof. It is clear that d_L is invariant by action of diffeomorphisms homotopic to the identity since $L_g = L_{\psi^* g}$ for such diffeomorphisms ψ. Now let g_1, g_2, g_3 three metrics. We have

$$\begin{aligned}\limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_2}(c_j)}\right| &= \limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_3}(c_j)}\frac{L_{g_3}(c_j)}{L_{g_2}(c_j)}\right| \\ &\leq \limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_3}(c_j)}\right| + \limsup_{j\to\infty}\left|\log\frac{L_{g_3}(c_j)}{L_{g_2}(c_j)}\right|,\end{aligned}$$

thus d_L satisfies the triangle inequality. Finally, by Theorem 1.3, if $d_L(g_1, g_2) = 0$ with g_1, g_2 in the $C^{k,\alpha}$ neighborhood $\mathcal{U}_{g_0}$ of Theorem 1.3, we have g_1 isometric to g_2, showing that d_L produces a distance on the quotient of $\mathcal{U}_{g_0}$ by diffeomorphisms. □

We also note that Theorem 1.3 states that there is $C_{g_0} > 0$ such that for each $g_1, g_2 \in C^{k,\alpha}(M; S^2T^*M)$ close to g_0 there is a diffeomorphism such that

$$d_L(g_1, g_2)^{1/2} \geq C_{g_0}\|\psi^* g_1 - g_2\|_{H^{-1/2}}.$$

5.2. *Thurston's distance.* We also introduce the Thurston distance on metrics with topological entropy 1, generalizing the distance introduced by Thurston in [**Thu98**] for surfaces on Teichmüller space (all hyperbolic metrics on surface have topological entropy equal to 1). We denote by $\mathcal{E}$ (respectively, $\mathcal{E}^{k,\alpha}$) the space of negatively curved metrics in $\mathcal{M}$ (respectively, in $\mathcal{M}^{k,\alpha}$) with topological entropy $\mathbf{h}_{\text{top}}(g) = 1$. (Let us also recall here for the sake of clarity that $\mathbf{h}_{\text{top}}(\lambda^2 g) = \mathbf{h}_{\text{top}}(g)/\lambda$, for $\lambda > 0$.) With the same arguments as in Lemma 3.6, this is a codimension-one submanifold of $\mathcal{M}$ and if $g_0 \in \mathcal{E}^{k,\alpha}$, we have

$$T_{g_0}\mathcal{E}^{k,\alpha} := \left\{h \in C^{k,\alpha}(M; S^2T^*M) \mid \int_{S_{g_0}M} \pi_2^* h \, d\mu_{g_0}^{\text{BM}} = 0\right\}. \tag{5.1}$$

Definition 5.3 We define the Thurston non-symmetric distance map $d_T : \mathcal{E}^{k,\alpha} \times \mathcal{E}^{k,\alpha} \to \mathbb{R}^+$ by

$$d_T(g_1, g_2) := \limsup_{j\to\infty} \log\frac{L_{g_2}(c_j)}{L_{g_1}(c_j)}.$$

Note that the finiteness of the previous quantity also follows from the same argument as the one justifying the finiteness of Definition 5.1. Its non-negativity will be a consequence of Lemma 5.5, where it is proved that this can be expressed in terms of the geodesic stretch. We will prove the following proposition.

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PROPOSITION 5.4. *The map d_T descends to the set of isometry classes of metrics in $\mathcal{E}^{k,\alpha}$ (for $k \in \mathbb{N}$ large enough, $\alpha \in (0, 1)$) with topological entropy equal to 1 and defines a non-symmetric distance in a small $C^{k,\alpha}$-neighborhood of the diagonal.*

Moreover, this distance is non-symmetric in the pair (g_1, g_2) which is also the case of the original distance introduced by Thurston [**Thu98**], but this is just an artificial limitation (Thurston [**Thu98**]): 'It would be easy to replace L (in Thurston's notation, $L(g, h) = \limsup_{j\to\infty} \log(L_g(c_j)/L_h(c_j)))$ by its symmetrization $\frac{1}{2}(L(g, h) + L(h, g))$, but it seems that, because of its direct geometric interpretations, L is more useful just as it is.' In order to justify that this is a distance, we start with the following lemma.

LEMMA 5.5. *Let $g_1, g_2 \in \mathcal{M}$ be negatively curved. Then*

$$\limsup_{j\to\infty} \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} = \sup_{m\in\mathfrak{M}_{inv,erg}} I_m(g_1, g_2) \geq 0.$$

Note that there is no need to assume g_1 and g_2 are close in this lemma: this follows from Appendix B, where we discuss the fact that the stretch (and the time reparametrization) is well defined despite the fact that the metrics may not be close. Here m is seen as an invariant ergodic measure for the flow $\varphi_t^{g_1}$ living on $S_{g_1}M$. However, writing $M = \Gamma\backslash\widetilde{M}$ with $\Gamma \simeq \pi_1(M, x_0)$ for $x_0 \in M$, it can also be identified with a geodesic current on $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$, that is, a Γ-invariant Borel measure, also invariant by the flip $(\xi, \eta) \mapsto (\eta, \xi)$ on $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$. This point of view has the advantage of being independent of g_1 (see [**STar**]).

Proof. First of all, we claim that (as pointed out to us by one of the referees, the map $\mathfrak{M}_{\rm inv} \ni m \mapsto I_m(g_1, g_2)$ is continuous and linear on a compact convex set; it thus achieves its maximum on the extremal points of the convex sets (the ergodic measures) so the argument could be shortened)

$$\sup_{m\in\mathfrak{M}_{\rm inv,erg}} I_m(g_1, g_2) = \sup_{m\in\mathfrak{M}_{\rm inv}} I_m(g_1, g_2).$$

Of course, it is clear that $\sup_{m\in\mathfrak{M}_{\rm inv,erg}} I_m(g_1, g_2) \leq \sup_{m\in\mathfrak{M}_{\rm inv}} I_m(g_1, g_2)$ and thus we are left to prove the reverse inequality. By compactness, we can consider a measure $m_0 \in \mathfrak{M}_{\rm inv}$ realizing $\sup_{m\in\mathfrak{M}_{\rm inv}} I_m(g_1, g_2)$. By the Choquet representation theorem (see [**Wal82**, pp. 153]), there exists a (unique) probability measure τ on $\mathfrak{M}_{\rm inv,erg}$ such that m_0 admits the ergodic decomposition $m_0 = \int_{\mathfrak{M}_{\rm inv,erg}} m\, d\tau(m)$. Thus

$$\begin{aligned} I_{m_0}(g_1, g_2) &= \int_{S_{g_1}M} a_{g_1,g_2}\, dm_0 \\ &= \int_{\mathfrak{M}_{\rm inv,erg}} \int_{S_{g_1}M} a_{g_1,g_2}\, dm\, d\tau(m) \\ &\leq \sup_{m\in\mathfrak{M}_{\rm inv,erg}} \int_{S_{g_1}M} a_{g_1,g_2}\, dm \int_{\mathfrak{M}_{\rm inv,erg}} d\tau(m) = \sup_{m\in\mathfrak{M}_{\rm inv,erg}} I_m(g_1, g_2), \end{aligned}$$

which eventually proves the claim.

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Let $(c_j)_{j\in\mathbb{N}}$ be a subsequence such that $\lim_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j)$ realizes the lim sup. Then, by compactness, we can extract a subsequence such that $\delta_{g_1}(c_j) \rightharpoonup m \in \mathfrak{M}_{\mathrm{inv}}$. Thus:

$$L_{g_2}(c_j)/L_{g_1}(c_j) = \langle \delta_{g_1}(c_j), a_{g_1,g_2}\rangle \to_{j\to+\infty} \langle m, a_{g_1,g_2}\rangle = I_m(g_1, g_2),$$

which proves, using our preliminary remark, that

$$\limsup_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2).$$

To prove the reverse inequality, we consider a measure $m_0 \in \mathfrak{M}_{\mathrm{inv,erg}}$ such that $I_{m_0}(g_1, g_2) = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2)$ (which is always possible by compactness). Since m_0 is invariant and ergodic, there exists a sequence of free homotopy classes $(c_j)_{j\in\mathbb{N}}$ such that $\delta_{g_1}(c_j) \rightharpoonup m_0$ (by [Sig72]). Then, as previously, we have

$$I_{m_0}(g_1, g_2) = \lim_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \limsup_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j),$$

which provides the reverse inequality. □

We can now prove Proposition 5.4.

Proof of Proposition 5.4.. By (2.17), for $g_1, g_2 \in \mathcal{E}^{k,\alpha}$, we have that $I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) \geq 1$ and thus, by Lemma 5.5, we obtain that $d_T(g_1, g_2) \geq 0$ (note that g_1 and g_2 do not need to be close for this property to hold). Moreover, the triangle inequality is immediate for this distance. Eventually, if $d_T(g_1, g_2) = 0$, then $0 \leq \log I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) \leq d_T(g_1, g_2) = 0$, that is, $I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) = 1$ and, by Theorem 2.9, it implies that g_1 is isometric to g_2 if g_2 is close enough to g_1 in the $C^{k,\alpha}$ topology (note that this neighborhood depends on g_1). □

We now investigate in more detail the structure of the distance d_T. A consequence of Lemma 5.5 is the following expression of the Thurston Finsler norm.

LEMMA 5.6. *Let $g_0 \in \mathcal{E}^{k,\alpha}$ and $(g_t)_{t\in[0,\varepsilon)}$ be a smooth family of metrics and let $f := \partial_t g_t|_{t=0}$. Then*

$$\|f\|_T := \frac{d}{dt} d_T(g_0, g_t)\Big|_{t=0} = \frac{1}{2} \sup_{m\in\mathfrak{M}_{inv,erg}} \int_{S_{g_0}M} \pi_2^* f \, dm \tag{5.2}$$

The norm $\|\cdot\|_T$ is a Finsler norm on $T_{g_0}\mathcal{E}^{k,\alpha} \cap \ker D^*_{g_0}$

Proof. We introduce

$$u(t) := e^{d_T(g_0,g_t)} = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_0, g_t)$$

and write $a_t := a_{g_0,g_t}$ for the time reparametrization (as in (2.14)). Then

$$\begin{aligned}\lim_{t\to 0} \frac{u(t)-u(0)}{t} &= \lim_{t\to 0} \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \frac{a_t - 1}{t} dm = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \dot{a}_0 \, dm \\ &= \frac{1}{2} \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \pi_2^* f \, dm = u'(0) = \frac{d}{dt} d_T(g_0, g_t)\Big|_{t=0},\end{aligned}$$

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since $\dot{a}_0 = \partial_t a_t|_{t=0}$ and $\pi_2^* f$ are cohomologous by Lemma 3.3. This also shows that the derivative exists. The inversion of the limit and the sup follows from the fact that, writing $F_t(m) := \int_{S_{g_0}M} (a_t - 1)/t \ \ dm$, we have $\sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} |F_t(m) - F_0(m)| \to_{t \to 0} 0$. Note that, up to taking a large $k \in \mathbb{N}$ and iterating the same computation for higher-order derivatives, this shows that $t \mapsto u(t)$ (thus $t \mapsto d_T(g_0, g_t)$) is at least C^2.

We now prove that this is a Finsler norm in a neighborhood of the diagonal. We fix $g_0 \in \mathcal{E}^{k,\alpha}$. By Lemma 2.4, isometry classes near g_0 can be represented by solenoidal tensors, namely, there exists a $C^{k,\alpha}$-neighborhood $\mathcal{U}$ of g_0 such that for any $g \in \mathcal{U}$, there exists a (unique) $\psi \in \mathcal{D}_0^{k+1,\alpha}$ such that $D_{g_0}^* \psi^* g = 0$. Moreover, if $g \in \mathcal{E}^{k,\alpha}$, then $\psi^* g \in \mathcal{E}^{k,\alpha}$. As a consequence, using (5.1), the statement now boils down to proving that (5.2) is a norm for solenoidal tensors $f \in C^{k,\alpha}(M; S^2 T^* M)$ such that $\int_{S_{g_0}M} \pi_2^* f \, d\mu_{g_0}^{\mathrm{BM}} = 0$. Since the triangle inequality, $\mathbb{R}_+$-scaling and non-negativity are immediate, we simply need to show that $\|f\|_T = 0$ implies $f = 0$. Now, for such a tensor f, we have

$$\begin{aligned}\mathbf{P}(\pi_2^* f) &= \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \int_{S_{g_0}M} \pi_2^* f \, dm \\ &\leq \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \pi_2^* f \, dm = \underbrace{\mathbf{h}_{\mathrm{top}}(\varphi_1^{g_0})}_{=1} + 0,\end{aligned}$$

and this supremum is achieved for $m = \mu_{g_0}^{\mathrm{BM}}$ and $\mathbf{P}(\pi_2^* f) = 1$. As a consequence, the equilibrium state associated to the potential $\pi_2^* f$ is the Bowen–Margulis measure $\mu_{g_0}^{\mathrm{BM}}$ (the equilibrium state associated to the potential 0) and thus $\pi_2^* f$ is cohomologous to a constant $c \in \mathbb{R}$ (see [**HF19**, Theorem 9.3.16]) which has to be $c = 0$ since the average of $\pi_2^* f$ with respect to Bowen–Margulis is equal to 0, that is, there exists a Hölder-continuous function u such that $\pi_2^* f = Xu$. Since $f \in \ker D_{g_0}^*$, the s-injectivity of the X-ray transform $I_2^{g_0}$ implies that $f \equiv 0$. □

The asymmetric Finsler norm $\|\cdot\|_T$ induces a distance d_F between isometry classes, namely,

$$d_F(g_1, g_2) = \inf_{\gamma : [0,1] \to \mathcal{E}, \gamma(0) = g_1, \gamma(1) = g_2} \int_0^1 \|\dot{\gamma}(t)\|_T \, dt.$$

It is easy to prove that $d_T(g_1, g_2) \leq d_F(g_1, g_2)$, which shows that d_F is indeed a distance in a neighborhood of the diagonal, just like d_T. Indeed, consider a C^1-path $\gamma : [0, 1] \to \mathcal{E}$ such that $\gamma(0) = g_1$, $\gamma(1) = g_2$. Then, considering $N \in \mathbb{N}$, $t_i := i/N$, we have, by the triangle inequality,

$$\begin{aligned}d_T(g_1, g_2) &\leq \sum_{i=0}^{N-1} d_T(\gamma(t_i), \gamma(t_{i+1})) \\ &= \sum_{i=0}^{N-1} \|\dot{\gamma}(t_i)\|_T (t_{i+1} - t_i) + \mathcal{O}(|t_{i+1} - t_i|^2) \to_{N \to +\infty} \int_0^1 \|\dot{\gamma}(t)\|_T \, dt,\end{aligned}$$

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which proves the claim (note that we here use the fact that $t \mapsto d_T(g_0, g_t)$ is at least C^2). In [**Thu98**], Thurston proves that, on restriction to Teichmüller space, the asymmetric Finsler norm induces the distance d_T, that is, $d_T = d_F$. We make the following conjecture.

CONJECTURE 5.7. *The distances d_T coincide with d_F for isometry classes of negatively curved metrics with topological entropy equal to* 1.

This conjecture would imply the marked length spectrum rigidity conjecture. Indeed, as mentioned just after Theorem 2.9, two metrics with the same marked length spectrum have the same topological entropy and there is no harm (up to a scaling of the metrics) in assuming that this topological entropy is equal to 1. Then, if the previous conjecture is true, using that their Thurston distance d_T is zero, we obtain that their Finsler distance d_F is zero. But this implies that the metrics are isometric.

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A. *Appendix. Asymptotic marked length spectrum*

In this appendix, we show the following lemma (the proof was communicated to us by one of the referees).

LEMMA A.1. *Let g and g_0 be two metrics with Anosov geodesic flows on a fixed manifold M and assume that g is close to g_0 in $C^{k,\alpha}$ norm. Assume that for all sequences $(c_j)_{j\geq 0}$ in $\mathcal{C}$, $L_g(c_j)/L_{g_0}(c_j) \to_{j\to+\infty} 1$. Then $L_g = L_{g_0}$.*

Proof. By Sigmund [**Sig72**, Theorem 1], the set $\mathfrak{D} := \{\delta_{g_0}(c) \mid c \in \mathcal{C}\}$ is dense in $\mathfrak{M}_{\mathrm{inv}}$ (the set of invariant measures by the g_0-geodesic flow on $S_{g_0}M$). If $\mu \in \mathfrak{M}_{\mathrm{inv}} \setminus \mathfrak{D}$, we can therefore find a sequence such that $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu$ and $L_{g_0}(c_j) \to +\infty$. (Indeed, if $L_{g_0}(c_j) \leq C$ for some $C \geq 0$, then the sequence $(\delta_{g_0}(c_j))_{j\geq 0}$ only achieves a finite number of measures, which would imply that μ is a Dirac mass on a closed orbit and this is excluded since $\mu \notin \mathfrak{D}$.) Then, the condition $L_g/L_{g_0} \to 1$ immediately implies that

$$I_\mu(g_0, g) = \int_{S_{g_0}M} a_g(z)\, d\mu(z) = 1.$$

Now, for $c \in \mathcal{C}$ and $t > 0$ small, the linear combination $t\mu + (1-t)\delta_{g_0}(c) \notin \mathfrak{D}$. Indeed, if not, we would have $t\mu + (1-t)\delta_{g_0}(c) = \delta_{g_0}(c_t)$ but by continuity, $c_t = c_0$ for t small,

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which contradicts $\mu \notin \mathfrak{D}$. Therefore

$$t \int_{S_{g_0}M} a_g(z)\, d\mu(z) + (1-t)\frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} a_g(\varphi_s^{g_0}(z))\, ds = 1,$$

that is,

$$\frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} a_g(\varphi_s^{g_0}(z))\, ds = \frac{L_g(c)}{L_{g_0}(c)} = 1. \qquad \square$$

B. *Appendix. Global conjugacy for Riemannian Anosov flows*

Let (M, g) be a closed Riemannian manifold whose geodesic flow is Anosov. As has been shown by Klingenberg [**Kli74**] the geodesic flow has no conjugate points. Let $(\widetilde{M}, g)$ be the universal cover of M where for simplicity the lifted metric is also denoted by g. Let Γ be the group of deck transformations. As has been remarked in [**Kni12**], the universal cover $\widetilde{M}$ is Gromov hyperbolic (see [**BH99**, Section III.H.1] for a definition of Gromov hyperbolicity). Denote by $\partial_\infty \widetilde{M}$ the Gromov boundary which is equipped with the visibility topology (see, for example, [**Kni02**] for more details). For $\xi \in \partial_\infty \widetilde{M}$ and $x_0 \in \widetilde{M}$, the Busemann function $x \mapsto b_\xi^g(x_0, x)$ is defined by

$$b_\xi^g(x_0, x) := \lim_{z \to \xi} d_g(x_0, z) - d_g(x, z). \tag{B.1}$$

It has the following properties:

$$b_\xi^g(x_0, x) = b_\xi^g(x_0, x_1) + b_\xi^g(x_1, x) \quad \text{(cocycle property)} \tag{B.2}$$

and

$$b_{\gamma(\xi)}^g(\gamma(x_0), \gamma(x)) = b_\xi^g(x_0, x) \quad (\Gamma\text{-equivariance}) \tag{B.3}$$

for all $\gamma \in \Gamma$. We introduce the gradient of the Busemann function $B^g(x, \xi) := \nabla_x b_\xi^g(x_0, x)$ which is independent of x_0 by property (B.2). Also observe that $B^g(x, \xi) \in S_g\widetilde{M}$ by the very definition (B.1). Here, $S_g\widetilde{M}$ is the unit tangent bundle on the universal cover and $\pi : S_g\widetilde{M} \to \widetilde{M}$ denotes the projection. Given $z = (x, v) \in S_g\widetilde{M}$, we introduce $c_g(z, t) := \pi(\varphi_t^g(x, v))$, where $(\varphi_t^g)_{t\in\mathbb{R}}$ is the (lift of) the geodesic flow on $\widetilde{M}$. We set $z_\pm^g = c_g(z, \pm\infty) \in \partial_\infty \widetilde{M}$.

For $\xi = z_+^g$ the submanifolds $W^{ss}(z) = \{(x, -B^g(x, \xi)) \in S_g\widetilde{M} \mid b_\xi^g(x_0, x) = b_\xi^g(x_0, \pi z)\}$ and $W^{uu}(z) = \{(x, B^g(x, \xi)) \in S_g\widetilde{M} \mid b_\xi^g(x_0, x) = b_\xi^g(x_0, \pi z)\}$ are the lifts of the leafs of strong stable and unstable foliations through $z \in S_g\widetilde{M}$. Since the leafs are smooth and the foliations are Hölder continuous, the Busemann functions $(x, \xi) \mapsto b_\xi^g(x_0, x)$ are smooth with respect to x and Hölder continuous with respect to ξ. The following lemma was proved in [**STar**] (see also [**Gro00**]) in negative curvature.

LEMMA B.1. *Let $M = \widetilde{M}/\Gamma$ be a closed manifold, and let g_1, g_2 be two Riemannian metrics with Anosov geodesic flow. Consider the map $\psi_{g_1,g_2} : S_{g_1}\widetilde{M} \to S_{g_2}\widetilde{M}$ defined by $\psi_{g_1,g_2}(z) = w$ where $w \in S^{g_2}\widetilde{M}$ is the unique vector with $w_+^{g_2} = z_+^{g_1}$ and $w_-^{g_2} = z_-^{g_1}$ and*

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$b^{g_2}_{z^{g_1}_+}(\pi(z), \pi(w)) = 0$. *Then* ψ_{g_1,g_2} *is a Hölder-continuous surjective map with*

$$\widetilde{\varphi}^{g_2}_{\tau(z,t)}\psi_{g_1,g_2}(z) = \psi_{g_1,g_2}(\widetilde{\varphi}^{g_1}_t(z)),$$

where

$$\tau(z, t) = b^{g_2}_{z^{g_1}_+}(\pi(z), \pi(\widetilde{\varphi}^{g_1}_t(z))) = \int_0^t g_2(B^{g_2}(\pi(\widetilde{\varphi}^{g_1}_s(z)), z^{g_1}_+), \widetilde{\varphi}^{g_1}_s(z))\, ds$$

for all $z \in S^{g_1}\widetilde{M}$. *Furthermore, for all* $\gamma \in \Gamma$ *we have*

$$\gamma_*\psi_{g_1,g_2}(z) = \psi_{g_1,g_2}(\gamma_* z)$$

and $\tau(\gamma_* z, t) = \tau(z, t)$ *and therefore* ψ_{g_1,g_2} *descends to a map between the quotients* $S_{g_i}M$.

Proof. We show first that for each $(z, t) \in S_{g_1}\widetilde{M} \times \mathbb{R}$ we have

$$\widetilde{\varphi}^{g_2}_{\tau(z,t)}\psi_{g_1,g_2}(z) = \psi_{g_1,g_2}(\widetilde{\varphi}^{g_1}_t(z)),$$

where $\tau(z, t) = b^{g_2}_{z^{g_1}_+}(\pi(z), \pi(\widetilde{\varphi}^{g_1}_t(z)))$. From the cocycle property (B.2) of the Busemann function we obtain

$$\begin{aligned}
&b^{g_2}_{z^{g_1}_+}(\pi(\widetilde{\varphi}^{g_1}_t(z)), \pi(\widetilde{\varphi}^{g_2}_{\tau(z,t)}\psi_{g_1,g_2}(z)))\\
&\quad= b^{g_2}_{z^{g_1}_+}(\pi(\widetilde{\varphi}^{g_1}_t(z)), \pi(\psi_{g_1,g_2}(z))) + b^{g_2}_{z^{g_1}_+}(\pi(\psi_{g_1,g_2}(z)), \pi(\widetilde{\varphi}^{g_2}_{\tau(z,t)}\psi_{g_1,g_2}(z)))\\
&\quad= b^{g_2}_{z^{g_1}_+}(\pi(\widetilde{\varphi}^{g_1}_t(z)), \pi(\psi_{g_1,g_2}(z))) + \tau(z, t)\\
&\quad= b^{g_2}_{z^{g_1}_+}(\pi(\widetilde{\varphi}^{g_1}_t(z)), \pi(z)) + b^{g_2}_{z^{g_1}_+}(\pi(z), \pi(\psi_{g_1,g_2}(z))) + \tau(z, t)\\
&\quad= -b^{g_2}_{z^{g_1}_+}(\pi(z), \pi(\widetilde{\varphi}^{g_1}_t(z))) + \tau(z, t) = 0.
\end{aligned}$$

By the definition of ψ_{g_1,g_2} this yields $\widetilde{\varphi}^{g_2}_{\tau(z,t)}\psi_{g_1,g_2}(z) = \psi_{g_1,g_2}(\widetilde{\varphi}^{g_1}_t(z))$. The regularity of the Busemann function shows that ψ_{g_1,g_2} is Hölder continuous. The remaining assertions follow from the Γ-equivariance (B.3) of the Busemann function. □

Remark B.2. Note that $\psi_{g_1,g_2} : S_{g_1}M \to S_{g_2}M$ maps orbits of the geodesic flow of g_1 surjectively onto orbits of the geodesic flow of g_2 but is not necessarily injective. To obtain a injective map the following modification due to Gromov [**Gro00**] (see also [**Kni02**]) can be made. Choose $r_0 > 0$ such that $\tau(z, r_0) > 0$ for all $z \in S_{g_1}M$. Define

$$r(z) = \frac{1}{r_0}\int_0^{r_0} \tau(z, s)\, ds$$

and consider $\psi^r_{g_1,g_2}(z) := \varphi^{g_2}_{r(z)} \circ \psi_{g_1,g_2}(z)$ Then

$$\psi^r_{g_1,g_2}(\varphi^{g_1}_t(z)) = \varphi^{g_2}_{r(\varphi^{g_1}_t(z))+\tau(z,t)}\psi_{g_1,g_2}(z).$$

Since

$$\widehat{\tau}(z, t) := r(\varphi^{g_1}_t(z)) + \tau(z, t) = \frac{1}{r_0}\int_0^{r_0} \tau(z, t + s)\, ds$$

and

$$\frac{d}{dt}\widehat{\tau}(z,t) = \frac{1}{r_0}(\tau(z,t+r_0) - \tau(z,r_0)) = \frac{1}{r_0}\tau(\varphi_t^{g_1}(z), r_0) > 0,$$

the map $t \mapsto \hat{\tau}(z,t)$ is strictly monotone increasing and therefore $\psi^r_{g_1,g_2}$ is injective and yields a conjugacy between the geodesic flows.

C. *Appendix. Anosov stability*

The proof of the Anosov stability theorem is given using the implicit function theorem in [**dlLMM86**] in the C^0 category; the extension to the Hölder setting (with the same method) appears in [**KKPW89**]. We need the continuity with respect to the two metrics here; the proof of [**dlLMM86**, **KKPW89**] indeed shows this, as we explain below. Let $\nu \in (0,1)$. Then if X is a C^k vector field for $k \geq 4$ with flow φ_t^X, we will denote by $C^\nu_X(\mathcal{M},\mathcal{M})$ the space of C^ν maps ψ on a closed manifold $\mathcal{M}$ so that $d\psi.X := \partial_t(\psi\circ\varphi_t^X)|_{t=0}$ exists and belongs to $C^\nu(\mathcal{M};T\mathcal{M})$. This is a Banach manifold [**KKPW89**, Proposition 2.2].

PROPOSITION C.1. *Let g_0 be a smooth metric, and assume that X_{g_0} its geodesic vector field on $\mathcal{M} := S_{g_0}M$ is Anosov. We view all geodesic vector fields X_g associated to g near g_0 as vector fields on $\mathcal{M}$ (by pulling back from S_gM to $S_{g_0}M$). For $k \geq 4$, there exist $\nu > 0$ and two open neighborhoods $\mathcal{U}_0 \subset \mathcal{U}$ of X_{g_0} in $C^{k+1}(\mathcal{M};T\mathcal{M})$ such that, for each $Y \in \mathcal{U}$ and each $g \in C^{k+2}(M;S^2T^*M)$ so that $X_g \in \mathcal{U}_0$, there exist a homeomorphism $\psi_{g,Y} \in C^\nu_{X_g}(\mathcal{M},\mathcal{M})$ and $a_{g,Y} \in C^\nu(\mathcal{M},\mathbb{R}^+)$ such that*

$$\text{for all } x \in \mathcal{M},\quad d\psi_{g,Y}(x)X_g(x) = a_{g,Y}(x)Y(\psi_{g,Y}(x)),$$

where X_g is the geodesic vector field of g. Moreover, $Y \in \mathcal{U} \mapsto a_{g,Y} \in C^\nu(\mathcal{M},\mathbb{R}^+)$ and $Y \mapsto \psi_{g,Y} \in C^\nu_{X_g}(\mathcal{M},\mathcal{M})$ are C^k, and each derivative of order $\ell \leq k$ with respect to Y is continuous with respect to (g, Y) with values in C^ν.

Proof. The proof is essentially contained in [**KKPW89**, Proposition 2.2], except for the statement about the continuity with respect to X_g. Consider, for $\nu \in (0,1)$ and $E := C^{k+1}(\mathcal{M};T\mathcal{M}) \times C^\nu(\mathcal{M},T\mathcal{M})$, the map

$$F^{X_g} : C^{k+1}(\mathcal{M};T\mathcal{M}) \times C^\nu_{X_g}(\mathcal{M},\mathcal{M}) \times C^\nu(\mathcal{M}) \to E$$

defined by

$$F^{X_g}(Y,u,\gamma) := (Y, \gamma du.X_g - Y\circ u).$$

This is a C^k map between Banach manifolds. The differential of F^{X_g} at $(X_g, \mathrm{Id}, 1)$ is given (as in [**KKPW89**]) by

$$dF^{X_g}_{(X_g,\mathrm{Id},1)}(Y,V,\gamma) = (Y, -Y + \mathcal{L}_{X_g}V + \gamma X_g), \tag{C.1}$$

where $V \in C^\nu_{X_g}(\mathcal{M};T\mathcal{M}) := \{V \in C^\nu(\mathcal{M};T\mathcal{M}) \mid \mathcal{L}_{X_g}V \in C^\nu\}$. Let α_g be the contact form of g, so that $\ker\alpha_g = E_u(g) \oplus E_s(g)$ is the smooth bundle of stable or unstable vectors for g. By [**KKPW89**, Proposition 2.2. and Lemma 2.3], the operator $\mathcal{L}_{X_g} : V \mapsto \mathcal{L}_{X_g}V$ is invertible from $C^\nu_{X_g}(\mathcal{M};\ker\alpha_g) \to C^\nu(\mathcal{M};\ker\alpha_g)$ for some ν depending on the maximal/minimal expansion rates of the flow $\varphi_t^{X_g}$. The inverse is given by

$$\mathcal{L}_{X_g}^{-1} : V = V_u + V_s \mapsto \mathcal{L}_{X_g}^{-1} V = -\int_0^{+\infty} d\varphi_{-t}^{X_g} V_u \circ \varphi_t^{X_g}\, dt + \int_0^{+\infty} d\varphi_t^{X_g} V_s \circ \varphi_{-t}^{X_g}\, dt,$$

where the integrals converge due to the contraction of the differential: for all $t \geq 0$,

$$C^{-1} e^{-\lambda_+ t} \leq \|d\varphi_t^{X_g}|_{E_s(g)}\| \leq C e^{-\lambda_- t}, \quad C^{-1} e^{-\lambda_+ t} \leq \|d\varphi_{-t}^{X_g}|_{E_u(g)}\| \leq C e^{-\lambda_- t}. \quad \text{(C.2)}$$

This operator maps continuously $C^\nu(\mathcal{M}; \ker \alpha_g)$ to $C^\nu_{X_g}(\mathcal{M}; \ker \alpha_g)$ if $\nu > 0$ is small enough, depending on λ_+ and $\|\varphi_T^{X_g}\|_{C^2}$ for $T > 0$ large (see below). Moreover, by continuity of the bundles $E_s(g)$, $E_u(g)$ with respect to g [**HP68**, Theorem 3.2], for g close enough to g_0 in C^{k+5}, $E_u(g)$ and $E_s(g)$ are contained in a small conic neighborhood of $E_u(g_0)$ and $E_s(g_0)$ respectively, and the contraction exponents $\lambda_\pm(g)$ are also close to $\lambda_\pm(g_0)$ (see, for example, [**Bon20**, Lemma 3]), so this will give the boundedness of $\mathcal{L}_{X_g}^{-1}$ in C^ν for some fixed $\nu > 0$ for g close enough to g_0 in C^{k+5}. From the expression of $\mathcal{L}_{X_g}^{-1}$, and the fact that (C.2) holds uniformly for g close to g_0 for some $0 < \lambda_- < \lambda_+$ (and similarly on $E_u(g)$), we claim that, if $\pi_g : T\mathcal{M} \to \ker \alpha_g$ is the projection given by $\pi_g(V) = V - \alpha_g(V) X_g$, then

$$\mathcal{L}_{X_g}^{-1} \pi_g : C^\nu(\mathcal{M}; T\mathcal{M}) \to C^\nu(\mathcal{M}; T\mathcal{M})$$

is continuous with respect to g (in C^{k+5}) for $\nu > 0$ small enough. To prove this, we rewrite $\mathcal{L}_{X_g}^{-1} \pi_g$ as

$$\mathcal{L}_X^{-1} \pi_g = \int_0^\infty e^{-t\mathcal{L}_{X_g}} \pi_g^s\, dt - \int_0^\infty e^{t\mathcal{L}_{X_g}} \pi_g^u\, dt \quad \text{(C.3)}$$

where $\pi_g^u : C^\nu(\mathcal{M}; T\mathcal{M}) \to C^\nu(\mathcal{M}; T\mathcal{M})$ is the projection on E_u parallel to E_s and $\pi_g^s : C^\nu(\mathcal{M}; T\mathcal{M}) \to C^\nu(\mathcal{M}; T\mathcal{M})$ is the projection on E_s parallel to E_u, and $e^{t\mathcal{L}_{X_g}} Y := d\varphi_{-t}^{X_g} Y \circ \varphi_t^{X_g}$ is the propagator. Here ν is chosen small so that E_u and E_s are C^ν bundles (see [**HP68**]), and by [**Con92**] the maps $g \mapsto \pi_g^u$ and $g \mapsto \pi_g^s$ are continuous (actually C^r for some r depending on the smoothness of g). Next, there exist $C > 0$ and $\Lambda > 0$ such that, for all t, $\|\varphi_t^{X_g}\|_{C^2} \leq C e^{\Lambda|t|}$ for all g near g_0 in C^{k+5}, which implies that, for all $V \in C^1(\mathcal{M}; T\mathcal{M})$,

$$\text{for all } t, \quad \|e^{t\mathcal{L}_{X_g}} V\|_{C^1} \leq C e^{2\Lambda|t|} \|V\|_{C^1}, \quad \|e^{t\mathcal{L}_{X_g}} V\|_{C^0} \leq C e^{\Lambda|t|} \|V\|_{C^0},$$

thus if $\nu_0 \in (0, 1)$ is such that $E_u \in C^{\nu_0}$, we have by interpolation that $\|e^{t\mathcal{L}_{X_g}}\|_{\mathcal{L}(C^\nu)} \leq C e^{(1+\nu_0)\Lambda|t|}$ for each $\nu \leq \nu_0$. Since $\|e^{t\mathcal{L}_{X_g}} \pi_g^u\|_{\mathcal{L}(C^0)} + \|e^{-t\mathcal{L}_{X_g}} \pi_g^s\|_{\mathcal{L}(C^0)} \leq C e^{-\lambda_- t}$ for all $t \geq 0$, we obtain by interpolating C^ν between the spaces C^0 and C^{ν_0} with $\nu = \theta \times 0 + (1 - \theta) \times \nu_0$ (for $\theta \in (0, 1)$) that, for all $t \geq 0$,

$$\|e^{t\mathcal{L}_{X_g}} \pi_g^u\|_{\mathcal{L}(C^\nu)} + \|e^{-t\mathcal{L}_{X_g}} \pi_g^s\|_{\mathcal{L}(C^\nu)} \leq C e^{(-\theta\lambda_- + (1-\theta)(1+\nu_0)\Lambda)t}.$$

We can now fix ν small enough (that is, θ close enough to 1) to guarantee

$$-\theta\lambda_- + (1 - \theta)(1 + \nu_0)\Lambda < 0,$$

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which implies that (C.3) is uniformly converging with respect to g near g_0 in C^{k+5}. Since $g \mapsto e^{t\mathcal{L}_{X_g}}\pi_g^u$ and $g \mapsto e^{-t\mathcal{L}_{X_g}}\pi_g^s$ are continuous for each $t \geq 0$, we can apply the Lebesgue theorem to deduce the continuity of $g \mapsto \mathcal{L}_{X_g}^{-1}\pi_g \in \mathcal{L}(C^\nu)$ for $\nu > 0$ small enough.

Next, we consider the map $\widetilde{F}^{X_g} : E \to E$ defined by

$$\begin{aligned}\widetilde{F}^{X_g}(Y, V) &:= F^{X_g}(Y, \exp_{g_0}(\mathcal{L}_{X_g}^{-1}\pi_g(Y+V)), \alpha_g(Y+V)) \\ &= (Y, \alpha_g(Y+V)d(\exp_{g_0}(\mathcal{L}_{X_g}^{-1}\pi_g(Y+V))).X_g - Y \circ \exp_{g_0}(\mathcal{L}_{X_g}^{-1}\pi_g(Y+V))),\end{aligned}$$

where we recall that $E = C^{k+1}(\mathcal{M}; T\mathcal{M}) \times C^\nu(\mathcal{M}, T\mathcal{M})$ and $\exp_{g_0}$ is the exponential map of g_0. This map satisfies $\widetilde{F}^{X_g}(X_g, 0) = (X_g, 0)$. We want to apply the inverse function theorem to find a pre-image to each $(Y, 0)$ close to $(X_g, 0)$. As in [**KKPW89**, Proposition 2.2] (see also [**dlLMM86**, Appendix A]), the map $\widetilde{F}^{X_g}$ is C^k, and moreover it depends continuously on $g \in C^{k+5}(M; S^2T^*M)$, with all its derivatives of order $\ell \leq k$ being also continuous with respect to g, due to the continuity of $g \mapsto \mathcal{L}_{X_g}^{-1}\pi_g$ as a map $C^{k+5}(M; S^2T^*M) \to \mathcal{L}(C^\nu(\mathcal{M}; T\mathcal{M}))$. Now we have

$$d\widetilde{F}^{X_g}(X_g, 0) = \mathrm{Id},$$

by using (C.1) and $\pi_g(X_g) = 0$. In particular, there is $\varepsilon > 0$ such that if $\|g - g_0\|_{C^{k+5}} < \varepsilon$, $\|Y - X_g\|_{C^{k+1}} < \varepsilon$ and $\|V\|_{C^\nu} < \varepsilon$, then

$$\|d\widetilde{F}^{X_g}_{(Y,V)} - \mathrm{Id}\|_{\mathcal{L}(C^\nu(\mathcal{M};T\mathcal{M}))} < 1/4.$$

For each Y close to X_g, we can then apply the fixed point theorem (as in the proof of the inverse function theorem) to the map $(Z, V) \in E \mapsto (Z + Y, V) - \widetilde{F}^{X_g}(Z, V)$ and obtain that there is a unique $(Y, V(Y))$ such that $(Y, 0) = \widetilde{F}^{X_g}(Y, V(Y))$, and $V(Y) \in C^\nu(\mathcal{M}; T\mathcal{M})$ depends in a C^k fashion on Y and is continuous with respect to g. Moreover, the usual argument in the inverse function theorem used to prove the C^k property of $Y \mapsto V(Y)$ also shows that the derivatives of order $\ell \leq k$ are continuous with respect to (X_g, Y), by using the continuity of $\widetilde{F}^{X_g}$ and its derivatives with respect to g. This shows that for each Y close to X_g in C^{k+1} norm and g close to g_0 in C^{k+5} norm, there is a

$$u = \exp_{g_0}(\mathcal{L}_{X_g}^{-1}\pi_g(Y+V)) \in C^\nu_{X_g}(\mathcal{M}, \mathcal{M}), \quad \gamma = \alpha_g(Y+V) \in C^\nu(\mathcal{M})$$

so that $\gamma du.X_g = Y \circ u$, with

$$C^{k+1}(\mathcal{M}; T\mathcal{M}) \ni Y \mapsto (u, \gamma) \in C^\nu(\mathcal{M} \times \mathcal{M}) \times C^\nu(\mathcal{M}),$$

and where C^k and all the derivatives of order $\ell \leq k$ are continuous in (g, Y) (with values in $C^\nu(\mathcal{M}, \mathcal{M}) \times C^\nu(\mathcal{M})$). □

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