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Fine Selmer groups of congruent p-adic Galois representations

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Abstract. We compare the Pontryagin duals of fine Selmer groups of two congruent p-adic Galois representations over admissible pro-p, p-adic Lie extensions K_∞ of number fields K. We prove that in several natural settings the π-primary submodules of the Pontryagin duals are pseudo-isomorphic over the Iwasawa algebra; if the coranks of the fine Selmer groups are not equal, then we can still prove inequalities between the μ-invariants. In the special case of a $\mathbb{Z}_p$-extension K_∞/K, we also compare the Iwasawa λ-invariants of the fine Selmer groups, even in situations where the μ-invariants are nonzero. Finally, we prove similar results for certain abelian non-p-extensions.

1 Introduction

Let p be a prime, and let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and uniformizing element π. Suppose that V_1 and V_2 denote two F-representations of the absolute Galois group of a fixed number field K, and that $T_1 \subseteq V_1$ and $T_2 \subseteq V_2$ are two Galois stable sublattices. We let $A_1 = V_1/T_1$ and $A_2 = V_2/T_2$ and we assume that $A_1[\pi^l]$ and $A_2[\pi^l]$ are isomorphic as Galois modules for some $l \in \mathbb{N}$. In this article, we study the Pontryagin duals of the fine Selmer groups of A_1 and A_2 over (strongly) admissible p-adic Lie extensions, and we compare their ranks and Iwasawa invariants.

By an *admissible p-adic Lie extension*, we mean a normal extension K_∞/K such that only finitely many primes of K ramify in K_∞ and such that $G = \mathrm{Gal}(K_\infty/K)$ is a compact, pro-p, p-adic Lie group without p-torsion. For any finite set Σ of primes of K, an admissible p-adic Lie extension K_∞/K shall be called *strongly Σ-admissible* if K_∞ contains a $\mathbb{Z}_p$-extension L of K such that no prime $v \in \Sigma$ and no prime of K which ramifies in K_∞ is completely split in L (see also Section 2; in the literature usually only the case of the cyclotomic $\mathbb{Z}_p$-extension $L = K_\infty^c$ of K is considered, see for example [Lim17a]—in this article, we typically focus on complementary cases).

The comparison of *Selmer groups* of congruent p-adic representations goes back to the seminal work of Greenberg and Vatsal (see [GV00]), who considered elliptic curves defined over $\mathbb{Q}$ with good and ordinary reduction at some odd prime p (in fact the Selmer groups were studied more generally in the context of Galois representations). The main issue dealt with in the article [GV00] is the relation between algebraically and analytically (i.e., via p-adic L-functions) defined Iwasawa

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invariants. Roughly speaking, Greenberg and Vatsal treated the $\mu = 0$ case and only considered the cyclotomic $\mathbb{Z}_p$-extension.

Over the last years, the results in [GV00] have been generalized in many different ways and we only mention a few exemplary results. For the comparison of analytical invariants of congruent elliptic curves defined over $\mathbb{Q}$, we refer to [Hat17]; in the present article, we stick to the algebraic side. Most authors have focused on the $\mu = 0$ setting from [GV00]: if $\mu = 0$ for the Selmer group of A_1, then the same holds true for the Selmer group of A_2. Moreover, over $\mathbb{Z}_p$-extensions one can then often prove equality of λ-invariants (we refer to Section 2 for the definition of the Iwasawa invariants). Analogous results have been obtained for Selmer groups of Galois representations over the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic base field K (see [HL19]) and for *signed Selmer groups* of Galois representations over the cyclotomic $\mathbb{Z}_p$-extension of a number field in the non-ordinary setting (see, e.g., [Pon20, Section 3]). Moreover, there exist vast generalisations to Selmer groups attached to families of modular forms (see, e.g., [EPW06, Sha09, Bar13]).

Situations where $\mu \neq 0$ have been studied, for example, in [AS15, BS10]. In these articles, the authors considered congruent elliptic curves E_1 and E_2 over $\mathbb{Q}$ at primes $p > 2$ of good ordinary reduction. Under the additional assumption that $E_j(\mathbb{Q})[p^\infty] = \{0\}$ for $j \in \{1, 2\}$, the authors deduced the equality of λ-invariants (see [AS15]), respectively μ-invariants (see [BS10]) from a sufficiently high congruence relation $E_1[p^l] \cong E_2[p^l]$. Much more generally, Lim studied the Selmer groups of Galois representations over admissible p-adic Lie extensions in [Lim17a]. In particular, he obtained the following result: if A_1 and A_2 are attached to two p-adic Galois representations and $A_1[\pi^l] \cong A_2[\pi^l]$ for some sufficiently large l, then the π-primary submodules of the Pontryagin duals of the associated Selmer groups are pseudo-isomorphic. This comparison statement is much stronger than the previous results. We are able to prove a similar result for fine Selmer groups (see Theorem 1.1). Lim also studied *strict Selmer groups*, as introduced by Greenberg in [Gre89]. These strict Selmer groups of p-adic Galois representations have also been studied by Hachimori in [Hac11].

In the present article, our main objective is the comparison of *fine Selmer groups* of congruent p-adic Galois representations over admissible p-adic Lie extensions. These objects have previously been investigated by Lim and Sujatha in [LS18], who obtained a comparison result in the $\mu = 0$ setting under a stronger condition on the decomposition of primes in K_∞/K (see [LS18, Theorem 3.5]). Moreover, Jha studied in [Jha12] the invariance of several arithmetic properties of fine Selmer groups of modular forms in a branch of a Hida family in the $\mu = 0$ setting.

Our first main result is an analogue of the strong result of Lim in [Lim17a] for fine Selmer groups over strongly admissible p-adic Lie extensions, which is not restricted to the case $\mu = 0$. We note that it is conjectured that the μ-invariant of fine Selmer groups over the cyclotomic $\mathbb{Z}_p$-extension of any number field should vanish (see [CS05, Conjecture A]), and this propagates to p-adic Lie extensions containing the cyclotomic $\mathbb{Z}_p$-extension (cf. e.g., [Lim15]). On the other hand, fine Selmer groups with nonzero μ-invariant do occur naturally. For example, let K be an imaginary-quadratic number field, and let K_∞^a be the *anticyclotomic* $\mathbb{Z}_p$-extension of K. Choose at least $t = 2(p-1) + 1$ primes $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ of K which do not split in $K/\mathbb{Q}$, and let $\alpha \in K$ be

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divisible by each of theses primes exactly once. Then Iwasawa proved that the classical Iwasawa μ-invariant of the shifted anticyclotomic $\mathbb{Z}_p$-extension $K'_\infty = K^a_\infty \cdot K'$ of $K' := K(\mu_p, \sqrt[p]{\alpha})$ is nonzero (see [Lan90, Theorems 13.5.1 and 13.5.2]). Now let A be any abelian variety defined over K such that each prime of bad reduction is coprime with p and splits in $K/\mathbb{Q}$. Then each of these primes is finitely split in K'_∞/K' by work of Brink (see [Bri07, Theorem 2]), and the same holds for the primes above p. Now, we enlarge the base field further and let $L = K'(A[p])$ and $L_\infty = K'_\infty L$. It then follows from work of Lim and Murty (see [LKM16, Theorem 5.1]) that the μ-invariant of the fine Selmer group of A over L_∞ is nontrivial. In fact, with a little more work, one can produce examples with arbitrarily large μ-invariant (see [Kun21, Sections 3 and 4]).

For an admissible p-adic Lie extension K_∞ of K and an F-representation V of the absolute Galois group of K, we let T denote a Galois stable $\mathcal{O}$-lattice in V and set $A = V/T$. Let Σ be a finite set of primes of K containing all the primes above p and each prime where V is ramified. Then $Y^{(K_\infty)}_{A,\Sigma}$ shall denote the Pontryagin dual of the Σ-fine Selmer group of A over K_∞ (see Section 2.2 for the precise definition).

Theorem 1.1 *Let A_1 and A_2 be associated with two F-representations V_1 and V_2 of the absolute Galois group of the number field K. If $p = 2$, then we assume that K is totally imaginary. Let Σ be a finite set of primes of K which contains the primes above p and the sets of primes of K where either V_1 or V_2 is ramified. Let K_∞/K be a strongly Σ-admissible p-adic Lie extension, and let $G = Gal(K_\infty/K)$.*

We let $r_j = rank_{\mathcal{O}[[G]]}\left(Y^{(K_\infty)}_{A_j,\Sigma}\right)$, $1 \le j \le 2$. Let l be the minimal integer such that $\left(\pi^l Y^{(K_\infty)}_{A_1,\Sigma}\right)[\pi]$ is pseudo-null over $\mathcal{O}[[G]]$ in the sense of Section 2.1. Then the following statements hold.

(a) *If $A_1[\pi^l] \cong A_2[\pi^l]$ as G_K-modules and $r_2 \le r_1$, then $\mu\left(Y^{(K_\infty)}_{A_1,\Sigma}\right) \le \mu\left(Y^{(K_\infty)}_{A_2,\Sigma}\right)$.*

(b) *If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$, then $r_2 \le r_1$. If moreover $r_2 = r_1$, then*

$$\mu\left(Y^{(K_\infty)}_{A_1,\Sigma}\right) = \mu\left(Y^{(K_\infty)}_{A_2,\Sigma}\right)$$

and the modules $Y^{(K_\infty)}_{A_1,\Sigma}[\pi^\infty]$ and $Y^{(K_\infty)}_{A_2,\Sigma}[\pi^\infty]$ are pseudo-isomorphic in the sense of Section 2.1.

(c) *In particular, if $A_1[\pi] \cong A_2[\pi]$ and $r_2 = r_1$, then $\mu\left(Y^{(K_\infty)}_{A_1,\Sigma}\right) = 0$ holds if and only if $\mu\left(Y^{(K_\infty)}_{A_2,\Sigma}\right) = 0$.*

(d) *If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ for some integer l such that both $\left(\pi^l Y^{(K_\infty)}_{A_j,\Sigma}\right)[\pi]$, $1 \le j \le 2$, are pseudo-null, then $r_2 = r_1$ and $\mu\left(Y^{(K_\infty)}_{A_1,\Sigma}\right) = \mu\left(Y^{(K_\infty)}_{A_2,\Sigma}\right)$.*

Note that $Y^{(K_\infty)}_{A,\Sigma}$ may contain quite complicated pseudo-null submodules and it may be difficult in practice to determine the right value for l and to prove the isomorphisms $A_1[\pi^l] \cong A_2[\pi^l]$ needed in Theorem 1.1 (see [BS10, Section 3] for a concrete example for 9-congruent elliptic curves)—if we want to apply Theorem 1.1 in a non-cyclotomic setting, then the additional problem occurs of how to determine or at least estimate one of the two μ-invariants in order to derive information about the second one.

The *weak Leopoldt conjecture* for A over K_∞ holds if and only if $Y_{A,\Sigma}^{(K_\infty)}$ is a torsion $\mathcal{O}[[G]]$-module (see also [Lim17b, Lemma 7.1]). The authors are not aware of any example where this conjecture is known to fail. Nevertheless, we paid attention to proving Theorem 1.1 also in the higher rank setting, since this allows a formulation which is unconditional.

Theorem 1.1 will be proved in Section 3.1. The basic idea of the proof is to relate the π^k-torsion subgroups of the fine Selmer groups of A_j, $k \in \mathbb{N}$, to certain π^k-fine Selmer groups (defined in Section 2) which depend only on $A_j[\pi^k]$. In the case of admissible p-adic Lie extensions K_∞/K which are not strongly admissible, we can derive similar results under the hypothesis that $A_j(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$ and $j \in \{1, 2\}$ (see Theorem 3.7). In order to obtain this result, we use an argument which goes back to the paper of Greenberg and Vatsal (see [GV00, Proposition 2.8]). It also appears in work of Mazur and Rubin (see [MR04, Lemma 3.5.3]) and has been used in, e.g., [BS10, Pon20]. This approach is of particular interest if one wants to treat $\mathbb{Z}_p$-extensions K_∞ of K in which some prime above p or a ramified prime is completely split. In the special case of $\mathbb{Z}_p$-extensions, and under the additional hypotheses on the π-torsion which have been mentioned above, we can in fact go one step further and obtain results on the λ-invariants, provided that the $\mathcal{O}[[G]]$-modules $Y_{A_i,\Sigma}^{(K_\infty)}$ both are torsion:

Theorem 1.2 *In the setting of Theorem 3.7, suppose that $G \cong \mathbb{Z}_p$ and that both ranks r_1 and r_2 are zero. Then, in addition to the assertions of Theorem 3.7, the following two statements hold:*

(a) *If $l \in \mathbb{N}$ is large enough such that $\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}\right)[\pi] = \{0\}$ and $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$, then $\lambda\left(Y_{A_2,\Sigma}^{(K_\infty)}\right) \le \lambda\left(Y_{A_1,\Sigma}^{(K_\infty)}\right)$.*

(b) *If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ for some l such that both $\left(\pi^l Y_{A_j,\Sigma}^{(K_\infty)}\right)[\pi] = \{0\}$, $1 \le j \le 2$, then $\lambda\left(Y_{A_1,\Sigma}^{(K_\infty)}\right) = \lambda\left(Y_{A_2,\Sigma}^{(K_\infty)}\right)$.*

We remark that we do not have to assume that the μ-invariants vanish in Theorem 1.2.

Finally, in Section 4, we consider certain abelian non-p-extensions K_∞ of K. In two different settings (inspired by the two different cases treated in Section 3), we compare the $\mathcal{O}$-ranks of $Y_{A_1,\Sigma}^{(K_\infty)}$ and $Y_{A_2,\Sigma}^{(K_\infty)}$ and derive (in-)equalities analogous to those in Theorem 1.2. We also remark that the group ring $\mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$ is not well-behaved in this situation and the $\mathcal{O}$-rank is the natural substitute for the notion of Iwasawa λ-invariants.

2 Background and notation

2.1 Admissible *p*-adic Lie extensions and Iwasawa modules

We fix once and for all a rational prime p. Let F be a finite extension of $\mathbb{Q}_p$. We denote its ring of integers by $\mathcal{O}$ and a generator of its maximal ideal by π. Note that $\mathcal{O}/(\pi)$

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is a finite field with $q = p^f$ elements, where f is the inertia degree of p in $F/\mathbb{Q}_p$. For any Noetherian $\mathcal{O}$-module G, we denote by $G[\pi^\infty]$ the subgroup of π-power torsion elements; for any $i \in \mathbb{N}$,[1] $G[\pi^i]$ shall denote the subgroup of elements which are annihilated by π^i.

In this article, an *admissible p-adic Lie extension* K_∞ of a number field K will always be a normal extension K_∞/K such that

- $G := \mathrm{Gal}(K_\infty/K)$ is a compact pro-p, p-adic Lie group,
- $G[p^\infty] = \{0\}$, i.e., G does not contain any p-torsion elements, and
- the set $S_{\mathrm{ram}}(K_\infty/K)$ of primes of K ramifying in K_∞ is finite.

Let Σ be a finite set of finite primes of K. The pro-p-extension K_∞/K is called *strongly Σ-admissible* if it is admissible and moreover contains a $\mathbb{Z}_p$-extension L of K such that no prime in $\Sigma \cup S_{\mathrm{ram}}(K_\infty/K)$ is completely split in L. In this case, we fix L and denote by $H \subseteq G$ the subgroup fixing L. Note that any strongly Σ-admissible p-adic Lie extension K_∞/K is strongly $\Sigma \cup S_{\mathrm{ram}}(K_\infty/K)$-admissible. By abuse of notation, we always assume that Σ contains $S_{\mathrm{ram}}(K_\infty/K)$ if K_∞/K is a strongly Σ-admissible p-adic Lie extension.

An admissible p-adic Lie extension K_∞/K is called *strongly admissible* if it contains the cyclotomic $\mathbb{Z}_p$-extension of K. Since no prime of K splits completely in the cyclotomic $\mathbb{Z}_p$-extension, a strongly admissible p-adic Lie extension is strongly Σ-admissible for every finite set Σ.

If K_∞/K is an admissible p-adic Lie extension, then the completed group ring $\mathcal{O}[[G]] = \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G]]$ is a Noetherian domain (see [CH01, Theorem 2.3]), and we can define the $\mathcal{O}[[G]]$-rank of a finitely generated $\mathcal{O}[[G]]$-module X by

$$\mathrm{rank}_{\mathcal{O}[[G]]}(X) = \dim_{\mathcal{F}(G)}(\mathcal{F}(G) \otimes_{\mathcal{O}[[G]]} X),$$

where $\mathcal{F}(G)$ denotes the skew field of fractions of $\mathcal{O}[[G]]$ (see [GW04, Chapter 10]).

A finitely generated $\mathcal{O}[[G]]$-module X is called *pseudo-null* if X is torsion and $\mathrm{Ext}^1_{\mathcal{O}[[G]]}(X, \mathcal{O}[[G]]) = \{0\}$. Moreover, following Howson (see [How02, (33)]), we define the μ-invariant of a finitely generated $\mathcal{O}[[G]]$-module X as

$$\mu(X) = \sum_{i \geq 0} \mathrm{rank}_{\mathbb{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1} X[\pi^\infty]); \tag{2.1}$$

this is a finite sum as X is Noetherian.

Remark 2.1 Let X be a Noetherian π-primary $\mathcal{O}[[G]]$-module. Then there exists an integer m such that $\pi^m X = \{0\}$. Suppose now that $\mathrm{rank}_{\mathbb{F}_q[[G]]}(X[\pi]) = 0$. Then there exists an annihilator $f \in \mathcal{O}[[G]] \setminus \pi\mathcal{O}[[G]]$ of $X[\pi]$. In particular,

$$\pi^{m-1} f X = \{0\} \quad \text{and} \quad \pi^{m-2} f X \subseteq X[\pi].$$

Thus, we inductively obtain that $f^m X = \{0\}$. Therefore $\mathrm{rank}_{\mathbb{F}_q[[G]]}(X/\pi X) = 0$.

[1] Note that for us, the smallest natural number is 0.

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Lemma 2.2 (Lim) *Let G be a compact pro-pp-adic Lie group without p-torsion, and let X be a finitely generated $\mathcal{O}[[G]]$-module. Then*

$$rank_{\mathbb{F}_q[[G]]}(X/\pi X) = rank_{\mathbb{F}_q[[G]]}(X[\pi]) + rank_{\mathcal{O}[[G]]}(X).$$

Proof. This is [Lim17b, Proposition 4.12]. ∎

Corollary 2.3 *Let G be as in Lemma 2.2, and let X be a finitely generated $\mathcal{O}[[G]]$-module of rank r. Then*

$$rank_{\mathbb{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty]) = rank_{\mathbb{F}_q[[G]]}(\pi^i X/\pi^{i+1}X) - r$$

for each $i \in \mathbb{N}$.

Proof. By applying Lemma 2.2 to the $\mathcal{O}[[G]]$-modules $\pi^i X$ and $\pi^i X[\pi^\infty]$, we obtain that

$$\mathrm{rank}_{\mathbb{F}_q[[G]]}(\pi^i X/\pi^{i+1}X) = \mathrm{rank}_{\mathbb{F}_q[[G]]}((\pi^i X)[\pi]) + \mathrm{rank}_{\mathcal{O}[[G]]}(\pi^i X)$$

and

$$\begin{aligned}\mathrm{rank}_{\mathbb{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty]) &= \mathrm{rank}_{\mathbb{F}_q[[G]]}((\pi^i X[\pi^\infty])[\pi]) \\ &\quad + \mathrm{rank}_{\mathcal{O}[[G]]}(\pi^i X[\pi^\infty]).\end{aligned}$$

Now $(\pi^i X)[\pi] = (\pi^i X[\pi^\infty])[\pi]$, $\mathrm{rank}_{\mathcal{O}[[G]]}(\pi^i X) = r$ and $\mathrm{rank}_{\mathcal{O}[[G]]}(\pi^i X[\pi^\infty]) = 0$, and therefore, starting from the second equation,

$$\begin{aligned}\mathrm{rank}_{\mathbb{F}_q[[G]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty]) &= \mathrm{rank}_{\mathbb{F}_q[[G]]}((\pi^i X[\pi^\infty])[\pi]) \\ &= \mathrm{rank}_{\mathbb{F}_q[[G]]}((\pi^i X)[\pi]) \\ &= \mathrm{rank}_{\mathbb{F}_q[[G]]}(\pi^i X/\pi^{i+1}X) - r.\end{aligned}$$ ∎

The most important class of admissible p-adic Lie extensions are the $\mathbb{Z}_p$-extensions. A *$\mathbb{Z}_p$-extension* K_∞/K is a normal extension such that $G = \mathrm{Gal}(K_\infty/K)$ is isomorphic to the additive group of p-adic integers. In this special case, the theory of finitely generated $\mathcal{O}[[G]]$-modules is well understood: the completed group ring $\mathcal{O}[[G]]$ is isomorphic to the ring $\Lambda := \mathcal{O}[[T]]$ of formal power series in one variable. Each finitely generated Λ-module X is pseudo-isomorphic to an *elementary* Λ-module of the form

$$E_X = \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/(\pi^{e_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(h_j),$$

where $h_1, \ldots, h_t \in \Lambda$ are so-called distinguished polynomials. Here, *pseudo-isomorphic* means that there exists a Λ-module homomorphism $\varphi : X \longrightarrow E_X$ with finite kernel and cokernel. One defines the *(classical) Iwasawa invariants* of X by $\mu(X) := \sum_{i=1}^{s} e_i$ and $\lambda(X) := \sum_{j=1}^{t} \deg(h_j)$. This notation is well-defined since the classical μ-invariant coincides with the invariant $\mu(X)$ given in (2.1) in the special case of $\mathbb{Z}_p$-extensions:

Lemma 2.4 *Let X be a finitely generated Λ-module. Then the classical Iwasawa μ-invariant is equal to*

$$\sum_{i=0}^{\infty} rank_{\mathbb{F}_q[[T]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty]).$$

Proof. This proof is well-known (see, e.g., [Ven02, Section 3.4]), but we recall it for the convenience of the reader. Let E_X be an elementary Λ-module that is pseudo-isomorphic to X. Then, we can write $E_X = \Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/(\pi^{e_i}) \oplus E_\lambda$ for a torsion Λ-module E_λ which is a finitely generated free $\mathcal{O}$-module. Therefore, the classical Iwasawa invariants can be computed as

$$\mu(X) = \mu(E_X) = \sum_{i=0}^{\infty} |\{k \mid e_k \geq i+1\}| = \sum_{i=0}^{\infty} \mathrm{rank}_{\mathbb{F}_q[[T]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty])$$

because

$$\begin{aligned}\mathrm{rank}_{\mathbb{F}_q[[T]]}(\pi^i X[\pi^\infty]/\pi^{i+1}X[\pi^\infty]) &= \mathrm{rank}_{\mathbb{F}_q[[T]]}(\pi^i E_X[\pi^\infty]/\pi^{i+1}E_X[\pi^\infty]) \\ &= |\{k \mid e_k \geq i+1\}|\end{aligned}$$

for every $i \in \mathbb{N}$ ■.

2.2 Fine Selmer groups

For any discrete $\mathbb{Z}_p$-module M, we define the *Pontryagin dual* of M as

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$$

(i.e., the set of continuous homomorphisms).

If K is a number field and v denotes any prime of K, then K_v will always denote the completion of K at v. We denote by G_K the Galois group $\mathrm{Gal}(\overline{K}/K)$, where $\overline{K}$ denotes a fixed algebraic closure of K. If M is any G_K-module, then we let $H^i(K,M) := H^i(G_K, M)$ denote the corresponding Galois cohomology groups, $i \in \mathbb{N}$. Moreover, if L/K is an algebraic extension, then we write $H^i(L/K, M) = H^i(\mathrm{Gal}(L/K), M)$ for brevity.

Now fix a number field K. Let V be a finite dimensional F-vector space with a continuous action of $\mathrm{Gal}(\overline{K}/K)$ for some fixed algebraic closure $\overline{K}$ of K. Let T be a Galois stable $\mathcal{O}$-lattice in V and write $A = V/T$. Note that, as an $\mathcal{O}$-module, A is isomorphic to $(F/\mathcal{O})^d$ for some non-negative integer $d = \dim(V)$. In particular, $A = A[\pi^\infty]$ is π-primary, i.e., each element of the $\mathcal{O}$-module A is annihilated by some power of π. By abuse of terminology we will also refer to d as the dimension of A.

We denote by S_p and $S_{\mathrm{ram}}(A)$ the set of primes of K over p and the set of primes of K where V is ramified. For any algebraic extension $L \supseteq K$ we denote by $A(L)$ the maximal submodule of A on which $\mathrm{Gal}(\overline{K}/L)$ acts trivially. If L_v is the completion of a number field $L \supseteq K$ at some prime v, then we denote by $A(L_v)$ the maximal submodule of A on which the local absolute Galois group G_{L_v} acts trivially (here G_{L_v} is embedded canonically into the absolute Galois group G_L).

We mention an important and classical special case: let A be an abelian variety defined over the number field K. We assume that $F = \mathbb{Q}_p$, i.e., $\mathcal{O} = \mathbb{Z}_p$. Let $T = T_p(A) = \varprojlim_n A[p^n]$ be the Tate module of A and $V = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; then $V/T \cong A[p^\infty]$. In this setting, for any field L as above, the group $A(L)[p^\infty]$ is the usual group of L-rational p-power torsion points on A. Moreover, the ramified primes correspond to the primes of K where A has bad reduction, by the criterion of Néron–Ogg–Shafarevich (see [Lan97, Theorem IV.4.1]).

For the number field K, $A = V/T$ as above and a prime number p, we define, following [CS05], the (π-primary part of the) *fine Selmer group* of A over K as

$$\mathrm{Sel}_{0,A}(K) = \ker\left(H^1(K,A) \longrightarrow \prod_v H^1(K_v,A)\right).$$

In our applications, it will be more convenient to work with the following definition:

$$\mathrm{Sel}_{0,A,\Sigma}(K) = \ker\left(H^1(K_\Sigma/K,A) \longrightarrow \prod_{v\in\Sigma} H^1(K_v,A)\right)$$

for suitable (usually finite) sets Σ of primes of K containing all the ramified primes of the representation V and all primes above p. Here, we let K_Σ be the maximal algebraic extension of K unramified outside the primes in Σ. If $L \subseteq K_\Sigma$ is any, non-necessarily finite, extension, then we define

$$\mathrm{Sel}_{0,A,\Sigma}(L) = \varinjlim_{K\subseteq L'\subseteq L} \mathrm{Sel}_{0,A,\Sigma}(L'),$$

where L' runs through all finite subextensions $K \subseteq L' \subseteq L$. Here, we note that $K_\Sigma = L'_\Sigma$, since L/K is unramified outside of Σ, and therefore each $\mathrm{Sel}_{0,A,\Sigma}(L')$ is a subgroup of $H^1(K_\Sigma/K,A)$.

A priori, this definition depends on the choice of Σ. But if the cyclotomic $\mathbb{Z}_p$-extension of K, denoted by K_∞^c, is contained in L, then the definition becomes independent of the set Σ by a result of Sujatha and Witte (see [SW18, Section 3]). They also show that in this case the two definitions of the Selmer group given above coincide. In fact, their proof depends only on the fact that none of the primes in Σ is totally split in K_∞^c/K. Therefore, the definition of the fine Selmer group does not depend on the choice of Σ if we consider strongly Σ-admissible extensions K_∞/K.

Finally, we define π^i-fine Selmer groups, $i \in \mathbb{N}$, as

$$\mathrm{Sel}_{0,A[\pi^i],\Sigma}(K) = \ker\left(H^1(K_\Sigma/K,A[\pi^i]) \longrightarrow \prod_{v\in\Sigma} H^1(K_v,A[\pi^i])\right),$$

where Σ is as above. Note: these π^i-fine Selmer groups may depend on the choice of Σ even for algebraic extensions L of K which contain the cyclotomic $\mathbb{Z}_p$-extension K_∞^c (see [LKM16, proof of Theorem 5.1] for an example for abelian varieties).

Now let K_∞/K be an admissible p-adic Lie extension, and let Σ be a finite set of primes of K which contains $S_{\mathrm{ram}}(K_\infty/K) \cup S_p \cup S_{\mathrm{ram}}(A)$ (if $p = 2$, then we assume that K is totally imaginary). Then we can define fine Selmer groups of A over each number field $L \subseteq K_\infty$ containing K. We denote the corresponding Pontryagin duals by

$$Y_{A,\Sigma}^{(L)} = \mathrm{Sel}_{0,A,\Sigma}(L)^\vee,$$

and we define the projective limit

$$Y_{A,\Sigma}^{(K_\infty)} = \varprojlim_{K \subseteq L \subseteq K_\infty} Y_{A,\Sigma}^{(L)}$$

with respect to the corestriction maps (where L runs over the finite subextensions of K_∞/K).

3 Fine Selmer groups of congruent representations

The aim of this section is to study the relation between the Iwasawa invariants of the fine Selmer groups associated with two representations V_1 and V_2 defined over the same number field K. The representations we consider will always satisfy a congruence condition, meaning that $A_1[\pi^l]$ and $A_2[\pi^l]$ are isomorphic as G_K-modules for some integer l (where $A_i = V_i/T_i$ as usual). Note that this implies that the two representations have the same dimension d. We will always fix a set Σ of primes in K containing all ramified places for A_1 and A_2, and all places above p. Let K_∞/K be an admissible p-adic Lie extension. We consider two cases:

i) K_∞/K is strongly Σ-admissible (Section 3.1).
ii) K_∞/K is admissible and $A(K_v)[\pi] = 0$ for all $v \in \Sigma$ (Section 3.2).

Note that case ii) only becomes relevant if a prime of Σ is completely split in K_∞/K.

3.1 The generic case

In this section, we prove Theorem 1.1. The main ingredient in the proof is a relation between $\mathrm{Sel}_{0,A}(L)[\pi^l]$ and $\mathrm{Sel}_{0,A[\pi^l]}(L)$ for any finite subextension $K \subseteq L \subseteq K_\infty$ of the p-adic Lie extension K_∞/K.

Lemma 3.1 *Let A be associated with a representation of G_K of dimension d and let Σ be a finite set of primes of K containing S_p and $S_{\mathrm{ram}}(A)$. If $p = 2$, then we assume that K is totally imaginary. Let L/K be a finite extension which is contained in K_Σ. Then*

$$|v_p(|Sel_{0,A,\Sigma}(L))[\pi^k]|) - v_p(|Sel_{0,A[\pi^k],\Sigma}(L)|)| \le fdk(1 + |\Sigma(L)|)$$

for each integer $k \ge 1$, where $\Sigma(L)$ denotes the set of primes of L above Σ and f is the inertia degree of p in $F/\mathbb{Q}_p$.

Remark 3.2 Note that if A is the p-primary part of an abelian variety of dimension d then the corresponding representation has dimension $2d$.

Proof. We start with the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{0,A[\pi^k]}(L) & \longrightarrow & H^1(K_\Sigma/L, A[\pi^k]) & \longrightarrow & \bigoplus_{v\in\Sigma(L)} H^1(L_v, A[\pi^k]) \\ & \downarrow s & & \downarrow h & & \downarrow g \\ 0 \longrightarrow & \mathrm{Sel}_{0,A}(L)[\pi^k] & \longrightarrow & H^1(K_\Sigma/L, A)[\pi^k] & \longrightarrow & \bigoplus_{v\in\Sigma(L)} H^1(L_v, A)[\pi^k] \end{array}$$

Consider the exact sequence

$$0 \longrightarrow A[\pi^k] \longrightarrow A \xrightarrow{\cdot\pi^k} A \longrightarrow 0.$$

The surjectivity follows from the fact that A is divisible as $\mathcal{O}$-module. Note further that the representation V is unramified outside Σ. Thus, there is a well-defined action of $\mathrm{Gal}(K_\Sigma/L)$ on A and we can take K_Σ/L-cohomology of the exact sequence in order to see that the map h is surjective. Moreover,

$$\begin{aligned} \ker(h) &\cong \mathrm{coker}(\pi^k\colon H^0(K_\Sigma/L, A) \longrightarrow H^0(K_\Sigma/L, A)) \\ &= A(L)/\pi^k A(L). \end{aligned}$$

The last equality is due to the fact that all ramified primes are contained in Σ. Analogously, we see that g is surjective and that

$$\ker(g) \cong \bigoplus_{v\in\Sigma(L)} A(L_v)/\pi^k A(L_v).$$

We obtain the bounds $v_p(|\ker(h)|) \le dkf$ and $v_p(|\ker(g)|) \le dkf|\Sigma(L)|$. Using the exact sequence

$$0 \longrightarrow \ker(s) \longrightarrow \mathrm{Sel}_{0,A[\pi^k],\Sigma}(L) \longrightarrow \mathrm{Sel}_{0,A,\Sigma}(L)[\pi^k] \longrightarrow \mathrm{coker}(s) \longrightarrow 0, \tag{3.1}$$

we may conclude that $|v_p(|\mathrm{Sel}_{0,A[\pi^k],\Sigma}(L)|) - v_p(|\mathrm{Sel}_{0,A,\Sigma}(L)[\pi^k]|)|$ is bounded by

$$\begin{aligned} v_p(|\ker(s)|) + v_p(|\mathrm{coker}(s)|) &\le v_p(|\ker(h)|) + v_p(|\ker(g)|) \\ &\le fdk + dfk|\Sigma(L)|. \end{aligned}$$

■

Corollary 3.3 *Let A be associated with a representation of G_K, and let Σ be a finite set of primes of K containing S_p and $S_{ram}(A)$. If $p = 2$, then we assume that K is totally imaginary. Let K_∞/K be a strongly Σ-admissible p-adic Lie extension. Then $rank_{\mathbb{F}_q[[G]]}(\pi^i Y_{A,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A,\Sigma}^{(K_\infty)})$ equals*

$$rank_{\mathbb{F}_q[[G]]}\left(\varprojlim_{K\subseteq L\subseteq K_\infty} Sel_{0,A[\pi^{i+1}],\Sigma}(L)^\vee / \varprojlim_{K\subseteq L\subseteq K_\infty} Sel_{0,A[\pi^i],\Sigma}(L)^\vee\right)$$

for every $i \in \mathbb{N}$, where L runs over the finite subfields of K_∞/K.

Proof. Let $k \in \mathbb{N}$. For every finite subextension $L \subseteq K_\infty$ of K, we consider the exact sequence

$$0 \longrightarrow M^{(L)} \longrightarrow \mathrm{Sel}_{0,A,\Sigma}(L)[\pi^k]^\vee \longrightarrow \mathrm{Sel}_{0,A[\pi^k],\Sigma}(L)^\vee \longrightarrow N^{(L)} \longrightarrow 0,$$

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which is obtained from (3.1) by taking Pontryagin duals. In particular, $N^{(L)}$ is a finite abelian group of order at most p^{dkf}, and $M^{(L)} = \bigoplus_{v \in \Sigma(L)} G_v^{(L)}$, where each $G_v^{(L)}$ is a finite abelian group of order at most p^{dkf}.

Taking the projective limits along the $L \subseteq K_\infty$, we obtain an exact sequence

$$0 \longrightarrow M \longrightarrow Y_{A,\Sigma}^{(K_\infty)}/\pi^k Y_{A,\Sigma}^{(K_\infty)} \longrightarrow \varprojlim_{K \subseteq L \subseteq K_\infty} \mathrm{Sel}_{0,A[\pi^k],\Sigma}(L)^\vee \longrightarrow N \longrightarrow 0, \tag{3.2}$$

where N is a finite abelian group and where M is finitely generated over $\mathcal{O}[[H]]$ because no prime $v \in \Sigma$ splits completely in the $\mathbb{Z}_p$-extension K_∞^H of K which is fixed by $H \subseteq G$. In fact, replacing K by a finite subextension of K_∞^H if necessary (this does not affect the projective limit), we may assume that actually the primes $v \in \Sigma$ do not split at all in K_∞^H/K.

Letting $\Gamma := G/H \cong \mathbb{Z}_p$, the group ring $\mathcal{O}[[\Gamma]]$ can be identified with the ring $\Lambda = \mathcal{O}[[T]]$. Since M is finitely generated over $\mathcal{O}[[H]]$, there exists a nonconstant annihilator of M in $\mathcal{O}[[G]] \cong \mathcal{O}[[H]][[T]]$ by [CFK+ 05, Proposition 2.3 and Theorem 2.4]; in particular, the annihilator is not a power of π (note: the result in [CFK+ 05] is formulated for the case $\mathcal{O} = \mathbb{Z}_p$, but the proof goes through in our more general setting).

Considering now $k = i$ and $k = i + 1$, we may conclude that there exists a nonconstant annihilator in $\mathcal{O}[[G]]$ of the cokernels and kernels of both maps

$$Y_{A,\Sigma}^{(K_\infty)}/\pi^i Y_{A,\Sigma}^{(K_\infty)} \longrightarrow \varprojlim_{K \subseteq L \subseteq K_\infty} \mathrm{Sel}_{0,A[\pi^i],\Sigma}(L)^\vee$$

and

$$Y_{A,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A,\Sigma}^{(K_\infty)} \longrightarrow \varprojlim_{K \subseteq L \subseteq K_\infty} \mathrm{Sel}_{0,A[\pi^{i+1}],\Sigma}(L)^\vee.$$

Taking quotients proves the assertion of the corollary. ∎

We need one final auxiliary

Lemma 3.4 *Let A_1 and A_2 be associated with two representations V_1 and V_2 of G_K, and let Σ be a finite set of primes of K which contains $S_p \cup S_{ram}(A_1) \cup S_{ram}(A_2)$. If $p = 2$, then we assume that K is totally imaginary. We assume that $A_1[\pi^i]$ and $A_2[\pi^i]$ are isomorphic as G_K-modules for some $i \in \mathbb{N}$, $i \geq 1$.*

Then $Sel_{0,A_1[\pi^i],\Sigma}(L) \cong Sel_{0,A_2[\pi^i],\Sigma}(L)$ for every finite extension $L \subseteq K_\Sigma$ of K.

Proof. Let $\phi: A_1[\pi^i] \longrightarrow A_2[\pi^i]$ be a G_K-module homomorphism. As V_1 and V_2 are unramified outside of Σ, the group $\mathrm{Gal}(\overline{K}/K_\Sigma)$ acts trivially on A_1 and A_2 and we can interpret ϕ as a $\mathrm{Gal}(K_\Sigma/K)$-isomorphism. Then ϕ induces an isomorphism

$$\phi: H^1(K_\Sigma/L, A_1[\pi^i]) \longrightarrow H^1(K_\Sigma/L, A_2[\pi^i])$$

of G_K-modules.

For any prime v of L, the inclusion $G_{L_v} \hookrightarrow G_L$ of the local absolute Galois group at the completion L_v of L at v induces an isomorphism

$$H^1(L_v, A_1[\pi^i]) \longrightarrow H^1(L_v, A_2[\pi^i]).$$

The corresponding isomorphism between fine Selmer groups is now immediate. ■

Now we turn to the proof of our first main result.

Proof of Theorem 1.1 Let l be such that $(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi]$ is pseudo-null. By definition of the μ-invariant (see (2.1)) and Corollary 2.3, we have

$$\begin{aligned}\mu(Y_{A_1,\Sigma}^{(K_\infty)}) &= \sum_{i=0}^{\infty}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_1,\Sigma}^{(K_\infty)}\right) - r_1\right)\\ &= \sum_{i=0}^{l-1}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_1,\Sigma}^{(K_\infty)}\right) - r_1\right). \qquad (3.3)\end{aligned}$$

Now Corollary 3.3 implies that for both $j = 1$ and $j = 2$ and every $i \in \mathbb{N}$, the $\mathbb{F}_q[[G]]$-rank of $\pi^i Y_{A_j,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_j,\Sigma}^{(K_\infty)}$ equals

$$\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_j[\pi^{i+1}],\Sigma}(L)^\vee/\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_j[\pi^{i}],\Sigma}(L)^\vee\right).$$

Using that $A_1[\pi^l] \cong A_2[\pi^l]$, Lemma 3.4 implies that

$$\mathrm{Sel}_{0,A_1[\pi^i],\Sigma}(L) \cong \mathrm{Sel}_{0,A_2[\pi^i],\Sigma}(L)$$

for every $i \le l$. By (3.3), we may conclude that

$$\begin{aligned}\mu(Y_{A_1,\Sigma}^{(K_\infty)}) &= \sum_{i=0}^{l-1}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_1,\Sigma}^{(K_\infty)}\right) - r_1\right)\\ &= \sum_{i=0}^{l-1}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_2,\Sigma}^{(K_\infty)}\right) - r_1\right)\\ &\le \sum_{i=0}^{\infty}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{i+1}Y_{A_2,\Sigma}^{(K_\infty)}\right) - r_2\right) = \mu\left(Y_{A_2,\Sigma}^{(K_\infty)}\right).\end{aligned}$$

Here, we used the hypothesis $r_2 \le r_1$, i.e., $-r_1 \le -r_2$.

Now we prove assertion (b). In the following, we abbreviate $\mathrm{rank}_{\mathbb{F}_q[[G]]}$ to r. If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ then

$$\begin{aligned}r\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1}Y_{A_1,\Sigma}^{(K_\infty)}\right) &= r\left(\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_1[\pi^{l+1}],\Sigma}(L)^\vee/\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_1[\pi^{l}],\Sigma}(L)^\vee\right)\\ &= r\left(\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_2[\pi^{l+1}],\Sigma}(L)^\vee/\varprojlim_{K\subseteq L\subseteq K_\infty}\mathrm{Sel}_{0,A_2[\pi^{l}],\Sigma}(L)^\vee\right)\\ &= r\left(\pi^l Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{l+1}Y_{A_2,\Sigma}^{(K_\infty)}\right).\end{aligned}$$

Using Lemma 2.2 and the definition of l, we obtain

$$\begin{aligned}r\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1}Y_{A_1,\Sigma}^{(K_\infty)}\right) &= r\left(\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}\right)[\pi]\right) + \mathrm{rank}_{\mathbb{Z}_q[[G]]}\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}\right)\\ &= 0 + r_1,\end{aligned}$$

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and similarly

$$r\left(\pi^l Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_2,\Sigma}^{(K_\infty)}\right) \geq r_2.$$

This proves the first claim of (b).

If $r_2 = r_1$, then it follows from the above that $\mathrm{rank}_{\mathbb{F}_q[[G]]}((\pi^l Y_{A_2,\Sigma}^{(K_\infty)})[\pi]) = 0$. In view of Remark 2.1 this implies that

$$\mu\left(Y_{A_2,\Sigma}^{(K_\infty)}\right) = \sum_{i=0}^{l-1}\left(\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2,\Sigma}^{(K_\infty)}\right) - r_2\right) = \mu\left(Y_{A_1,\Sigma}^{(K_\infty)}\right),$$

proving the second claim of (b). Using the equality of ranks derived above we obtain that

$$\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1,\Sigma}^{(K_\infty)}\right) = \mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2,\Sigma}^{(K_\infty)}\right)$$

for all $0 \leq i \leq l$. Let $E_j = \bigoplus_{i=1}^{s_j} \mathcal{O}[[G]]/(\pi^{e_i^j})$ be the elementary $\mathcal{O}[[G]]$-module associated to $Y_{A_j,\Sigma}^{(K_\infty)}[\pi^\infty]$ via [Ven02, Theorem 3.40] (Venjakob's result is proven only for $\mathcal{O} = \mathbb{Z}_p$, but it is valid in our more general setting) and define

$$f_i^j = |\{k \mid e_k^j > i\}|.$$

Since $r_1 = r_2 =: r$ we obtain

$$\mathrm{rank}_{\mathbb{F}_q[[G]]}\left(\pi^i Y_{A_j,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_j,\Sigma}^{(K_\infty)}\right) = r + f_i^j$$

by Corollary 2.3. Therefore, $f_i^1 = f_i^2$ for all $0 \leq i \leq l$ and thus $E_1 = E_2$ from which the claim is immediate.

The assertion (c) is a special case of (b). Finally, if $(\pi^l Y_{A_2,\Sigma}^{(K_\infty)})[\pi]$ is also pseudo-null, then we can exchange the roles of A_1 and A_2 and obtain equality of μ-invariants. ■

3.2 The completely split case

Now we treat admissible p-adic Lie extensions K_∞/K such that some $v \in \Sigma$ may be completely split in K_∞/K. In this case, we work under the restrictive assumption that $A_j(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$ and $j \in \{1,2\}$ (meaning that the subgroup of $A_j[\pi]$ fixed by $G_{K_v} \subseteq G_K$ is trivial). First, we derive several auxiliary results, starting with a lemma which will serve as a substitute for Lemma 3.1.

Lemma 3.5 *Let A be associated with a p-adic G_K-representation V and let Σ be a finite set of primes of K containing $S_p \cup S_{ram}(A)$. If $p = 2$, then we assume that K is totally imaginary. Assume that $A(K_v)[p] = \{0\}$ for every $v \in \Sigma$. Then*

$$Sel_{0,A,\Sigma}(L)[\pi^i] \cong Sel_{0,A[\pi^i],\Sigma}(L)$$

for every finite normal p-extension $L \subseteq K_\Sigma$ of K and each $i \in \mathbb{N}$, $i \geq 1$.

Proof. The assumptions imply that $A(K)[\pi] = \{0\}$. Since L/K is a p-extension, it follows from [NSW08, Corollary (1.6.13)] that also $H^0(L, A[\pi]) = A(L)[\pi] = \{0\}$.

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Hence, $H^0(L, A) = \{0\}$. As V is unramified outside of Σ we obtain that $H^0(K_\Sigma/L, A) = 0$. Now consider the exact sequence

$$0 \longrightarrow A[\pi^i] \longrightarrow A \xrightarrow{\cdot\pi^i} A \longrightarrow 0.$$

Taking K_Σ/L-cohomology we obtain a second exact sequence

$$0 \longrightarrow H^1(K_\Sigma/L, A[\pi^i]) \longrightarrow H^1(K_\Sigma/L, A) \longrightarrow H^1(K_\Sigma/L, A),$$

where the last homomorphism is multiplication by π^i. Hence, we obtain the isomorphism

$$H^1(K_\Sigma/L, A[\pi^i]) \cong H^1(K_\Sigma/L, A)[\pi^i].$$

Let now w be a place in L above a prime $v \in \Sigma$. Using analogous arguments, we can derive from the hypothesis $A(K_v)[\pi] = \{0\}$ that

$$H^1(L_w, A[\pi^i]) \cong H^1(L_w, A)[\pi^i].$$ ■

Corollary 3.6 *Let A be as above, let K_∞/K be an admissible p-adic Lie extension, and let Σ be a finite set of primes of K containing $S_{ram}(K_\infty/K) \cup S_p \cup S_{ram}(A)$. If $p = 2$, then we assume that K is totally imaginary. Assume that $A(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$. Then*

$$Y_{A,\Sigma}^{(K_\infty)}/\pi^i Y_{A,\Sigma}^{(K_\infty)} \cong \varprojlim_{K \subseteq L \subseteq K_\infty} Sel_{0,A[\pi^i],\Sigma}(L)^\vee$$

for every $i \in \mathbb{N}$, $i \geq 1$, where the projective limit is taken over the finite normal subextensions L of K_∞/K.

Proof. In view of Lemma 3.5, we have isomorphisms

$$\mathrm{Sel}_{0,A,\Sigma}(L)[\pi^i] \cong \mathrm{Sel}_{0,A[\pi^i],\Sigma}(L)$$

for each L. By duality, $\pi^i Y_{A,\Sigma}^{(L)}$ is precisely the group acting trivially on $\mathrm{Sel}_{0,A,\Sigma}(L)[\pi^i]$. Therefore,

$$Y_{A,\Sigma}^{(L)}/\pi^i Y_{A,\Sigma}^{(L)} \cong (\mathrm{Sel}_{0,A,\Sigma}(L)[\pi^i])^\vee \cong \mathrm{Sel}_{0,A[\pi^i],\Sigma}(L)^\vee.$$

The result now follows since

$$Y_{A,\Sigma}^{(K_\infty)}/\pi^i Y_{A,\Sigma}^{(K_\infty)} \cong \varprojlim_{K \subseteq L \subseteq K_\infty} Y_{A,\Sigma}^{(L)}/\pi^i Y_{A,\Sigma}^{(L)}.$$ ■

We can now prove an analogon of Theorem 1.1:

Theorem 3.7 *Let A_1 and A_2 be associated with two representations V_1 and V_2 of G_K. Let K_∞/K be an admissible p-adic Lie extension, and let $G = Gal(K_\infty/K)$. Let Σ be a finite set of primes of K which contains $S_{ram}(K_\infty/K)$, S_p and the sets of primes of K where either V_1 or V_2 is ramified. If $p = 2$, then we assume that K is totally imaginary.*

Suppose that $A_j(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$ and $j \in \{1, 2\}$. We let $r_j = rank_{\mathcal{O}[[G]]}(Y_{A_j,\Sigma}^{(K_\infty)})$, $1 \leq j \leq 2$. Let l be minimal such that $(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi]$ is pseudo-null.

Then the statements from Theorem 1.1 hold.

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Proof. Suppose that $A_1[\pi^l] \cong A_2[\pi^l]$ as G_K-modules. Then

$$\begin{aligned}\pi^i Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_1,\Sigma}^{(K_\infty)} &\cong \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_1[\pi^{i+1}],\Sigma}(L)^\vee / \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_1[\pi^i],\Sigma}(L)^\vee \\ &\cong \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_2[\pi^{i+1}],\Sigma}(L)^\vee / \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_2[\pi^i],\Sigma}(L)^\vee \\ &\cong \pi^i Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{i+1} Y_{A_2,\Sigma}^{(K_\infty)}\end{aligned}$$

for every $i < l$, by Corollary 3.6 and Lemma 3.4. In particular, both $\mathbb{F}_q[[G]]$-modules have the same rank. Therefore, we can proceed as in the proof of Theorem 1.1. ∎

Proof of Theorem 1.2 The hypothesis in (a) implies that $\pi^l Y_{A_1,\Sigma}^{(K_\infty)}$ is $\mathcal{O}$-free. Let E be an elementary Λ-module pseudo-isomorphic to $\pi^l Y_{A_1,\Sigma}^{(K_\infty)}$. For any finitely generated $\mathcal{O}$-module M, we denote by $\mathrm{rank}_q(M)$ the dimension of $M/\pi M$ as $\mathbb{F}_q$-vector space. The following auxiliary lemma follows by using an argument given in the proof of [Kle17, Proposition 3.4(i)].

Lemma 3.8 *We have* $\mathrm{rank}_q(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}) = \mathrm{rank}_q(E)$. ∎

Proof. Since $\pi^l Y_{A_1,\Sigma}^{(K_\infty)}$ is $\mathcal{O}$-free, the maximal finite submodule of $Y_{A_1,\Sigma}^{(K_\infty)}$ is annihilated by π^l. Therefore, we have an injection $\varphi\colon \pi^l Y_{A_1,\Sigma}^{(K_\infty)} \longrightarrow E$ with finite cokernel. Moreover, since multiplication by π is injective on E, the quotients $E/\mathrm{im}(\varphi)$ and $\pi E/\pi\mathrm{im}(\varphi)$ are isomorphic, proving that

$$\mathrm{rank}_q(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}) = \mathrm{rank}_q(\mathrm{im}(\varphi))$$

indeed equals $\mathrm{rank}_q(E)$. ∎

Therefore,

$$|\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_1,\Sigma}^{(K_\infty)}| = q^{\mathrm{rank}_q(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})} = q^{\mathrm{rank}_q(E)} = q^{\lambda(E)} = q^{\lambda(Y_{A_1,\Sigma}^{(K_\infty)})}.$$

On the other hand, the maximal finite Λ-submodule of $Y_{A_2,\Sigma}^{(K_\infty)}$ need not be annihilated by π^l; therefore

$$|\pi^l Y_{A_2,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_2,\Sigma}, \Sigma^{(K_\infty)}| \geq q^{\lambda(Y_{A_2,\Sigma}^{(K_\infty)})}.$$

This concludes the proof of (a). If both $\pi^l Y_{A_1,\Sigma}^{(K_\infty)}$ and $\pi^l Y_{A_2,\Sigma}^{(K_\infty)}$ are $\mathcal{O}$-free, then we can exchange the roles of A_1 and A_2 and obtain equality of λ-invariants.

4 Non *p*-extensions

In this final section, we study the growth of fine Selmer groups of congruent Galois representations over normal algebraic extensions of K which are the compositum of finite r-extensions for suitable primes $r \neq p$. If $p = 2$, then we always assume that K is totally imaginary. Similarly as in Sections 3.1 and 3.2, we distinguish between two different settings, starting with one resembling the case which has been studied in Section 3.2.

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Theorem 4.1 *Let p be a fixed prime, let A_1 and A_2 be associated with two representations of G_K, and let K_∞/K be a normal algebraic extension. Let $\mathcal{P}$ be the set of primes r such that K_∞/K contains a finite subextension of degree r over K. Let Σ be a finite set of primes of K which contains S_p, $S_{ram}(K_\infty/K)$ and $S_{ram}(A_j)$, $j \in \{1,2\}$.*

We assume that $\dim(A_1) = \dim(A_2) =: d$, that $r \geq q^d$ for each $r \in \mathcal{P}$, and that $A_j(K_v)[\pi] = \{0\}$ for $j \in \{1,2\}$ and every $v \in \Sigma$. Then the following statements hold:

(a) *If $A_1[\pi] \cong A_2[\pi]$ as G_K-modules, then $Y_{A_1,\Sigma}^{(K_\infty)}$ is a finitely generated $\mathcal{O}$-module if and only if $Y_{A_2,\Sigma}^{(K_\infty)}$ is finitely generated over $\mathcal{O}$.*

(b) *Suppose that both $Y_{A_j,\Sigma}^{(K_\infty)}$, $j \in \{1,2\}$, are finitely generated over $\mathcal{O}$. Let $l \in \mathbb{N}$ be large enough such that $(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi] = \{0\}$. If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ as G_K-modules, then $\mathrm{rank}_{\mathcal{O}}(Y_{A_2,\Sigma}^{(K_\infty)}) \leq \mathrm{rank}_{\mathcal{O}}(Y_{A_1,\Sigma}^{(K_\infty)})$.*

(c) *In the setting of (b), suppose that $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ for some l such that both $(\pi^l Y_{A_j,\Sigma}^{(K_\infty)})[\pi]$ are trivial. Then $\mathrm{rank}_{\mathcal{O}}(Y_{A_2,\Sigma}^{(K_\infty)}) = \mathrm{rank}_{\mathcal{O}}(Y_{A_1,\Sigma}^{(K_\infty)})$.*

Proof. The proof is analogous to the proofs of Theorems 3.7 and 1.2. Note that $Y_{A_j,\Sigma}^{(K_\infty)}$ is a finitely generated $\mathcal{O}$-module if and only if $Y_{A_j,\Sigma}^{(K_\infty)}/\pi Y_{A_j,\Sigma}^{(K_\infty)}$ is finite. Recall that $Y_{A_j,\Sigma}^{(K_\infty)}/\pi Y_{A_j,\Sigma}^{(K_\infty)} = \varprojlim_{K \subseteq L \subseteq K_\infty} \mathrm{Sel}_{0,A_j,\Sigma}(L)[\pi]^\vee$. Claim (a) now follows from the fact that $A_1[\pi] \cong A_2[\pi]$, by using the following

Lemma 4.2 *Let A be associated with a G_K-representation of dimension d, let Σ be a finite set of primes of K containing $S_p \cup S_{ram}(A)$. If $p = 2$, then we assume that K is totally imaginary. Assume that $A(K_v)[\pi] = \{0\}$ for every $v \in \Sigma$.*

Let $L \subseteq K_\Sigma$ be a finite normal extension of K such that each prime number r dividing $[L:K]$ satisfies $r \geq q^d$. Then

$$Sel_{0,A,\Sigma}(L)[\pi^i] \cong Sel_{0,A[\pi^i],\Sigma}(L)$$

for each $i \in \mathbb{N}$, $i \geq 1$. ■

Proof. By assumption $A(K)[\pi] = \{0\}$. We mimic the proof of [NSW08, Corollary (1.6.13)] and show that also $H^0(L, A[\pi]) = A(L)[\pi] = \{0\}$. Let r be the smallest prime number dividing $[L:K]$. Since $A(L)[\pi] \setminus A(K)[\pi]$ is the disjoint union of $\mathrm{Gal}(L/K)$-orbits with more than one element, the cardinality of each such orbit is divisible by some prime $r' \geq r$. Thus, if there exists at least one orbit containing more than one element, then

$$|A(L)[\pi]| \geq |A(K)[\pi]| + r' \geq |A(K)[\pi]| + r.$$

On the other hand, $|A(L)[\pi]| \leq q^d$. Since $r \geq q^d$ by assumption, we obtain that such a nontrivial orbit cannot exist. Therefore, $A(L)[\pi] = \{0\}$. Now we can proceed as in the proof of Lemma 3.5. ■

For points (b) and (c), we note that $\pi^l Y_{A_1,\Sigma}^{(K_\infty)}$ is a free $\mathcal{O}$-module with the property that $|\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_1,\Sigma}^{(K_\infty)}| = q^{\mathrm{rank}_q(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})} = q^{\mathrm{rank}_{\mathcal{O}}(Y_{A_1,\Sigma}^{(K_\infty)})}$, where rank_q is defined

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as in the proof of Theorem 1.2. As the maximal finite submodule of $Y_{A_2,\Sigma}^{(K_\infty)}$ is not necessarily annihilated by π^l, claim (b) follows. In the situation of claim (c), we can interchange the roles of A_1 and A_2 in order to obtain equality.

Now we turn to the second result for non-p-extensions. As in Theorem 4.1, we let $\mathcal{P} = \mathcal{P}(K_\infty)$ be the set of prime numbers r such that K_∞ contains an extension of K of degree r.

Theorem 4.3 *Let p be a fixed prime, let A_1 and A_2 be associated with two G_K-representations, and let K_∞/K be an abelian algebraic extension such that $p \notin \mathcal{P}(K_\infty)$. Let Σ be a finite set of primes of K which contains S_p, $S_{ram}(K_\infty/K)$ and $S_{ram}(A_j)$, $j \in \{1,2\}$.*

We assume that each prime $v \in \Sigma$ is finitely split in K_∞/K.

(a) *If $A_1[\pi] \cong A_2[\pi]$ as G_K-modules, then $Y_{A_1,\Sigma}^{(K_\infty)}$ is a finitely generated $\mathcal{O}$-module if and only if $Y_{A_2,\Sigma}^{(K_\infty)}$ is finitely generated over $\mathcal{O}$.*

Suppose now that for each $j \in \{1,2\}$ and every $w \in \Sigma(K_\infty)$, the group $A_j(K_{\infty,w})[\pi^\infty]$ is finite. Then also the following statements hold:

(b) *Suppose that both $Y_{A_j,\Sigma}^{(K_\infty)}$, $j \in \{1,2\}$, are finitely generated over $\mathcal{O}$. Let $l \in \mathbb{N}$ be large enough such that $(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi] = \{0\}$ and $\pi^l A_1(K_{\infty,w})[\pi^\infty] = \{0\}$ for every $w \in \Sigma(K_\infty)$. If $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ as G_K-modules, then $rank_{\mathcal{O}}(Y_{A_2,\Sigma}^{(K_\infty)}) \le rank_{\mathcal{O}}(Y_{A_1,\Sigma}^{(K_\infty)})$.*

(c) *In the setting of (b), if $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$ for some l such that both $(\pi^l Y_{A_j,\Sigma}^{(K_\infty)})[\pi]$ and all the groups $\pi^l A_j(K_{\infty,w})[\pi^\infty]$, $j \in \{1,2\}$, are trivial, then*

$$rank_{\mathcal{O}}\left(Y_{A_2,\Sigma}^{(K_\infty)}\right) = rank_{\mathcal{O}}\left(Y_{A_1,\Sigma}^{(K_\infty)}\right).$$

Proof. We first note that $Y_{A_j,\Sigma}^{(K_\infty)}$ is finitely generated over $\mathcal{O}$ if and only if the quotient $Y_{A_j,\Sigma}^{(K_\infty)}/\pi Y_{A_j,\Sigma}^{(K_\infty)}$ is finite. The exact sequence (3.2) from the proof of Corollary 3.3 implies that the kernels and cokernels of the maps

$$Y_{A_j,\Sigma}^{(K_\infty)}/\pi Y_{A_j,\Sigma}^{(K_\infty)} \longrightarrow \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_j[\pi],\Sigma}(L)^\vee$$

are finite for both $j = 1$ and $j = 2$ (here, we use the hypothesis that each $v \in \Sigma$ is finitely split in K_∞/K). The assertion (a) therefore follows from the isomorphism $A_1[\pi] \cong A_2[\pi]$.

More generally, we have exact sequences

$$0 \longrightarrow M_k \longrightarrow Y_{A_j,\Sigma}^{(K_\infty)}/\pi^k Y_{A_j,\Sigma}^{(K_\infty)} \longrightarrow \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_j[\pi^k],\Sigma}(L)^\vee \longrightarrow N_k \longrightarrow 0$$

for finite abelian groups M_k and N_k, $k \in \mathbb{N}$. Moreover, from the proof of Lemma 3.1, we obtain exact sequences

$$(4.1)\quad \begin{aligned} &0 \longrightarrow N_k \longrightarrow A_j(K_\infty)[\pi^\infty]/\pi^k A_j(K_\infty)[\pi^\infty] \longrightarrow C_k \longrightarrow 0,\\ &0 \longrightarrow C_k \longrightarrow \bigoplus_{w\in\Sigma(K_\infty)} A_j(K_{\infty,w})[\pi^\infty]/\pi^k A_j(K_{\infty,w})[\pi^\infty] \longrightarrow M_k \longrightarrow 0 \end{aligned}$$

for every $k \in \mathbb{N}$ and $j \in \{1,2\}$ (note that $M_k \cong M_k^\vee$ and $N_k \cong N_k^\vee$). Using the assumption that $A_j(K_{\infty,w})[\pi^\infty]$ is finite, we can deduce from (4.1) the following equality:

$$|N_k| = \frac{|A_j(K_\infty)[\pi^\infty]/\pi^k A_j(K_\infty)[\pi^\infty]|}{|\bigoplus_{w\in\Sigma(K_\infty)} A_j(K_{\infty,w})[\pi^\infty]/\pi^k A_j(K_{\infty,w})[\pi^\infty]|}|M_k|.$$

To simplify notation we write $B_k = \bigoplus_{w\in\Sigma(K_\infty)} A_j(K_{\infty,w})[\pi^\infty]/\pi^k A_j(K_{\infty,w})[\pi^\infty]$ and $C_k = A_j(K_\infty)[\pi^\infty]/\pi^k A_j(K_\infty)[\pi^\infty]$. Clearly, C_k can be seen as a subgroup of each direct term of B_k. Hence,

$$\frac{|N_{k+1}|}{|N_k|} = \frac{|M_{k+1}|}{|M_k|}\frac{|C_{k+1}||B_k|}{|C_k||B_{k+1}|} \le \frac{|M_{k+1}|}{|M_k|}.$$

Therefore, $|\pi^k Y_{A_j,\Sigma}^{(K_\infty)}/\pi^{k+1} Y_{A_j,\Sigma}^{(K_\infty)}|$ differs from

$$|\varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_j[\pi^{k+1}],\Sigma}(L)^\vee / \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_j[\pi^k],\Sigma}(L)^\vee|$$

by a factor $\frac{|M_k||N_{k+1}|}{|N_k||M_{k+1}|}$ which is smaller than or equal to 1. In fact, for $k = l$ and $j = 1$, this factor is 1 by our hypotheses. Therefore,

$$|\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_1,\Sigma}^{(K_\infty)}| = |\varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_2[\pi^{l+1}],\Sigma}(L)^\vee / \varprojlim_{K\subseteq L\subseteq K_\infty} \mathrm{Sel}_{0,A_2[\pi^l],\Sigma}(L)^\vee|$$

because $A_1[\pi^{l+1}] \cong A_2[\pi^{l+1}]$. Note that the factor $\frac{|M_k||N_{k+1}|}{|N_k||M_{k+1}|}$ can be strictly smaller than 1 for A_2. This happens if π^l does not annihilate the π-primary subgroups of the $A_2(K_{\infty,w})$. We have thus shown that

$$\mathrm{rank}_q\left(\pi^l Y_{A_2,\Sigma}^{(K_\infty)}\right) \le \mathrm{rank}_q\left(\pi^l Y_{A_1,\Sigma}^{(K_\infty)}\right),$$

where rank_q is defined as in the proof of Theorem 1.2. The assertion (b) follows since

$$|\pi^l Y_{A_1,\Sigma}^{(K_\infty)}/\pi^{l+1} Y_{A_1,\Sigma}^{(K_\infty)}| = q^{\mathrm{rank}_{\mathcal{O}}(Y_{A_1,\Sigma}^{(K_\infty)})}$$

because $(\pi^l Y_{A_1,\Sigma}^{(K_\infty)})[\pi] = \{0\}$ by assumption; the $\mathcal{O}$-rank of $Y_{A_2,\Sigma}^{(K_\infty)}$ can be strictly smaller than the corresponding q-rank, as in the proof of Theorem 1.2.

Finally, (c) follows by interchanging the roles of A_1 and A_2 in the previous proof. ∎

Remark 4.4 Going through the proof of the theorem, one sees that actually the finiteness of $A_2(K_{\infty,w})[\pi^\infty]$ is needed only for at least one $w \in \Sigma(K_\infty)$. Moreover, if one assumes that $A(K_\infty)[\pi] = \{0\}$, then one can drop completely the condition that the group $A_2(K_{\infty,w})[\pi^\infty]$ is finite for every $w \in \Sigma(K_\infty)$ in point (b) of the above theorem.

Remark 4.5 In order to give some evidence for the finiteness assumptions in the last two parts of Theorem 4.3, we mention some known results in the special setting of abelian varieties. In the following, we let A be an abelian variety defined over the number field K, and we consider $\mathcal{O} = \mathbb{Z}_p$.

Actually the following conditions are sufficient for ensuring finite torsion groups, i.e., not only finite p-torsion for some fixed prime p.

(i) If K_∞/K is a finite extension, then the torsion subgroup of $A(K_{\infty,w})$ is finite for each prime w by the theorem of Mattuck (see [Mat55]).

(ii) If A has potentially good and ordinary reduction at some prime q, then the torsion subgroup of $A(K_{\infty,w})$ is finite for each $w \mid q$ if K_∞ is a finite extension of the cyclotomic $\mathbb{Z}_q$-extension of K (see [Ima75]).

(iii) For global fields, more is known: let Ω be the field obtained from K by adjoining *all* roots of unity in some fixed algebraic closure of K (i.e., Ω contains the cyclotomic $\mathbb{Z}_q$-extensions for all primes q). Then it follows from results of Ribet (see [KL81, Appendix, Theorem 1]) that the torsion group of $\Omega(A)$ is finite.

We conclude by mentioning a special setting, namely of an elliptic curve $A = E$ defined over K, in which $Y_{A,\Sigma}^{(K_\infty)}$ is known to be finitely generated over $\mathcal{O} = \mathbb{Z}_p$ for an infinite non-p-extension K_∞ of K.

Example 4.6 Let N be an imaginary quadratic number field, and let E be an elliptic curve defined over N with complex multiplication by the ring of integers $\mathcal{O}_N$ of N. Let $q > 3$ be a prime of good reduction which splits in N, $q\mathcal{O}_N = \mathfrak{q}\overline{\mathfrak{q}}$. Let K be an abelian extension of N which is tamely ramified at $\mathfrak{q}$, and let $K_\infty = K \cdot N(E[\mathfrak{q}^\infty])$.

Now suppose that $p \neq q$ is a prime number which is co-prime with $6[K : N]$. We assume that p splits in $N/\mathbb{Q}$, does not ramify in K/N and that E has good reduction at the primes of N above p. Let Σ be a finite set of primes of K which contains $S_{\mathrm{ram}}(K_\infty/K)$, S_p and $S_{\mathrm{ram}}(E)$. If $E(K)[p] = \{0\}$ and $E(K_v)[p] = \{0\}$ for every $v \in \Sigma$, then $\mathrm{rank}_{\mathbb{Z}_p}(Y_{E',\Sigma}^{(K_\infty)})$ is finite for every elliptic curve E' which is defined over K and satisfies $S_{\mathrm{ram}}(E') \subseteq \Sigma$ and $E'[p] \cong E[p]$ as G_K-modules.

Indeed, by [Lam15, Theorem 1.2], the hypotheses of the above example imply that in fact the Pontryagin dual $X_E^{(K_\infty)}$ of the (ordinary) Selmer group over K_∞ is finitely generated over $\mathbb{Z}_p$. Now we can apply Theorem 4.1.

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