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TP_2 DEPENDENCE OF SAMPLE SPACINGS WITH APPLICATIONS

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This article investigates TP_2 dependence of sample spacings. It is proved that TP_2 (RR_2) dependence between a general spacing and a nonadjacent order statistic might be characterized by the DLR (ILR) property of the parent distribution, and TP_2 dependence between any pair of consecutive spacings might be characterized by the DLR aging property of the population. Furthermore, TP_2 dependence between any two consecutive spacings in multiple outliers exponential models is also derived. In addition, some applications in reliability and business auction are presented as well.

1. INTRODUCTION

Sample spacings have received tremendous attention from numerous researchers during the past several decades because they play important roles in reliability theory, life testing, data analysis, goodness-of-fit tests, and other related areas. The past two decades witnessed an extensively amount of stochastic comparisons among sample spacings. Readers can refer to Barlow and Proschan [3], Kochar [14, 15], Misra and van der Meulen [20], Hu, Wang, and Zhu [6], Hu and Zhuang [7], and Xu and Li [24] for those detailed statements. On the other hand, dependence among sample spacings are also of great interest for the reason that, in actuarial science, reliability theory, survival analysis, system safety, and so forth, it is more important to judge whether an increase of some sample space tends to incur the increase or decrease of another one. In the recent two decades, some authors, for example, Barlow and Proschan [3], Kim and David [13], and Khaledi and Kochar [11], devoted themselves to study the dependence among sample spacings and many interesting results have been built. The purpose of this article is to further

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investigate TP_2 properties of general spacings. For convenience, the term *increasing* is used instead of monotone nondecreasing and the term *decreasing* is used instead of monotone nonincreasing throughout this article; it is also implicitly assumed that all random variables under consideration are nonnegative, are absolutely continuous and have zero as the common left end point of the supports.

Recall that a bivariate function $h(x, y)$ is said to be *sign-regular* (SR_2) of order 2 if, for ε_1 and ε_2 equal to $+1$ or -1, $\varepsilon_1 h(x, y) \geq 0$ and $\varepsilon_2[h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1)] \geq 0$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$. For two random variables X and Y with joint probability density or mass function $f(x, y)$, (X, Y), or $f(x, y)$ is said to be *totally positive* (TP_2) of order 2, denoted by $TP_2(X, Y)$, if $\varepsilon_2 = 1$; (X, Y) or $f(x, y)$ is said to be *reversely regular* (RR_2) of order 2, denoted by $RR_2(X, Y)$, if $\varepsilon_2 = -1$. For more details on SR_2, readers can refer to Karlin [8]. As a stronger notion of dependence between random variables, TP_2 and RR_2 also have their multivariate versions, called the *multivariate total positive* of order 2 (MTP_2) and *multivariate reversely regular* of order 2 (MRR_2), respectively. A random vector $(X_1, \ldots, X_n)$ is said to be MTP_2 (MRR_2) if its probability density or mass function $f(x_1, \ldots, x_n)$ satisfies $f(\mathrm{x} \vee \mathrm{y})f(\mathrm{x} \wedge \mathrm{y}) \geq (\leq) f(\mathrm{x})f(\mathrm{y})$ for all $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_n)$; here, $\mathrm{x} \vee \mathrm{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$, $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$, $x_1 \vee y_1 = \max\{x_1, y_1\}$, and $x_1 \wedge y_1 = \min\{x_1, y_1\}$. One can refer to Karlin and Rinott [9, 10] for further discussions on MTP_2 and MRR_2.

The stochastic monotone property is also useful to characterize the weaker dependence of random variables. A random variable Y is said to be *stochastically increasing* in X, denoted by $SI(Y|X)$, if $P(Y > y|X = x)$ is increasing in x for all y; X is said to be *stochastically decreasing* in X, denoted by $SD(Y|X)$, if $P(Y > y|X = x)$ is decreasing in x for all y. It has been shown that

$$TP_2(X, Y) \Longrightarrow SI(Y|X) \quad \text{and} \quad RR_2(X, Y) \Longrightarrow SD(Y|X).$$

Readers can refer to Müller and Stoyan [21] for more detailed statements.

The main results in this article are also closely related to some aging notions. A random variable X with distribution F and density f is said to be of *decreasing likelihood ratio* (DLR) if $\log f(x)$ is convex; it is said to be of *increasing likelihood ratio* (ILR) if $\log f(x)$ is concave; X is said to be of *decreasing failure rate* (DFR) if $f(t)/\bar{F}(t)$ decreases on its interval of support; it is said to be of *increasing failure rate* (IFR) if $f(t)/\bar{F}(t)$ increases on its interval of support. The following chains of implication are well known:

$$\text{ILR} \Longrightarrow \text{IFR} \quad \text{and} \quad \text{DLR} \Longrightarrow \text{DFR}.$$

Aging conceptions play important roles in maintenance and degradation; we refer readers to Barlow and Proschan [3] and Müller and Stoyan [21] for comprehensive statements.

For an independent sample $X_1, \ldots, X_n$, let $0 \equiv X_{0:n} \leq X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. Denote X the population random variable when $X_1, \ldots, X_n$ are identical. Some authors have made efforts to characterize particular classes of life distributions based on order statistics and sample spacings; pioneering work is due to Langberg, Leon, and Prochan [16], who presented the following characterizations for IFR and its dual notion:

$$X \text{ is of DFR (IFR)} \Longleftrightarrow \text{SI (SD)}(X_{s+1:n} - X_{s:n}|X_{s:n}) \quad \text{for } n > s \geq 1. \tag{1.1}$$

Barlow and Proschan [3] presented the following dependent result of sample spacings:

$$X \text{ is of DFR} \Longrightarrow \text{SI}(X_{s+p:n} - X_{s:n}|X_{s:n}) \quad \text{for } n \geq s+p > s \geq 1. \tag{1.2}$$

Subsequently, Kim and David [13] showed that

$$\begin{aligned} &X \text{ is of DFR (IFR)} \Longrightarrow \text{SI (SD)}(X_{r+p:n} - X_{r:n}|X_{s:n}) \\ &\text{for } n \geq r+p > r \geq s \geq 1. \end{aligned} \tag{1.3}$$

In combination with (1.1)–(1.3), it can be concluded that

$$\begin{aligned} &X \text{ is of DFR (IFR)} \Longleftrightarrow \text{SI (SD)}(X_{r+p:n} - X_{r:n}|X_{s:n}) \\ &\text{for } n \geq r+p > r \geq s \geq 1. \end{aligned} \tag{1.4}$$

On the other hand, other researchers devoted themselves to the dependence of sample spacings. Karlin and Rinott [9] were among the first to point out that

$$\text{MTP}_2(X_{1:n} - X_{0:n}, X_{2:n} - X_{1:n}, \ldots, X_{n-1:n} - X_{n:n}) \tag{1.5}$$

provided the DLR property of the population X. As a direct consequence, it holds that, for an independent and identically distributed (i.i.d.) sample from a DLR population,

$$\text{TP}_2(X_{i+1:n} - X_{i:n}, X_{i:n} - X_{i-1:n}) \quad \text{for } n > i \geq 1. \tag{1.6}$$

At a later time, Khaledi and Kochar [11] proved that (1.6) also holds in multiple outliers exponential models, where, for some $m \in \{1, \ldots, n\}$, $X_1, \ldots, X_m$ are exponential with parameter λ and $X_{m+1}, \ldots, X_n$ are exponential with another parameter λ^*. Recently, Hu et al. [6, Them. 3.1] further showed that (1.5) remains valid in multiple outliers exponential models and, hence, (1.6) can be viewed as a special case.

The rest of this article is organized as follows. Section 2 deals with distributions having a monotone likelihood ratio. In parallel to (1.4), Theorem 2.4 develops a characterization result, which claims that X is of DLR (ILR) property if and only if $X_{r+p:n} - X_{r:n}$ and $X_{s:n}$ are TP_2 (RR_2). Theorem 2.6 further proves that in fact X is of DLR (ILR) property if and only if any pair of $X_{j:n} - X_{k:n}$ and $X_{k:n} - X_{l:n}$ for any

$n \geq j > k > l \geq 0$ are TP_2 dependent; this forms a strengthened version of (1.6). Section 3 provides a discussion on the dependence of sample spacings in the context of multiple outliers exponential models. Theorem 3.2 builds the TP_2 dependence between $X_{j:n} - X_{k:n}$ and $X_{k:n} - X_{l:n}$ for any $n \geq j > k > l \geq 1$, which can be regarded as an extension of (1.6).

2. SAMPLE SPACINGS FROM ILR AND DLR OBSERVATIONS

Before proceeding to the main results, let us introduce the following three lemmas, which will be repeatedly used in the proof of our main theorems in sequel. More details on them can be found in Karlin [8] and Khaledi and Kochar [12].

LEMMA 2.1 [8]: *Let A, B, and C be subsets of the real line, let L(x, z) be SR_2 for $x \in A$ and $z \in B$ and let M(z, y) be SR_2 for $z \in B$ and $y \in C$. Then, for a σ-finite measure μ,*

$$K(x, y) = \int_B L(x, z)M(z, y)d\mu(z)$$

is also SR_2 for $x \in A$ and $y \in C$ and $\varepsilon_i(K) = \varepsilon_i(L)\varepsilon_i(M)$ for $i = 1, 2$.

LEMMA 2.2 [8]: *Suppose λ, ω, and γ traverse the ordered sets Λ, Ω, and Γ, respectively. If $f(\lambda, \omega, \gamma) > 0$ is TP_2 in each pair of its variables with the other one fixed and $g(\lambda, \gamma)$ is TP_2, then, for a σ-finite measure μ,*

$$h(\lambda, \omega) = \int_\Gamma f(\lambda, \omega, \gamma)g(\lambda, \gamma)d\mu(\gamma)$$

is also TP_2 in $(\lambda, \omega) \in \Lambda \times \Omega$.

LEMMA 2.3 [12]: *Suppose λ, ω, and γ traverse the ordered sets Λ, Ω, and Γ, respectively. If $f(\lambda, \omega, \gamma) > 0$ and $g(\lambda, \gamma)$ are TP_2 in (λ, γ) and $f(\lambda, \omega, \gamma)$ is RR_2 in (λ, ω) and (ω, γ), then, for a σ-finite measure μ,*

$$h(\lambda, \omega) = \int_\Gamma f(\lambda, \omega, \gamma)g(\lambda, \gamma)d\mu(\gamma)$$

is also RR_2 in $(\lambda, \omega) \in \Lambda \times \Omega$.

Now, we are ready to state the main results.

THEOREM 2.4:

(*i*) *X is of DLR if and only if*

$$\mathrm{TP}_2(X_{r+p:n} - X_{r:n}, X_{s:n}) \quad \textit{for any}\ n \geq r + p > r \geq s \geq 1.$$

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(*ii*) *X is of ILR if and only if*

$$\mathrm{RR}_2(X_{r+p:n} - X_{r:n}, X_{s:n}) \quad \textit{for any}\, n \geq r+p > r \geq s \geq 1.$$

PROOF:

(i) $\Rightarrow$Let X have the distribution function F and density function f; $\bar{F} = 1 - F$ is the survival function. Denote by $f_{s,r,r+p}(x, y, z)$ and $f_{s,r+p-r}(x, y)$ the joint density functions of $(X_{s:n}, X_{r:n}, X_{r+p:n})$ and $(X_{r+p:n} - X_{r:n}, X_{s:n})$, respectively.

For $r > s$,

$$\begin{aligned} f_{s,r+p-r}(x, y) &= \int_y^\infty f_{s,r,r+p}(y, t, x+t)dt \\ &= C_1 \int_y^\infty F^{s-1}(y)[F(t) - F(y)]^{r-s-1}[F(x+t) - F(t)]^{p-1} \\ &\qquad \times \bar{F}^{n-r-p}(x+t)f(y)f(t)f(x+t)dt \\ &= C_1 \int_0^\infty L(x, t)M(t, y)dt, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} C_1 &= \frac{n!}{(s-1)!(r-s-1)!(p-1)!(n-r-p)!}, \\ L(x, t) &= [F(x+t) - F(t)]^{p-1}\bar{F}^{n-r-p}(x+t)f(x+t), \\ M(t, y) &= F^{s-1}(y)[F(t) - F(y)]^{r-s-1}f(t)f(y)I(t \geq y). \end{aligned}$$

Since X is of DLR, by Lemma 2.1(ii) in Misra and van der Meulen [20], $L(x, t)$ is TP_2 in $(x, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ for $p \geq 1$. Similarly, $M(t, y)$ is also TP_2 in $(t, y) \in \mathbb{R}^+ \times \mathbb{R}^+$. So, from Lemma 2.1, the TP_2 property between $X_{r+p:n} - X_{r:n}$ and $X_{s:n}$ follows immediately.

For $r = s$, the joint density function of $X_{r+p:n} - X_{r:n}$ and $X_{r:n}$ is

$$\begin{aligned} f_{r,r+p-r}(x, y) = C_2[F(y)]^{r-1}[F(x+y) - F(y)]^{p-1} \\ [\bar{F}(x+y)]^{n-r-p}f(y)f(x+y), \end{aligned} \tag{2.2}$$

where

$$C_2 = \frac{n!}{(r-1)!(p-1)!(n-r-p)!}.$$

For $x_1 \le x_2$,

$$\frac{f_{r,r+p-r}(x_2, y)}{f_{r,r+p-r}(x_1, y)} = \left[\frac{F(x_2+y)-F(y)}{F(x_1+y)-F(y)}\right]^{p-1}\left[\frac{\bar{F}(x_2+y)}{\bar{F}(x_1+y)}\right]^{n-r-p}\left[\frac{f(x_2+y)}{f(x_1+y)}\right].$$

It is easy to verify that both the second term and the third term above are increasing in y if X is of DLR. By Lemma 2.1(ii) in Misra and van der Meulen [20], the first term is also increasing in y. Thus, $f_{r,r+p-r}(x, y)$ is TP_2 in $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ for $p \ge 1$.

$\Leftarrow$ For $0 \le t' \le t$ and $1 \le s \le r < r+p \le n$, it can be directly verified that TP_2 $(X_{r+p:n} - X_{r:n}, X_{s:n})$ is equivalent to

$$[X_{r+p:n} - X_{r:n} | X_{s:n} = t] \ge_{\mathrm{lr}} [X_{r+p:n} - X_{r:n} | X_{s:n} = t']. \tag{2.3}$$

Setting $r = s = n-1$ and $p = 1$ in (2.3) reduces to

$$[X_{n:n} - X_{n-1:n} | X_{n-1:n} = t] \ge_{\mathrm{lr}} [X_{n:n} - X_{n-1:n} | X_{n-1:n} = t']. \tag{2.4}$$

By the Markovain property,

$$[X_{n:n} - X_{n-1:n} | X_{n-1:n} = t] \stackrel{\mathrm{st}}{=} [X - t | X > t] \quad \text{for any } t \ge 0.$$

Thus, (2.4) is equivalent to

$$[X - t | X > t] \ge_{\mathrm{lr}} [X - t' | X > t'] \quad \text{for all } t \ge t' \ge 0.$$

That is to say, X is of DLR.

(ii) $\Rightarrow$For $r > s$, set in (2.1)

$$K(x, y, t) = [F(x+t) - F(t)]^{p-1} \bar{F}^{n-r-p}(x+t) f(x+t) I(t \ge y),$$
$$N(y, t) = F^{s-1}(y)[F(t) - F(y)]^{r-s-1} f(t) f(y).$$

If X is of ILR, it can be verified that both $K(x, y, t) > 0$ and $N(y, t)$ are TP_2 in (y, t) and that $K(x, y, t)$ is RR_2 both in (x, y) and in (x, t). So, the RR_2 property can be shown from Lemma 2.3 immediately.

For $r = s$, by the ILR property of X and Lemma 2.1(ii) in Misra and van der Meulen [20], the joint density $f_{r,r+p-r}(x, y)$ in (2.2) is RR_2 in $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ for $p \ge 1$.

$\Leftarrow$ The proof is similar to that of the sufficiency in (i) and, hence, is omitted here for briefness. ■

The k-out-of-n system is a very popular fault-tolerant system in industrial and military systems; it functions if and only if at least k components function. The lifetime of a k-out-of-n system of n components with lifetimes $X_1, \ldots, X_n$ corresponds to the $(n-k+1)$st statistic $X_{n-k+1:n}$. Thus, the study of lifetime of the k-out-of-n system is equivalent to the study of the stochastic properties of order statistics. In practice situations, it is of great interest to study how the time of a failure of a component affects the lifetime of a k-out-of-n system with i.i.d. components. In fact, given that the $(n-k)$th failure occurs at time $t \geq 0$, the residual life of a k-out-of-n system can be represented by $[X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t]$. Langberg et al. [16] first proved that X is of DFR (IFR) if and only if, for $1 \leq k < n$ and all $t \geq t' \geq 0$,

$$[X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t] \geq_{\mathrm{st}} (\leq_{\mathrm{st}}) [X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t'].$$

In this spirit, Belzunce, Franco, and Ruiz [4] provided some equivalent characterizations of IFR in terms of dispersive order and failure rate order. Recently, Li and Zuo [19] and Li and Chen [18] further investigated the behavior of the residual life of such a system with i.i.d. components and that with independent but nonidentical components, respectively.

It is worthwhile to point out that Theorem 2.4 leads to a very nice supplement to those results developed in the literature.

Corollary 2.5: *X is of DLR (ILR) if and only if, for $1 \leq k < n$ and all $t \geq t' \geq 0$,*

$$[X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t] \geq_{\mathrm{lr}} (\leq_{\mathrm{lr}}) [X_{n-k+1:n} - X_{n-k:n} | X_{n-k:n} = t'].$$

To end this section, the next theorem addresses the TP$_2$ dependence of two consecutive general spacings based on a sample of a DLR population.

Theorem 2.6: *X is of DLR if and only if*

$$\mathrm{TP}_2(X_{j:n} - X_{k:n}, X_{k:n} - X_{l:n}) \quad \textit{for } n \geq j > k > l \geq 0.$$

Proof:

$\Rightarrow$The case of $n \geq j > k > l = 0$ is a direct consequence of Theorem 2.4(i). For $n \geq j > k > l \geq 1$, the joint density of $X_{j:n} - X_{k:n}$ and $X_{k:n} - X_{l:n}$ is

$$\begin{aligned} f_{k-l,j-k}(x, y) &= \int_0^\infty f_{l,k,j}(u, u+x, u+x+y)du \\ &= C_3 \int_0^\infty F^{l-1}(u)[F(u+x) - F(u)]^{k-l-1}[F(u+x+y) - F(u+x)]^{j-k-1} \\ &\qquad \times \bar{F}^{n-j}(u+x+y) f(u) f(x+u) f(x+y+u) du, \end{aligned}$$

where

$$C_3 = \frac{n!}{(l-1)!(k-l-1)!(j-k-1)!(n-j)!}.$$

Let

$$g(x, u) = F^{l-1}(u)[F(u+x) - F(u)]^{k-l-1} f(u) f(x+u),$$
$$h(x, y, u) = [F(u+x+y) - F(u+x)]^{j-k-1} \bar{F}^{n-j}(u+x+y) f(x+y+u).$$

Since X is of DLR, by Lemma 2.1(ii) in Misra and van der Meulen [20] again, $g(x, u)$ is TP_2 in $(x, u) \in \mathbb{R}^+ \times \mathbb{R}^+$, for $k > l$.

For any fixed y and $u_2 > u_1$,

$$\frac{h(x, y, u_2)}{h(x, y, u_1)} = \left[\frac{F(u_2+x+y) - F(u_2+x)}{F(u_1+x+y) - F(u_1+x)}\right]^{j-k-1} \left[\frac{\bar{F}(u_2+x+y)}{\bar{F}(u_1+x+y)}\right]^{n-j} \frac{f(x+y+u_2)}{f(x+y+u_1)}.$$

It is easy to verify that both the second term and the third term above are increasing in x for $n > j$ under the assumption that X is of DLR. Note that, for any fixed $y \geq 0$,

$$h_1(x, u) = \int_0^y f(w+x+u)dw.$$

X is of DLR; by Lemma 2.2, $h_1(x, u)$ is TP_2 in $(x, u) \in \mathbb{R}^+ \times \mathbb{R}^+$. Then, it follows that $h(x, y, u)$ is TP_2 in $(x, u) \in \mathbb{R}^+ \times \mathbb{R}^+$.

For any fixed $x \geq 0$,

$$h_2(y, u) = \int_0^\infty f(w+x+u)\mathrm{I}(w \leq y)dw.$$

Likewise, it can be verified that $h(x, y, u)$ is TP_2 in $(y, u) \in \mathbb{R}^+ \times \mathbb{R}^+$ under the assumption that X is of DLR.

For any fixed $u \geq 0$, by Lemma 2.1(ii) of Misra and van der Meulen [20],

$$h_3(x, y) = F(u+x+y) - F(u+x)$$

is TP_2 in $(x, y) \in \mathbb{R} \times \mathbb{R}$. Thus, by the DLR property of X, we have $h(x, y, u)$ is TP_2 in $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Now, from Lemma 2.2 again, the desired result follows immediately.

$\Leftarrow$ Setting $l = 0$, TP_2 $(X_{j:n} - X_{k:n}, X_{k:n} - X_{l:n})$ for any $n \geq j > k \geq 1$ is just $TP_2(X_{j:n} - X_{k:n}, X_{k:n})$ for any $n \geq j > k \geq 1$. From Theorem 2.4(i), the DLR property of X follows directly.

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Remark: To the best of our knowledge, we do not know whether there exists some implications between ILR property of the population X and TP$_2(X_{j:n} - X_{k:n}, X_{k:n} - X_{l:n})$ for any $n \geq j > k \geq 1$.

3. SAMPLE SPACINGS IN MULTIPLE OUTLIERS EXPONENTIAL MODELS

In the multiple outliers exponential model, $X_1, \ldots, X_p$ are assumed to be exponential with common failure rate λ and $X_{p+1}, \ldots, X_n$ are assumed to be exponential with failure rate λ^*, $q = n - p \geq 1$. Since the joint density function of order statistics from an independent but not identical sample can be represented as permanent, let us digress for a moment to permanent; one can refer to Bapat and Beg [1] and Bapat and Kochar [2] for more related results.

The *permanent* of a $n \times n$ matrix $\mathbf{A} = (a_{i,j})$ is defined as $\sum_\sigma \prod_{i=1}^n a_{i,\sigma(i)}$, where the summation is taken over all permutations $\sigma = (\sigma(1), \ldots, \sigma(n))$ of $(1, \ldots, n)$. If $\mathbf{d}_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n$, we will denote by $[\mathbf{d}_1, \ldots, \mathbf{d}_n]$ the permanent of the $n \times n$ matrix $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$. The permanent

$$\left[\underbrace{\mathbf{d}_1}_{r_1}, \underbrace{\mathbf{d}_2}_{r_2}, \cdots\right]$$

is obtained by taking r_1 copies of $\mathbf{d}_1$, r_2 copies of $\mathbf{d}_2$, and so on. If $r_i = 1$, it is omitted in the notation above. If $r_i = 0$, then it is understood that d_i does not appear in the permanent; If $r_i < 0$, for some i, the permanent is defined to be zero. For mutually independent random variables $X_1, \ldots, X_n$, let f_i, F_i, and $\bar{F}_i$ be the density, distribution, and survival functions, respectively, of X_i, $i = 1, \ldots, n$. The column vector $(f_1(x), \ldots, f_n(x))'$ will be denoted simply by $\mathbf{f}(x)$; vectors $\boldsymbol{\lambda}$, $\mathbf{F}(x)$, and $\bar{\mathrm{F}}(x)$ can be similarly defined.

For $1 \leq \pi_1 < \cdots < \pi_j \leq n$, the joint density function of $(X_{\pi_1:n}, \ldots, X_{\pi_j:n})$ is

$$f_{\pi_1:n,\ldots,\pi_j:n}(s_1,\ldots,s_j) = K_{\pi_1,\ldots,\pi_j:n} \times \left[\underbrace{\mathbf{F}(s_1)}_{\pi_1-1}, \mathbf{f}(s_1), \underbrace{\mathbf{F}(s_2)-\mathbf{F}(s_1)}_{\pi_2-\pi_1-1}, \mathbf{f}(s_2), \ldots, \underbrace{\mathbf{F}(s_j)-\mathbf{F}(s_{j-1})}_{\pi_j-\pi_{j-1}-1}, \mathbf{f}(s_j), \underbrace{\bar{\mathbf{F}}(s_j)}_{n-\pi_j}\right], \tag{3.1}$$

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for $s_1 < \cdots < s_j$, where

$$K^{-1}_{\pi_1,\cdots,\pi_j:n} = (\pi_1 - 1)!(n - \pi_j)! \prod_{i=2}^{j} (\pi_i - \pi_{i-1} - 1)!.$$

For convenience, denote

$$[i, j, k, l]_{p,q} = \Bigg[\underbrace{e^{\boldsymbol{\lambda} u} - 1}_{i}, \underbrace{\boldsymbol{\lambda} e^{\boldsymbol{\lambda} u}}_{j}, \underbrace{\boldsymbol{\lambda}}_{k}, \underbrace{\mathbf{F}(y)}_{l}\Bigg]_{p,q}.$$

The next lemma, which can be proved in a completely similar manner to Lemma 3.3 and Theorem 1.1 in Wen, Lu, and Hu [23], will be used to reach the main conclusion.

LEMMA 3.1: *Let* $\mathfrak{G} = \{i: \max\{k - p - 1, 0\} \le i \le \min\{k - 1, q\}\}$; *then, for* $\lambda \ge (\le)\lambda^*$,

$$\Bigg[\boldsymbol{\lambda}, \underbrace{\mathbf{F}(y)}_{j-k-1}, \mathbf{f}(y), \underbrace{\bar{\mathbf{F}}(y)}_{n-j}\Bigg]_{p+i-k+1,q-i}$$

and

$$\Phi_i(y) = \exp\{[\lambda(k - i - 1) + \lambda^* i]y\} \int_0^{\infty} [l - 1, 0, 1, k - l - 1]_{k-i-1,i} e^{-[\lambda p + \lambda^* q]u} du$$

are both RR_2 (TP_2) in $(i, y) \in \mathfrak{G} \times \mathbb{R}^+$.

Now, let us prove the following result.

THEOREM 3.2: $X_{j:n} - X_{k:n}$ *and* $X_{k:n} - X_{l:n}$ *are* TP_2 *dependent for* $n \ge j > k > l \ge 1$.

PROOF: The joint distribution function of $X_{j:n} - X_{k:n}$ and $X_{k:n} - X_{l:n}$ is

$$f_{j-k,k-l}(x, y) = \int_y^{\infty} f_{l,k,j}(u - y, u, u + x)du \quad \text{for } x, y \ge 0.$$

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Applying the Laplace expansion along the first $k-1$ columns of the permanent below, it holds that, for $u \geq y \geq 0$,

$$f_{l,k,j}(u-y,u,u+x)$$

$$= C_4 \left[\underbrace{\mathbf{F}(u-y)}_{l-1}, \mathbf{f}(u-y), \underbrace{\mathbf{F}(u)-\mathbf{F}(u-y)}_{k-l-1}, \mathbf{f}(u), \underbrace{\mathbf{F}(u+x)-\mathbf{F}(u)}_{j-k-1}, \mathbf{f}(u+x), \underbrace{\bar{\mathbf{F}}(u+x)}_{n-j} \right]_{p,q}$$

$$= C_4 \sum_{i\in\mathfrak{G}} \binom{q}{i}\binom{p}{k-i-1} \left[\underbrace{\mathbf{F}(u-y)}_{l-1}, \mathbf{f}(u-y), \underbrace{\mathbf{F}(u)-\mathbf{F}(u-y)}_{k-l-1} \right]_{k-i-1,i}$$

$$\cdot \left[\mathbf{f}(u), \underbrace{\bar{\mathbf{F}}(u)\mathbf{F}(x)}_{j-k-1}, \mathbf{f}(u+x), \underbrace{\bar{\mathbf{F}}(u+x)}_{n-j} \right]_{p+i+1-k,q-i}$$

$$= C_4 \sum_{i\in\mathfrak{G}} \binom{q}{i}\binom{p}{k-i-1} \left[\underbrace{\mathbf{F}(u-y)}_{l-1}, \mathbf{f}(u-y), \underbrace{\mathbf{F}(u)-\mathbf{F}(u-y)}_{k-l-1} \right]_{k-i-1,i}$$

$$\exp\{-[\lambda(p+i+1-k)+\lambda^*(q-i)]u\} \cdot \left[\boldsymbol{\lambda}, \underbrace{\mathbf{F}(x)}_{j-k-1}, \mathbf{f}(x), \underbrace{\bar{\mathbf{F}}(x)}_{n-j} \right]_{p+i+1-k,q-i},$$

where

$$C_4 = \frac{n!}{(l-1)!(k-l-1)!(j-k-1)!(n-j)!}.$$

Since

$$f_{j-k,k-l}(x,y) = \sum_{i\in\mathfrak{G}} a_i(x) b_i(y),$$

for $i = 1, 2, \ldots, n$,

$$a_i(x) = \left[\boldsymbol{\lambda}, \underbrace{\mathbf{F}(x)}_{j-k-1}, \mathbf{f}(x), \underbrace{\bar{\mathbf{F}}(x)}_{n-j} \right]_{p+i+1-k,q-i}$$

and letting

$$c_i = \binom{q}{i}\binom{p}{k-i-1},$$

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we have

$$\begin{aligned}
b_i(y) &= c_i \int_y^\infty \Big[\underbrace{\mathbf{F}(u-y)}_{l-1}\mathbf{f}(u-y), \underbrace{\mathbf{F}(u)-\mathbf{F}(u-y)}_{k-l-1}\Big]_{k-i-1,i} e^{-[\lambda(p+i-k+1)+\lambda^*(q-i)]u}du \\
&= c_i \int_0^\infty \Big[\underbrace{\mathbf{F}(u)}_{l-1}, \mathbf{f}(u), \underbrace{\bar{\mathbf{F}}(u)(1-\bar{\mathbf{F}}(y))}_{k-l-1}\Big]_{k-i-1,i} e^{-[\lambda(p+i-k+1)+\lambda^*(q-i)](u+y)}du \\
&= c_i \int_0^\infty \Big[\underbrace{\mathbf{F}(u)}_{l-1}, \mathbf{f}(u), \underbrace{\bar{\mathbf{F}}(u)(1-\bar{\mathbf{F}}(y))}_{k-l-1}\Big]_{k-i-1,i} e^{-[\lambda p+\lambda^* q](u+y)}e^{[\lambda(k-i-1)+\lambda^* i](u+y)}du \\
&= c_i e^{-[\lambda p+\lambda^* q]y} \cdot \int_0^\infty [l-1,0,1,k-l-1]_{k-i-1,i} e^{-[\lambda p+\lambda^* q]u} e^{[\lambda(k-i-1)+\lambda^* i]y}du \\
&= c_i \Phi_i(y) e^{-[\lambda p+\lambda^* q]y}.
\end{aligned}$$

From Lemma 3.1, it follows that $a_i(x)$ is RR_2 (TP_2) in $(i, x) \in \mathfrak{G} \times \mathbb{R}^+$, and $b_i(x)$ is RR_2 (TP_2) in $(i, y) \in \mathfrak{G}_1 \times \mathbb{R}^+$ for $\lambda \geq (\leq)\lambda^*$. By the basic composition formula in Karlin [8], we have that $f_{j-k,k-l}(x, y)$ is RR_2 (TP_2) in $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$; hence, we complete the proof. ■

It is remarkable to point out that (1.6) follows immediately if we set $j = k + 1$ and $l = k - 1$ in Theorem 3.2.

The stochastic model of a business auction is a very active research area in recent years. Recently, some authors paid their attention to the winner's rent in a k-price auction and derive several interesting conclusions. For more details, please refer to Bulow and Klemperer [5], Paul and Gutierrez [22], and Li [17].

In the $(n - k + 1)$-price buyer's auction, a seller and a number of buyers gather to the auction of some good, and the rent of the winner is $X_{n:n} - X_{k:n}$; whereas in the k-price reverse auction, a buyer and a number of sellers gather to the auction of some good, and the rent of the winner is $X_{k:n} - X_{1:n}$. Some researchers focused on how a variation or increase of the bid has an effect on the winner's rent, but few of them investigated the relationship between the winner's rent in a buyer's auction and that in a reverse auction. In fact, observing that

$$[X_{n:n} - X_{k:n}] + [X_{k:n} - X_{1:n}] \equiv X_{n:n} - X_{1:n},$$

we often jump to the conclusion that the above two kinds of winner's rents are negative dependent through instinct. However, the following two results claim that this is not necessarily true.

By setting $j = n$ and $l = 1$ in Theorem 2.6, we have the following.

COROLLARY 3.3: *The winner's rent in the $(n - k + 1)$-price buyer's auction and that in the k-price reverse auction are TP_2 dependent, provided that all bids are i.i.d. with a DLR distribution.*

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According to Theorem 3.2, this conclusion also holds even when bids are composed of two subgroups of i.i.d. exponential random variables.

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