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Characterizations of stable model semantics for logic programs with arbitrary constraint atoms

YI-DONG SHEN

State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100190, China
(*e-mail:* ydshen@ios.ac.cn)

JIA-HUAI YOU and LI-YAN YUAN

Department of Computing Science, University of Alberta, Edmonton, Alberta, T6G 2H1 Canada
(*e-mail:* {you,yuan}@cs.ualberta.ca)

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Abstract

This paper studies the stable model semantics of logic programs with (abstract) constraint atoms and their properties. We introduce a succinct abstract representation of these constraint atoms in which a constraint atom is represented compactly. We show two applications. First, under this representation of constraint atoms, we generalize the Gelfond–Lifschitz transformation and apply it to define stable models (also called answer sets) for logic programs with arbitrary constraint atoms. The resulting semantics turns out to coincide with the one defined by Son *et al.* (2007), which is based on a fixpoint approach. One advantage of our approach is that it can be applied, in a natural way, to define stable models for disjunctive logic programs with constraint atoms, which may appear in the disjunctive head as well as in the body of a rule. As a result, our approach to the stable model semantics for logic programs with constraint atoms generalizes a number of previous approaches. Second, we show that our abstract representation of constraint atoms provides a means to characterize dependencies of atoms in a program with constraint atoms, so that some standard characterizations and properties relying on these dependencies in the past for logic programs with ordinary atoms can be extended to logic programs with constraint atoms.

KEYWORDS: answer set programing, abstract constraint atoms, stable model semantics, Gelfond–Lifschitz transformation

1 Introduction

Answer set programing (ASP) as an alternative logic programing paradigm has been demonstrated to be an effective knowledge representation formalism for solving combinatorial search problems arising in many application areas such as planning, reasoning about actions, diagnosis, abduction, and so on (Marek and Truszczynski 1999; Niemela 1999; Gelfond and Leone 2002; Lifschitz 2002; Baral 2003). In recent years, researchers have paid particular attention to extensions of ASP with means to model aggregate constraints in particular, and constraints on sets in general (Denecker *et al.* 2001; Simons *et al.* 2002; Dell'Armi *et al.* 2003; Pelov *et al.*

2003; Elkabani *et al.* 2004; Faber *et al.* 2004; Marek and Remmel 2004; Marek and Truszczynski 2004; Pelov 2004; Pelov and Truszczynski 2004; Calimeri *et al.* 2005; Elkabani *et al.* 2005; Ferraris 2005; Liu and Truszczynski 2005; Liu and Truszczynski 2006; Son *et al.* 2006; Liu *et al.* 2007; Pelov *et al.* 2007; Shen and You 2007; Son and Pontelli 2007; Son *et al.* 2007; Marek *et al.* 2008). Logic programs with constraint atoms were introduced as a general framework for representing, and reasoning with, sets of atoms (Marek and Remmel 2004; Marek and Truszczynski 2004; Marek *et al.* 2008). This is in contrast with traditional logic programs, which are used primarily to reason with individuals.

The abstract form of a constraint atom takes the form (D, C), where D is a finite set of atoms and C a collection of subsets from the power set of D, which expresses a constraint on the domain D with the collection C of admissible solutions.

By allowing constraint atoms to appear anywhere in a rule, the framework of logic programs has become a highly expressive knowledge representation language. For example, many constraints can be conveniently and compactly represented with constraint atoms such as weight and cardinality constraints and aggregates (see, e.g., Denecker *et al.* 2001; Simons *et al.* 2002; Dell'Armi *et al.* 2003; Faber *et al.* 2004; Marek and Truszczynski 2004; Pelov 2004; Calimeri *et al.* 2005). In fact, any constraint studied in the context of constraint satisfaction problem (CSP) can be represented by a constraint atom. In this way, the framework of logic programs with constraint atoms can express complex constraint satisfaction problems, such as those involving conditional constraints (Mittal and Falkenhainer 1990) (called *dynamic CSPs*), which are useful in modeling configuration and design problems. When the head of a rule is allowed to be a disjunction of constraint atoms, logic programs become capable of expressing, not only conditional constraints, but also disjunctive constraints, both of which have been investigated by the constraint programing community outside of logic programing (see, e.g., Baptistie and Pape 1996; Cohen *et al.* 2000; Marriott *et al.* 2001). For example, disjunctive constraints have been widely used in scheduling to ensure that the time intervals over which activities require the same resource do not overlap in time (Baptiste and Pape 1996). Although practical ASP languages and systems typically incorporate concrete, predefined constraint atoms, such as weight constraint atoms (Simons *et al.* 2002) and aggregate atoms (Dell'Armi *et al.* 2003), the adoption of the abstract form of constraint atoms has made it possible to study the semantics and properties of these programs in an abstract setting.

In this paper, we characterize and define stable models for logic programs with constraint atoms by introducing a succinct abstract representation of constraint atoms. In the current literature as mentioned above, a constraint atom is expressed as a pair (D, C), where D is a finite set of ground atoms and C a collection of sets of atoms in D. We call this a *power set form* representation (w.r.t. D) of constraint atoms, as C may involve the whole power set 2^D of D. This is the case even for special classes of constraint atoms such as monotone constraint atoms (a constraint atom (D, C) is *monotone* if for any $S \subset D$, whenever $S \in C$ all of its supersets in 2^D are in C).

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For instance, suppose we have a monotone constraint atom $A_1 = (D, 2^D)$. Semantically, this constraint atom is a tautology, since for any set I of atoms, it is a fact that I satisfies A_1, in the sense that $I \cap D \in 2^D$. A clever representation would just need to express two pieces of information, the "bottom element" $\emptyset$ and the "top element" D; two elements together implicitly cover all the sets in between. So, instead of using the power set representation to express all the admissible solutions of this constraint atom, we could use a pair of sets. As another example, consider a monotone constraint atom $A_2 = (D, 2^D \setminus \{\emptyset\})$. A minimal element (set inclusive) in $2^D \setminus \{\emptyset\}$ is a singleton in 2^D. In this case, any minimal element B in 2^D and D form a pair with B being the bottom element and D being the top. So, we could represent this constraint atom by a collection of pairs, one for each singleton in D. The number of such pairs in this case equals to the size of D.

In this paper, we introduce such an abstract representation. In general, the abstract representation of a constraint atom (A_d, A_c) is expressed as (A_d, A_c^*), where A_c^* consists of what will be called *abstract prefixed power sets*, denoted $W \uplus V$, with $W, V \subseteq A_d$ and $W \cap V = \emptyset$. Intuitively, $W \uplus V$ represents a collection of sets of the form $W \cup S$ with $S \in 2^V$, all of which are in A_c.

The abstract representation of constraint atoms not only yields a compact representation, but also captures the essential information embodied in constraint atoms appearing in the bodies of rules. To substantiate this claim, we show two applications.

In the first application, we restore the power of the Gelfond–Lifschitz transformation by generalizing it for logic programs with constraint atoms. The key idea is that given an interpretation I, each constraint atom $A = (A_d, A_c^*)$ under our abstract representation can be concisely characterized by a set of *abstract satisfiable sets* of the form $W \uplus V \in A_c^*$ such that $W \uplus V$ covers $I \cap A_d$. Therefore, the standard Gelfond–Lifschitz transformation can be naturally generalized to logic programs with constraint atoms by representing each constraint atom by its abstract satisfiable sets. We then use the generalized Gelfond–Lifschitz transformation to define stable models for disjunctive logic programs with constraint atoms. It turns out that, for nondisjunctive logic programs with arbitrary constraint atoms, the stable models defined this way are precisely those defined by Son *et al.* (2006, 2007) for logic programs with arbitrary constraint atoms, and the equivalent semantics, called the *ultimate stable semantics*, for aggregate logic programs (Denecker *et al.* 2001). These semantics are defined by a substantially different approach, the fixpoint approach.

One advantage of our approach is that the semantics is defined for disjunctive programs where a constraint atom can appear anywhere in a disjunctive rule. This is due to the power of the Gelfond–Lifschitz transformation. Roughly speaking, for a nondisjunctive program with constraint atoms, a stable model M is just the least model of the reduct by the generalized Gelfond–Lifschitz transformation, while for a disjunctive program with constraint atoms, a stable model M is a minimal model of the reduct. We show that logic programs whose constraint atoms appearing in disjunctive rule heads are elementary possess the minimality property; i.e., for such logic programs, all stable models under the generalized

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Gelfond–Lifschitz transformation are minimal models. Thus, by the known relationships among different definitions of stable models, the stable model semantics defined in this paper extends the semantics of conventional disjunctive logic programs (Gelfond and Lifschitz 1991), the semantics defined by Marek and Truszczynski (2004) for nondisjunctive logic programs with monotone constraint atoms, the semantics by Son *et al.* (2006, 2007), and others equivalent to it (Denecker *et al.* 2001; Pelov *et al.* 2003).

We note that disjunctive programs with aggregates have been studied previously by Faber *et al.* (2004) and Pelov and Truszczynski (2004), where aggregates do not appear in the heads of program rules.

In the second application, we show that our abstract representation of constraint atoms provides a means to characterize the *dependency relation* among ordinary atoms in a program with constraint atoms. This dependency relation in the past is defined using a *dependency graph*. One question for logic programs with constraint atoms is how this dependency graph may be constructed so that the means to characterize the properties of programs by *odd cycles*, *even cycles*, *call-consistency*, *acyclic programs* in the traditional context is applicable to the new context. We show that the information captured in our abstract representation is essential in constructing the dependency graph for a program with constraint atoms. As we will see, this is due to a simple way to represent a logic program with constraint atoms by a normal logic program.

To summarize, the main contributions of this paper are:

- We introduce an abstract representation of constraint atoms, independently of any programs in which they appear.
- Using this abstract representation, we present a generalized form of Gelfond–Lifschitz transformation and apply it to define stable models for disjunctive logic programs with constraint atoms. For nondisjunctive programs, the semantics defined this way coincides with the one based on conditional satisfaction (Son *et al.* 2006, 2007), and with the ones equivalent to it (Denecker *et al.* 2001).
- We show that our abstract representation of constraint atoms encodes the information needed to capture the atom dependency relation in a given program, thus the means to characterize the properties for normal programs can still be applied to programs with constraint atoms, and in the process, the unfolding approach (Son and Pontelli 2007) is made simple.

The paper is structured as follows. Following the preliminaries in the next section, in Section 3 we present our abstract representation of constraint atoms. In Section 4, we show some characterization of constraint atoms under this abstract representation. In Section 5, we introduce a generalized Gelfond–Lifschitz transformation and apply it to define stable models for disjunctive logic programs with constraint atoms. In Section 6, we prove the relationship of our approach with Son *et al.* (2006)'s fixpoint approach. Then in Section 7, we show that our abstract representation of constraint atoms encodes precisely the needed information to define the dependency graph of a program with constraint atoms. In Section 8, we

discuss the related approaches. Finally in Section 9, we provide conclusions and discuss future work.

Proofs of theorems and lemmas will be delayed to Appendix.

Some results of this paper have been reported in (Shen and You 2007). The current paper, however, contains substantial new results.

2 Preliminaries

We consider propositional (ground) logic programs and assume a fixed propositional language with a countable set $\mathscr{V}$ of propositional atoms (atoms for short). Any subset I of $\mathscr{V}$ is called an *interpretation*. A literal is an atom A (a *positive literal*) or its negation *not* A (a *negative literal*). For a set $S = \{A_1, \ldots, A_m\}$ of atoms, we use *not* S to denote $\{not\ A_1, \ldots, not\ A_m\}$ and $|S|$ to denote the size of S. For convenience, when S appears in a logic expression, it represents a conjunction $A_1 \wedge \ldots \wedge A_m$; when *not* S appears in a logic expression, it represents a conjunction $not\ A_1 \wedge \ldots \wedge not\ A_m$.

An *abstract constraint atom* (or *c-atom* following Son *et al.* 2006, 2007) A is a pair (D, C), where D is a finite set of atoms in $\mathscr{V}$ and C a collection of sets of atoms in D, i.e., $C \subseteq 2^D$. For convenience, we use A_d and A_c to refer to the components D and C of A, respectively. As a general framework, c-atoms can be used to represent any constraints with a finite set A_c of admissible solutions over a finite domain A_d.

A c-atom A is *elementary* if it is of the form $(\{a\}, \{\{a\}\})$, where a is an atom. Due to the equivalence in satisfaction, an elementary c-atom may be simply written by the atom in it. A is *monotone* if it has the property that for any $S \subset A_d$, if $S \in A_c$ then all of its supersets in 2^{A_d} are in A_c. A is *nonmonotone* if it is not monotone. A is *antimonotone* if A_c is closed under subsets, i.e., for every $X, Y \subseteq A_d$, if $Y \in A_c$ and $X \subseteq Y$ then $X \in A_c$. A is *convex* if for any $S_1, S, S_2 \subseteq A_d$ such that $S_1 \subseteq S \subseteq S_2$ and $S_1, S_2 \in A_c$, we have $S \in A_c$.

A *disjunctive constraint program*, or a *disjunctive (logic) program with c-atoms*, is a finite set of rules of the form

$$H_1 \vee \ldots \vee H_k \leftarrow A_1, \ldots, A_m, not\ B_1, \ldots, not\ B_n$$

where $k \geqslant 1$, $m, n \geqslant 0$ and H_i, A_i and B_i are either an atom or a c-atom ("$\leftarrow$" is omitted when $m = n = 0$). P is a *normal constraint program* if $k = 1$ for all of its rules; P is a *positive constraint program* if $n = 0$ for all of its rules; P is a *positive basic program* if $n = 0$ and $k = 1$ with H_1 being an elementary c-atom for all of its rules. As usual, P is a *normal program* if P is a normal constraint program where all c-atoms are elementary; P is a *disjunctive program* if P is a disjunctive constraint program where all c-atoms are elementary.

In the sequel, if not specifically quantified, a logic program (or simply a program) refers to a disjunctive constraint program. To make it explicit, when a program contains only elementary c-atoms, it may be called a program with *ordinary* atoms, or just a program without c-atoms.

Let r be a rule of the above form. We define

$$head(r) = \{H_1, \ldots, H_k\}$$
$$body(r) = \{A_1, \ldots, A_m, not\ B_1, \ldots, not\ B_n\}$$

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which will be referred to as the *head* and the *body* of the rule, respectively, where $body(r)$ denotes the conjunction of the elements in the set and $head(r)$ the disjunction. Without confusion, we may use the set notation in a rule to express the body as well as the head. For example, given a rule r, we may write $head(r) \leftarrow body(r)$.

We will use $At(P)$ to denote the set of atoms that appear in a program P.

The *satisfaction* relation is defined as follows. An interpretation $I \subseteq \mathscr{V}$ satisfies an atom a if $a \in I$; *not* a if $a \notin I$. I satisfies a c-atom A if $I \cap A_d \in A_c$; *not* A if $I \cap A_d \notin A_c$. This relation extends to arbitrary expressions F mentioning negation *not*, conjunction $\wedge$ and disjunction $\vee$, in a usual way. We will use $I \models F$ to denote that I satisfies F, and $I \not\models F$ to denote that I does not satisfy F. We say F is *true* (resp. *false*) in I if and only if I satisfies (resp. does not satisfy) F.

I satisfies a rule r if it satisfies $head(r)$ or it does not satisfy $body(r)$. I is a *model* of a logic program P if it satisfies all rules of P. I is a *minimal model* of P if it is a model of P and there is no proper subset of I which is also a model of P. I is a *supported* model of P if for any $a \in I$, there is $r \in P$ such that $a \in head(r)$ and $I \models body(r)$.

As commented earlier, atoms can be viewed as elementary c-atoms. This is due to the fact that for an atom a, an interpretation I satisfies a iff $a \in I$ iff $I \models (\{a\}, \{\{a\}\})$.

Sometimes we say *a model M restricted to the atoms appearing in a program P*. By this we mean $M \cap At(P)$, and denote it by $M|_{At(P)}$.

Note that c-atoms of the form $(D, \emptyset)$ are not satisfied by any interpretation. We will use a special symbol $\perp$ to denote any such c-atom.

Following Son *et al.* (2007), for any c-atom $A = (A_d, A_c)$, its negation *not* A is interpreted by its complement, which is a c-atom (A_d, A_c^-) where $A_c^- = 2^{A_d} \setminus A_c$[1]. So a logic program with negated c-atoms can be rewritten to a logic program free of negated c-atoms by replacing all occurrences of negated c-atoms with their respective complement c-atoms. Due to this assumption, in the sequel we only consider logic programs without negated c-atoms in rule bodies.

Given a disjunctive program P (where c-atoms are elementary) and an interpretation I, the standard *Gelfond–Lifschitz transformation* of P w.r.t. I, written as P^I, is obtained from P by performing two operations: (1) remove from P all rules whose bodies contain a negative literal *not* A such that $I \not\models$ *not* A, and (2) remove from the remaining rules all negative literals. Since P^I is a positive constraint program where c-atoms are elementary, it has a set of minimal models. I is defined to be a stable model of P if it is a minimal model of P^I (Gelfond and Lifschitz 1988; Gelfond and Lifschitz 1991; Przymusinski 1991).

The cardinality and weight constraints can be represented by c-atoms. In some of the example programs of this paper we may write weight constraints instead of c-atoms. We will adopt the notation proposed in (Simons *et al.* 2002). A *weight constraint* is an expression of the form

$$l\{a_1 = w_{a_1}, \ldots, a_n = w_{a_n}, not\ b_1 = w_{b_1}, \ldots, not\ b_m = w_{b_m}\}u$$

[1] Note that this is consistent with our definition of satisfaction of negated c-atoms. But not all semantics are based on the complement approach. A detailed comparison can be found in (Son *et al.* 2007).

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where a_i and b_j are atoms and w_{a_i} is the weight of atom a_i and w_{b_j} is the weight of negative literal *not* b_j. The numbers l and u are lower and upper bounds of the constraint, respectively. A weight constraint is satisfied by a set of atoms S if the sum of the weights of the literals in the set $\{a_1, \ldots, a_n, not\ b_1, \ldots, not\ b_m\}$ that are satisfied by S is between l and u (inclusive).

A *cardinality constraint* is a special case of weight constraint where each weight is one. In writing a cardinality constraint, we will omit the weights. A *choice constraint* is a cardinality constraint of the form $l\{a_1, \ldots, a_n\}u$, where $l = 0$ and $u = n$. In writing a choice constraint, we will omit the bounds.

3 Abstract representation of constraint atoms

In this section, we present a compact representation of c-atoms. In the current literature, for any c-atom A its admissible solutions are all explicitly enumerated and written in A_c. In many cases, A_c may involve a large portion of A_d. It is then of great interest if we can represent A_c using some abstract structure so that its size can be substantially compressed. We begin by introducing a notion of prefixed power sets.

Definition 3.1
Let $I = \{a_1, \ldots, a_m\}$ and $J = \{b_1, \ldots, b_n\}$ $(m, n \geqslant 0)$ be two sets of atoms.

(1) The *I-prefixed power set* of J, denoted by $I \uplus J$, is the collection $\{I \cup J_{sub} | J_{sub} \in 2^J\}$; i.e., each set in the collection consists of all a_is in I plus zero or more b_is in J. For any set of atoms S, we say S is *covered by* $I \uplus J$ (or $I \uplus J$ *covers* S) if $I \subseteq S$ and $S \subseteq I \cup J$.
(2) For any two abstract prefixed power sets $I \uplus J$ and $I' \uplus J'$, $I \uplus J$ *is included* in $I' \uplus J'$ if any set covered by $I \uplus J$ is covered by $I' \uplus J'$.

Theorem 3.1
When $I \uplus J$ is included in $I_1 \uplus J_1$, we have $I_1 \subseteq I$ and $I \cup J \subseteq I_1 \cup J_1$. If $I \uplus J$ is included in $I_1 \uplus J_1$ and $I_1 \uplus J_1$ is included in $I_2 \uplus J_2$, then $I \uplus J$ is included in $I_2 \uplus J_2$.

Given a c-atom A, let $I \in A_c$ and $J \subseteq A_d \setminus I$. $I \uplus J$ is called *I-maximal in A* (or just *maximal*) if all sets covered by $I \uplus J$ are in A_c and there is no J' with $J \subset J' \subseteq A_d \setminus I$ such that all sets covered by $I \uplus J'$ are in A_c.

Definition 3.2
Let A be a c-atom and $S \in A_c$. The collection of *abstract S-prefixed power sets* of A is $\{S \uplus J \mid S \uplus J$ is S-maximal in $A\}$.

For instance, consider a c-atom A, where

$$A_d = \{a, b, c, d\}$$
$$A_c = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}.$$

For $\emptyset \in A_c$, the collection of abstract $\emptyset$-prefixed power sets of A is $\{\emptyset \uplus \{b, c\}\}$; for $\{b\} \in A_c$, the collection is $\{\{b\} \uplus \{c\}\}$; for $\{c\} \in A_c$, the collection is $\{\{c\} \uplus \{a, b\}, \{c\} \uplus \{b, d\}\}$. Note that $\{b\} \uplus \{c\}$ is included in $\emptyset \uplus \{b, c\}$. It is easy to check that all abstract

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prefixed power sets for $\{a,c\},\{b,c\},\{a,b,c\} \in A_c$ are included in $\{c\} \uplus \{a,b\}$ and all those for $\{b,c\},\{c,d\},\{b,c,d\} \in A_c$ are included in $\{c\} \uplus \{b,d\}$.

When a collection contains two abstract prefixed power sets, $I \uplus J$ and $I' \uplus J'$ with $I \uplus J$ being included in $I' \uplus J'$, we say $I \uplus J$ is *redundant* in this collection.

For instance, consider $I \uplus J$ where $I = \{a,b\}$ and $J = \{c\}$, and $I' \uplus J'$ where $I' = \{a\}$ and $J' = \{b,c\}$. Then, $I \uplus J$ is redundant in a collection that contains $I' \uplus J'$, since every set covered by $I \uplus J$ is covered by $I' \uplus J'$.

Definition 3.3
The *abstract representation* A^* of a c-atom A is a pair (A_d, A_c^*) where A_c^* is the collection $\bigcup_{S \in A_c} C_S$, where C_S is the collection of abstract S-prefixed power sets of A, with all redundant abstract prefixed power sets removed.

Observe that when $W \uplus V$ is in A_c^*, all sets in the collection $\{W \cup V_{sub} | V_{sub} \in 2^V\}$ are in A_c. Conversely, when $\{W \cup V_{sub} | V_{sub} \in 2^V\} \subseteq A_c$, there exist $W', V' \subseteq A_d$ such that $W' \subseteq W$ and $W \cup V \subseteq W' \cup V'$, and $W' \uplus V' \in A_c^*$, i.e., $W \uplus V$ is included in $W' \uplus V' \in A_c^*$. In other words, A_c^* is the collection of maximal sublattices of the lattice $(2^{A_d}, \subseteq)$, of which all elements are in A_c. For such a maximal sublattice $W \uplus V$, the bottom element is W and the top element is $W \cup V$.

Consider the above example c-atom A again. Its abstract representation is (A_d, A_c^*) with $A_c^* = \{\emptyset \uplus \{b,c\}, \{c\} \uplus \{a,b\}, \{c\} \uplus \{b,d\}\}$.

Theorem 3.2
Let $A = (A_d, A_c)$ be a c-atom.

(1) A has a unique abstract form (A_d, A_c^*).
(2) For any interpretation I, $I \models A$ if and only if A_c^* contains an abstract prefixed power set $W \uplus V$ covering $I \cap A_d$.

For some special classes of c-atoms, their abstract representations are much simpler and can be stated more structurally.

We need a terminology: given a set S of sets, we say that $I \in S$ is *minimal* (resp. *maximal*) in S if there is no $I' \in S$ such that $I' \subset I$ (resp. $I' \supset I$).

Theorem 3.3
Let A be a c-atom.

(1) A is monotone if and only if $A_c^* = \{B \uplus A_d \setminus B : B \text{ is minimal in } A_c\}$ if and only if $|W| + |V| = |A_d|$ for each $W \uplus V \in A_c^*$.
(2) A is antimonotone if and only if $A_c^* = \{\emptyset \uplus T : T \text{ is maximal in } A_c\}$ if and only if $W = \emptyset$ for each $W \uplus V \in A_c^*$.
(3) A is convex if and only if $A_c^* = \{B \uplus T : B \text{ is minimal and } B \cup T \text{ is maximal in } A_c\}$.

By this theorem, given A^*, to check if A is monotone (resp. antimonotone) it suffices to check if $|W| + |V| = |A_d|$ (resp. $W = \emptyset$) for each $W \uplus V \in A_c^*$. This process takes linear time in the size of A_c^*. Let $S_1 = \{W \mid W \uplus V \in A_c^*\}$ and $S_2 = \{W \cup V \mid W \uplus V \in A_c^*\}$. To check if A is convex, it suffices to check (a) there are no $p,q \in S_1$ with $p \subset q$, and (b) there are no $p,q \in S_2$ with $p \subset q$. Case (a)

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guarantees that W is minimal while case (b) guarantees $W \cup V$ is maximal in A_c, for each $W \uplus V \in A_c^*$. The time for the two cases is bounded by $O(|A_c^*|^2 * |A_d|^2)$, where each subset check is assumed to take time $|A_d|^2$. This leads to the following complexity result.

Theorem 3.4
Given the abstract representation A^* of a c-atom A, the time to check if A is monotone or antimonotone is linear in the size of A_c^*, while the time to check if A is convex is bounded by $O(|A_c^*|^2 * |A_d|^2)$.

We now discuss the issue of compactness. Given a c-atom A, its abstract representation A^* is more compact than A when A_c contains some relatively large abstract prefixed power sets. This can be seen from the special classes of c-atoms in Theorem 3.3. Since the admissible solutions in such a c-atom are tightly clustered together, they easily form large abstract prefixed power sets. For example, since a monotone c-atom is closed under its supersets in A_c, for any minimal set S in A_c, all the sets in between S and A_d must be in A_c. Therefore, $S \uplus A_d \setminus S$ is an abstract S-prefixed power set. The bigger is the difference between S and A_d, the more information is captured compactly. As another example, we know that weight constraints without negative literals or negative weights are convex. That is, these constraints are of the form $l\{a_1 = w_{a_1}, \ldots, a_n = w_{a_n}\}u$, where $w_{a_i} \geqslant 0$, for all $1 \leqslant i \leqslant n$. Let A denote such a weight constraint. Then, $A_d = \{a_1, \ldots, a_n\}$ and A_c consists of all subsets of A_d where the sum of the weights of the atoms in such a subset is between l and u. Thus, if the sets B and T are such that $B \subseteq T \subseteq A_d$, and B is minimal and T is maximal in A_c, then $B \uplus T \setminus B$ forms an abstract B-prefixed power set, representing all the sets in between.

Apparently, c-atoms that are *nearly* monotone (or antimonotone or convex) can greatly benefit from the abstract representation. For example, given a set $S = \{a_1, \ldots, a_n\}$, a c-atom that expresses all subsets of S except some V in between $\emptyset$ and S can easily fall outside of the above special classes. For instance, suppose $S = \{a, b, c\}$ and let $A = (S, 2^S \setminus \{\{a, b\}\})$. Then $A^* = (S, \{\emptyset \uplus \{a, c\}, \emptyset \uplus \{b, c\}, \{a, c\} \uplus \{b\}, \{b, c\} \uplus \{a\}\})$.

It should also be clear that there are situations where A^* may not be strictly more compact than A. This is typically the case where the admissible solutions in A_c are largely unrelated. We say that two sets I and J are *unrelated* if either no one is a subset of the other, or $I \subseteq J$ and $J \setminus I$ is not singleton.

For example, consider a c-atom A where A_c consists of all subsets of A_d with an equal size. In this case, no set in A_c is a subset of another in A_c. The abstract representation of such a c-atom A is (A_d, A_c^*) where $A_c^* = \{I \uplus \emptyset : I \in A_c\}$, which trivially enumerates all admissible solutions in A_c. As another example, consider a c-atom $A = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, b, c, d\}\})$. In this case, for any $I, J \in A_c$, if J is a superset of I, then $J \setminus I$ is not singleton. The abstract representation of A is (A_d, A_c^*), where $A_c^* = \{\emptyset \uplus \emptyset, \{a, b\} \uplus \emptyset, \{a, b, c, d\} \uplus \emptyset\}$. Again, A_c^* essentially enlists all admissible solutions in A_c.

Although all the evidence indicates that for any c-atom A the number of abstract prefixed power sets in A_c^* is less than or equal to the number of admissible solutions

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in A_c, i.e., $|A_c^*| \leqslant |A_c|$, a rigorous proof for this claim seems challenging. We leave this proof as an interesting open problem.

Finally in this section, we comment that for a c-atom A, it takes polynomial time in the size of A to construct A^*. This result will be useful in determining the complexity of the semantics defined by the generalized Gelfond–Lifschitz transformation later in this paper.

Below, we give a bound for the construction.

Theorem 3.5
Let A be a c-atom. The time to construct A^* from A is bounded by $O(|A_c|^4 * |A_d|^2)$.

4 Characterizations of c-atoms under abstract representation

In this section, we present some characterizations of c-atoms under the abstract representation. Essentially, these characterizations are related to the fact that a c-atom can be semantically represented by a propositional formula.

Recall that the standard semantics of a c-atom A is defined by its satisfaction: for any set of atoms M, $M \models A$ if and only if $M \cap A_d \in A_c$. For nonmonotone c-atoms, a difficulty with this interpretation of the meaning of a c-atom is that the iterative construction by the *one-step provability operator* (Liu and Truszczynski 2006; Marek *et al.* 2008) may lead to an undesirable situation—there is no guarantee that once a c-atom is satisfied by a set of atoms I, it remains to be satisfied by an extension of I.

However, by definition, a set of atoms M satisfies a c-atom A if and only if M satisfies the propositional formula that corresponds to the admissible solutions in A_c. This formula is a disjunction of conjunctions, each of which represents an admissible solution in A_c. As a propositional formula, it can be simplified to a logically equivalent one. It turns out that this simplification process is significant as it reveals the nature of the information encoded in our abstract representation. Therefore, the main result of this section is to show that the abstract representation of a c-atom encodes the "simplest" propositional formula, in the form of a disjunctive normal form (DNF). We then use this insight to define what are called *abstract satisfiable sets*, which make it possible to define a new form of Gelfond–Lifschitz transformation.

Below, we make it precise as what the formula is, and state some facts which easily follow from the definition.

Proposition 4.1
Let $A = (A_d, A_c)$ be a c-atom with $A_c = \{S_1, \ldots, S_m\}$, and I be an interpretation. The DNF $C_1 \vee \ldots \vee C_m$ for A is defined as: each C_i is a conjunction $S_i \wedge \mathit{not}\ (A_d \setminus S_i)$.

(1) I satisfies A if and only if $C_1 \vee \ldots \vee C_m$ is true in I.
(2) I satisfies *not* A if and only if *not* $(C_1 \vee \ldots \vee C_m)$ is true in I.

Given a c-atom A, the DNF $C_1 \vee \ldots \vee C_m$ for A can be simplified. In propositional logic, we have $(S \wedge \neg F) \vee (S \wedge F) \equiv S$, for any formulas S and F.

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Example 4.1
Consider a monotone c-atom $A = (\{a, b\}, \{\{a\}, \{b\}, \{a, b\}\})$. Its corresponding DNF is $(a \wedge not\ b) \vee (b \wedge not\ a) \vee (a \wedge b)$, which can be simplified as follows:

$$\begin{aligned}&(a \wedge not\ b) \vee (b \wedge not\ a) \vee (a \wedge b)\\ &\equiv \underline{(a \wedge not\ b) \vee (a \wedge b)} \ \vee \ \underline{(b \wedge not\ a) \vee (a \wedge b)}\\ &\equiv a \vee b.\end{aligned}$$

Note that in the second line above, a disjunct in the previous DNF is added.

What is interesting is that the resulting propositional formula corresponds to the abstract representation of A, where $A_c^* = \{\{a\} \uplus \{b\}, \{b\} \uplus \{a\}\}$. This correspondence is made precise in the following theorem.

Theorem 4.2
Let A be a c-atom and M be a set of atoms. $M \models A$ if and only if M satisfies

$$\bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V)). \tag{1}$$

The proof of this theorem requires the following lemma.

Lemma 4.3
Let $E = \{a_1, \ldots, a_m\}$ be a set of atoms and F be a DNF covering all possible interpretations on the a_is, i.e.,

$$F = \bigvee_{1 \leqslant i \leqslant m,\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_m.$$

F can be simplified to *true* in propositional logic by applying the following rule:

$$\text{For any } S_1 \text{ and } S_2, (S_1 \wedge L \wedge S_2) \vee (S_1 \wedge not\ L \wedge S_2) \equiv S_1 \wedge S_2. \tag{2}$$

Note that rule (2) is like the *resolution rule* in its underlying pattern, but it applies to a DNF while resolution applies to CNFs.

Theorem 4.2 shows that the satisfaction of a c-atom A can be simplified to (1) in terms of its abstract representation by applying rule (2).

As a slightly more involved example, consider a c-atom

$$B = (\{a, b, c, d\}, \{\{d\}, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}).$$

The DNF for this c-atom is:

$$\begin{aligned}&(d \wedge not\ a \wedge not\ b \wedge not\ c) \vee (a \wedge not\ b \wedge not\ c \wedge not\ d) \vee\\ &(a \wedge b \wedge not\ c \wedge not\ d) \vee (a \wedge c \wedge not\ b \wedge not\ d) \vee (a \wedge b \wedge c \wedge not\ d).\end{aligned}$$

which can be simplified to

$$(d \wedge not\ a \wedge not\ b \wedge not\ c) \vee (a \wedge not\ d),$$

each disjunct of which corresponds to a prefixed power set in the abstract representation of B, i.e., $B_c^* = \{\{d\} \uplus \emptyset, \{a\} \uplus \{b, c\}\}$.

We say that a DNF is *maximally simplified* if it cannot be further simplified by applying rule (2).

The following theorem shows that (1) is maximally simplified.

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Theorem 4.4
The semantic characterization (1) of a c-atom A is maximally simplified.

Theorems 4.2 and 4.4 suggest that the satisfaction of c-atom A can be described by its simplest DNF given in (1), independently of any interpretations. When we generalize the standard Gelfond–Lifschitz transformation for constraint programs, we can apply a given interpretation to further simplify this DNF. In the following, and in the rest of the paper, given an interpretation I, for any c-atom A we use T_A^I to denote $I \cap A_d$ and F_A^I to denote $A_d \setminus T_A^I$.

We are ready to define *abstract satisfiable sets.*

Definition 4.1
Let A be a c-atom and I an interpretation. $W \uplus V \in A_c^*$ is an *abstract satisfiable set* of A w.r.t. I if $W \uplus V$ covers T_A^I. In this case, W is called a *satisfiable set* of A w.r.t. T_A^I. We use A_s^I to denote the set of abstract satisfiable sets of A w.r.t. I.

The next two theorems characterize some properties of abstract satisfiable sets as well as satisfiable sets.

Theorem 4.5
Let A be a c-atom and I an interpretation. $I \models A$ if and only if $I \models \bigvee_{W \uplus V \in A_s^I} W \wedge$ *not* $(A_d \setminus (W \cup V))$.

Theorem 4.6
Let A be a c-atom and I an interpretation. If S is a satisfiable set of A w.r.t. T_A^I, then for every S' with $S \subseteq S' \subseteq T_A^I$, we have $S' \in A_c$.

5 A generalization of the Gelfond–Lifschitz transformation

In this section we show that the characterizations of c-atoms presented in the last section can be used to generalize the standard Gelfond–Lifschitz transformation for logic programs with c-atoms.

In the following, special atoms of the forms θ_A, β_A and $\bot$ will be used, where A is a c-atom. Unless otherwise stated, we assume that these special atoms will not occur in any given logic programs or interpretations. Let Γ_θ and Γ_β be the sets of special atoms prefixed with θ and β, respectively. Let $\Gamma = \Gamma_\theta \cup \Gamma_\beta$.

Definition 5.1
Given a logic program P and an interpretation I, the *generalized Gelfond–Lifschitz transformation* of P w.r.t. I, written as P^I, is obtained from P by performing the following four operations:

(1) Remove from P all rules whose bodies contain either a negative literal *not* A such that $I \not\models$ *not* A or a c-atom A such that $I \not\models A$.
(2) Remove from the remaining rules all negative literals.
(3) Replace each c-atom A in the body of a rule with a special atom θ_A and introduce a new rule $\theta_A \leftarrow A_1, \ldots, A_m$ for each satisfiable set $\{A_1, \ldots, A_m\}$ of A w.r.t. T_A^I.

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(4) Replace each c-atom A in the head of a rule with $\bot$ if $I \not\models A$, or replace it with a special atom β_A and introduce a new rule $B \leftarrow \beta_A$ for each $B \in T_A^I$, a new rule $\bot \leftarrow B, \beta_A$ for each $B \in F_A^I$, and a new rule $\beta_A \leftarrow T_A^I$.

In the first operation, we remove all rules whose bodies are not satisfied in I because of the presence of a negative literal or a c-atom that is not satisfied in I. In the second operation, we remove all negative literals because they are satisfied in I. The last two operations transform c-atoms in the body and head of each rule, respectively.

Each c-atom A in the body of a rule is substituted by a special atom θ_A. By Theorem 4.5, θ_A can be defined by introducing a new rule $\theta_A \leftarrow W \wedge not\ (A_d \backslash (W \cup V))$ for each abstract satisfiable set $W \uplus V$. Since the negative part $not\ (A_d \backslash (W \cup V))$ is true in I, it can be removed from the rule body following the standard Gelfond–Lifschitz transformation. Note that the remaining part W is a satisfiable set. Therefore, in the third operation, θ_A is defined by introducing a new rule $\theta_A \leftarrow A_1, \ldots, A_m$ for each satisfiable set $\{A_1, \ldots, A_m\}$ of A w.r.t. T_A^I.

When $I \models A$, each c-atom A in the head of a rule is replaced by a special atom β_A. Note that β_A represents a conclusion that every $B \in T_A^I$ is true and every $B \in F_A^I$ is false in I. Such a conclusion is formulated, in the fourth operation, by introducing a new rule $B \leftarrow \beta_A$ for each $B \in T_A^I$, a new rule $\bot \leftarrow B, \beta_A$ for each $B \in F_A^I$, and a new rule $\beta_A \leftarrow T_A^I$. $\bot$ is a special atom meaning $false$. The last rule comes from the rule $\beta_A \leftarrow T_A^I \wedge not\ F_A^I$, where the negative part $not\ F_A^I$ is true in I and thus is removed following the standard Gelfond–Lifschitz transformation. When $I \not\models A$, we replace A with $\bot$. In the case that $\bot$ appears in a disjunction $B_1 \vee \ldots \vee \bot \vee \ldots \vee B_m$ with $m > 0$, $\bot$ can be removed, as the satisfaction of the disjunction is determined by the B_is.

Apparently, the generalized Gelfond–Lifschitz transformation coincides with the standard Gelfond–Lifschitz transformation when P contains no c-atoms.

Since the generalized transformation P^I is a positive logic program without c-atoms, it has minimal models. We then define the stable model semantics of a constraint program in the same way as that of a logic program with ordinary atoms.

Definition 5.2
For any logic program P, an interpretation I is a *stable model* of P if $I = M \setminus \Gamma$, where M is a minimal model of the generalized Gelfond–Lifschitz transformation P^I.

Immediately, if P is a normal constraint program, then I is a stable model of P if $I = M \setminus \Gamma$ and M is the least model of the generalized Gelfond–Lifschitz transformation P^I. In other words, the extension to disjunctive constraint programs from normal constraint programs follows the same way as the extension to disjunctive programs from normal programs.

Again, stable models of P under the generalized Gelfond–Lifschitz transformation coincide with stable models under the standard Gelfond–Lifschitz transformation when P has no c-atoms. In the following, unless otherwise stated, by stable models we refer to stable models under the generalized Gelfond–Lifschitz transformation.

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Example 5.1
Consider the following program:

$$\begin{aligned} P_1: \quad & p(1). \\ & p(-1) \leftarrow p(2). \\ & p(2) \leftarrow \mathrm{SUM}(\{X|p(X)\}) \geqslant 1. \end{aligned}$$

The aggregate constraint $\mathrm{SUM}(\{X|p(X)\}) \geqslant 1$ can be represented by a c-atom A where

$$\begin{aligned} A_d &= \{p(-1), p(1), p(2)\}, \\ A_c &= \{\{p(1)\}, \{p(2)\}, \{p(-1), p(2)\}, \{p(1), p(2)\}, \{p(-1), p(1),\ p(2)\}\}. \end{aligned}$$

Its abstract representation is (A_d, A_c^*) with

$$A_c^* = \{\{p(1)\} \uplus \{p(2)\}, \{p(2)\} \uplus \{p(-1), p(1)\}\}.$$

Let us check if $I = \{p(-1), p(1), p(2)\}$ is a stable model of P_1 using the generalized Gelfond–Lifschitz transformation. The first two operations do not apply. Since $I \models A$ with $T_A^I = I \cap A_d = \{p(-1), p(1), p(2)\}$, A has only one abstract satisfiable set $\{p(2)\} \uplus \{p(-1), p(1)\}$, and thus it has only one satisfiable set $\{p(2)\}$ w.r.t. T_A^I. So, in the third operation A is replaced by a special atom θ_A, followed by a new rule $\theta_A \leftarrow p(2)$. Hence we have

$$\begin{aligned} P_1^I: \quad & p(1). \\ & p(-1) \leftarrow p(2). \\ & p(2) \leftarrow \theta_A. \\ & \theta_A \leftarrow p(2). \end{aligned}$$

The only minimal model of P_1^I is $\{p(1)\}$, so I is not a stable model of P_1.

It is easy to check that this program has no stable model.

Example 5.2
Consider a disjunctive constraint program:

$$\begin{aligned} P_2: \quad & A \vee B. \\ & a \leftarrow b. \end{aligned}$$

where A is a c-atom $(\{a\}, \{\{a\} \uplus \emptyset\})$ and $B = (\{b\}, \{\{b\} \uplus \emptyset\})$.

(1) Let $I_1 = \{a, b\}$. After performing the fourth operation, we obtain

$$\begin{aligned} P_2^{I_1}: \quad & \beta_A \vee \beta_B. \\ & a \leftarrow \beta_A. \\ & \beta_A \leftarrow a. \\ & b \leftarrow \beta_B. \\ & \beta_B \leftarrow b. \\ & a \leftarrow b. \end{aligned}$$

$P_2^{I_1}$ has only one minimal model, $M = \{a, \beta_A\}$; hence, I_1 is not a stable model of P_2.

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(2) Let $I_2 = \{a\}$. After performing the fourth operation, we obtain

$$\begin{aligned} P_2^{I_2}: \quad & \beta_A. \\ & a \leftarrow \beta_A. \\ & \beta_A \leftarrow a. \\ & a \leftarrow b. \end{aligned}$$

$P_2^{I_2}$ has one minimal model, $M = \{a, \beta_A\}$; hence, I_2 is a stable model of P_2.

The introduction of disjunction into the head of a rule increases the expressiveness of the language, and allows natural representation using disjunction.

Example 5.3
In scheduling, combinatorial counting or grouping is often needed. For example, a shift either has a in it, or not. If a is in it, then either a goes along with exactly one in $\{b, c\}$, or any two in $\{d, e, f\}$. This can be represented by a disjunctive program with cardinality constraints:

$$1\{a, not\ a\}1.$$
$$1\{b, c\}1 \vee 2\{d, e, f\}2 \leftarrow a.$$

The semantics of this program can be understood by the semantics of the corresponding constraint program:

$$(\{a\}, \{\emptyset, \{a\}\}).$$
$$(\{b, c\}, \{\{b\}, \{a\}\}) \vee (\{d, e, f\}, \{\{d, e\}, \{d, f\}, \{e, f\}\}) \leftarrow a.$$

This program has the following stable models: $\emptyset$, $\{a, b\}$, $\{a, c\}$, $\{a, d, e\}$, $\{a, d, f\}$, and $\{a, f, e\}$.

Once c-atoms are allowed to appear in the disjunctive head of a rule, disjunctive aggregates may be expressed.

Example 5.4
Suppose the set of atoms in our propositional language is $\{p(-1), p(1), p(2)\}$[2]. Consider the following program:

$$p(1) \vee p(-1).$$
$$\mathrm{SUM}(X|p(X)) \geqslant 3 \vee \mathrm{SUM}(X|p(X)) \leqslant 0 \leftarrow \mathrm{COUNT}(X|p(X)) \geqslant 1.$$

Its stable models are: $\{p(1), p(2)\}$, $\{p(-1), p(1)\}$, and $\{p(-1)\}$.

As commented in Simons *et al.* (2002), a weight constraint can be transformed to one with negative weights but without negative literals. The weight constraints of this kind in fact express linear inequations. Thus, a disjunction of weight constraints can be viewed as specifying a disjunction of linear inequations. For instance, the second rule in the above example can be expressed using weight constraints. To encode the

[2] Note that we assume a fixed propositional language that includes all the atoms appearing in a given program.

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SUM aggregate constraint above, let $l\ \Sigma\ u$ denote $l\{p(-1) = -1, p(1) = 1, p(2) = 2\}u$, where l and u are the lower and upper bounds, respectively. When l (resp. u) is omitted, it means $-\infty$ (resp. ∞). Then, we can write the following rule

$$3\ \Sigma \vee \Sigma\ 2 \leftarrow 1\{p(-1) = 1, p(1) = 1, p(2) = 1\}$$

where the right-hand side encodes the COUNT aggregate constraint.

We argue that disjunctive logic programing with constraint atoms provides a rich knowledge representation language for modeling *conditional* as well as *disjunctive* constraints, which have been studied in the past in constraint programing (see, e.g., Baptiste and Pape 1996; Cohen *et al.* 2000; Marriott *et al.* 2001)[3].

5.1 Properties of stable models

We now show some properties of stable models.

Theorem 5.1
Any stable model M of a logic program P is a model of P.

A stable model may not be a minimal model for some constraint programs. To illustrate, consider a logic program

$$P: \quad (\{a, b\}, \{\{a\} \uplus \{b\}, \{b\} \uplus \{a\}\}).$$

It is easy to check that $\{a\}$, $\{b\}$ and $\{a, b\}$ are all stable models of P. We see that $\{a, b\}$ is not minimal.

It turns out that logic programs whose c-atoms appearing in rule heads are elementary possess the minimality property.

Theorem 5.2
Let P be a logic program such that c-atoms appearing in the heads of its rules are all elementary. Any stable model of P is a minimal model of P.

Recall that any atom A can be expressed as a c-atom $A' = (\{A\}, \{\{A\}\})$ and any negative literal *not* A can be expressed as a c-atom $A'' = (\{A\}, \{\emptyset\})$, such that for any interpretation I, $I \models A$ (resp. $I \models$ *not* A) if and only if $I \models A'$ (resp. $I \models A''$). The following result further justifies our generalization of the standard stable model semantics to logic programs with c-atoms.

Theorem 5.3
Let P be a logic program with ordinary atoms and P' be P with each positive literal A replaced by a c-atom $(\{A\}, \{\{A\}\})$, and each negative literal *not* A replaced by a c-atom $(\{A\}, \{\emptyset\})$. An interpretation I is a stable model of P if and only if it is a stable model of P'.

If all c-atoms are coded in the abstract representation, the time complexity of the generalized Gelfond–Lifschitz transformation is as follows.

[3] But note that disjunction in rule heads is *epistemic disjunction* (Gelfond and Lifschitz 1991), not the classic disjunction in propositional logic.

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Theorem 5.4
Let P be a logic program with n different c-atoms that are coded in the abstract representation and I be an interpretation. Let A be a c-atom such that $I \models A$.

(1) The time complexity of computing all satisfiable sets of A w.r.t. T_A^I is linear in the size of A_c^*.
(2) The time complexity of the generalized Gelfond–Lifschitz transformation is bounded by $O(|P| + n * (2M_{A_c^*} + M_{A_d} + 1))$, where $M_{A_c^*}$ and M_{A_d} are the maximum sizes of A_c^* and A_d of a c-atom in P, respectively.

The following result is immediate.

Corollary 5.5
The size of P^I is bounded by $O(|P| + n * (M_{A_c^*} + M_{A_d} + 1))$.

Finally, we show the complexity of the major decision problem, namely the stable model existence problem. In the following, we assume the explicit representation of c-atoms A in the form (A_d, A_c) in a given program P.

Theorem 5.6
(1) The problem of deciding whether a stable model exists for a normal constraint program P is NP-complete.
(2) The problem of deciding whether a stable model exists for a disjunctive constraint program P is Σ_P^2-complete.

6 Relationship to conditional satisfaction

Recently, Son *et al.* (2006) proposed a fixpoint definition of stable models for logic programs with c-atoms. They introduce a key concept termed *conditional satisfaction.*

Definition 6.1 (*Son et al. 2006*)
Let R and S be two sets of atoms. The set R conditionally satisfies a c-atom A w.r.t. S, denoted $R \models_S A$, if $R \models A$ and for every S' such that $R \cap A_d \subseteq S'$ and $S' \subseteq S \cap A_d$, we have $S' \in A_c$.

An immediate consequence operator $T_P(R, S)$ is introduced, which evaluates each c-atom using the conditional satisfaction $\models_S$ instead of the standard satisfaction $\models$.

Definition 6.2 (*Son et al. 2006*)
Let P be a positive basic logic program and R and S be two sets of atoms. Define

$$T_P(R, S) = \left\{ A \;\middle|\; \begin{array}{l} \exists r \in P : R \models_S body(r), \\ head(r) = (\{A\}, \{\{A\}\}) \end{array} \right\}.$$

When the second argument is a model of P, T_P is monotone w.r.t. the first argument. In particular, given a model M and let $R \subseteq U \subseteq M$, then $T_P(R, M) \subseteq T_P(U, M) \subseteq M$. Thus, for any model I, the sequence $T_P^i(\emptyset, I)$ with $T_P^0(\emptyset, I) = \emptyset$ and $T_P^{i+1}(\emptyset, I) = T_P(T_P^i(\emptyset, I), I)$, converges to a fixpoint $T_P^\infty(\emptyset, I)$. I is defined to be a stable model if it is the same as the fixpoint.

The following result reveals the relationship between conditional satisfaction and satisfiable sets.

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Theorem 6.1
Let A be a c-atom and R and I be two interpretations with $R \subseteq I$. Let $T_A^I = I \cap A_d$. $R \models_I A$ if and only if A_c^* has an abstract prefixed power set $W \uplus V$ such that $R \cap A_d \uplus T_A^I \setminus (R \cap A_d)$ is included in $W \uplus V$ (thus W is a satisfiable set of A w.r.t. T_A^I and $W \subseteq R \cap A_d$).

Theorem 6.1 leads us to the conclusion that Son *et al.* (2006)'s fixpoint definition and our definition of stable models are semantically equivalent for positive basic programs, as stated formally by the following theorem.

Theorem 6.2
Let P be a positive basic program and I a model of P. I is a stable model under Son *et al.* (2006)'s fixpoint definition if and only if it is a stable model derived from the generalized Gelfond–Lifschitz transformation.

Note that by Theorem 5.2, any stable model of a positive basic program is a minimal model.

When the head A of a rule r is not elementary, given an interpretation I, Son *et al.* (2006) transform r into the following set of rules:

$$\begin{array}{ll} B \leftarrow body(r), & \text{for each } B \in T_A^I \\ \bot \leftarrow B, body(r), & \text{for each } B \in F_A^I. \end{array}$$

Under our generalized Gelfond–Lifschitz transformation, r is transformed into the following set of rules:

$$\begin{array}{ll} \beta_A \leftarrow body(r), & \\ B \leftarrow \beta_A, & \text{for each } B \in T_A^I \\ \bot \leftarrow B, \beta_A, & \text{for each } B \in F_A^I \\ \beta_A \leftarrow T_A^I. & \end{array}$$

Apparently, the two transformations are semantically equivalent in that when $body(r)$ is true, they derive the same conclusions except for the special atoms. Combining with Theorem 6.2, we then conclude that Son *et al.*(2006)'s fixpoint definition and our definition of stable models under the generalized Gelfond–Lifschitz transformation are semantically equivalent for normal constraint programs.

Note that any normal constraint program can be transformed into a positive basic program by replacing each negative literal *not* B with a c-atom $(\{B\}, \{\emptyset\})$ and replacing each negative c-atom *not* A with the complement $(A_d, 2^{A_d} \setminus A_c)$ of A. Therefore, our approach with the generalized Gelfond–Lifschitz transformation is semantically equivalent to Son *et al.* (2006)'s approach for normal constraint programs, as stated by the following result.

Corollary 6.3
Let P be a normal constraint program and I a model of P. Let P' be P with each negative literal *not* B being replaced by $(\{B\}, \{\emptyset\})$ and each negative c-atom *not* A replaced by $(A_d, 2^{A_d} \setminus A_c)$. I is a stable model of P' under Son *et al.* (2006)'s approach if and only if it is a stable model of P' derived from the generalized Gelfond–Lifschitz transformation.

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7 Properties based on dependency relation

In normal logic programing, the *dependency relation* over the atoms in a program is an essential notion based on which a number of important properties are characterized (see, e.g., Sato 1990; Fages 1994; You and Yuan 1994). In this section, we extend these characterizations to normal constraint programs. A central question here is what should be the *dependency graph* for a given program. We will see that our abstract representation of c-atoms in the bodies of rules is precisely what is needed to construct such a dependency graph, for the semantics defined by (Son *et al.* 2006).

In this section, a *basic program* P refers to a collection of rules of the form

$$H \leftarrow A_1, \ldots, A_n \tag{3}$$

where H is either $\bot$ or an elementary c-atom, and A_i are arbitrary c-atoms. Each rule in a basic program is also called a *basic rule*.

To be consistent with the original definition of stable model (Gelfond and Lifschitz 1988), we assume that a rule of the form

$$\bot \leftarrow body$$

in a basic program is already replaced by a rule with an elementary head

$$f \leftarrow body, (\{f\}, \{\emptyset\})$$

where f is a new symbol representing the elementary c-atom $(\{f\}, \{\{f\}\})$ and the c-atom $(\{f\}, \{\emptyset\})$ in the body is its complement.

The proof of the main result of this section is based on a method of representing a basic program by a normal program, directly using the abstract representation of c-atoms, while preserving the stable model semantics. Since the material is of interest on its own, we will first present it in the next subsection.

7.1 Representing basic programs by normal programs

The semantics of logic programs with c-atoms or aggregates have been studied by the unfolding approach (Pelov *et al.* 2003; Son and Pontelli 2007). It turns out, under our abstract representation of c-atoms, the unfolding approach can be made simple.

Let P be a basic program. The *normal program translation* of P, denoted P_n, is a normal program defined as follows. For each rule in P

$$H \leftarrow A_1, \ldots, A_n, \tag{4}$$

we have a rule

$$H \leftarrow \theta_{A_1}, \ldots, \theta_{A_n}, \tag{5}$$

in P_n, where θ_{A_i} are new symbols, plus the following rules: for each $1 \leqslant i \leqslant n$,

$$\theta_{A_i} \leftarrow W, not\ d_1, \ldots, not\ d_k \qquad \text{for each } W \uplus V \in A^*_{i_c},$$
$$\text{where } \{d_1, \ldots, d_k\} = A_{i_d} \setminus W \cup V.$$

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Example 7.1
Consider the program P_2 in Example 5.1 again, which consists of the following rules

$$p(1).$$
$$p(-1) \leftarrow p(2).$$
$$p(2) \leftarrow \mathrm{SUM}(\{X \mid p(X)\}) \geqslant 1.$$

Let A denote the aggregate in P_2. Recall that $A_c^* = \{\{p(1)\} \uplus \{p(2)\}, \{p(2)\} \uplus \{p(-1), p(1)\}\}$. Thus, P_n consists of

$$p(1).$$
$$p(-1) \leftarrow p(2).$$
$$p(2) \leftarrow \theta.$$
$$\theta \leftarrow p(1), not\ p(-1).$$
$$\theta \leftarrow p(2).$$

It is clear that this normal program has no stable models.

A distinct feature of our translation, as compared with the previous unfolding approach (Pelov *et al.* 2003; Son and Pontelli 2007), is that the abstract representation of c-atoms is defined independently of any given program, while in (Pelov *et al.* 2003; Son and Pontelli 2007), the translation to a normal program is an integrated process. This difference contributes to the simplicity of our approach.

The use of the abstract representation of c-atoms is essential. The following example shows that a simple enumeration of admissible solutions in a c-atom does not work. This is the case even for logic programs with only monotone c-atoms.

Example 7.2
Suppose a program P that consists of a single rule

$$a \leftarrow A$$

where $A = (\{a\}, \{\emptyset, \{a\}\})$. Note that A is monotone, as well as a tautology, and P has a unique stable model $\{a\}$. Since $A_c^* = \{\emptyset \uplus \{a\}\}$, P_n consists of

$$a \leftarrow \theta.$$
$$\theta \leftarrow .$$

Without the information encoded in the prefixed power set above, it may appear that a natural normal program encoding is to split admissible solutions as conditions into different rules. If we adopt this strategy, we will get the following normal program:

$$a \leftarrow \theta.$$
$$\theta \leftarrow not\ a.$$
$$\theta \leftarrow a.$$

This program has no stable model.

We now show that our translation preserves the stable models semantics. Though the result is presented as a lemma for proving Theorem 7.2 of the next subsection, it is obviously of independent interest.

Below, given a program P, we denote by $ST(P)$ the set of stable models of P.

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Lemma 7.1
Let P be a basic program and P_n be its normal program translation. Then, $ST(P) = \{M_{|At(P)} | M \text{ is a stable model of } P_n\}$.

7.2 Dependency relation-based characterizations

We are now ready to extend some of the well-known characterizations for normal programs to normal constraint programs. The key is the notion of a dependency graph for normal constraint programs.

Definition 7.1
Let P be a basic program. The *dependency graph* G_P is a graph (V, E), where $V = At(P)$ and E is the set of positive and negative edges defined as the follows: there is a *positive edge* from u to v, denoted $u \rightarrow^+ v$, if there is a rule r of the form (3) in P such that $head(r) = u$, and for some $A_i \in body(r)$ and $W \uplus V \in A^*_{i_c}$, $v \in W$; there is a *negative edge* from u to v, denoted $u \rightarrow^- v$, if there is a rule r of the form (3) in P such that $head(r) = u$, and for some $A_i \in body(r)$ and $W \uplus V \in A^*_{i_c}$, $v \in A_{i_d} \setminus W \cup V$.

It is important to notice that, in the definition above, for an abstract prefixed power set $W \uplus V \in A^*_{i_c}$, although we know that for any I such that $W \subseteq I \subseteq W \cup V$ we have $I \in A_{i_c}$, positive edges are only into atoms in W, not into any atom in $I \setminus W$. Also, in normal logic programing, negative edges are only into negative literals in rule bodies, but here a negative edge may result from a positive c-atom in the body of a rule.

Example 7.3
Suppose program P consists of a single rule

$$a \leftarrow (\{a, b, c\}, \{\emptyset, \{b\}, \{b, c\}\}).$$

Let A be the c-atom in the body of the above rule. Since $A^*_c = \{\emptyset \uplus \{b\}, \{b\} \uplus \{c\}\}$, we have $a \rightarrow^- a$, $a \rightarrow^- c$, and $a \rightarrow^+ b$.

We say that P has an *positive cycle* if there is a path in G_P from an atom to itself via only positive edges. P has an *odd cycle* if there is a path in G_P from an atom to itself via an odd number of negative edges, and P has an *even cycle* if there is a path in G_P from an atom to itself via an even number of negative edges. P is said to be *call-consistent* if P has no odd cycles. P is *acyclic* if it has no cycle of any kind.

We remark that our definition of dependency graph reduces to the standard one for normal programs. Recall that the dependency graph for a normal program is defined as: for each normal rule in a normal program P

$$a \leftarrow b_1, \ldots, b_k, not\ c_1, \ldots, not\ c_m, \tag{6}$$

there is a positive edge $a \rightarrow^+ b_i$ in G_P for each i, and a negative edge $a \rightarrow^- c_j$ for each j.

A normal program is in fact a basic program, in the sense that each positive literal b_i in the rule above is replaced by an elementary c-atom $(\{b_i\}, \{\{b_i\}\})$ and each

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negative literal *not* c_i replaced by $(\{c_i\}, \{\emptyset\})$, i.e., the complement of $(\{c_i\}, \{\{c_i\}\})$. Let the resulting program be P'. Since if $C_i = (\{c_i\}, \{\emptyset\})$ then $C_i^* = (\{c_i\}, \{\emptyset \uplus \emptyset\})$, by Definition 7.1, there is a negative edge $a \rightarrow^- c_i$ in $G_{P'}$.

The following theorem shows that the well-known properties based on the dependency graphs for normal programs as shown in You and Yuan (1994) remain to hold for normal constraint programs under the new definition of dependency graph for the latter.

Theorem 7.2
Let P be a basic program.

(1) P has a stable model if P is call-consistent.
(2) P has more than one stable model only if P has an even loop.
(3) P has a unique stable model if P is acyclic.
(4) If P has no positive cycles, then every supported model of P is a stable model of P.

Example 7.4
To illustrate the point (2) above, consider the following program.

$$p \leftarrow .$$
$$a \leftarrow (\{p, b\}, \{\{p\}\}).$$
$$b \leftarrow (\{p, a\}, \{\{p\}\}).$$

The program has two stable models $\{p, a\}$ and $\{p, b\}$. Then, according to the theorem, there must exist an even loop in its dependency graph. Indeed, the edges $a \rightarrow^- b$ and $b \rightarrow^- a$ form such an even cycle.

8 Related work

The notion of logic programs with c-atoms is introduced in Marek and Remmel (2004) and Marek and Truszczynski (2004), and further developed in Liu and Truszczynski (2005), Marek *et al.* (2008), Son and Pontelli (2007), and Son *et al.* (2007). As we mentioned earlier, major existing approaches can be roughly classified into three types: unfolding approaches, fixpoint approaches, and minimal model approaches.

Representative *unfolding* approaches to handling c-atoms include Pelov *et al.* (2003) and Son and Pontelli (2007), where a notion of aggregate solutions (or solutions) is introduced. Informally, a solution of a c-atom $A = (A_d, A_c)$ is a pair $\langle S_1, S_2 \rangle$ of disjoint sets of atoms of A_d such that for every interpretation I, if $S_1 \subseteq I$ and $S_2 \cap I = \emptyset$ then $I \models A$. This definition is given by Son and Pontelli (2007). Pelov *et al.* (2003) define an aggregate solution A as a pair $\langle S_1, S_2 \rangle$ with $S_1 \subseteq S_2 \subseteq A_d$ such that for every interpretation I, if $S_1 \subseteq I$ and $(A_d \setminus S_2) \cap I = \emptyset$ then $I \models A$. In the following, we use the former definition.

It turns out that each $W \uplus V \in A_c^*$ corresponds to a minimal solution $\langle W, A_d \setminus (W \cup V) \rangle$ of A. A solution $\langle S_1, S_2 \rangle$ of A is minimal if for no $S_3 \subset S_1$ nor $S_4 \subset S_2$, $\langle S_3, S_2 \rangle$ or $\langle S_1, S_4 \rangle$ is a solution of A. First, $\langle W, A_d \setminus (W \cup V) \rangle$ is a solution of A; by

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Theorem 4.2 for any interpretation I, if $W \subseteq I$ and $(A_d \setminus (W \cup V)) \cap I = \emptyset$ then $I \models A$. Second, $\langle W, A_d \setminus (W \cup V)\rangle$ is a minimal solution of A, as by Theorem 4.4, $W \wedge not\ (A_d \setminus (W \cup V))$ cannot be further simplified.

Representative *fixpoint* approaches include Liu *et al.* (2007), Marek *et al.* (2008), Marek and Truszczynski (2004), Pelov (2004), Pelov and Truszczynski (2004), and Son *et al.* (2006, 2007). Son *et al.* (2006; 2007) can handle arbitrary c-atoms, while Marek *et al.* (2008), Marek and Truszczynski (2004), and Pelov and Truszczynski (2004) apply only to monotone c-atoms. Liu *et al.* (2007) extend Liu and Truszczynski (2005), Marek *et al.* (2008), Marek and Truszczynski (2004), and Pelov and Truszczynski (2004) for arbitrary c-atoms based on a concept of *computation*. Son *et al.* (2006, 2007) show that their fixpoint approach is semantically equivalent to that of Marek and Truszczynski (2004) for normal logic programs with monotone c-atoms; equivalent to that of Faber *et al.* (2004) and Ferraris (2005) for positive basic logic programs with monotone c-atoms; equivalent to that of Denecker *et al.* (2001) and Pelov *et al.* (2003) for positive basic logic programs with arbitrary c-atoms. In Section 6, we show that our approach using the generalized Gelfond–Lifschitz transformation is semantically equivalent to the approach of Son *et al.* (2007) for normal logic programs with arbitrary c-atoms. Therefore, the stable model semantics defined in this paper for disjunctive logic programs with arbitrary c-atoms extends these existing semantics.

Faber *et al.* (2004) propose a *minimal model* approach. To check if an interpretation I is a stable model of P, they first remove all rules in P whose bodies are not satisfied by I, then define I to be a stable model if it is a minimal model of the simplified program. They consider the class of disjunctive logic programs whose rule heads are a disjunction of ordinary atoms. Stable models of P under this semantics are minimal models of P. Ferraris (2005) defines a stable model semantics in a different way, which (when negated c-atoms are treated as their complement c-atoms) agrees with the minimal-model based one on this class of programs. Son *et al.* (2006) show that for normal logic programs whose c-atoms appearing in rule heads are elementary, stable models under their semantics are stable models under the semantics of Faber *et al.* (2004) and Ferraris (2005). It immediately follows that for such normal logic programs, stable models under our semantics are stable models under the semantics of Faber *et al.* and Ferraris. However, the converse is not necessarily true, even for positive basic logic programs. Consider the positive basic logic program P:

$$b \leftarrow c.$$
$$c \leftarrow d.$$
$$d \leftarrow (\{b, c\}, \{\emptyset, \{b\}, \{b, c\}\}).$$

P has only one model $I = \{b, c, d\}$. It is easy to check that I is not a stable model under the semantics of Son *et al.* (2006) and ours. However, I is a stable model under the semantics of Faber *et al.* (2004) and Ferraris (2005). Observe that the truth of b, c, d can only be inferred via a self-supporting loop:

$$b \rightarrow d \rightarrow c \rightarrow b.$$

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This example program indicates that both the semantics of Faber *et al.* (2004) and that of Ferraris allow self-supporting loops.

9 Conclusions and future work

In this paper we have introduced an abstract representation of c-atoms. To substantiate the claim that the abstract representation captures the essential information correctly and compactly, we showed two applications. In the first one, we show that the semantics based on conditional satisfaction (Son *et al.* 2006; Son *et al.* 2007), and the one equivalent to it (Denecker *et al.* 2001), can be defined by a generalized form of Gelfond–Lifschitz transformation, thus demonstrating that Gelfond–Lifschitz transformation can still play an important role in the study of semantics for logic programs with arbitrary c-atoms. In the second application, we show that our abstract representation of c-atoms encodes the information needed to define the atom dependency relation in a given program. As a result, the properties known to normal programs can be extended to programs with c-atoms. In this process, the unfolding approach (Son and Pontelli 2007) is made simple.

Several interesting tasks remain open. One is the possibility of showing that other semantics may be characterized by our abstract representation of c-atoms. This is because prefixed power sets identify "monotone components" of c-atoms. Another task is to develop new algorithms for efficiently constructing the abstract form of c-atoms from the power set form representation. Finally, methods for computing the stable models (under our generalized Gelfond–Lifschitz transformation) of logic programs with arbitrary c-atoms remain a challenging open problem.

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A Proof of theorems and lemmas

Proof of Theorem 3.1: Assume that $I \uplus J$ is included in $I_1 \uplus J_1$. We first prove $I_1 \subseteq I$. If on the contrary $I_1 \not\subseteq I$, there is an atom a such that $a \in I_1$ and $a \notin I$. This means that every S covered by $I_1 \uplus J_1$ must contain a. Since I is covered by $I \uplus J$, I is covered by $I_1 \uplus J_1$. But I does not contain a, a contradiction. We now prove $I \cup J \subseteq I_1 \cup J_1$. If on the contrary $I \cup J \not\subseteq I_1 \cup J_1$, $I \cup J$ is not covered by $I_1 \uplus J_1$. This means $I \uplus J$ is not included in $I_1 \uplus J_1$, a contradiction.

Next, assume that $I \uplus J$ is included in $I_1 \uplus J_1$ and $I_1 \uplus J_1$ is included in $I_2 \uplus J_2$. We have $I_2 \subseteq I_1 \subseteq I$, and $I \cup J \subseteq I_1 \cup J_1 \subseteq I_2 \cup J_2$. This means all sets covered by $I \uplus J$ are covered by $I_2 \uplus J_2$. That is, $I \uplus J$ is included in $I_2 \uplus J_2$. □

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Proof of Theorem 3.2: (1) For each $S \in A_c$, the collection C_S of abstract S-prefixed power sets of A is uniquely defined by Definition 3.2, thus A_c^* is uniquely defined by Definition 3.3.

(2) Assume $I \models A$, i.e., $I \cap A_d = S \in A_c$. By Definition 3.2, the collection C_S of abstract S-prefixed power sets of A contains $S \uplus S_i$ covering S. By Definition 3.3, A_c^* has an abstract prefixed power set $W \uplus V$ such that either $W \uplus V = S \uplus S_i$ or $S \uplus S_i$ is included in $W \uplus V$. This means that $W \uplus V$ covers S.

Conversely, assume that A_c^* has an abstract prefixed power set $W \uplus V$ covering $I \cap A_d$. By Definition 3.3, $W \uplus V$ is an abstract W-prefixed power set of A with $W \in A_c$. By Definition 3.2, all sets covered by $W \uplus V$ are in A_c. This means $I \cap A_d \in A_c$, and hence $I \models A$. □

Proof of Theorem 3.3: Let $G = \bigcup_{S \in A_c} C_S$, where C_S is the collection of abstract S-prefixed power sets of A. By Definition 3.3, A_c^* is G with all redundants removed.

(1) ($\Longrightarrow$) Assume that A is monotone. Then, all supersets of $S \in A_c$ from 2^{A_d} are in A_c, so all abstract S-prefixed power sets in G must be of the form $S \uplus A_d \setminus S$. If S is not minimal in A_c, $S \uplus A_d \setminus S$ is redundant in G since for some $S' \subset S$, which is minimal in A_c, $S' \uplus A_d \setminus S'$ is in G. Therefore, $A_c^* = \{B \uplus A_d \setminus B : B \text{ is minimal in } A_c\}$. Clearly, $|W| + |V| = |A_d|$ for each $W \uplus V \in A_c^*$.
($\Longleftarrow$) Assume that for every $W \uplus V \in A_c^*$, we have $|W| + |V| = |A_d|$; i.e., $V = A_d \setminus W$. Every abstract S-prefixed power set in G must be of the form $S \uplus A_d \setminus S$, for otherwise, there is one $W \uplus V \in A_c^*$ with $W \subseteq S$ and $V \subset A_d \setminus W$. As shown above, in this case every $S \uplus A_d \setminus S$ in G is redundant unless S is minimal in A_c. Therefore, A_c^* is G with all $S \uplus A_d \setminus S$ removed, where S is not minimal in A_c. That is, $A_c^* = \{B \uplus A_d \setminus B : B \text{ is minimal in } A_c\}$. This shows that for any S' which is minimal in A_c, all supersets of S' are in A_c. For any $S \in A_c$, there is some $S' \subseteq S$, which is minimal in A_c. Since all supersets of S' are in A_c, all supersets of S are in A_c. This shows that A is monotone.

(2) ($\Longrightarrow$) Assume that A is antimonotone. Every abstract S-prefixed power set in G must be of the form $\emptyset \uplus T$. By Definition 3.2, T is maximal in A_c. That is, $A_c^* = \{\emptyset \uplus T : T \text{ is maximal in } A_c\}$. Clearly, $W = \emptyset$ for each $W \uplus V \in A_c^*$.
($\Longleftarrow$) Assume that every abstract prefixed power set in A_c^* is of the form $\emptyset \uplus T$. By Definition 3.2, T is maximal in A_c. That is, $A_c^* = \{\emptyset \uplus T : T \text{ is maximal in } A_c\}$. Clearly, for any $T \in A_c$ all subsets of T are in A_c. This shows that A is antimonotone.

(3) ($\Longrightarrow$) Assume that A is convex. Consider $B \uplus T$ in G. If B is not minimal in A_c, since A is convex $B \uplus T$ is included in $B' \uplus T$, where $B' \subset B$ is minimal in A_c. For the same reason, if $B \cup T$ is not maximal in A_c, $B \uplus T$ is included in $B \uplus T'$, where $T' \supset T$ and $B \cup T'$ is maximal in A_c. In both cases, $B \uplus T$ is redundant in G. Therefore, A_c^* is G with all $B \uplus T$ removed, where either B is not minimal or $B \cup T$ is not maximal in A_c. That is, $A_c^* = \{B \uplus T : B \text{ is minimal and } B \cup T \text{ is maximal in } A_c\}$.
($\Longleftarrow$) Assume $A_c^* = \{B \uplus T : B \text{ is minimal and } B \cup T \text{ is maximal in } A_c\}$. Then, for any $S_1, S_2 \in A_c$ with $S_1 \subset S_2$, there is some $B \uplus T$ in A_c^*, which covers

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all S with $B \subseteq S_1 \subseteq S \subseteq S_2 \subseteq B \cup T$. This means that all subsets in between S_1 and S_2 are in A_c. That is, A is convex. □

Proof of Theorem 3.5: Let A be a c-atom. We use a simple algorithm to construct A_c^*. The algorithm returns a set, say Π, which is set to $\emptyset$ at the beginning.

Note that for any $p \in A_c$, $|p| \leqslant |A_d|$. Therefore, for any $p, q \in A_c$ it takes $O(|A_d|^2)$ time to determine if p is a subset of q. Moreover, when $p \subset q$, there are at most $2^{|q \setminus p|} - 2$ sets w such that $p \subset w \subset q$.

For each pair (p, q), where $p, q \in A_c$ and $p \subset q$, let S be the set of all $w \in A_c$ such that $p \subset w \subset q$. If $|S| = 2^{|q \setminus p|} - 2$, we add $p \uplus q \setminus p$ to Π. Since there are at most $O(|A_c|^2)$ such pairs to check, and for each, it takes $O(|A_c| * |A_d|^2)$ time to perform the test (i.e., for each $w \in A_c$ we check if $p \subset w \subset q$), the time for the above process is bounded by $O(|A_c|^3 * |A_d|^2)$. Note that $|\Pi|$ is bounded by $O(|A_c|^2)$.

After the above process, all possible abstract prefixed power sets of A are in the resulting Π. Then, we remove all (redundant) π from Π if π is included in some other $\xi \in \Pi$. By Theorem 3.1, it takes $O(|A_d|^2)$ time to check if π is included in ξ. Therefore, the time for this redundancy removing process is bounded by $O(|A_c|^4 * |A_d|^2)$.

As a result, Π consists of all nonredundant abstract prefixed power sets of A. By Definition 3.3, Π is A_c^*. In total, it takes $O(|A_c|^4 * |A_d|^2)$ time to construct A^* from A. □

Proof of Proposition 4.1: (1) Assume that I satisfies A; i.e., $A_d \cap I = S_i \in A_c$. Then, we have $C_i = S_i \wedge not\ (A_d \setminus S_i)$ with $S_i \subseteq I$ and $(A_d \setminus S_i) \cap I = \emptyset$. This means that both S_i and $not\ (A_d \setminus S_i)$ are true in I. Hence, C_i is true in I and thus $C_1 \vee ... \vee C_m$ is true in I. Conversely, assume that $C_1 \vee \ldots \vee C_m$ is true in I. Some $C_i = S_i \wedge not\ (A_d \setminus S_i)$ must be true in I, meaning that $S_i \subseteq I$ and $(A_d \setminus S_i) \cap I = \emptyset$. This shows that $A_d \cap I = S_i$. Since $S_i \in A_c$, I satisfies A.

(2) Assume that I satisfies $not\ A$; i.e., $A_d \cap I \notin A_c$. Then, every $C_i = S_i \wedge not\ (A_d \setminus S_i)$ is false in I because either $S_i \nsubseteq I$ or $(A_d \setminus S_i) \cap I \neq \emptyset$. Thus, $not\ (C_1 \vee \ldots \vee C_m)$ is true in I. Conversely, assume that $not\ (C_1 \vee \ldots \vee C_m)$ is true in I; i.e., every $C_i = S_i \wedge not\ (A_d \setminus S_i)$ is false in I. This means that for each $S_i \in A_c$, either $S_i \nsubseteq I$ or $(A_d \setminus S_i) \cap I \neq \emptyset$; therefore, $A_d \cap I \neq S_i$. This shows $A_d \cap I \notin A_c$; thus I satisfies $not\ A$. □

Proof of Lemma 4.3: The proof is by induction on k with $1 \leqslant k \leqslant m$. When $k = 1$ (induction basis), $F = a_1 \vee not\ a_1 \equiv true$. For the induction hypothesis, assume that $F = \bigvee_{1 \leqslant i \leqslant k,\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_k$ can be simplified to $true$ by applying rule (2) for any $k < m$. This holds for $k = m$, as shown below:

$$
\begin{aligned}
F &= \textstyle\bigvee_{1 \leqslant i \leqslant m,\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_m \\
&\equiv [\textstyle\bigvee_{1 \leqslant i \leqslant (m-1),\ L_i \in \{a_i, not\ a_i\}} (L_1 \wedge \ldots \wedge L_{m-1}) \wedge a_m] \vee \\
&\quad [\textstyle\bigvee_{1 \leqslant i \leqslant (m-1),\ L_i \in \{a_i, not\ a_i\}} (L_1 \wedge \ldots \wedge L_{m-1}) \wedge not\ a_m] \\
&\equiv a_m \wedge [\textstyle\bigvee_{1 \leqslant i \leqslant (m-1),\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_{m-1}] \vee \\
&\quad not\ a_m \wedge [\textstyle\bigvee_{1 \leqslant i \leqslant (m-1),\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_{m-1}] \\
&\equiv a_m \vee not\ a_m \text{ (by the induction hypothesis)} \\
&\equiv true
\end{aligned}
$$

□

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Proof of Theorem 4.2: By Theorem 3.2, A_c and A_c^* express the same set of admissible solutions to A in that for any $S \subseteq A_d$, $S \in A_c$ if and only if A_c^* contains an abstract prefixed power set $W \uplus V$ covering S. Let $V = \{a_1, \ldots, a_m\}$. Note that each $W \uplus V$ in A_c^* exactly covers the set $\{W \cup S | S \subseteq V\}$ of items in A_c, and all items in A_c^* exactly cover all items in A_c. Since the semantics of each $S \in A_c$ is $S \wedge not\ (A_d \setminus S)$, the semantics of each $W \uplus V$ in A_c^* is

$$\begin{aligned}&\bigvee_{1 \leqslant i \leqslant m,\ L_i \in \{a_i, not\ a_i\}} W \wedge (L_1 \wedge \ldots \wedge L_m) \wedge not\ (A_d \setminus (W \cup V)) \\ &\equiv W \wedge not\ (A_d \setminus (W \cup V)) \wedge [\bigvee_{1 \leqslant i \leqslant m,\ L_i \in \{a_i, not\ a_i\}} L_1 \wedge \ldots \wedge L_m]\end{aligned}$$

which, by Lemma 4.3, can be simplified to $W \wedge not\ (A_d \setminus (W \cup V))$ by applying rule (2). Thus, we have

$$\begin{aligned}A &\equiv \bigvee_{S \in A_c} S \wedge not\ (A_d \setminus S) \\ &\equiv \bigvee_{W \uplus V \in A_c^*} \bigvee_{1 \leqslant i \leqslant m,\ \ L_i \in \{a_i, not\ a_i\}} W \wedge (L_1 \wedge \ldots \wedge L_m) \wedge not\ (A_d \setminus (W \cup V)) \\ &\equiv \bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V))\end{aligned}$$

□

Proof of Theorem 4.4: For any two $W_1 \uplus V_1, W_2 \uplus V_2 \in A_c^*$, we distinguish between three cases: (1) if $W_1 = W_2$, then the two conjunctions $W_1 \wedge not\ (A_d \setminus (W_1 \cup V_1))$ and $W_2 \wedge not\ (A_d \setminus (W_2 \cup V_2))$ have no conflicting literals, thus they cannot be pairwise simplified using rule (2); (2) if $W_1 \subset W_2$ with $|W_2| - |W_1| = 1$, then $V_1 \neq V_2$ (otherwise, $W_1 \uplus V_1 \cup (W_2 \setminus W_1)$ should be in A_c^* so that $W_1 \uplus V_1$ is not in A_c^*), which means that the two conjunctions $W_1 \wedge not\ (A_d \setminus (W_1 \cup V_1))$ and $W_2 \wedge not\ (A_d \setminus (W_2 \cup V_2))$ have at least two different literals, one in their positive part and another in their negative part, so that they cannot be pairwise simplified using rule (2); (3) otherwise (i.e., $W_1 \neq W_2$ and $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$, or $W_1 \subset W_2$ with $|W_2| - |W_1| > 1$), the two conjunctions $W_1 \wedge not\ (A_d \setminus (W_1 \cup V_1))$ and $W_2 \wedge not\ (A_d \setminus (W_2 \cup V_2))$ have at least two different positive literals, thus they cannot be pairwise simplified using rule (2). □

Proof of Theorem 4.5: By Proposition 4.1, $I \models A$ if and only if $I \models \bigvee_{S \in A_c} S \wedge not\ (A_d \setminus S)$, and by Theorem 4.2, if and only if I satisfies $\bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V))$. For each $W \uplus V \in A_c^* \setminus A_s^I$, since it does not cover T_A^I, $W \wedge not\ (A_d \setminus (W \cup V))$ is false in I. This means that $\bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V))$ is true in I if and only if $\bigvee_{W \uplus V \in A_s^I} W \wedge not\ (A_d \setminus (W \cup V))$ is true in I. Therefore, $I \models A$ if and only if $I \models \bigvee_{W \uplus V \in A_s^I} W \wedge not\ (A_d \setminus (W \cup V))$. □

Proof of Theorem 4.6: When S is a satisfiable set, there is an abstract S-prefixed power set $S \uplus S_1$ in A_c^* such that T_A^I is covered by $S \uplus S_1$. By the definition of an abstract prefixed power set, every S' with $S \subseteq S' \subseteq T_A^I$ is covered by $S \uplus S_1$. By Definition 3.2, every such S' is in A_c. □

Proof of Theorem 5.1: Let M be a stable model of P obtained by applying the generalized Gelfond–Lifschitz transformation. Note that $\perp$ is not in M. To prove that M is a model of P is to prove that for any rule r in P we have $M \models r$. By definition, if $M \models head(r)$ or $M \not\models body(r)$ then $M \models r$. Assume that $M \not\models head(r)$

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and, on the contrary, that $M \models body(r)$. Let r take the form

$$H_1 \vee ... \vee H_k \leftarrow B_1, \ldots, B_m, A_1, \ldots, A_n, not\ C_1, \ldots, not\ C_l$$

where each B_i or C_i is an atom and each A_i is a c-atom. H_i can be an atom or a c-atom. We then have $M \models B_i$, $M \models A_i$, $M \models not\ C_i$ and $M \not\models H_i$.

For every negative literal $not\ C_i$ in $body(r)$, since $M \models not\ C_i$ it will be removed in the second operation of the generalized Gelfond–Lifschitz transformation. For every c-atom A_i in $body(r)$, since $M \models A_i$ it will be replaced in the third operation by a special atom θ_{A_i} along with a new rule $\theta_{A_i} \leftarrow D_1, \ldots, D_t$ for each satisfiable set $\{D_1, \ldots, D_t\}$ of A_i w.r.t. $T^M_{A_i}$. As a result, the generalized Gelfond–Lifschitz transformation P^M contains the following rules derived from r:

$$H'_1 \vee \ldots \vee H'_k \leftarrow B_1, \ldots, B_m, \theta_{A_1}, \ldots, \theta_{A_n},$$
$$\theta_{A_i} \leftarrow D_1, \ldots, D_t, \quad \text{for each c-atom } A_i \text{ and each satisfiable set } \{D_1, \ldots, D_t\} \text{ of } A_i \text{ w.r.t. } T^M_{A_i}$$

Here, H'_i is H_i if H_i is an atom; or when H_i is a c-atom, H'_i is $\bot$ because $M \not\models H_i$ (H_i is replaced by $\bot$ in the fourth operation).

Let N be a minimal model of P^M with $M = N \setminus \Gamma$ (which leads to M being a stable model of P). For each c-atom A_i, we have $M \cap A_{i_d} = N \cap A_{i_d} = T^M_{A_i}$. Since each satisfiable set $\{D_1, \ldots, D_t\}$ of A_i is a subset of $T^M_{A_i}$, we have $\{D_1, \ldots, D_t\} \subseteq M \subseteq N$. This means that for each A_i, the body of the rule

$$\theta_{A_i} \leftarrow D_1, \ldots, D_t$$

in P^M is satisfied in N. Since N is a minimal model of P^M, the head θ_{A_i} of the above rule must be in N. As a result, the body of the rule

$$H'_1 \vee \ldots \vee H'_k \leftarrow B_1, \ldots, B_m, \theta_{A_1}, \ldots, \theta_{A_n}$$

in P^M is satisfied in N, thus some H'_j in the head is in N. Since no H'_i is a special atom prefixed with θ or β, H'_j is also in M. Since $\bot$ is not in M, H'_j must be H_j in the rule r. This means that M satisfies $head(r)$, contradicting the assumption $M \not\models head(r)$. We then conclude that M is a model of P. □

Proof of Theorem 5.2: Let I be a stable model of P and M be a minimal model of the generalized Gelfond–Lifschitz transformation P^I with $I = M \setminus \Gamma$. Let $(P^I)^i$ be obtained from P after performing the ith operation ($i = 1, \ldots, 4$) in Definition 5.1. Note that $P^I = (P^I)^4$.

Since every c-atom A appearing in each rule head is an elementary c-atom of the form $(\{a\}, \{\{a\}\})$, the semantics of P will not be changed if we replace A in the head with a new symbol β_A and define β_A by the two rules $\beta_A \leftarrow a$ and $a \leftarrow \beta_A$ (expressing $\beta_A \equiv a$). This means that when c-atoms in the rule heads are all elementary, performing the fourth operation in Definition 5.1 does not change the semantics of P. Therefore, since M is a minimal model of $(P^I)^4$, $M \setminus \Gamma_\beta$ is a minimal model of $(P^I)^3$.

Note that for each rule (introduced in the third operation) of the form $\theta_A \leftarrow W$, where $W = \{A_1, \ldots, A_m\}$ is a satisfiable set, we have $W \subset M \setminus \Gamma_\beta$ and $\theta_A \in M \setminus \Gamma_\beta$.

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Let Q be the set of rules in $(P^I)^3$ whose heads are not special atoms prefixed with θ. For any nonempty set S of $M \setminus \Gamma$, $M \setminus (\Gamma_\beta \cup S)$ will not satisfy Q; otherwise, $M \setminus \Gamma_\beta$ would not be a minimal model of $(P^I)^3$.

Let Q_1 be $(P^I)^3$ such that all rules $\theta_A \leftarrow W$ with the same head θ_A are replaced by a compact rule

$$\theta_A \leftarrow \bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V))$$

Since $M \setminus \Gamma_\beta$ is a minimal model of $(P^I)^3$, $M \setminus \Gamma_\beta$ is a minimal model of Q_1.

Now let Q_2 be Q_1 obtained by first replacing all occurrences of each θ_A in rule bodies with the body of the above compact rule, then removing all compact rules. Since $M \setminus \Gamma_\beta$ is a minimal model of Q_1, $M \setminus \Gamma$ is a minimal model of Q_2. Note $I = M \setminus \Gamma$.

By Theorem 4.2, we can replace each $\bigvee_{W \uplus V \in A_c^*} W \wedge not\ (A_d \setminus (W \cup V))$ in Q_2 with c-atom A without changing the semantics of Q_2. This transforms Q_2 into $(P^I)^2$. Therefore, I is a minimal model of $(P^I)^2$.

$(P^I)^2$ is $(P^I)^1$ with all negative literals removed. Since all such negative literals are satisfied by I, that I is a minimal model of $(P^I)^2$ implies I is a minimal model of $(P^I)^1$.

$(P^I)^1$ is P with those rules removed whose bodies are no satisfied by I. Assume, on the contrary, that some $M \subset I$ is a model of P. Since I is a minimal model of $(P^I)^1$, $(P^I)^1$ is not satisfied by M. Since $(P^I)^1 \subseteq P$, P is not satisfied by M, a contradiction. As a result, I is a minimal model of P. This concludes the proof. □

The following lemma is required for the proof of Theorem 5.3.

Lemma Appendix A.1
Let P be a positive logic program with ordinary atoms and A be a literal in P. Let P' be P with each occurrence of A in rule bodies replaced by a special atom θ_A, and each occurrence of A in rule heads replaced by a special atom β_A, where θ_A is defined in P' by a rule $\theta_A \leftarrow A$, and β_A is defined in P' by two rules $A \leftarrow \beta_A$ and $\beta_A \leftarrow A$. An interpretation I is a stable model of P if and only if M is a stable model of P' with $I = M \setminus \{\theta_A, \beta_A\}$.

Proof: Since θ_A is used only to replace A in rule bodies, it can be derived from P' only by applying the rule $\theta_A \leftarrow A$. That is, if θ_A is in a stable model of P', A must be in the model. The converse also holds. Therefore, replacing θ_A with A does not change the semantics of P'.

For β_A, the two rules $A \leftarrow \beta_A$ and $\beta_A \leftarrow A$ express $A \equiv \beta_A$. Thus, replacing β_A with A does not change the semantics of P'.

After the above replacement, we transform P' to P. Therefore, P' and P have the same stable models. □

Proof of Theorem 5.3: Let *not* A be a negative literal in the body of a rule r of P, which is replaced in P' by a c-atom $A' = (\{A\}, \{\emptyset\})$. When $I \not\models not\ A$ (i.e.,

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$A \in I$), we have $I \not\models A'$; when $I \models not\ A$ (i.e., $A \notin I$), we have $I \models A'$. For the former case, r will be removed in the first operation, from P under the standard Gelfond–Lifschitz transformation, and from P' under the generalized Gelfond–Lifschitz transformation. For the latter case, *not* A will be removed from r under the standard Gelfond–Lifschitz transformation, while A' will be replaced, under the generalized Gelfond–Lifschitz transformation, by a special atom $\theta_{A'}$, where $\theta_{A'}$ is defined by a bodiless rule $\theta_{A'}$ in P'. In this case, $\theta_{A'}$ can be removed from P'. Let P^I be the standard Gelfond–Lifschitz transformation of P w.r.t. I. We can further remove all rules from P^I whose body contains a positive literal $A \notin I$, since if I is a stable model, A will not be derived from P^I and thus these rules will not be applicable. These rules will also be removed from P' in the first operation of the generalized Gelfond–Lifschitz transformation, as $A \notin I$ implies $I \not\models (\{A\}, \{\{A\}\})$. As a result, the resulting standard transformation P^I of P is the same as P'^I obtained by applying to P' the first two operations of the generalized Gelfond–Lifschitz transformation, except that each atom A in P^I is replaced in P'^I by a c-atom $(\{A\}, \{\{A\}\})$. Then, after applying to P'^I the third and fourth operations of the generalized Gelfond–Lifschitz transformation, P'^I becomes P^I except that for each literal A in P^I, each occurrence of A in rule bodies are replaced by a special atom θ_A, and each occurrence of A in rule heads replaced by a special atom β_A, where θ_A is defined in P'^I by a rule $\theta_A \leftarrow A$, and β_A is defined in P'^I by two rules $A \leftarrow \beta_A$ and $\beta_A \leftarrow A$. By Lemma A, I is a stable model of P^I if and only if M is a stable model of P'^I with $I = M \setminus \Gamma$. This means that I is a stable model of P if and only if it is a stable model of P'. □

Proof of Theorem 5.4: The first part of the theorem is straightforward, as all satisfiable sets of A w.r.t. T_A^I can be obtained simply by comparing each $W \uplus V$ in A_c^* with T_A^I to see if it covers T_A^I.

For the second part of the theorem, the time complexity of the generalized Gelfond–Lifschitz transformation consists of the following three parts: (a) The time complexity of the standard Gelfond–Lifschitz transformation of P with all c-atoms ignored. This is linear in the number $|P|$ of rules in P. (b) The time complexity of computing all satisfiable sets of all n c-atoms. As just proved above, it is bounded by $O(n * M_{A_c^*})$. (c) The time complexity of introducing new rules for all n c-atoms. Assume that it takes constant time to introduce a new rule for a special atom θ_A or β_A (see the third and fourth operations). Then, the time complexity of this part is bounded by $O(n * (M_{A_c^*} + M_{A_d} + 1))$, as the generalized Gelfond–Lifschitz transformation introduces at most $2 * n$ special atoms (one θ_A and one β_A for each c-atom A), each accompanied by at most $M_{A_c^*}$ (for θ_A) or $M_{A_d} + 1$ (for β_A) new rules. The total time complexity of the generalized Gelfond–Lifschitz transformation is then bounded by $O(|P| + n * (2M_{A_c^*} + M_{A_d} + 1))$. □

Proof of Theorem 5.6: For normal constraint programs, since our stable model semantics coincides with that of Son *et al.* (2007), the complexity of the latter semantics applies, which is known to be NP-complete (stated in Liu *et al.* 2007 as part of computation-based semantics and proved in You *et al.* 2007).

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It is known that the decision problem for disjunctive programs (without c-atoms) is Σ_P^2-complete (Eiter and Gottlob 1993). Since disjunctive programs are disjunctive constraint programs, the decision problem is at least as hard as for disjunctive programs, i.e., it is Σ_P^2-hard. To see that the problem is in Σ_P^2, we first note that replacing c-atoms by their abstract representations takes polynomial time, in the size of P (cf. Theorem 3.5), so does the generalized Gelfond–Lifschitz transformation (cf. Theorem 5.4) for a given interpretation I. Then, to determine whether M is a minimal model of the generalized Gelfond–Lifschitz transformation P^I (cf. Definition 5.2) is to determine whether M is a minimal model of a positive disjunctive program. Therefore, the fact that the latter is in Σ_P^2 implies that the former is also in Σ_P^2. □

Proof of Theorem 6.1: ($\Longrightarrow$) Assume $R \models_I A$. By Definition 6.1, $R \models A$ and for every S' such that $R \cap A_d \subseteq S'$ and $S' \subseteq T_A^I$, we have $S' \in A_c$. By Definition 3.2, the collection of abstract $(R \cap A_d)$-prefixed power sets of A contains $R \cap A_d \uplus S_i$ with $S_i \supseteq T_A^I \setminus (R \cap A_d)$, which covers all S' with $R \cap A_d \subseteq S' \subseteq T_A^I$. By Definition 3.3, A_c^* contains an abstract prefixed power set $W \uplus V$ such that $R \cap A_d \uplus S_i$ is included in $W \uplus V$. Since $S_i \supseteq T_A^I \setminus (R \cap A_d)$, $R \cap A_d \uplus T_A^I \setminus (R \cap A_d)$ is included in $R \cap A_d \uplus S_i$, hence $R \cap A_d \uplus T_A^I \setminus (R \cap A_d)$ is included in $W \uplus V$. Note that in this case, $W \subseteq R \cap A_d$, and since $W \uplus V$ covers T_A^I, W is a satisfiable set of A w.r.t. T_A^I.

($\Longleftarrow$) Assume that A_c^* has an abstract prefixed power set $W \uplus V$ such that $R \cap A_d \uplus T_A^I \setminus (R \cap A_d)$ is included in $W \uplus V$. Then, $W \uplus V$ covers the whole collection covered by $R \cap A_d \uplus T_A^I \setminus (R \cap A_d)$. This means that $W \uplus V$ covers every S' with $R \cap A_d \subseteq S'$ and $S' \subseteq T_A^I$. Since $W \uplus V$ is in A_c^*, this collection covered by $W \uplus V$ is included in A_c and thus $R \models A$. By Definition 6.1, we have $R \models_I A$. Note again that in this case, $W \subseteq R \cap A_d$ and W is a satisfiable set of A w.r.t. T_A^I. □

Proof of Theorem 6.2: Let P^I be the generalized Gelfond–Lifschitz transformation. Since P^I is a positive normal logic program, it has a least model which is the fixpoint $T_{P^I}^{\infty}(\emptyset)$ with $T_{P^I}^0(\emptyset) = \emptyset$ and $T_{P^I}^{i+1}(\emptyset) = T_{P^I}(T_{P^I}^i(\emptyset))$, where the operator T_{P^I} is defined by

$$T_{P^I}(R) = \left\{ A \;\middle|\; \begin{array}{l} \exists r \in P^I : R \models body(r), \\ head(r) = A \end{array} \right\}.$$

We want to prove, by induction on $i \geqslant 0$, that $T_P^i(\emptyset, I) = T_{P^I}^{3i}(\emptyset) \setminus \Gamma$. As induction basis, when $i = 0$, $T_P^0(\emptyset, I) = T_{P^I}^{3*0}(\emptyset) = \emptyset$. For induction hypothesis, assume that for any $i \leqslant k$ we have $T_P^i(\emptyset, I) = T_{P^I}^{3i}(\emptyset) \setminus \Gamma$. Now consider $i = k + 1$.

($\Longrightarrow$) Assume that I is a stable model under the fixpoint definition of Son *et al.* (2007). We first prove that for each atom B derived in $T_P^{k+1}(\emptyset, I)$ (i.e., $B \in T_P^{k+1}(\emptyset, I)$ but $B \notin T_P^k(\emptyset, I)$), we have $B \in T_{P^I}^{3(k+1)}(\emptyset)$. By Definition 6.2, there is a rule r in P of the form

$$r : \quad (\{B\}, \{\{B\}\}) \leftarrow A_1, \ldots, A_m$$

such that $T_P^k(\emptyset, I) \models_I body(r)$. Consider an arbitrary c-atom A_j in $body(r)$. Note that $T_P^k(\emptyset, I) \models_I A_j$. By Theorem 6.1, there is a satisfiable set W of A_j w.r.t. $T_{A_j}^I = I \cap A_{j_d}$

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such that $W \subseteq T_P^k(\emptyset, I) \cap A_{j_d}$. Let $W = \{D_1, \ldots, D_t\} \subseteq T_P^k(\emptyset, I)$. The generalized Gelfond–Lifschitz transformation P^I must contain the following rules:

(1) $\beta_B \leftarrow \theta_{A_1}, \ldots, \theta_{A_m}$,
(2) $B \leftarrow \beta_B$,
(3) $\theta_{A_j} \leftarrow D_1, \ldots, D_t$.

By the induction hypothesis, $\{D_1, \ldots, D_t\} \subseteq T_{P^I}^{3k}(\emptyset)$. Due to this, rule (3) can be applied, leading to $\theta_{A_j} \in T_{P^I}^{3k+1}(\emptyset)$. This process applies to all c-atoms A_j in $body(r)$ so that $\theta_{A_1}, \ldots, \theta_{A_m}$ are all in $T_{P^I}^{3k+1}(\emptyset)$. Rule (1) is then applied, leading to $\beta_B \in T_{P^I}^{3k+2}(\emptyset)$. Then, rule (2) is applied, leading to $B \in T_{P^I}^{3k+3}(\emptyset)$.

The above induction shows that for any atom $B \in T_P^i(\emptyset, I)$, we have $B \in T_{P^I}^{3i}(\emptyset)$. When $i \to \infty$, $T_P^\infty(\emptyset, I) \subseteq T_{P^I}^\infty(\emptyset)$. Since I is a stable model under Son *et al.*'s fixpoint definition with $T_P^\infty(\emptyset, I) = I$ and contains no special atoms, we have $I \subseteq T_{P^I}^\infty(\emptyset) \setminus \Gamma$.

Next, we prove that when I is a stable model under Son *et al.*'s fixpoint definition, we have $T_{P^I}^\infty(\emptyset) \setminus \Gamma \subseteq I$. For any (nonspecial) atom B derived in $T_{P^I}^{3(k+1)}(\emptyset)$, there must be a rule r as above in P and a rule of form (2) in P^I derived from r such that β_B is derived in $T_{P^I}^{3k+2}(\emptyset)$ by applying rule (1) where each θ_{A_j} is satisfiable in $T_{P^I}^{3k+1}(\emptyset)$ and at least one θ_{A_j} is derived in $T_{P^I}^{3k+1}(\emptyset)$ by applying rule (3) where each atom D_j is satisfiable in $T_{P^I}^{3k}(\emptyset)$. By the induction hypothesis, $T_P^k(\emptyset, I) = T_{P^I}^{3k}(\emptyset) \setminus \Gamma$, so $T_P^k(\emptyset, I) \models \{D_1, \ldots, D_t\}$. Let $W = \{D_1, \ldots, D_t\}$. Since W comes from rule (3), it is a satisfiable set of A_j w.r.t. $T_{A_j}^I = I \cap A_{j_d}$. By Definition 4.1, $A_{j_c}^*$ contains an abstract W-prefixed power set $W \uplus V$ covering $T_{A_j}^I$. So, $W \uplus T_{A_j}^I \setminus W$ is included in $W \uplus V$. Since $T_P^k(\emptyset, I) \subseteq T_P^\infty(\emptyset, I) = I$, we have $W \subseteq T_P^k(\emptyset, I) \cap A_{j_d} \subseteq T_{A_j}^I$. By Theorem 3.1, $T_P^k(\emptyset, I) \cap A_{j_d} \uplus T_{A_j}^I \setminus (T_P^k(\emptyset, I) \cap A_{j_d})$ is included in $W \uplus T_{A_j}^I \setminus W$, thus it is included in $W \uplus V$. By Theorem 6.1, $T_P^k(\emptyset, I) \models_I A_j$. This holds for all A_j in $body(r)$. By Definition 6.2, B is in $T_P^{k+1}(\emptyset, I)$. This induction shows that for any nonspecial atom $B \in T_{P^I}^{3i}(\emptyset)$, we have $B \in T_P^i(\emptyset, I)$. When $i \to \infty$, $T_{P^I}^\infty(\emptyset) \setminus \Gamma \subseteq T_P^\infty(\emptyset, I) = I$.

The above proof concludes that when I is a stable model under Son *et al.*'s fixpoint definition, $T_{P^I}^\infty(\emptyset) \setminus \Gamma = I$. Hence, by Definition 5.2 I is a stable model derived from the generalized Gelfond–Lifschitz transformation.

($\Longleftarrow$) Assume that I is a stable model, with $T_{P^I}^\infty(\emptyset) \setminus \Gamma = I$, derived from the generalized Gelfond–Lifschitz transformation. Copying the same proof as the first part above, we can prove that any nonspecial atom B derived in $T_P^{k+1}(\emptyset, I)$ is in $T_{P^I}^{3(k+1)}(\emptyset)$. That is, $T_P^\infty(\emptyset, I) \subseteq T_{P^I}^\infty(\emptyset) \setminus \Gamma = I$. Next, we prove the converse part: $I \subseteq T_P^\infty(\emptyset, I)$.

For any (nonspecial) atom B derived in $T_{P^I}^{3(k+1)}(\emptyset)$, there must be a rule r as above in P and a rule of form (2) in P^I derived from r such that β_B is derived in $T_{P^I}^{3k+2}(\emptyset)$ by applying rule (1) where each θ_{A_j} is satisfiable in $T_{P^I}^{3k+1}(\emptyset)$ and at least one θ_{A_j} is derived in $T_{P^I}^{3k+1}(\emptyset)$ by applying rule (3) where each atom D_j is satisfiable in $T_{P^I}^{3k}(\emptyset)$. By the induction hypothesis, $T_P^k(\emptyset, I) = T_{P^I}^{3k}(\emptyset) \setminus \Gamma$, so $T_P^k(\emptyset, I) \models \{D_1, \ldots, D_t\}$. Let $W = \{D_1, \ldots, D_t\}$. Since W comes from rule (3), it is a satisfiable set of A_j w.r.t. $T_{A_j}^I = I \cap A_{j_d}$. By Definition 4.1, $A_{j_c}^*$ contains an abstract W-prefixed power set $W \uplus V$ covering $T_{A_j}^I$. So, $W \uplus T_{A_j}^I \setminus W$ is included in $W \uplus V$. Note that $T_P^k(\emptyset, I) \subseteq I$ because $T_{P^I}^{3k}(\emptyset) \setminus \Gamma \subseteq T_{P^I}^\infty(\emptyset) \setminus \Gamma = I$. Then, we have $W \subseteq T_P^k(\emptyset, I) \cap A_{j_d} \subseteq T_{A_j}^I$. By

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Theorem 3.1, $T_P^k(\emptyset, I) \cap A_{j_d} \uplus T_{A_j}^I \setminus (T_P^k(\emptyset, I) \cap A_{j_d})$ is included in $W \uplus T_{A_j}^I \setminus W$, thus it is included in $W \uplus V$. By Theorem 6.1, $T_P^k(\emptyset, I) \models_I A_j$. This holds for all A_j in $body(r)$. By Definition 6.2, B is in $T_P^{k+1}(\emptyset, I)$. This induction shows that for any nonspecial atom $B \in T_{P^I}^{3i}(\emptyset)$, we have $B \in T_P^i(\emptyset, I)$. When $i \to \infty$, $T_{P^I}^{\infty}(\emptyset) \setminus \Gamma \subseteq T_P^{\infty}(\emptyset, I)$. That is, $I \subseteq T_P^{\infty}(\emptyset, I)$.

The above proof concludes that when I is a stable model derived from the generalized Gelfond–Lifschitz transformation, $T_P^{\infty}(\emptyset, I) = I$. Hence, I is also a stable model under Son *et al.*'s fixpoint definition. □

Proof of Lemma 7.1: First, we note that, under the assumptions of basic programs in this section, part 4 in Definition 5.2 can be omitted. Thus, that I is a stable model of P if and only if $M = I \cup \Gamma_\theta$ is the least model of the generalized Gelfond–Lifschitz transformation P^I, if and only if M is the least model of the standard Gelfond–Lifschitz transformation P_n^M. □

Proof of Theorem 7.2: We know that the same claims hold for normal programs ((1), (3), and (4) are due to Fages 1994, and (2) due to You and Yuan 1994), where the dependency graph is defined as: for each rule $a \leftarrow b_1, \ldots, b_m, not\ c_1, \ldots, not\ c_n$ in a normal program, there is a positive edge from a to each b_i, $1 \leqslant i \leqslant m$, and a negative edge from a to each c_j, $1 \leqslant j \leqslant n$. Let us denote by G_P^N the dependency graph for a normal program P. Recall that we use G_P to denote the dependency graph for a basic program P.

Let P be a basic program and P_n be its normal program translation. By definition, for any positive edge $u \to^+ v$ in G_P, there is a path $u \to^+ \theta_{A_i} \to^+ v$ in $G_{P_n}^N$, for some new symbol θ_{A_i}, and vice versa. Similarly, for any negative edge $u \to^- v$ in G_P, there is a path $u \to^+ \theta_{A_i} \to^- v$ in $G_{P_n}^N$, and vice versa. Therefore, for any loop L in G_P, there is a loop L' in $G_{P_n}^N$ with some additional positive edges to new symbols, and vice versa. Therefore, there is a one-to-one correspondence between loops in G_P and those in $G_{P_n}^N$, modulo the new symbols θ_{A_i}.

Notice that the extra positive edges have no effect on the type of the loops based on negative dependency; i.e., for any odd cycle in G_P, the same odd cycle with some additional positive edges is in $G_{P_n}^N$, and vice versa; similarly for even cycles.

Let P be a basic program. Suppose P is call-consistent, i.e., P has no odd cycles in G_P. By the one-to-one correspondence between cycles, P_n has no odd cycles in $G_{P_n}^N$. Thus, according to Fages (1994), a stable model, say M, exists for P_n. By Lemma 7.1, $M_{|At(P)}$ is a stable model of P. This proves claim (1). Now assume P has more than one stable model, say M_1 and M_2 (and possibly others). By Lemma 7.1, P_n has stable models S_1 and S_2 such that $S_{1|At(P)} = M_1$ and $S_{2|At(P)} = M_2$. Thus, according to (You and Yuan 1994), P_n has an even loop in $G_{P_n}^N$, and it follows that P has an even loop in G_P. This proves claim (2). Now assume P is acyclic in G_P. Then P_n is acyclic in $G_{P_n}^N$. By Lemma 7.1 again, that P_n has a unique stable model implies the same for P. This shows claim (3). Finally, suppose P has no positive cycles in G_P. Let M be a supported model of P. We can extend M to be a supported model of P_n by adding extra symbols δ_{A_i} in the following way: whenever a rule of the form

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(4) in P supports atom H in M, add θ_{A_i} $(1 \leqslant i \leqslant n)$ of the rule (5) in P_n to M. Let the resulting set be S. That is, $M = S_{|At(P)}$. Clearly, S is a supported model of P_n. Since P_n has no positive cycle in $G^N_{P_n}$, S is a stable model of P_n, and by Lemma 7.1, M is a stable model of P. This proves claim (4). □

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