Proceedings of the Edinburgh Mathematical Society (2020) **63**, 956–970
doi:10.1017/S0013091520000279

A CLASS OF SIMPLE NON-WEIGHT MODULES OVER THE VIRASORO ALGEBRA

HAIBO CHEN[1] AND JIANZHI HAN[2]

[1] *School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China* (rebel1025@126.com)
[2] *School of Mathematical Sciences, Tongji University, Shanghai 200092, China* (jzhan@tongji.edu.cn)

(Received 12 October 2019; first published online 7 August 2020)

Abstract The Virasoro algebra $\mathcal{L}$ is an infinite-dimensional Lie algebra with basis $\{L_m, C \mid m \in \mathbb{Z}\}$ and relations $[L_m, L_n] = (n-m)L_{m+n} + \delta_{m+n,0}((m^3-m)/12)C$, $[L_m, C] = 0$ for $m, n \in \mathbb{Z}$. Let $\mathfrak{a}$ be the subalgebra of $\mathcal{L}$ spanned by L_i for $i \geq -1$. For any triple (μ, λ, α) of complex numbers with $\mu \neq 0, \lambda \neq 0$ and any non-trivial $\mathfrak{a}$-module V satisfying the condition: for any $v \in V$ there exists a non-negative integer m such that $L_i v = 0$ for all $i \geq m$, non-weight $\mathcal{L}$-modules on the linear tensor product of V and $\mathbb{C}[\partial]$, denoted by $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ ($\Omega(\lambda, \alpha) = \mathbb{C}[\partial]$ as vector spaces), are constructed in this paper. We prove that $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ is simple if and only if $\mu \neq 1, \lambda \neq 0, \alpha \neq 0$. We also give necessary and sufficient conditions for two such simple $\mathcal{L}$-modules being isomorphic. Finally, these simple $\mathcal{L}$-modules $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ are proved to be new for V not being the highest weight $\mathfrak{a}$-module whose highest weight is non-zero.

Keywords: Virasoro algebra; non-weight module; simple module

2010 Mathematics subject classification: Primary 17B10
Secondary 17B65; 17B68

1. Introduction

The most important infinite-dimensional Lie algebra in mathematics and mathematical physics is the Virasoro algebra $\mathcal{L}$, which has a basis $\{L_m := t^{m+1}(\mathrm{d}/\mathrm{d}t), C \mid m \in \mathbb{Z}\}$ subject to the following Lie brackets:

$$\left[t^{m+1}\frac{\mathrm{d}}{\mathrm{d}t}, t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}\right] = (n-m)t^{m+n+1}\frac{\mathrm{d}}{\mathrm{d}t} + \delta_{m+n,0}\frac{m^3-m}{12}C$$

$$\text{and} \quad \left[t^{m+1}\frac{\mathrm{d}}{\mathrm{d}t}, C\right] = 0 \quad \text{for } m, n \in \mathbb{Z}.$$

The theory of weight modules over $\mathcal{L}$ is well developed (see, e.g. [8]). A weight $\mathcal{L}$-module whose weight subspaces are all finite dimensional is called a *Harish–Chandra module*. The classification of simple Harish–Chandra modules over $\mathcal{L}$ was obtained in [17]. In fact, any simple weight module over the Virasoro algebra with a non-zero finite-dimensional weight space is a Harish–Chandra module (see [19]). After that, the study of weight modules turns to modules with an infinite-dimensional weight space (see, e.g. [5, 12, 15]). Such a module was first constructed by taking the tensor product of a highest weight modules and some intermediate series module (see [24]), whose simplicity was determined in [4].

Non-weight $\mathcal{L}$-modules, as the other component of the representation theory of the Virasoro algebra, have drawn much attention in the past few years, such as Whittaker modules, $\mathbb{C}[L_0]$-free modules, simple module from Weyl modules and a class of non-weight modules including highest-weight-like modules (see, e.g. [2, 6, 11, 14, 16, 18, 20–23]). Unfortunately, it is far from classifying all simple non-weight $\mathcal{L}$-modules. So it is important to construct such modules. In the present paper, we shall study non-weight $\mathcal{L}$-modules. To be more precise, we are going to construct a family of new simple $\mathcal{L}$-modules from tensor products of $\mathcal{L}$-modules $\Omega(\lambda, \alpha) = \mathbb{C}[\partial]$ (see [14]) and $\mathfrak{a}$-modules, where $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$ and $\mathfrak{a} = \text{span}\{L_i \mid i \geq -1\}$. And we also give a necessary and sufficient condition for two of these modules being isomorphic.

Here follows a brief summary of this paper. In § 2, we construct a class of non-weight modules $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha)) := V \otimes \Omega(\lambda, \alpha)$ associated to $\mathfrak{a}$-modules V and $\mathcal{L}$-modules $\Omega(\lambda, \alpha)$, where $\mu \in \mathbb{C}^*$. The simplicity of modules in this class is determined in § 3. We show that $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ is simple if and only if $\mu \neq 1$ and $\alpha \neq 0$. Section 4 is devoted to giving a necessary and sufficient condition for two simple $\mathcal{L}$-modules $\mathcal{M}(V_1, \mu_1, \Omega(\lambda_1, \alpha_1))$ and $\mathcal{M}(V_2, \mu_2, \Omega(\lambda_2, \alpha_2))$ being isomorphic. In § 5, we compare simple $\mathcal{L}$-modules constructed in the present paper with the known simple non-weight $\mathcal{L}$-modules and show that all simple $\mathcal{L}$-modules $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ (V is not a highest weight module with highest weight in $\mathbb{C}^*$) are new.

Throughout this paper, we respectively denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}$, $\mathbb{Z}_+$ and $\mathcal{U}(\mathfrak{g})$ the sets of complex numbers, non-zero complex numbers, integers, non-negative integers and the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. All vector spaces are assumed to be over $\mathbb{C}$.

2. Non-weight modules

In this section, we shall recall some known results and approach to a class of Virasoro modules from twisted Heisenberg–Virasoro modules.

Let $\mathbb{C}[[t]]$ be the algebra of formal power series in t with coefficients in $\mathbb{C}$. Denote

$$e^{mt} = \sum_{i=0}^{\infty} \frac{(mt)^i}{i!} \in \mathbb{C}[[t]] \quad \text{for } m \in \mathbb{Z}.$$

Then $e^{mt}(\mathrm{d}/\mathrm{d}t) = \sum_{i=0}^{\infty}(m^i/i!)L_{i-1}$ and for $m, n \in \mathbb{Z}$,

$$\left[e^{mt}\frac{\mathrm{d}}{\mathrm{d}t}, e^{nt}\frac{\mathrm{d}}{\mathrm{d}t}\right] = (n-m)e^{(m+n)t}\frac{\mathrm{d}}{\mathrm{d}t}.$$

For any $\mu \in \mathbb{C}^*$, set $g^\mu(m) = (\mu^m e^{mt} - 1)(\mathrm{d}/\mathrm{d}t)$ for $m \in \mathbb{Z}$. Denote by $\mathcal{G}^\mu$ the Lie algebra (see [11]) with basis $\{g^\mu(m) \mid m \in \mathbb{Z}\}$ and relations

$$[g^\mu(m), g^\mu(n)] = mg^\mu(m) - ng^\mu(n) + (n-m)g^\mu(m+n).$$

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

Consider the following two subalgebras of $\mathcal{L}$:

$$\mathfrak{a} = \text{span}\{L_i \mid i \geq -1\}, \quad \mathfrak{b} = \text{span}\{L_i \mid i \geq 0\}.$$

The following lemma can be easily checked.

Lemma 2.1. *Let V be an $\mathfrak{a}$-module such that for any $v \in V, L_{m-1}v = 0$ for all but finitely many of m (≥ 0). Then V naturally becomes a $\mathcal{G}^\mu$-module for any $\mu \in \mathbb{C}^*$.*

Besides the Virasoro algebra, another important infinite-dimensional Lie algebra is the Heisenberg–Virasoro algebra $\mathcal{H}$, which has a basis $\{L_m, I_m, C_i \mid C_1 = C, m \in \mathbb{Z}, i = 1, 2, 3\}$ subject to the following Lie brackets, for $m, n \in \mathbb{Z}$ and $i = 1, 2, 3$,

$$\begin{aligned}
[L_m, L_n] &= (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C, \\
[L_m, I_n] &= nI_{m+n} + \delta_{m+n,0}(m^2+m)C_2, \\
[I_m, I_n] &= n\delta_{m+n,0}C_3, \quad [\mathcal{H}, C_i] = 0.
\end{aligned}$$

Let V be a $\mathcal{G}^\mu$-module and W a vector space on which $I_{m+n} = I_m I_n$ for $m, n \in \mathbb{Z}$ hold. Define an $\mathcal{L}$-action on the vector space $\mathcal{M}(V, \mu, W) := V \otimes W$ as follows, for $m \in \mathbb{Z}$ and $v \in V, w \in W$,

$$L_m(v \otimes w) = v \otimes L_m w + g^\mu(m)v \otimes I_m w, \quad C(v \otimes w) = 0. \tag{2.1}$$

Proposition 2.2. *Let V, μ and W be as above. Then $\mathcal{M}(V, \mu, W)$ is an $\mathcal{L}$-module under the actions (2.1).*

Proof. We simply write $g^\mu(m)$ as $g(m)$ for any $m \in \mathbb{Z}$. Note for any $m, n \in \mathbb{Z}, v \in V$ and $w \in W$ that

$$\begin{aligned}
&(L_m L_n - L_n L_m)\big(v \otimes w\big) \\
&\quad = L_m\big(v \otimes L_n w + g(n)v \otimes I_n w\big) - L_n\big(v \otimes L_m w + g(m)v \otimes I_m w\big) \\
&\quad = v \otimes L_m L_n w + g(m)v \otimes I_m L_n w + g(n)v \otimes L_m I_n w + g(m)g(n)v \otimes I_m I_n w \\
&\qquad - v \otimes L_n L_m w - g(n)v \otimes I_n L_m w - g(m)v \otimes L_n I_m w - g(n)g(m)v \otimes I_n I_m w \\
&\quad = (n-m)v \otimes L_{m+n}w - mg(m)v \otimes I_{m+n}w + ng(n)v \otimes I_{m+n}w \\
&\qquad + g(m)g(n)v \otimes I_{m+n}w - g(n)g(m)v \otimes I_{m+n}w \\
&\quad = (n-m)L_{m+n}(v \otimes w).
\end{aligned}$$

That is, $\mathcal{M}(V, \mu, W)$ is an $\mathcal{L}$-module. □

Example 2.3. Here we give two explicit examples of $\mathcal{M}(V, \mu, W)$.

(1) Take W as the weight $\mathcal{H}$-module $A_{\alpha,\beta}$ $(\alpha, \beta \in \mathbb{C})$ (see [9, 10]) of intermediate series, which has a basis $\{v_i \mid i \in \mathbb{Z}\}$ with trivial central actions and

$$L_m v_n = \big(\alpha + n + m\beta\big)v_{m+n}, \quad I_m v_n = v_{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

Then $\mathcal{M}(V, \mu, A_{\alpha,\beta})$ is a weight $\mathcal{L}$-module, which was studied in [15].

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

(2) Take W as the non-weight $\mathcal{H}$-module $\Omega(\lambda,\alpha) := \mathbb{C}[\partial]$ ($\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$) (see [3, 14]), on which the action is given for $i = 1, 2, 3$, $f(\partial) \in \mathbb{C}[\partial]$ and $m \in \mathbb{Z}$, by

$$L_m f(\partial) = \lambda^m(\partial - m\alpha) f(\partial - m), \quad I_m f(\partial) = \lambda^m f(\partial - m), \quad C_i f(\partial) = 0. \quad (2.2)$$

Then $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ is a non-weight $\mathcal{L}$-module, which will be studied in this paper.

A module V over a Lie algebra $\mathfrak{g}$ is called *trivial* if $xV = 0$ for any $x \in \mathfrak{g}$, and *non-trivial* otherwise.

Remark 2.4. (1) Let M be an $\mathfrak{a}$-module for which there exists $r \in \mathbb{Z}_+$ such that $L_{r+i}M = 0$ for all $i \geq 1$. Then $\mathcal{M}(M, 1, \Omega(\lambda, \alpha))$ is, in fact, the module studied in [11].

(2) If V is a trivial simple $\mathfrak{a}$-module, then $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha)) \cong \Omega(\lambda, \alpha)$ as $\mathcal{L}$-modules.

3. Simplicity

From now on, let $\mathcal{C}$ denote the class of all non-trivial $\mathfrak{a}$-modules V satisfying the following condition:

for any $0 \neq v \in V$ **there exists** $r \in \mathbb{Z}_+$ **such that** $L_{r+i}v = 0$ **for all** $i \geq 1$.

The minimal such r is called *the order of* v, denoted by $\mathrm{ord}(v)$. Define $\mathrm{ord}(v) = 0$ if $v = 0$. Choose $s \in \mathbb{Z}_+$ minimal such that the set $\{v \in V \mid L_{s+i}v = 0 \text{ for all } i \geq 1\}$ is non-zero. Denote this set by $V_{\mathfrak{b}}$ and denote s by $\mathrm{ord}(V_{\mathfrak{b}})$.

The following lemma can be found in [15, Lemmas 1 and 2].

Lemma 3.1. *Suppose that V and W are simple modules in $\mathcal{C}$. Then*

(1) *$V_{\mathfrak{b}}$ is a simple $\mathfrak{b}$-module;*

(2) *$V \cong \mathcal{U}(\mathfrak{a}) \otimes_{\mathcal{U}(\mathfrak{b})} V_{\mathfrak{b}}$ ($\cong \mathbb{C}[L_{-1}] \otimes V_{\mathfrak{b}}$ as vector spaces) as $\mathfrak{a}$-modules;*

(3) *L_r acts bijectively on $V_{\mathfrak{b}}$ and $\mathrm{ord}(f(L_{-1})v) = \deg f + r$ if $V_{\mathfrak{b}}$ is a non-trivial $\mathfrak{b}$-module, where $r = \mathrm{ord}(V_{\mathfrak{b}}), v \in V_{\mathfrak{b}}$ and $0 \neq f(x) \in \mathbb{C}[x]$;*

(4) *$V \cong W$ as $\mathfrak{a}$-modules if and only if $V_{\mathfrak{b}} \cong W_{\mathfrak{b}}$ as $\mathfrak{b}$-modules.*

For any $m \in \mathbb{Z}$, we denote

$$J_m^0 = 1 \quad \text{and} \quad J_m^n = \prod_{j=m+1}^{m+n} (\partial - j) \quad \text{for } n > 0.$$

Note that $\{J_m^n \mid n \in \mathbb{Z}_+\}$ forms a basis of $\Omega(\lambda, \alpha)$ for any $m \in \mathbb{Z}$. By (2.2), it is easy to check for $m, n \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$,

$$L_m J_n^k = \lambda^m(\partial - m\alpha) J_{m+n}^k.$$

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

Recall from § 2 that the $\mathcal{L}$-module structure on $\mathcal{M}(V,\mu,\Omega(\lambda,\alpha)) = V\otimes\mathbb{C}[\partial]$ is given for $m\in\mathbb{Z}, v\in V, f(\partial)\in\mathbb{C}[\partial]$ by

$$\begin{aligned} &L_m\big(v\otimes f(\partial)\big) = v\otimes\lambda^m(\partial - m\alpha)f(\partial-m) + (\mu^m e^{mt}-1)\frac{\mathrm{d}}{\mathrm{d}t}v\otimes\lambda^m f(\partial-m),\\ &C\big(v\otimes f(\partial)\big) = 0. \end{aligned} \tag{3.1}$$

Lemma 3.2. *Let $\lambda\in\mathbb{C}^*, \alpha\in\mathbb{C}$ and let V be an $\mathfrak{a}$-module in $\mathcal{C}$. Then $\mathcal{M}(V,1,\Omega(\lambda,\alpha))$ has a series of $\mathcal{L}$-submodules*

$$V_{\mathfrak{b}}^{(0)}\subsetneq V_{\mathfrak{b}}^{(1)}\subsetneq\cdots\subsetneq V_{\mathfrak{b}}^{(p)}\subsetneq\cdots,$$

where $V_{\mathfrak{b}}^{(p)} = \sum_{i=0}^{p} L_{-1}^i V_{\mathfrak{b}}\otimes\mathbb{C}[\partial]$.

Proof. We proceed by using induction on n to show that each $V_{\mathfrak{b}}^{(n)}$ is an $\mathcal{L}$-submodule. It is easy to check that $V_{\mathfrak{b}}^{(0)}$ is a submodule of $\mathcal{M}(V,1,\Omega(\lambda,\ \alpha))$. Suppose that $V_{\mathfrak{b}}^{(0)},\ldots,V_{\mathfrak{b}}^{(n-1)}$ are all $\mathcal{L}$-submodules. Take $L_{-1}^n v\otimes J_0^k + v^{(n-1)}\in V_{\mathfrak{b}}^{(n)}$, where $v\in V, k\in\mathbb{Z}_+$ and $v^{(n-1)}\in V_{\mathfrak{b}}^{(n-1)}$. Then by (3.1) and the induction hypothesis,

$$\begin{aligned} &L_m(L_{-1}^n v\otimes J_0^k + v^{(n-1)})\\ &\quad\equiv L_{-1}^n v\otimes\lambda^m(\partial-m\alpha)J_m^k + (e^{mt}-1)\frac{\mathrm{d}}{\mathrm{d}t}L_{-1}^n v\otimes\lambda^m J_m^k (\mathrm{mod}\ V_{\mathfrak{b}}^{(n-1)})\\ &\quad\equiv L_{-1}^n v\otimes\lambda^m(\partial-m\alpha)J_m^k + (L_{-1}-m)^n e^{mt}\frac{\mathrm{d}}{\mathrm{d}t}v\otimes\lambda^m J_m^k\\ &\qquad - L_{-1}^{n+1}v\otimes\lambda^m J_m^k (\mathrm{mod}\ V_{\mathfrak{b}}^{(n-1)})\\ &\quad\equiv L_{-1}^n v\otimes\lambda^m(\partial-m\alpha)J_m^k + L_{-1}^n\Big(\sum_{i\geq 1}\frac{m^i}{i!}L_{i-1} - mn\Big)v\otimes\lambda^m J_m^k (\mathrm{mod}\ V_{\mathfrak{b}}^{(n-1)})\\ &\quad\equiv 0 (\mathrm{mod}\ V_{\mathfrak{b}}^{(n)}), \end{aligned}$$

where of course we have used the formula:

$$e^{kt}\frac{\mathrm{d}}{\mathrm{d}t}L_{-1}^i = \big(L_{-1}-k\big)^i e^{kt}\frac{\mathrm{d}}{\mathrm{d}t}\quad\text{for } i\in\mathbb{Z}_+, k\in\mathbb{Z}. \tag{3.2}$$

This shows that $V_{\mathfrak{b}}^{(n)}$ is a submodule of $\mathcal{M}(V,1,\Omega(\lambda,\alpha))$. □

Lemma 3.3. *Let $\lambda,\mu\in\mathbb{C}^*$ and let V be an $\mathfrak{a}$-module in $\mathcal{C}$. Then the space*

$$\tilde{\mathcal{M}}\big(V,\mu,\Omega(\lambda,0)\big) = \{L_{-1}v\otimes J_0^k - v\otimes\partial J_0^k \mid k\in\mathbb{Z}_+, v\in V\}$$

is an $\mathcal{L}$-submodule of $\mathcal{M}(V,\mu,\Omega(\lambda,0))$ isomorphic to $\mathcal{M}(V,\mu,\Omega(\lambda,1))$.

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

Proof. Without loss of generality, assume $\lambda = 1$. Note for any $n \in \mathbb{Z}, k \in \mathbb{Z}_+$ and $v \in V$ that

$$\begin{aligned}
&L_n(L_{-1}v \otimes J_0^k - v \otimes \partial J_0^k) \\
&= L_{-1}v \otimes \partial J_n^k + (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} L_{-1}v \otimes J_n^k \\
&\quad - v \otimes \partial(\partial - n)J_n^k - (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes (\partial - n)J_n^k \\
&= L_{-1}v \otimes \partial J_n^k + (L_{-1} - n)\mu^n e^{nt}\frac{\mathrm{d}}{\mathrm{d}t} v \otimes J_n^k - L_{-1}^2 v \otimes J_n^k \\
&\quad - v \otimes \partial(\partial - n)J_n^k - (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes \partial J_n^k + n(\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes J_n^k \\
&= L_{-1}v \otimes (\partial - n)J_n^k - v \otimes \partial(\partial - n)J_n^k \\
&\quad + L_{-1}(\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes J_n^k - (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes \partial J_n^k \in \tilde{\mathcal{M}}\big(V, \mu, \Omega(1,0)\big).
\end{aligned}$$

Thus, $\tilde{\mathcal{M}}(V, \mu, \Omega(1,0))$ does form an $\mathcal{L}$-submodule of $\mathcal{M}(V, \mu, \Omega(1,0))$.

Let $\tau : \mathcal{M}(V, \mu, \Omega(1,1)) \to \tilde{\mathcal{M}}(V, \mu, \Omega(1,0))$ be a linear isomorphism given by

$$\tau(v \otimes J_0^k) = L_{-1}v \otimes J_0^k - v \otimes \partial J_0^k \quad \text{for } k \in \mathbb{Z}_+, v \in V.$$

Then a direct computation

$$\begin{aligned}
\tau\big(L_n(v \otimes J_0^k)\big) &= \tau\left(v \otimes (\partial - n)J_n^k + (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes J_n^k\right) \\
&= L_{-1}v \otimes (\partial - n)J_n^k - v \otimes \partial(\partial - n)J_n^k \\
&\quad + L_{-1}(\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes J_n^k - (\mu^n e^{nt} - 1)\frac{\mathrm{d}}{\mathrm{d}t} v \otimes \partial J_n^k \\
&= L_n \tau(v \otimes J_0^k)
\end{aligned}$$

shows that τ is a homomorphism and therefore an isomorphism of $\mathcal{L}$-modules. □

The following result is useful, which will be frequently used in this paper.

Proposition 3.4 ([15, Proposition 7]). *Let P be a vector space over $\mathbb{C}$ and P_1 a subspace of P. Assume that $\mu_1, \mu_2, \ldots, \mu_s \in \mathbb{C}^*$ are pairwise distinct, $v_{i,j} \in P$ and $f_{i,j}(t) \in \mathbb{C}[t]$ with $\deg f_{i,j}(t) = j$ for $i = 1, 2, \ldots, s; j = 0, 1, 2, \ldots, k$. If*

$$\sum_{i=1}^{s}\sum_{j=0}^{k} \mu_i^m f_{i,j}(m) v_{i,j} \in P_1 \quad \textit{for } K < m \in \mathbb{Z} \quad (K \textit{ is any fixed element in } \mathbb{Z} \cup \{-\infty\}),$$

then $v_{i,j} \in P_1$ for all i, j.

Remark 3.5. Though the statement of Proposition 3.4 is slightly different from [15, Proposition 7] (in which K is taken to be $-\infty$), it follows from the proof there that the proposition above also holds.

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

Lemma 3.6. *Let $\lambda, \alpha, 1 \neq \mu \in \mathbb{C}^*$, and let V be a simple $\mathfrak{a}$-module in $\mathcal{C}$ and W an $\mathcal{L}$-submodule of $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$. Suppose $0 \neq u \otimes f(\partial) \in W$ for some $u \in V$ and $f(\partial) \in \mathbb{C}[\partial]$. Then $W = \mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$.*

Proof. Let $r \geq -1$ be the maximal integer such that $L_r u \neq 0$. Note for any $m \in \mathbb{Z}$ that

$$\begin{aligned} W &\ni L_m\big(u \otimes f(\partial)\big) \\ &= \lambda^m\Bigg(u \otimes (\partial - m\alpha) f(\partial - m) + \Bigg(\sum_{j=0}^{r+1} \frac{\mu^m m^j}{j!} L_{j-1} - L_{-1}\Bigg) u \otimes f(\partial - m)\Bigg). \end{aligned}$$

Applying Proposition 3.4 here one has $0 \neq L_r u \otimes 1 \in W$. Then by using $L_0^i(v \otimes 1) = v \otimes \partial^i$ for all $v \in V$ and $i \in \mathbb{Z}_+$ we see that $L_r u \otimes \Omega(\lambda, \alpha) \subseteq W$. Set $M = \{w \in V \mid w \otimes \Omega(\lambda, \alpha) \subseteq W\}$. Replacing u by w in the above procedure and using Proposition 3.4 again we have $L_i w \otimes \Omega(\lambda, \alpha) \subseteq W$ for any $i \geq -1$ and $w \in M$. Thus, M is a non-zero $\mathfrak{a}$-submodule of V. Then the simplicity of V implies $M = V$ and therefore $W = \mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$. □

Now we are ready to state the first main result of this paper.

Theorem 3.7. *Let $\lambda, \mu \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and let V be a simple $\mathfrak{a}$-module in $\mathcal{C}$. Then the $\mathcal{L}$-module $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ is simple if and only if $\alpha \neq 0$ and $\mu \neq 1$.*

Proof. By Lemmas 3.2 and 3.3, $\mathcal{M}(V, 1, \Omega(\lambda, \alpha))$ and $\mathcal{M}(V, \mu, \Omega(\lambda, 0))$ are not simple. Consider now $\alpha \neq 0$ and $\mu \neq 1$. Let W be a non-zero submodule of $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$. Take a non-zero element $u = \sum_{(i,j,k) \in I} a_{ijk} L_{-1}^i u_j \otimes \partial^k \in W \subseteq \mathbb{C}[L_{-1}] V_{\mathfrak{b}} \otimes \mathbb{C}[\partial]$ (see Lemma 3.1 (2)) such that all a_{ijk} are non-zero complex numbers and that these u_j's are linearly independent in $V_{\mathfrak{b}}$, where I is a finite subset of $\mathbb{Z}_+^3$. Then by (3.2),

$$\begin{aligned} L_m(u) = \lambda^m \sum_{(i,j,k) \in I} a_{ijk} \Bigg(& L_{-1}^i u_j \otimes (\partial - m\alpha)(\partial - m)^k \\ & + \Bigg(\mu^m (L_{-1} - m)^i \sum_{p=0}^{r+1} \frac{m^p}{p!} L_{p-1} - L_{-1}^{i+1}\Bigg) u_j \otimes (\partial - m)^k\Bigg), \end{aligned}$$

which together with Lemma 3.1 (3) and Proposition 3.4 gives $0 \neq (\sum_{(i_0,j,k_0) \in I} a_{i_0 j k_0} L_r u_j) \otimes 1 \in W$, where $i_0 = \max\{i \mid (i,j,k) \in I\}$, $k_0 = \max\{k \mid (i,j,k) \in I\}$ and $r = \operatorname{ord}(V_{\mathfrak{b}})$. Then by Lemma 3.6, $W = \mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$. □

4. Isomorphism classes

The second main result of this paper is to give the isomorphisms between modules of the forms $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$.

Lemma 4.1. *Let $\lambda, \alpha_1, \alpha_2, 1 \neq \mu \in \mathbb{C}^*$ and let V_1, V_2 be simple highest weight $\mathfrak{a}$-modules with highest weights $-\alpha_2, -\alpha_1$, respectively. Then the linear map*

$$\phi : \mathcal{M}\big(V_1, \mu, \Omega(\lambda, \alpha_1)\big) \to \mathcal{M}\big(V_2, \mu^{-1}, \Omega(\mu\lambda, \alpha_2)\big),$$
$$L_{-1}^i v_1 \otimes f(\partial) \mapsto \sum_{p=0}^{i} (-1)^p \binom{i}{p} L_{-1}^p v_2 \otimes \partial^{i-p} f(\partial)$$
$$\text{for } f(\partial) \in \mathbb{C}[\partial] \quad \text{and} \quad i \in \mathbb{Z}_+,$$

is an isomorphism of $\mathcal{L}$-modules, where v_j is the highest weight vector of V_j for $j = 1, 2$.

Proof. It suffices to show that ϕ is a homomorphism. Assume $\lambda = 1$. On the one hand,

$$\begin{aligned}
&L_n \phi(L_{-1}^i v_1 \otimes J_0^k) \\
&= L_n \left(\sum_{p=0}^{i} (-1)^p \binom{i}{p} L_{-1}^p v_2 \otimes \partial^{i-p} J_0^k \right) \\
&= \sum_{p=0}^{i} (-1)^p \binom{i}{p} \Big(\mu^n L_{-1}^p v_2 \otimes (\partial - n\alpha_2)(\partial - n)^{i-p} J_n^k \\
&\quad + (e^{nt} - \mu^n) \frac{\mathrm{d}}{\mathrm{d}t} L_{-1}^p v_2 \otimes (\partial - n)^{i-p} J_n^k \Big) \\
&= \sum_{p=0}^{i} (-1)^p \binom{i}{p} (\mu^n (L_{-1}^p v_2 \otimes \partial(\partial - n)^{i-p} J_n^k - L_{-1}^{p+1} v_2 \otimes (\partial - n)^{i-p} J_n^k \\
&\quad - n\alpha_2 L_{-1}^p v_2 \otimes (\partial - n)^{i-p} J_n^k) + (L_{-1} - n)^p L_{-1} v_2 \otimes (\partial - n)^{i-p} J_n^k \\
&\quad - n\alpha_1 (L_{-1} - n)^p v_2 \otimes (\partial - n)^{i-p} J_n^k),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\phi(L_n(L_{-1}^i v_1 \otimes J_0^k)) &= \phi\left(L_{-1}^i v_1 \otimes (\partial - n\alpha_1) J_n^k + (\mu^n e^{nt} - 1) \frac{\mathrm{d}}{\mathrm{d}t} L_{-1}^i v_1 \otimes J_n^k \right) \\
&= \phi\big(L_{-1}^i v_1 \otimes (\partial - n\alpha_1) J_n^k \\
&\quad + \mu^n (L_{-1} - n)^i (L_{-1} - n\alpha_2) v_1 \otimes J_n^k - L_{-1}^{i+1} v_1 \otimes J_n^k\big) \\
&= \phi\Big(\mu^n \Big((L_{-1} - n)^i L_{-1} v_1 \otimes J_n^k - n\alpha_2 (L_{-1} - n)^i v_1 \otimes J_n^k \Big) \\
&\quad + L_{-1}^i v_1 \otimes \partial J_n^k - L_{-1}^{i+1} v_1 \otimes J_n^k - n\alpha_1 L_{-1}^i v_1 \otimes J_n^k \Big).
\end{aligned}$$

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

The following four formulae will be good enough to make $\phi(L_n(L_{-1}^i v_1 \otimes J_0^k)) = L_n\phi(L_{-1}^i v_1 \otimes J_0^k)$ hold:

$$\phi\big((L_{-1}-n)^i v_1 \otimes J_n^k\big) = \sum_{p=0}^{i}(-1)^p\binom{i}{p}L_{-1}^p v_2 \otimes (\partial - n)^{i-p}J_n^k, \tag{4.1}$$

$$\phi\big(L_{-1}^i v_1 \otimes J_n^k\big) = \sum_{p=0}^{i}(-1)^p\binom{i}{p}(L_{-1}-n)^p v_2 \otimes (\partial - n)^{i-p}J_n^k, \tag{4.2}$$

$$\phi((L_{-1}-n)^i L_{-1}v_1 \otimes J_n^k) = \sum_{p=0}^{i}(-1)^p\binom{i}{p}(L_{-1}^p v_2 \otimes \partial(\partial - n)^{i-p}J_n^k - L_{-1}^{p+1}v_2 \otimes (\partial - n)^{i-p}J_n^k),$$

$$\phi\big(L_{-1}^i v_1 \otimes \partial J_n^k - L_{-1}^{i+1}v_1 \otimes J_n^k\big) = \sum_{p=0}^{i}(-1)^p\binom{i}{p}(L_{-1}-n)^p L_{-1}v_2 \otimes (\partial - n)^{i-p}J_n^k.$$

We only prove (4.1) and (4.2) as the other two follow these ones.

Note that $\mathcal{M}(V_2, \mu^{-1}, \Omega(\mu\lambda, \alpha_2))$ carries a natural module structure over $\mathbb{C}[L_{-1}, \partial]$ (the polynomial algebra on variables L_{-1}, ∂), since it is linearly isomorphic to $\mathbb{C}[L_{-1}, \partial]$. By this action, we have

$$\sum_{p=0}^{i}(-1)^p\binom{i}{p}\big((L_{-1}-n)^p v_2 \otimes (\partial - n)^{i-p}J_n^k\big) = \sum_{p=0}^{i}\binom{i}{p}(n-L_{-1})^p v_2 \otimes (\partial - n)^{i-p}J_n^k$$
$$= \big((n-L_{-1}) + (\partial - n)\big)^i(v_2 \otimes J_n^k) = (\partial - L_{-1})^i(v_2 \otimes J_n^k)$$
$$= \sum_{p=0}^{i}(-1)^p\binom{i}{p}L_{-1}^p v_2 \otimes \partial^{i-p}J_n^k = \phi\big(L_{-1}^i v_1 \otimes J_n^k\big),$$

proving (4.2). And (4.1) also follows from a direct computation, completing the proof. □

Theorem 4.2. *Let $\lambda_i, \alpha_i, 1 \neq \mu_i \in \mathbb{C}^*$ and let V_i be a simple $\mathfrak{a}$-module in $\mathcal{C}$ for $i = 1, 2$. Then*

$$\mathcal{M}\big(V_1, \mu_1, \Omega(\lambda_1, \alpha_1)\big) \cong \mathcal{M}\big(V_2, \mu_2, \Omega(\lambda_2, \alpha_2)\big)$$

as $\mathcal{L}$-modules if and only if one of the following conditions holds

(a) *$(\mu_1, \lambda_1, \alpha_1) = (\mu_2, \lambda_2, \alpha_2)$ and $V_1 \cong V_2$ as $\mathfrak{a}$-modules;*

(b) *$\mu_1 = \mu_2^{-1} = \lambda_2/\lambda_1$; V_1 and V_2 are highest weight $\mathfrak{a}$-module with highest weights $-\alpha_2$ and $-\alpha_1$, respectively.*

Proof. It suffices to show the 'only if' part, as the 'if' part follows from Lemma 4.1. Let $r_i = \operatorname{ord}(V_{i\mathfrak{b}})$ for $i = 1, 2$. Without loss of generality, assume that $r_2 \geq r_1$ and that

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

$V_{2\mathfrak{b}}$ is non-trivial if one of $V_{1\mathfrak{b}}$ and $V_{2\mathfrak{b}}$ is a non-trivial $\mathfrak{b}$-module. Assume that

$$\phi : \mathcal{M}\big(V_1, \mu_1, \Omega(\lambda_1, \alpha_1)\big) \to \mathcal{M}\big(V_2, \mu_2, \Omega(\lambda_2, \alpha_2)\big)$$

is an isomorphism of $\mathcal{L}$-modules. Take any $0 \neq w \in V_{1\mathfrak{b}}$ and assume $\phi(w \otimes 1) = \sum_{i=0}^{p} u_i \otimes \partial^i \in \mathcal{M}(V_2, \mu_2, \Omega(\lambda_2, \alpha_2))$ with $u_p \neq 0$. Then we have $\phi(w \otimes f(\partial)) = \sum_{i=0}^{p} u_i \otimes f(\partial)\partial^i$ for $f(\partial) \in \mathbb{C}[\partial]$ by repeatedly using the action of L_0. It follows from this and $\phi(L_m (w \otimes 1)) = L_m\phi(w \otimes 1)$ that

$$\begin{aligned}
&\sum_i u_i \otimes (\partial - m\alpha_1)\partial^i + \phi\bigg(\bigg(\mu_1^m \sum_{j=0}^{r_1+1} \frac{m^j}{j!} L_{j-1} - L_{-1}\bigg) w \otimes 1\bigg) \\
&\quad = \left(\frac{\lambda_2}{\lambda_1}\right)^m \sum_i \bigg(u_i \otimes (\partial - m\alpha_2)(\partial - m)^i \\
&\qquad + \bigg(\mu_2^m \sum_{j=0}^{r^{(i)}+1} \frac{m^j}{j!} L_{j-1} - L_{-1}\bigg) u_i \otimes (\partial - m)^i\bigg),
\end{aligned} \tag{4.3}$$

where $r^{(i)} = \operatorname{ord}(u_i)$ (recall that the order of the zero element is defined as zero). By the definition of r_2 we see that $r^{(p)} \geq r_2$. Now by Proposition 3.4, we see that $p = 0$ and $r_1 = r_2 = r^{(0)}$. This allows us to define an injective linear map $\varphi : V_{1\mathfrak{b}} \to V_{2\mathfrak{b}}$ via $\phi(w \otimes 1) = \varphi(w) \otimes 1$ for $w \in V_{1\mathfrak{b}}$. Then (4.3) is simplified as

$$\begin{aligned}
&\varphi(w) \otimes (\partial - m\alpha_1) + \mu_1^m \varphi\bigg(\sum_{j=1}^{r_1+1} \frac{m^j}{j!} L_{j-1} w\bigg) \otimes 1 + (\mu_1^m - 1)\phi(L_{-1} w \otimes 1) \\
&= \left(\frac{\lambda_2}{\lambda_1}\right)^m \bigg(\varphi(w) \otimes (\partial - m\alpha_2) + \mu_2^m \sum_{j=1}^{r_1+1} \frac{m^j}{j!} L_{j-1}\varphi(w) \otimes 1 + (\mu_2^m - 1) L_{-1}\varphi(w) \otimes 1\bigg).
\end{aligned} \tag{4.4}$$

Consider first that $V_{2\mathfrak{b}}$ is a trivial $\mathfrak{b}$-module, then so is $V_{1\mathfrak{b}}$ by our assumption at the beginning of this proof. In particular, $V_{1\mathfrak{b}} \cong V_{2\mathfrak{b}}$ by Lemma 3.1 (1), and the second terms on both sides of (4.4) vanish. It then follows immediately from Proposition 3.4 that $(\mu_1, \lambda_1, \alpha_1) = (\mu_2, \lambda_2, \alpha_2)$.

Assume that $V_{2\mathfrak{b}}$ is non-trivial. Using Lemma 3.1 (3) and comparing the maximal order of the first factors of elements in (4.4) give

$$\lambda_2^m(\mu_2^m - 1) = \lambda_1^m(\mu_1^m - 1)c_w,$$

where $c_w \in \mathbb{C}^*$ satisfies

$$\phi(L_{-1} w \otimes 1) \equiv c_w L_{-1}\varphi(w) \otimes 1 \pmod{V_{2\mathfrak{b}} \otimes \mathbb{C}[\partial]}.$$

□

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

Then by Proposition 3.4 we can conclude the following two cases:

Case 1. $(\mu_2, \lambda_2) = (\mu_1, \lambda_1)$ and $c_w = 1$.

Inserting $(\mu_2, \lambda_2) = (\mu_1, \lambda_1)$ into (4.4) and using Proposition 3.4 yield $\alpha_2 = \alpha_1$ and $\varphi(L_i w) = L_i \varphi(w)$ for all $i \geq 0$. Thus,

$$(\mu_1, \lambda_1, \alpha_1) = (\mu_2, \lambda_2, \alpha_2) \text{ and } \varphi \text{ is a homomorphism of } \mathfrak{b}\text{-modules.}$$

Then the injectivity of φ and the simplicity of $V_{1\mathfrak{b}}, V_{2\mathfrak{b}}$ (see Lemma 3.1 (1)) imply that φ is an isomorphism. This and Lemma 3.1 (4) imply $V_1 \cong V_2$. This is (a).

Case 2. $\lambda_2 = \mu_1 \lambda_1$ and $c_w = -1$.

Inserting $\lambda_2 = \mu_1 \lambda_1$ into (4.4) and using Proposition 3.4 yield that $\lambda_1 = \mu_2 \lambda_2$ and that $V_{1\mathfrak{b}}, V_{2\mathfrak{b}}$ are highest weight $\mathfrak{b}$-module with highest weights $-\alpha_2, -\alpha_1$, respectively. That is, we have (b).

5. New simple modules

We shall show that simple modules in Theorem 3.7 are new in the sense that they are not isomorphic to any other known non-weight $\mathcal{L}$-module. Let us first recall simple non-weight $\mathcal{L}$-modules from [7, 12–14, 20, 22].

For any $\lambda, \alpha \in \mathbb{C}^*$ and $h(t) \in \mathbb{C}[t]$ such that $\deg h(t) = 1$, recall from [2] that the polynomial algebra $\mathbb{C}[s,t]$ in two variables s and t carries the structure of a simple $\mathcal{L}$-module: $L_m f(s,t) = \sum_{j=0}^{\infty} \lambda^m (-m)^j S^j f(s,t)$ and $C f(s,t) = 0$ (see also [7]). Here for $j \in \mathbb{Z}_+$,

$$S^j = \frac{s}{j!}\partial_s^j - \frac{1}{(j-1)!}\partial_s^{j-1}\big(t(\eta - \partial_t) + h(\alpha)\big) - \frac{1}{(j-2)!}\partial_s^{j-2}\alpha(\eta - \partial_t),$$

where $\eta = ((h(t) - h(\alpha))/(t - \alpha)) \in \mathbb{C}^*$, $\partial_t = \partial/\partial t$, $\partial_s^k = (\partial/\partial s)^k$ if $k \in \mathbb{Z}_+$ and $\partial_s^k = 0$ if $k < 0$, and we have used the convention that $k! = 1$ for $k < 0$. In the following, we will also use the convention that $\binom{i}{j} = 0$ for $j > i$ or $j < 0$.

Let V be a simple $\mathcal{L}$-module for which there exists $R_V \in \mathbb{Z}_+$ such that L_m for all $m \geq R_V$ are locally nilpotent on V. In fact, such kind of simple $\mathcal{L}$-modules were classified in [20]. It was, respectively, shown in [7, 13, 22] that the tensor products $\mathcal{L}$-modules $\bigotimes_{i=1}^n \Phi(\lambda_i, \alpha_i, h_i(t)) \otimes V$, $\bigotimes_{i=1}^n \Phi(\lambda_i, \alpha_i, h_i(t)) \otimes \bigotimes_{i=1}^m \Omega(\mu_i, \beta_i) \otimes V$ and $\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i) \otimes V$ are simple if $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ are pairwise distinct.

Let $b \in \mathbb{C}$ and let A be a simple module over the associative algebra $\mathcal{K} = \mathbb{C}[t^{\pm 1}, t(\mathrm{d}/\mathrm{d}t)]$. The action of $\mathcal{L}$ on $A_b := A$ is given for $n \in \mathbb{Z}$ and $v \in A$ as

$$L_n v = \left(t^{n+1}\frac{\mathrm{d}}{\mathrm{d}t} + nbt^n\right)v, \quad Cv = 0.$$

It was proved in [14] that A_b is a simple $\mathcal{L}$-module if and only if one of the following conditions holds: (1) $b \neq 0$ or 1; (2.2) $b = 1$ and $t(\mathrm{d}/\mathrm{d}t)A = A$; (3.1) $b = 0$ and A is not isomorphic to the natural $\mathcal{K}$-module $\mathbb{C}[t, t^{-1}]$.

For any $r \in \mathbb{Z}_+$, let $\mathfrak{b}_r$ be the quotient algebra of $\mathfrak{b}$ by $\mathfrak{b}^{(r+1)} = \operatorname{span}\{L_i \mid i \geq r+1\}$. The classification of simple modules over $\mathfrak{b}_i$ for $i = 1, 2$ were, respectively, obtained in [1] and [20], and remains unsolved for $i \geq 3$. Let V be a $\mathfrak{b}_r$-module. For any $a \in \mathbb{C}, \gamma(t) =$

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

$\sum_i c_i t^i \in \mathbb{C}[t, t^{-1}]$, define the action of $\mathcal{L}$ on $V \otimes \mathbb{C}[t, t^{-1}]$ as follows: for $m, n \in \mathbb{Z}$ and $v \in V$,

$$L_m(v \otimes t^n) = \left(a + n + \sum_{i=0}^{r} \frac{m^{i+1}}{(i+1)!} L_i\right) v \otimes t^{m+n} + \sum_i c_i v \otimes t^{n+i}, \quad C(v \otimes t^n) = 0.$$

Then $V \otimes \mathbb{C}[t, t^{-1}]$ carries the structure of an $\mathcal{L}$-module under the above given actions, which is denoted by $\widetilde{\mathcal{M}}(V, \gamma(t))$. Note from [12] that $\widetilde{\mathcal{M}}(V, \gamma(t))$ is a weight $\mathcal{L}$-module if and only if $\gamma(t) \in \mathbb{C}$ and that $\widetilde{\mathcal{M}}(V, \gamma(t))$ is simple if and only if V is simple.

Proposition 5.1. *Let $1 \neq \mu, \alpha, \lambda, \alpha_i, \lambda_i \in \mathbb{C}^*$ for $i = 1, \ldots, n$ with $\lambda_1, \ldots, \lambda_n$ being pairwise distinct. Let V be a simple $\mathfrak{a}$-module in $\mathcal{C}$ and M a simple $\mathcal{L}$-module for which there exists $R_M \in \mathbb{Z}_+$ such that L_m for all $m \geq R_M$ are locally nilpotent on M. Then*

$$\mathcal{M}\big(V, \mu, \Omega(\lambda, \alpha)\big) \cong \bigotimes_{i=1}^{n} \Omega(\lambda_i, \alpha_i) \otimes M$$

if and only if M is a trivial $\mathcal{L}$-module, $V_{\mathfrak{b}}$ is a highest weight $\mathfrak{b}$-module with highest weight β and

$$(n, \lambda, \lambda\mu, \alpha, \beta) = (2, \lambda_{\sigma 1}, \lambda_{\sigma 2}, \alpha_{\sigma 1}, -\alpha_{\sigma 2}) \text{ for some } \sigma \in S_2,$$

where S_2 is the symmetric group on two letters.

Proof. Let $r \geq -1$ be the maximal integer such that L_r is injective on $V_{\mathfrak{b}}$. Assume $\lambda = 1$ for convenience. Let

$$\phi : \mathcal{M}\big(V, \mu, \Omega(1, \alpha)\big) \to \bigotimes_{i=1}^{n} \Omega(\lambda_i, \alpha_i) \otimes M$$

be an isomorphism of $\mathcal{L}$-modules. Take any $0 \neq v \in V_{\mathfrak{b}}$ and assume

$$\phi(v \otimes 1) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in I} \partial^{k_1} \otimes \cdots \otimes \partial^{k_n} \otimes v_{\mathbf{k}} \tag{5.1}$$

for some finite subset I of $\mathbb{Z}_+^n$ such that $\{\partial^{k_1} \otimes \cdots \otimes \partial^{k_n} \otimes v_{\mathbf{k}} \mid \mathbf{k} \in I\}$ is linearly independent. Choose p large enough so that $L_m v_{\mathbf{k}} = 0$ for all $m \geq p$ and $\mathbf{k} \in I$. It follows from $\phi(L_m(v \otimes 1)) = L_m \sum_{\mathbf{k}\in I} \partial^{k_1} \otimes \cdots \otimes \partial^{k_n} \otimes v_{\mathbf{k}}$ that

$$\begin{aligned}
&\phi\left(v \otimes (\partial - m\alpha) + \mu^m \sum_{j=0}^{r+1} \frac{m^j}{j!} L_{j-1} v \otimes 1 - L_{-1} v \otimes 1\right) \\
&= \sum_{\mathbf{k}\in I} \sum_{i=1}^{n} \partial^{k_1} \otimes \cdots \otimes \partial^{k_{i-1}} \otimes \lambda_i^m (\partial - m\alpha_i)(\partial - m)^{k_i} \\
&\quad \otimes \partial^{k_{i+1}} \otimes \cdots \otimes \partial^{k_n} \otimes v_{\mathbf{k}} \text{ for } m \geq p.
\end{aligned} \tag{5.2}$$

By Proposition 3.4, we know that $(n, \lambda_{\sigma 1}, \lambda_{\sigma 2}, \alpha_{\sigma 1}) = (2, 1, \mu, \alpha)$ for some $\sigma \in S_2$. Without loss of generality, we assume $\sigma = 1$, namely, $(\lambda_1, \lambda_2, \alpha_1) = (1, \mu, \alpha)$. Note also from

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

(5.2) on the one hand that $r \neq -1$, since otherwise $\mu^m m^i$ for some $i \geq 1$ would only be the coefficient of some non-zero term on the right-hand side; and on the other hand that $k_1 = 0, k_2 \leq r$ for all $\mathbf{k} = (k_1, k_2) \in I$ and $k_2 = r$ holds only for one $\mathbf{k}$. Thus (5.1) turns out to be $\phi(v \otimes 1) = \sum_{i=0}^{r} 1 \otimes \partial^i \otimes v_i$.

Consider first $r \geq 1$. It follows from $\phi(L_m^2(v \otimes 1)) = L_m^2 \sum_{i=0}^{r}(1 \otimes \partial^i \otimes v_i)$ that

$$\phi\bigg(v \otimes (\partial - m\alpha)(\partial - m\alpha - m) + (\mu^m e^{mt} - 1)\frac{\mathrm{d}}{\mathrm{d}t}v \otimes (\partial - m\alpha - m)$$

$$+ (\mu^m e^{mt} - 1)\frac{\mathrm{d}}{\mathrm{d}t}v \otimes (\partial - m\alpha) + (\mu^m e^{mt} - 1)\frac{\mathrm{d}}{\mathrm{d}t}(\mu^m e^{mt} - 1)\frac{\mathrm{d}}{\mathrm{d}t}v \otimes 1\bigg)$$

$$= \sum_{i=0}^{r}\Big((\partial - m\alpha)(\partial - m\alpha - m) \otimes \partial^i \otimes v_i + 2(\partial - m\alpha) \otimes \mu^m(\partial - m\alpha_2)(\partial - m)^r \otimes v_i$$

$$+ 1 \otimes \mu^{2m}(\partial - m\alpha_2)(\partial - m\alpha_2 - m)(\partial - 2m)^i \otimes v_i\Big),$$

which gives $\phi(L_r^2 v \otimes 1) = 0$ by comparing the coefficient of $\mu^{2m}m^{2r+2}$. This contradicts the injectivity of ϕ, since $L_r^2 v \otimes 1 \neq 0$. Thus, $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha)) \not\cong \bigotimes_{i=1}^{n} \Omega(\lambda_i, \alpha_i) \otimes M$ in this case.

The remaining case is $V_{\mathfrak{b}}$ being a highest weight module. Assume that its highest weight is $\beta \in \mathbb{C}^*$. Now it is not hard to deduce from $\phi(L_m(v \otimes 1)) = L_m(1 \otimes 1 \otimes v_0)$ that $\beta = -\alpha_2$ and that M is a trivial $\mathcal{L}$-module.

Conversely, suppose that M is a trivial $\mathcal{L}$-module, $V_{\mathfrak{b}}$ is a highest weight module with highest weight $-\alpha_2$ and $(n, 1, \mu, \alpha) = (2, \lambda_1, \lambda_2, \alpha_1)$ (this is the case $\sigma = 1$). Then following the proof of Lemma 4.1 one can check that the linear map $\varphi : \mathcal{M}(V, \mu, \Omega(1, \alpha)) \to \Omega(1, \alpha) \otimes \Omega(\mu, -\beta) \otimes \mathbb{C}$ sending $L_{-1}^i v \otimes \partial^j$ to $\sum_{p=0}^{j}\binom{j}{p}\partial^{j-p} \otimes \partial^{i+p} \otimes 1$ for any $i, j \in \mathbb{Z}_+$ is an isomorphism of $\mathcal{L}$-modules, where v is the highest weight vector of $V_{\mathfrak{b}}$. □

Proposition 5.2. *Let $\lambda, \alpha, 1 \neq \mu \in \mathbb{C}^*$ and let V be a simple $\mathfrak{a}$-module in $\mathcal{C}$. Then $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$ is not isomorphic to any of the following simple $\mathcal{L}$-modules:*

$$M, \bigotimes_{i=1}^{n} \Phi\big(\lambda_i, \alpha_i, h_i(t)\big) \otimes M, \bigotimes_{i=1}^{n} \Phi\big(\lambda_i, \alpha_i, h_i(t)\big) \otimes \bigotimes_{i=1}^{m} \Omega(\mu_i, \alpha_i) \otimes M, A_b, \widetilde{\mathcal{M}}\big(W, \gamma(t)\big),$$

where $m, n \geq 1, \lambda_i, \alpha_i \in \mathbb{C}^*, b \in \mathbb{C}, \gamma(t), h_i(t) \in \mathbb{C}[t]$ with $\deg h_i(t) = 1, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ being pairwise distinct, M is a simple $\mathcal{L}$-module for which there exists $R_M \in \mathbb{Z}_+$ such that L_m is locally nilpotent on M for all $m \geq R_M$, and W is a simple $\mathfrak{b}$-module.

Proof. Assume $\lambda = 1$ for convenience. Since for any large enough m, L_m is locally nilpotent on M but not on $\mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$, $M \not\cong \mathcal{M}(V, \mu, \Omega(\lambda, \alpha))$. Suppose that

$$\phi : \mathcal{M}\big(V, \mu, \Omega(1, \alpha)\big) \to \bigotimes_{i=1}^{n} \Phi\big(\lambda_i, \alpha_i, h_i(t)\big) \otimes M$$

is an isomorphism of $\mathcal{L}$-modules. Take $0 \neq u \in V$ and assume $\phi(u \otimes 1) = \sum_{i \in I} f_{1i} \otimes f_{2i} \otimes \cdots \otimes f_{ni} \otimes u_i$. Choose m to be large enough so that $L_m u_i = 0$ for all $i \in I$. Then it follows

https://doi.org/10.1017/S0013091520000279 Published online by Cambridge University Press

from $\phi(L_m(u\otimes 1)) = L_m\phi(u\otimes 1)$ that

$$\phi\left(u\otimes(\partial - m\alpha) + (\mu^m e^{mt}-1)\frac{\mathrm{d}}{\mathrm{d}t}u\otimes 1\right)$$
$$= \sum_{i\in I}\sum_{k=1}^{n}\sum_{j=0}^{\infty}\lambda_k^m(-m)^j \underbrace{f_{1i}\otimes\cdots\otimes}_{k-1} S^j f_{ki}\otimes\cdots\otimes u_i$$

Then in view of Proposition 3.4 we have $n=2$ and may assume $\lambda_1 = 1$ and $\lambda_2=\mu$; moreover, $\sum_{i\in I} S^j f_{1i}\otimes f_{2i}\otimes u_i = 0$ for all $j\geq 2$. But we see that $\sum_{i\in I} S^2 f_{1i}\otimes f_{2i}\otimes u_i \neq 0$ by choosing f_{1i} to be of the form t^{n_i} for $i\in I$, a contradiction. This shows $\mathcal{M}(V,\mu,\Omega(1,\alpha)) \ncong \bigotimes_{i=1}^n \Phi(\lambda_i,\alpha_i,h_i(t))\otimes M$. Similarly, one has

$$\mathcal{M}\big(V,\mu,\Omega(1,\alpha)\big) \ncong \bigotimes_{i=1}^{n}\Phi\big(\lambda_i,\alpha_i,h_i(t)\big)\otimes\bigotimes_{i=1}^{m}\Omega(\lambda_i,\alpha_i)\otimes M.$$

And the non-isomorphisms $\mathcal{M}(V,\mu,\,\Omega(1,\alpha))\ncong A_b$ and $\mathcal{M}(V,\mu,\Omega(1,\alpha))\ncong\widetilde{\mathcal{M}}(W,\gamma(t))$ follow immediately by using Proposition 3.4, completing the proof. □

Now we can conclude this section with the following corollary.

Corollary 5.3. *Let $\lambda,\alpha,1\neq\mu\in\mathbb{C}^*$ and let V be a simple $\mathfrak{a}$-module in $\mathcal{C}$ such that V is not a highest weight $\mathfrak{a}$-module whose highest weight is non-zero. Then $\mathcal{M}(V,\mu,\,\Omega(\lambda,\alpha))$ is not isomorphic to any simple $\mathcal{L}$-module in* [7, 13, 14, 20, 22].

Acknowledgments. The first author is supported by NSF of China (grant No. 11801369) and the CSC (grant No. 202008310011). The second author is supported by NSF of China (grant Nos. 11501417, 11671247) and by the CSC (grant No. 202006265002).

References

1. R. E. Block, The irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ and of the Weyl algebra, *Adv. Math.* **139** (1981), 69–110.
2. H. Chen and X. Guo, A new family of modules over the Virasoro algebra, *J. Algebra* **457** (2016), 73–105.
3. H. Chen and X. Guo, Non-weight modules over the Heisenberg-Virasoro algebra and the W algebra $W(2,2)$, *J. Algebra Appl.* **16** (2017), 1750097.
4. H. Chen, X. Guo and K. Zhao, Tensor product weight modules over the Virasoro algebra, *J. Lond. Math. Soc.* **88** (2013), 829–844.
5. C. H. Conley and C. Martin, A family of irreducible representations of the Witt Lie algebra with infinite-dimensional weight spaces, *Compositio Math.* **128** (2001), 153–175.
6. X. Guo, R. Lü and K. Zhao, Fraction representations and highest weight-like representations of the Virasoro algebra, *J. Algebra* **387** (2013), 68–86.
7. J. Han and Y. Su, Submodule structure of $\mathbb{C}[s,t]$ over $W(0,b)$ and a new class of irreducible modules over the Virasoro algebra, arXiv:1708.07272.
8. K. Iohara and Y. Koga, *Representation theory of the Virasoro algebra*, Springer Monographs in Mathematics (Springer, London, 2011).
9. V. G. Kac and A. K. Raina, *Bombay Lectures on Highest Weight Representations of Infinite-dimensional Lie Algebras*, Volume **2** (World Scientific Publishing, Teaneck, 1987).

10. D. Liu and C. Jiang, Harish-Chandra modules over the twisted Heisenberg-Virasoro algebra, *J. Math. Phys.* **49** (2008), 012901, 13 p.
11. G. Liu and Y. Zhao, Generalized polynomial modules over the Virasoro algebra, *Proc. Amer. Math. Soc.* **144** (2016), 5103–5112.
12. G. Liu, R. Lü and K. Zhao, A class of simple weight Virasoro modules, *J. Algebra* **424** (2015), 506–521.
13. X. Liu, X. Guo and J. Wang, A new class of irreducible Virasoro modules from tensor product, *J. Algebra* **541** (2020), 324–344.
14. R. Lü and K. Zhao, Irreducible Virasoro modules from irreducible Weyl modules, *J. Algebra* **414** (2014), 271–287.
15. R. Lü and K. Zhao, A family of simple weight Virasoro modules, *J. Algebra* **479** (2017), 437–460.
16. R. Lü, X. Guo and K. Zhao, Irreducible modules over the Virasoro algebra, *Doc. Math.* **16** (2011), 709–721.
17. O. Mathieu, Classification of Harish-Chandra modules over the Virasoro Lie algebra, *Invent. Math.* **107** (1992), 225–234.
18. V. Mazorchuk and E. Wiesner, Simple Virasoro modules induced from codimension one subalgebras of the positive part, *Proc. Amer. Math. Soc.* **142** (2014), 3695–3703.
19. V. Mazorchuk and K. Zhao, Classification of simple weight Virasoro modules with a finite-dimensional weight space, *J. Algebra* **307** (2007), 209–214.
20. V. Mazorchuk and K. Zhao, Simple Virasoro modules which are locally finite over a positive part, *Selecta Math. (N. S.)* **20** (2014), 839–854.
21. M. Ondrus and E. Wiesner, Whittaker modules for the Virasoro algebra, *J. Algebra Appl.* **8** (2009), 363–377.
22. H. Tan and K. Zhao, Irreducible Virasoro modules from tensor products II, *J. Algebra* **394** (2013), 357–373.
23. H. Tan and K. Zhao, Irreducible Virasoro modules from tensor products, *Ark. Mat.* **54** (2016), 181–200.
24. H. Zhang, A class of representations over the Virasoro algebra, *J. Algebra* **190** (1997), 1–10.