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COMPATIBILITY OPERATORS IN ABSTRACT ALGEBRAIC LOGIC

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Abstract. This paper presents a unified framework that explains and extends the already successful applications of the Leibniz operator, the Suszko operator, and the Tarski operator in recent developments in abstract algebraic logic. To this end, we refine Czelakowski's notion of an S-compatibility operator, and introduce the notion of coherent family of S-compatibility operators, for a sentential logic S. The notion of coherence is a restricted property of commutativity with inverse images by surjective homomorphisms, which is satisfied by both the Leibniz and the Suszko operators. We generalize several constructions and results already existing for the mentioned operators; in particular, the well-known classes of algebras associated with a logic through each of them, and the notions of full generalized model of a logic and a special kind of S-filters (which generalizes the less-known notion of Leibniz filter). We obtain a General Correspondence Theorem, extending the well-known one from the theory of protoalgebraic logics to arbitrary logics and to more general operators, and strengthening its formulation. We apply the general results to the Leibniz and the Suszko operators, and obtain several characterizations of the main classes of logics in the Leibniz hierarchy by the form of their full generalized models, by old and new properties of the Leibniz operator, and by the behaviour of the Suszko operator. Some of these characterizations complete or extend known ones, for some classes in the hierarchy, thus offering an integrated approach to the Leibniz hierarchy that uncovers some new, nice symmetries.

§1. Introduction. Abstract algebraic logic (AAL) is an area of algebraic logic that takes a global perspective on the algebraic study of the different logics, mainly propositional, that have been considered in several fields like philosophy, computer science, or the foundations of mathematics. Since the early 1930's (or even earlier) a plethora of logics have been introduced in those fields, for example intuitionistic logic, expansions of classical logic (such as the different modal, epistemic, temporal, deontic, dynamic logics and related systems), quantum logics, the broad family of substructural logics (which includes linear logic, relevance logics, many-valued logics such as Łukasiewicz's, and many others), etc.; and new logics continue to emerge. The many similarities in the existing algebraic studies of each one of them led to an abstract study of the very process of the algebraization of the different logics in itself. This study introduced general concepts and theorems in order to organize several of the results obtained for specific logics as particular cases of general results applicable to wide classes of logics. This approach can be traced back to the research of some Polish logicians in the 1960's and was definitely pursued and consolidated by the work of Willem J. Blok, Don Pigozzi, and Janusz Czelakowski in the 1980's. It led to the area of research that is now known as AAL. The perspective

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taken in AAL with respect to the algebraic study of logics is similar to the one universal algebra takes with respect to classical algebra; in fact, the connections between AAL and universal algebra are quite strong. As general references on AAL, see [5, 14, 15, 18, 20].

One of the main goals of abstract algebraic logic is to associate a class of algebras with every logic in a canonical way and to classify logics according to the kind of relation they have with their algebraic counterpart. This purpose has led to classify logics in the so-called Leibniz and Frege hierarchies. Another main goal of abstract algebraic logic is to obtain theorems that relate properties that a logic might have with properties that its algebraic counterpart has. For example, one of the main results, due to Blok and Pigozzi, is that a finitary and finitely algebraizable logic has a deduction-detachment theorem if and only if its algebraic counterpart has principal relative congruences equationally definable. Another one relates enjoying a version of Craig's interpolation theorem with the property that the algebras in the algebraic counterpart have the amalgamation property.

A methodology that has proven to be very fruitful in classifying logics in a theoretically useful way is the so-called "operator approach" [5, p. 5]; this methodology consists in considering operators that associate certain congruences with subsets of an algebra and in studying their behaviour on the lattices of filters of the logic on arbitrary algebras. The Leibniz hierarchy is defined (mainly) using the Leibniz operator, introduced and studied by Blok and Pigozzi [1, 2] and further studied by Herrmann [22, 23], Czelakowski [5], Czelakowski and Jansana [7], and Raftery [25]. The Leibniz operator associates with every logical filter F of a logic $\mathcal{S}$ on an algebra $\boldsymbol{A}$ of the appropriate type the largest congruence compatible with F, known as the Leibniz congruence of F. The study of this operator originated the systematic construction of a theory of the class of protoalgebraic logics and its subclasses (mainly, the classes of equivalential, weakly algebraizable, and algebraizable logics), and the definition of a new class, the class of truth-equational logics, which are not necessarily protoalgebraic. The classes so obtained form the Leibniz hierarchy. On the other hand, the study of the Suszko operator, initiated by Czelakowski [6] and continued by Raftery [25], has also been instrumental in the study of truth-equational logics. The Suszko operator associates with any logical filter F of a logic $\mathcal{S}$ on an algebra $\boldsymbol{A}$ the largest congruence compatible with all filters of the logic on $\boldsymbol{A}$ that include F. Finally, the Tarski operator, first considered by Font in [10] under the name of extended Leibniz operator and further studied by Font and Jansana [18] and by Font, Jansana and Pigozzi [21], led to the definition of the notion of full generalized model of a logic, to the proof of some results of general validity beyond the Leibniz hierarchy and to the establishment of the Frege hierarchy [12, 13], whose classes can be characterized by properties of the Tarski operator. Given a logic, the Tarski operator associates with every closure system of logical filters of an algebra the largest congruence compatible with all the members of the closure system. Note that the Tarski operator is of a higher type than the Leibniz and the Suszko operators: these two act on filters of a logic, while the Tarski operator acts on families of filters (i.e., generalized matrices) of the logic.

Many of the results on the mentioned operators obtained in abstract algebraic logic concern the definability of the Leibniz congruence by means of a set of formulas in two variables (and possibly parameters), a generalization of the situation encountered in several logics where the biimplication $x \leftrightarrow y$ defines the largest congruence compatible with a filter of the logic on the appropriate algebras. Other results consider the definability of the logical filters by means of a set of equations in one variable, a situation that generalizes the fact that in many logics the logical filters of the algebras associated with them are definable by an equation such as $x \approx 1$ or $x \wedge 1 \approx 1$. These definability properties are closely related to the issues typically considered in the operator approach, namely to the behaviour of the Leibniz and the Suszko operators as mappings from the lattice of logical filters on an algebra to the lattice of the congruences of the algebra relative to the algebraic counterpart of the logic under consideration. Examples of properties studied are whether the operator is monotonic, whether it respects arbitrary intersections or unions of up-directed families, etc.

There are significant differences between the behaviour of the Leibniz and of the Suszko operators: for example, the second is always monotone and the first is not; the first always commutes with inverse images by surjective homomorphisms while the second does not; etc. However, they share, as we will show, enough properties as to be fruitfully treated as instances of some general concepts that serve to unify their theories. This sharing of some properties is in part due to the fact that the Suszko operator is defined in a very specific way using the Leibniz one, and this forces that some properties we manage to isolate of the latter operator are inherited by the former; these properties serve to build a common mathematical ground for both.

One of our aims in this paper is to find this common ground and to start developing the mathematical theory necessary to obtain interesting results from which both the known results of the theory of the Leibniz operator and the theory of the Suszko operator will follow, and new ones will emerge. The general notions that allow us to do this are those of $\mathcal{S}$-*compatibility operator*, family of $\mathcal{S}$-compatibility operators (one for each algebra of the relevant type), and *coherence*, which is a restricted property of commutativity of the operators with inverse images by surjective homomorphisms. An $\mathcal{S}$-compatibility operator associates with any filter of a logic a congruence compatible with it; the Leibniz operator is the largest of these operators, and the Suszko operator is the largest of the order preserving ones among them.

In this paper we study, for an arbitrary family of $\mathcal{S}$-compatibility operators, several of the constructions that exist in the literature for either the Leibniz or the Suszko operator, or both, and several times we find that the assumption of coherence is crucial in making everything work in a smooth way and to obtain significant results. The main results in the paper are several Correspondence Theorems (which generalize and strengthen in several directions those obtained by Blok and Pigozzi using the Leibniz operator and by Czelakowski using the Suszko operator) and several characterizations of some of the already mentioned classes of logics in the Leibniz hierarchy in terms of the form of their full generalized models, in terms of properties of their Leibniz operator, and in terms of their Suszko operator. For example, since 1986 it is known that a logic is protoalgebraic if and only if the Leibniz operator is order preserving over its filters, a property that the Suszko operator always has. Here we establish (Theorem 6.29) the dual result that a logic is protoalgebraic if and only if the Suszko operator commutes with inverse images by surjective homomorphisms, a property that the Leibniz operator always has. This is however only one among the many outcomes of our approach to the study of the Leibniz and Suszko operators under the more general framework of $\mathcal{S}$-compatibility operators.

After recalling in Section 2 some key points of the terminology and notation to be used throughout the paper, in Section 3 we introduce the notion of an $\mathcal{S}$-operator on an algebra $\boldsymbol{A}$ as any map $\nabla^{\boldsymbol{A}}$ that associates a congruence of $\boldsymbol{A}$ with each filter of the logic $\mathcal{S}$ on $\boldsymbol{A}$. With each such operator we associate two more operators, its *lifting* $\widetilde{\nabla}^{\boldsymbol{A}}$ to the power set of $\mathcal{S}$-filters, and its *relativization* $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$, which is again an $\mathcal{S}$-operator, in the technical sense just introduced; these definitions generalize the way the Tarski and the Suszko operators, respectively, are obtained from the Leibniz operator. We use these operators to define a Galois connection between the power set of $\mathcal{S}$-filters and the lattice of congruences of $\boldsymbol{A}$, thus obtaining a dual order isomorphism between the sets of the fixed points of the associated closure operators; these fixed points, on each side of the connection, are called $\nabla^{\boldsymbol{A}}$-*full*. We show that in the case of the Leibniz operator $\boldsymbol{\Omega}$, the $\boldsymbol{\Omega}^{\boldsymbol{A}}$-full sets of $\mathcal{S}$-filters coincide with the full generalized models of $\mathcal{S}$ [18], and that the $\boldsymbol{\Omega}^{\boldsymbol{A}}$-full congruences are the congruences of $\boldsymbol{A}$ relative to the class $\mathsf{Alg}\mathcal{S}$; as a consequence, we see that the Isomorphism Theorem of [18, Theorem 2.30] is but one aspect of the mentioned Galois connection. We also introduce the notion of the $\nabla^{\boldsymbol{A}}$-*class* of an $\mathcal{S}$-filter F, which is the set of all filters whose Leibniz congruence includes $\nabla^{\boldsymbol{A}}(F)$, and that of a $\nabla^{\boldsymbol{A}}$-*filter*, which is an $\mathcal{S}$-filter that coincides with the smallest member of its own $\nabla^{\boldsymbol{A}}$-class (which always exists). The section also contains the main general properties of several of the notions introduced.

Section 4 is the central one in the paper as far as the general theory is regarded. In Section 4.1 Czelakowski's notion of an $\mathcal{S}$-*compatibility* operator is recalled; it is an $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ such that the congruence $\nabla^{\boldsymbol{A}}(F)$ is compatible with F, or, in other words, such that $\nabla^{\boldsymbol{A}}(F)$ is included in the Leibniz congruence of F, for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. After giving some general properties of these operators, we focus on global properties of a family ∇ that consists of one $\mathcal{S}$-compatibility operator $\nabla^{\boldsymbol{A}}$ for each algebra $\boldsymbol{A}$, without assuming that the operators on different algebras are defined in any particular or uniform way. We introduce the notions that such a family commutes with inverse images by surjective homomorphisms, or with inverse images by all homomorphisms, and a restricted version of this commutativity, which we call *coherence*. This notion happens to be the key to many of the subsequent results. It is noteworthy that both the Leibniz operator and the Suszko operator form coherent families of $\mathcal{S}$-compatibility operators, and that the former is the only family of $\mathcal{S}$-compatibility operators that commutes with inverse images by surjective homomorphisms (Theorem 4.6). The main result in this section is the General Correspondence Theorem 4.17, which states that if ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then for every surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that h is $\nabla^{\boldsymbol{A}}$-compatible with F (a technical notion introduced in this section), the extension of h to the power sets induces an order isomorphism between the $\nabla^{\boldsymbol{A}}$-class of F and the $\nabla^{\boldsymbol{B}}$-class of $h(F)$, whose inverse is given by h^{-1}. When we apply this result to the Leibniz and to the Suszko operators (Theorems 5.7 and 5.15, respectively), we will see that it generalizes and strengthens Blok and Pigozzi's well-known Correspondence

Theorem for protoalgebraic logics [1,2], and Czelakowski's theorem [6] for arbitrary logics, which establish order isomorphisms between the smaller sets of all $\mathcal{S}$-filters of $\boldsymbol{A}$ containing F and all $\mathcal{S}$-filters of $\boldsymbol{B}$ containing $h(F)$. The result also generalizes the first strengthening obtained for protoalgebraic logics by Font and Jansana [17].

In Section 4.2 we do in general, for any family ∇ of $\mathcal{S}$-operators, what is done in the standard theory of abstract algebraic logic for each of the three operators (Leibniz, Suszko and Tarski): we use the operators to associate classes of algebras with the logic $\mathcal{S}$ in several natural ways. Namely, given a family ∇, we consider the classes of algebras that support the (generalized) models of the logic that are reduced with respect to one of the operators $\nabla^{\boldsymbol{A}}$, $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$, and $\widetilde{\nabla}^{\boldsymbol{A}}$, and those that are obtained from arbitrary (generalized) models by a process of reduction under each operator; in all cases we close the classes under isomorphic images. In principle we obtain six classes of algebras, but we show (Corollary 4.30) that under the assumption of coherence these classes reduce to only two.

The general study is then instantiated for the Leibniz and the Suszko operators in Section 5, where special attention is payed to the relation of some of the studied notions with that of full generalized model of the logic. We see that the present notion of a *Leibniz filter* (the $\boldsymbol{\Omega}^{\boldsymbol{A}}$-filter of Section 3) coincides with that with the same name introduced in [17] for protoalgebraic logics (Lemma 5.2) and we characterize Leibniz filters as the least elements of the full generalized models of $\mathcal{S}$ (Theorem 5.5). As to the *Suszko filters* (the $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$-filters of Section 3), we see that every Suszko filter is a Leibniz filter but not conversely (Example 6.21), we characterize Suszko filters as the least elements of the full generalized models of $\mathcal{S}$ that are up-sets in the poset of all $\mathcal{S}$-filters, and we show that in fact there will be only one such model for every Suszko filter, namely that consisting of all the $\mathcal{S}$-filters containing it (Theorem 5.13).

The final Section 6 uses the results established in the previous section to give characterizations of several of the classes of logics in the Leibniz hierarchy. Section 6.1 gathers those concerning the notion of full generalized model and related ones. It is well known that a logic is protoalgebraic if and only if the Leibniz and the Suszko operators coincide on its logical filters, for every algebra. We add to this, among several results, that a logic is protoalgebraic if and only if all its full generalized models are up-sets, and actually determined by a Suszko filter (Theorem 6.5), and if and only if its full generalized models coincide with its Suszko-full ones (Proposition 6.6). We also show that a logic is truth-equational if and only if every filter is a Suszko filter (Theorem 6.10) and if and only if the up-set of all filters containing a given one is always full (Theorem 6.13). As a by-product, we obtain a new proof of Theorem 3.8 of [18], which states that a logic is weakly algebraizable if and only if its full generalized models are exactly those given by closure systems of filters that are up-sets. Theorems 6.18 and 6.19 end the section with characterizations of truth-equational logics and weakly algebraizable logics, respectively, in terms of a Correspondence Theorem they satisfy.

Section 6.2 gathers characterizations in terms of order-theoretic properties of the Leibniz operator, which complete several existing ones in the literature for the higher classes in the Leibniz hierarchy. The main result is Theorem 6.24, which says that a logic $\mathcal{S}$ is protoalgebraic if and only if for every algebra $\boldsymbol{A}$ the Leibniz operator restricts to an order isomorphism between the posets of all Suszko filters of $\mathcal{S}$ on

$\boldsymbol{A}$ and of the congruences of $\boldsymbol{A}$ relative to the class $\mathsf{Alg}^*\mathcal{S}$. After Corollary 6.25, we show that this result provides a new proof of Theorem 4.8 of [7], which states that a logic is weakly algebraizable if and only if the above result holds for the set of all filters instead.

Finally in Section 6.3 we undertake a similar study for the Suszko operator; this time there are more new results, as this operator has only been used up to now to characterize the class of truth-equational logics as those where the Suszko operator on every algebra is injective on the set of all filters (Theorem 28 of [25]); in fact we begin by giving a proof of this result inside our framework. Then we go on to show (Theorem 6.28) that a logic is protoalgebraic if and only if the Suszko operator commutes with inverse images by surjective homomorphisms, and if and only if, restricted to Suszko filters, it is surjective onto the set of congruences of the algebra relative to the class $\mathsf{Alg}\mathcal{S}$. This result is then extended in two directions, giving characterizations along two lines. One (Theorem 6.29) adds a characterization of the class of equivalential logics as those where the Suszko operator commutes with inverse images by arbitrary homomorphisms; from this, characterizations of weakly algebraizable logics and algebraizable logics are obtained by combining injectivity with the previous properties. The other line (Theorem 6.30) completes Theorem 6.28 by showing that a logic is protoalgebraic if and only if the Suszko operator restricts to an order isomorphism between the poset of all Suszko filters and the set of congruences of the algebra relative to the class $\mathsf{Alg}\mathcal{S}$. In the same result, truth-equational logics are characterized by the Suszko operator being an order embedding of the poset of all filters into the mentioned poset of congruences; from this, weakly algebraizable logics are obtained when the order embedding is turned into an order isomorphism, and finally algebraizable logics when commutativity with inverse images by arbitrary homomorphisms is added.

From this summary it becomes clear that, in our opinion, besides the intrinsic mathematical interest that the framework developed may have, the interest of the present study lies both in the new results obtained and in the realization of how several already known results fit in the general framework constructed. To some extent this can be taken as a deeper explanation of why they hold. We believe that it provides a novel view on the Leibniz hierarchy and on the relations between its main classes.

§2. Preliminaries.

In this paper we work in an arbitrary but fixed *logical language* $\mathcal{L}$, understood as a set of operation symbols, also called connectives, each of finite arity. We will always be working with $\mathcal{L}$-structures ($\mathcal{L}$-algebras, $\mathcal{L}$-matrices, $\mathcal{L}$-generalized matrices), so that generally the language $\mathcal{L}$ will not be mentioned, but phrases such as "for all algebras" or "for all logics" should be understood as limited to the language $\mathcal{L}$. In general, *algebras* will be denoted by $\boldsymbol{A}, \boldsymbol{B}, \dots$, and their universes by $A, B, \dots$ respectively. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are algebras, we write $h \colon \boldsymbol{A} \to \boldsymbol{B}$ to indicate that h is an *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$. The set of *congruences* of an algebra $\boldsymbol{A}$ is denoted by $\mathrm{Con}\boldsymbol{A}$; this set is well known to be a complete lattice when ordered under the subset relation. Given a class of algebras K and an algebra $\boldsymbol{A}$ (not necessarily in K), we shall denote the set of all K-*relative congruences* of $\boldsymbol{A}$ (i.e., the $\theta \in \mathrm{Con}\boldsymbol{A}$ such that $\boldsymbol{A}/\theta \in \mathsf{K}$) by $\mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$. Given $F \subseteq A$ and $\theta \in \mathrm{Con}\boldsymbol{A}$, we denote by π the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$, defined by $\pi(a) = a/\theta$ for all $a \in A$. The set $\{a/\theta : a \in F\}$ is interchangeably denoted either by F/θ or by $\pi(F)$.

Given any map $f : A \to B$ we denote its natural extension to the power sets with the same symbol; that is, we consider $f : \mathscr{P}(A) \to \mathscr{P}(B)$ defined, for each $X \subseteq A$, by $f(X) := \{f(a) : a \in X\} \subseteq B$. The associated "inverse image" map, which is usually denoted as $f^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)$, is defined, for each $Y \subseteq B$, by $f^{-1}(Y) := \{a \in A : f(a) \in Y\} \subseteq A$; this map is not the set-theoretic inverse of the extended map f, but its *residuum*, because it satisfies, for every $X \subseteq A$ and every $Y \subseteq B$, that $X \subseteq f^{-1}(Y)$ if and only if $f(X) \subseteq Y$. This implies that for every $X \subseteq A$, $X \subseteq f^{-1}\big(f(X)\big)$, and for every $Y \subseteq B$, $f\big(f^{-1}(Y)\big) \subseteq Y$. It is useful to recall that f is surjective if and only if for every $Y \subseteq B$, $f\big(f^{-1}(Y)\big) = Y$. This extension construction will be iterated in a natural way, still keeping the same symbol; for instance, for a family $\mathscr{C} \subseteq \mathscr{P}(A)$ we define $f(\mathscr{C}) := \big\{f(X) : X \in \mathscr{C}\big\}$, and for $\mathscr{D} \subseteq \mathscr{P}(Y)$, $f^{-1}(\mathscr{D}) := \big\{f^{-1}(Y) : Y \in \mathscr{D}\big\}$. We are sure that these notational simplifications will not cause any misunderstanding.

Similarly, a map $f : A \to B$ is extended to the cartesian powers component-wise; in particular $f : A \times A \to B \times B$ is defined as $f\big(\langle a, a'\rangle\big) = \langle f(a), f(a')\rangle$ for any $a, a' \in A$. This map can also be extended to the power sets as before, and we will use the property that the original map f is surjective if and only if for every $R \subseteq B \times B$, $f\big(f^{-1}(R)\big) = R$.

The following notion plays a central rôle in abstract algebraic logic, and in particular in this paper. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is *compatible* with a subset $F \subseteq A$ if for every $a, b \in A$, if $\langle a, b\rangle \in \theta$ and $a \in F$, then $b \in F$; that is, if θ does not identify elements in F with elements outside F. The following characterizations of compatibility should be born in mind, as we will make use of them without explicit mention.

LEMMA 2.1. *Let* $\boldsymbol{A}$ *be an algebra,* $\theta \in \mathrm{Con}\boldsymbol{A}$ *and* $F \subseteq A$. *The following conditions are equivalent.*

(i) θ *is compatible with* F.
(ii) $a \in F \Leftrightarrow a/\theta \in F/\theta$, *for every* $a \in A$.
(iii) $F = \pi^{-1}\big(\pi(F)\big)$.
(iv) $F = \bigcup_{a \in F} a/\theta$; *in other words,* F *is a union of blocks of* θ.

Recall that the *kernel* of $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is the congruence $\mathrm{Ker}(h) := \big\{\langle a, b\rangle \in A \times A : h(a) = h(b)\big\}$; it will be useful to record here two of its elementary properties.

LEMMA 2.2. *Let* $h\colon \boldsymbol{A} \to \boldsymbol{B}$.

1. *For every* $F \subseteq A$, $\mathrm{Ker}(h)$ *is compatible with* F *if and only if* $h^{-1}\big(h(F)\big) = F$.
2. *For every* $\theta \in \mathrm{Con}\boldsymbol{A}$, *if* $\mathrm{Ker}(h) \subseteq \theta$, *then* $h^{-1}\big(h(\theta)\big) = \theta$.

Let us also fix a countably infinite set of variables Var, disjoint from $\mathcal{L}$. The *formula algebra* ***Fm*** is the absolutely free algebra generated by the set Var over the language $\mathcal{L}$. Its universe is denoted by Fm, and its members are called ($\mathcal{L}$-)*terms* or ($\mathcal{L}$-)*formulas*. Every map from Var to Fm can be uniquely extended to an endomorphism of ***Fm***; such a map is called a *substitution*.

Order properties and Galois connections. Let $\mathbb{P}_1 = \langle P_1, \leq_1\rangle$ and $\mathbb{P}_2 = \langle P_2, \leq_2\rangle$ be two posets. A map $f : P_1 \to P_2$ is *order preserving* if for every $x, y \in P_1$, $x \leq_1 y$ implies $f(x) \leq_2 f(y)$; *order reversing*, if for every $x, y \in P_1$, $x \leq_1 y$ implies

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$f(y) \leq_2 f(x)$; *order reflecting*, if for every $x, y \in P_1$, $f(x) \leq_2 f(y)$ implies $x \leq_1 y$; and, assuming P_1 and P_2 are complete lattices, f is *completely order reflecting* if for every $\{x_i : i \in I\} \cup \{y\} \subseteq P_1$, $\bigwedge_{i \in I} f(x_i) \leq_2 f(y)$ implies $\bigwedge_{i \in I} x_i \leq_1 y$. A pair $\langle f, g \rangle$ of maps $f: P_1 \to P_2$ and $g: P_2 \to P_1$ establishes a *Galois connection* between $\mathbb{P}_1$ and $\mathbb{P}_2$ if for every $x \in P_1$ and every $y \in P_2$, $x \leq_1 g(y)$ if and only if $y \leq_2 f(x)$. If $\mathbb{P} = \langle P, \leq \rangle$ is a poset, a map $f: P \to P$ is a *closure on* $\mathbb{P}$ if it is expansive, order preserving and idempotent; that is, if for every $x, y \in P$, it satisfies

(a) $x \leq f(x)$;
(b) if $x \leq y$, then $f(x) \leq f(y)$; and
(c) $f\big(f(x)\big) = f(x)$.

We now state the basic properties of Galois connections; for the proofs, as well as general facts about them, see for example [8, Chapter 7] (though here the maps are order preserving rather than order reversing).

PROPOSITION 2.3. *Let $\mathbb{P}_1 = \langle P_1, \leq_1 \rangle$ and $\mathbb{P}_2 = \langle P_2, \leq_2 \rangle$ be two posets and let $f: P_1 \to P_2$ and $g: P_2 \to P_1$ establish a Galois connection between $\mathbb{P}_1$ and $\mathbb{P}_2$.*

1. *f and g are both order reversing.*
2. *The composition function $g \circ f$ is a closure on $\mathbb{P}_1$.*
3. *The composition function $f \circ g$ is a closure on $\mathbb{P}_2$.*
4. *The set of fixed points of $g \circ f$ is* $\mathrm{Ran}(g)$.
5. *The set of fixed points of $f \circ g$ is* $\mathrm{Ran}(f)$.
6. *The maps f and g restrict to mutually inverse dual order isomorphisms between the set of fixed points of $g \circ f$ and the set of fixed points of $f \circ g$.*

All the posets considered in this paper will be power sets or families of subsets of some universe (including a cartesian product, as in the case of the set of congruences of some algebra), and the order will always be the subset relation; in these cases the posets will be denoted by just their universe, and the order relation will not be specified.

Closure relations, closure operators, and closure systems. Let A be an arbitrary set. A relation $\vdash \subseteq \mathscr{P}(A) \times A$ is a *closure relation over A* if it satisfies, for every $X, Y \subseteq A$, the following properties.

(i) $X \vdash x$, for every $x \in X$.
(ii) If $X \vdash z$ and $X \subseteq Y$, then $Y \vdash z$, for every $z \in A$.
(iii) If $X \vdash y$, for every $y \in Y$, and $Y \vdash z$, then $X \vdash z$, for every $z \in A$.

For closure relations over the set Fm of terms, it makes sense to consider the following property.

(iv) If $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$, for every substitution σ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$.

A closure relation over Fm satisfying (iv) is said to be *structural*, or *substitution invariant*. A *consequence relation over* Fm is a structural closure relation over Fm. A *logic* (in the language $\mathcal{L}$) is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}}$ is a consequence relation over Fm.

A *closure operator over A* is a closure on the power set of A, when this is ordered under the subset relation; that is, a map $C: \mathscr{P}(A) \to \mathscr{P}(A)$ satisfying, for all $X, Y \subseteq A$,

(i') $X \subseteq C(X)$;
(ii') if $X \subseteq Y$, then $C(X) \subseteq C(Y)$; and
(iii') $C(C(X)) = C(X)$.

Given a closure relation $\vdash \subseteq \mathscr{P}(A) \times A$, the map $C\colon \mathscr{P}(A) \to \mathscr{P}(A)$ defined by $C(X) := \{z \in A : X \vdash z\}$ for every $X \subseteq A$, is a closure operator over A. Conversely, given a closure operator C over A, the relation $\vdash$ defined by $X \vdash z \iff z \in C(X)$ for every $X \subseteq A$ and every $z \in A$, is a closure relation over A. Furthermore, these correspondences are inverse of each other.

A *closure system* is a family of subsets of A containing A itself and closed under intersections of arbitrary nonempty families. It is well known that a closure system, when ordered under the subset relation, is a complete lattice, with intersection as meet. Every closure operator C over A (and hence, every closure relation as well) naturally induces a closure system, namely the set of all C-closed sets $\mathscr{C} := \{X \subseteq A : X = C(X)\}$. Conversely, given a closure system $\mathscr{C} \subseteq \mathscr{P}(A)$, the map C defined by $C(X) := \bigcap\{Y \in \mathscr{C} : X \subseteq Y\}$, for every $X \subseteq A$, is a closure operator over A. Once again, these two correspondences are inverse of each other.

The following notation will be used very often and in an essential way in the paper. For any family $\mathscr{C} \subseteq \mathscr{P}(A)$ and any $F \subseteq A$, we define $\mathscr{C}^F := \{G \in \mathscr{C} : F \subseteq G\}$. Note that such a family is always an up-set in the poset $\mathscr{C}$ (ordered under set inclusion), and when $\mathscr{C}$ is a closure system, $\mathscr{C}^F$ is one as well.

Filters, and the Leibniz, Suszko, Tarski and Frege operators. Let $\mathcal{S}$ be a logic and $\boldsymbol{A}$ an algebra. An *$\mathcal{S}$-filter of $\boldsymbol{A}$* is a subset $F \subseteq A$ such that, for every $h : \boldsymbol{Fm} \to \boldsymbol{A}$ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h(\Gamma) \subseteq F$, then $h(\varphi) \in F$. The set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. This set is easily seen to be a closure system, and the associated closure operator will be denoted by $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}$. According to a previous definition, the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ containing a given $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ will be denoted by $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$; sets of this form will play an important rôle in the paper. Note that a closure system $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is an up-set of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ with respect to set inclusion if and only if $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; in fact, only $F = \bigcap\mathscr{C}$ qualifies.

The next (easy) lemma, stating sufficient conditions for $\mathcal{S}$-filters to be preserved under images and inverse images by homomorphisms, will be crucial.

LEMMA 2.4. *Let $\mathcal{S}$ be a logic, $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h\colon \boldsymbol{A} \to \boldsymbol{B}$, and $G \subseteq B$.*

1. *If $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, then $h^{-1}(G) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*
2. *If h is surjective and $h^{-1}(G) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.*
3. *If h is surjective and* $\mathrm{Ker}(h)$ *is compatible with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $h(F) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.*

The set of all congruences on $\boldsymbol{A}$ compatible with a given $F \subseteq A$ forms a complete sublattice of the lattice $\mathrm{Con}\boldsymbol{A}$. Its least element is, of course, the identity congruence on $\boldsymbol{A}$. Its largest element, known as the *Leibniz congruence of F*, plays a prominent rôle in abstract algebraic logic, and is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Observe that $\theta \in \mathrm{Con}\boldsymbol{A}$ is compatible with $F \subseteq A$ if and only if $\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

Two other particular types of congruences, both important to abstract algebraic logic, arise naturally from the notion of a congruence being compatible with a set. The first, for $F \subseteq A$, is called the *Suszko congruence of F*, and can be defined as the largest congruence of $\boldsymbol{A}$ compatible with every $G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, or, equivalently, by

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \bigcap\{\boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq G\}. \tag{1}$$

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The second, for $\mathscr{C} \subseteq \mathscr{P}(A)$, is called the *Tarski congruence of* $\mathscr{C}$, and is defined as the largest congruence compatible with every $G \in \mathscr{C}$, or equivalently by

$$\widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C}) := \bigcap\{\boldsymbol{\Omega}^{A}(F) : F \in \mathscr{C}\}. \tag{2}$$

Notice that the Suszko congruence is strictly relative to the logic $\mathcal{S}$ under consideration, as reflected in the notation, while the other two congruences are independent of $\mathcal{S}$. Moreover, from (1) and (2) it follows that the Suszko congruence can be defined in terms of the Tarski congruence by the identity

$$\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{A}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}\big). \tag{3}$$

We can consider the map assigning to each subset $F \subseteq A$ its Leibniz congruence $\boldsymbol{\Omega}^{A}(F)$; when restricting its domain to the set of $\mathcal{S}$-filters of $\boldsymbol{A}$, we refer to the map $\boldsymbol{\Omega}^{A}\colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ as the *Leibniz operator on* $\boldsymbol{A}$. Similarly, the *Suszko operator on* $\boldsymbol{A}$ is the map $\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}}\colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ defined by $F \mapsto \widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}}(F)$. Finally, the *Tarski operator on* $\boldsymbol{A}$ is the map $\widetilde{\boldsymbol{\Omega}}\colon \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \to \mathrm{Con}\boldsymbol{A}$ defined by $\mathscr{C} \mapsto \widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C})$. Although formally defined with domain $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$, the Tarski operator is often restricted to the closure systems of $\mathcal{S}$-filters of $\boldsymbol{A}$, or even further restricted to the "full" closure systems of $\mathcal{S}$-filters of $\boldsymbol{A}$, a notion to be introduced later on.

Given that these congruences and operators are defined on every algebra, it is natural to consider the family $\boldsymbol{\Omega} := \{\boldsymbol{\Omega}^{A} : \boldsymbol{A} \text{ an algebra}\}$ and call it *the Leibniz operator*. Similarly, we call the family $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}} := \{\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}} : \boldsymbol{A} \text{ an algebra}\}$ *the Suszko operator*. This terminology makes it easy to name properties that necessarily involve the whole family, i.e., that relate the operators on different algebras (see Definitions 4.5 and 4.8). On the other hand, we will also deal with properties that involve just a single algebra; we say then that one of these operators *globally* has one such property when for each algebra $\boldsymbol{A}$, the operator on $\boldsymbol{A}$ has that property. For instance, it is obvious from the definition that the Suszko operator is globally order preserving.

It will be later useful, and rather intuitive to have in mind right from the start, the behaviour of each one of these operators with respect to inverse images by homomorphisms.

PROPOSITION 2.5. *Let $\mathcal{S}$ be a logic, let $\boldsymbol{A}, \boldsymbol{B}$ be algebras, and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$. For every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ it holds that $h^{-1}\big(\boldsymbol{\Omega}^{B}(G)\big) \subseteq \boldsymbol{\Omega}^{A}\big(h^{-1}(G)\big)$. If furthermore h is surjective, then for every $\mathscr{C} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, the following hold*:

1. $h^{-1}\big(\boldsymbol{\Omega}^{B}(G)\big) = \boldsymbol{\Omega}^{A}\big(h^{-1}(G)\big)$;
2. $h^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{B}(\mathscr{C})\big) = \widetilde{\boldsymbol{\Omega}}^{A}\big(h^{-1}(\mathscr{C})\big)$; *and*
3. $\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}}\big(h^{-1}(G)\big) \subseteq h^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{B}_{\mathcal{S}}(G)\big)$.

PROOF. The first statement and point 1 are well-known properties of the Leibniz operator; see [5, Proposition 0.5.5]. 2 follows from expression (2), using the property in 1, because h^{-1} commutes with intersections. To prove 3, observe that $h^{-1}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}\big) \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}$. Then, using the antimonotonicity of the Tarski operator, the expression (3) and the property in 2, we have

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}^{A}_{\mathcal{S}}\big(h^{-1}(G)\big) &= \widetilde{\boldsymbol{\Omega}}^{A}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}\big) \subseteq \widetilde{\boldsymbol{\Omega}}^{A}\big(h^{-1}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}\big)\big)\\ &= h^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{B}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}\big)\big) = h^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{B}_{\mathcal{S}}(G)\big).\end{aligned}$$ ⊣

In principle the Suszko operator does not seem to behave as well as the Leibniz and Tarski operators, at least with respect to inverse images by surjective homomorphisms, in the sense that $h^{-1}(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G))$ need not be equal to $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\big(h^{-1}(G)\big)$. To support this statement we must wait until Theorem 4.6, but it can already be foreseen that, when working with the Suszko operator, the usual arguments used with the Leibniz operator will not go as smoothly as one could hope. The quest for some weaker properties that are shared by the two operators is one of the leading ideas of this paper.

The *Frege relation of* $F \subseteq A$ *on* $\boldsymbol{A}$ (again, relative to $\mathcal{S}$) is

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \big\{\langle a,b\rangle \in A\times A : \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F,a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F,b)\big\}.$$

Notice that, unlike the previous operators we have seen so far, the equivalence relation $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is not necessarily a congruence. Nevertheless, for simplicity we call the map given by $F \mapsto \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, restricted to $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the *Frege operator* on $\boldsymbol{A}$. It can be proven that the largest congruence below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the Suszko congruence $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Another observation worth mentioning is that the Frege operator is always order preserving.

Matrices, generalized matrices, and models of a logic. A *matrix* (or *logical matrix*) is a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an algebra and $F \subseteq A$. Every matrix $\mathcal{M} = \langle \boldsymbol{A}, F\rangle$ induces a logic whose consequence relation $\vdash_{\mathcal{M}}$ is defined as follows: For every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$,

$$\Gamma \vdash_{\mathcal{M}} \varphi \iff \text{for all } h\colon \boldsymbol{Fm} \to \boldsymbol{A},\ h(\Gamma) \subseteq F \Rightarrow h(\varphi) \in F.$$

Similarly, every class M of matrices induces a logic whose consequence relation $\vdash_{\mathsf{M}}$ is defined by

$$\vdash_{\mathsf{M}} := \bigcap_{\mathcal{M}\in\mathsf{M}} \vdash_{\mathcal{M}}. \tag{4}$$

Let $\mathcal{S}$ be a logic. A matrix $\mathcal{M}$ is a *model of* $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{M}}$. It follows from the definition itself that $\langle \boldsymbol{A}, F\rangle$ is a model of $\mathcal{S}$ if and only if F is an $\mathcal{S}$-filter of $\boldsymbol{A}$.

The notion of matrix is in fact a particular case of a more general notion. A *generalized matrix*, or *g-matrix* for short, is a pair $\mathfrak{M} = \langle \boldsymbol{A}, \mathscr{C}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathscr{C} \subseteq \mathscr{P}(A)$ is a closure system. Every g-matrix $\mathfrak{M} = \langle \boldsymbol{A}, \mathscr{C}\rangle$ induces a consequence relation $\vdash_{\mathfrak{M}}$ as in (4) by taking the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : F \in \mathscr{C}\big\}$. A g-matrix $\mathfrak{M}$ is a *generalized model* (*g-model* for short) of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathfrak{M}}$. One can easily check that $\langle \boldsymbol{A}, \mathscr{C}\rangle$ is a g-model of $\mathcal{S}$ if and only if $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Often, for simplicity, the term "g-model" is applied to $\mathscr{C}$ rather than to the pair $\langle \boldsymbol{A}, \mathscr{C}\rangle$.

Among the g-models of a logic there are some of crucial importance in this research. They arise from the following notion, which in principle applies to arbitrary families of filters.

Definition 2.6. Let $\boldsymbol{A}$ be an algebra. A family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is *full* if $\mathscr{C} = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$.

This notion is obviously relative to the logic, but in general there will be no need to specify it. It is easy to see that every full family of $\mathcal{S}$-filters is a closure system, because a congruence compatible with every element of a family of subsets

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is compatible with its intersection. Therefore, when speaking of full families of $\mathcal{S}$-filters, we can equivalently speak of *full g-models*. Note also that, given an arbitrary family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it always holds that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ for every $G \in \mathscr{C}$. Thus, $\mathscr{C}$ is full when it is exactly the set of *all* the $\mathcal{S}$-filters on $\boldsymbol{A}$ with which $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$ is compatible.

We have chosen as definition of the notion of full g-model one among its many equivalent formulations, namely the one that will suit better within the framework we intend to set. Nevertheless, we shall make use of the following characterizations, one of which is the original definition [18, Definition 2.8].

PROPOSITION 2.7. *Let $\boldsymbol{A}$ be an algebra and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and let $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$ be the canonical projection. The following conditions are equivalent.*

(i) *$\mathscr{C}$ is full.*
(ii) $\pi(\mathscr{C}) = \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})\big)$.
(iii) $\mathscr{C} = \pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})\big)\big)$.
(iv) *$\mathscr{C} = h^{-1}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})$, for some algebra $\boldsymbol{B}$ and some surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$.*

Let $\mathcal{M} = \langle \boldsymbol{A}, F\rangle$ and $\mathcal{N} = \langle \boldsymbol{B}, G\rangle$ be matrices. A *matrix homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is an $h\colon \boldsymbol{A} \to \boldsymbol{B}$ such that $h^{-1}(G) \subseteq F$; and it is *strict* if $h^{-1}(G) = F$.

The classical reference for the theory of matrices is [26], but all we need can be also found in [5, Chapter 0] or in [18, Chapter 1].

The classes of algebras $\mathsf{Alg}^*\mathcal{S}$ *and* $\mathsf{Alg}\mathcal{S}$. The operators we have defined so far produce, when applied to models or g-models of a logic $\mathcal{S}$, several classes of algebras. A matrix $\langle \boldsymbol{A}, F\rangle$ is *Leibniz-reduced*, or simply *reduced*, if $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \mathrm{Id}_A$, where Id_A denotes the identity relation on A; and it is *Suszko-reduced* if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_A$. A g-matrix $\langle \boldsymbol{A}, \mathscr{C}\rangle$ is *reduced* if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A$. Then the following classes of algebras, supporting models or g-models that are reduced in one of these ways, are considered as naturally and intrinsically associated with the logic.

$$\mathsf{Alg}^*\mathcal{S} := \big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \mathrm{Id}_A\big\}, \tag{5}$$

$$\mathsf{Alg}^{\mathrm{Su}}\mathcal{S} := \big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_A\big\}, \tag{6}$$

$$\mathsf{Alg}\mathcal{S} := \big\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A\big\}, \tag{7}$$

Observe that, since the Tarski operator is order reversing, definition (7) is equivalent to the following.

$$\mathsf{Alg}\mathcal{S} := \big\{\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A\big\}. \tag{8}$$

The conditions in next lemma state the standard characterizations of these classes, as well as the known relations between them (notice that two of the classes always coincide). The first four conditions tell us that these classes can be equivalently obtained, instead of considering *reduced* (g-)models as above, by taking the *reductions* of arbitrary (g-)models of the logic, and closing under isomorphisms.

LEMMA 2.8. *Let $\mathcal{S}$ be a logic.*

1. $\mathsf{Alg}^*\mathcal{S} = \mathbb{I}\big\{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F) : \boldsymbol{A}$ *an algebra*, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$.
2. $\mathsf{Alg}^{\mathrm{Su}}\mathcal{S} = \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) : \boldsymbol{A}$ *an algebra*, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$.

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3. $\mathsf{Alg}\mathcal{S} = \mathbb{I}\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A} \text{ an algebra}, \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ full}\}$.
4. $\mathsf{Alg}\mathcal{S} = \mathbb{I}\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A} \text{ an algebra}, \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\}$.
5. $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$.
6. $\mathsf{Alg}\mathcal{S} = \mathbb{P}_{\mathrm{S}}(\mathsf{Alg}^{*}\mathcal{S})$.

($\mathbb{P}_{\mathrm{S}}$ *is the subdirect product operator and* $\mathbb{I}$ *is the isomorphic image operator.*)

The proofs, except the one of 4, can be found explicitly in [1, Lemma 1.3], [6, Theorem 3.2], [18, Section 2.2], [6, Theorem 5.6], and [18, Theorem 2.23], respectively. As to 4, which is easy to prove directly, we will see that it follows from 3 as a particular case of Lemma 4.26.1. Notice that, as a consequence of 6, $\mathsf{Alg}^{*}\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S}$.

The Leibniz hierarchy. We present here a set of definitions of those classes of logics in the Leibniz hierarchy that will appear in this paper. Among many equivalent characterizations of these classes, we have chosen to define them in terms of properties of the Leibniz operator, as these fit more naturally within the framework we wish to develop.

DEFINITION 2.9. Let $\mathcal{S}$ be a logic.

- $\mathcal{S}$ is *protoalgebraic* if the Leibniz operator is globally order preserving.
- $\mathcal{S}$ is *equivalential* if the Leibniz operator is globally order preserving and commutes with inverse images by homomorphisms (in the standard sense, recorded here in Definition 4.5).
- $\mathcal{S}$ is *truth-equational* if the Leibniz operator is globally completely order reflecting.
- $\mathcal{S}$ is *weakly algebraizable* if it is protoalgebraic and truth-equational.
- $\mathcal{S}$ is *algebraizable* if it is equivalential and truth-equational.

As it is obvious from the definitions, an equivalential logic is a fortiori protoalgebraic. As a consequence, a logic is algebraizable if and only if it is equivalential and weakly algebraizable; in some sense this is not a "best" characterization, but will be useful at several points in the paper. Figure 1 displays the main relations between these classes.

For an exhaustive study of the Leibniz hierarchy and other results in abstract algebraic logic, see [3, 5, 14, 15, 18, 20].

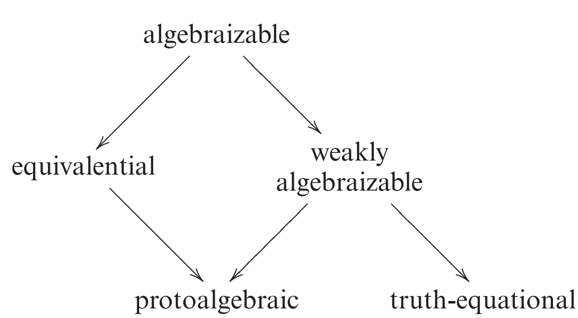

FIGURE 1. The fragment of the Leibniz hierarchy relevant for this paper.

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§**3.** $\mathcal{S}$**-operators.** In every statement in the paper where nothing is explicitly assumed about $\mathcal{S}$, $\boldsymbol{A}$, and $\boldsymbol{B}$, these symbols represent an arbitrary logic and arbitrary algebras respectively (over the same language), and the statement is understood as universally quantified over them.

3.1. General properties. The term "operator" is used in abstract algebraic logic to denote several kinds of maps that, given an algebra, assign one congruence either to a subset (e.g., the Leibniz operator, or the Suszko operator) or to a family of subsets (e.g., the Tarski operator); it is also used, in a more loose way, for the Frege operator, which maps a subset to an equivalence relation that needs not be a congruence. Now we want to be more specific.

Definition 3.1. An *$\mathcal{S}$-operator on $\boldsymbol{A}$* is a map $\nabla^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$. An $\mathcal{S}$-operator is *order preserving* when for all $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $F \subseteq G$, then $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$. For each $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ on $\boldsymbol{A}$ the following three maps associated with it are considered.

- The *lifting* of $\nabla^{\boldsymbol{A}}$ to the power set is the map $\widetilde{\nabla}^{\boldsymbol{A}} \colon \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \to \mathrm{Con}\boldsymbol{A}$, defined by $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) := \bigcap\{\nabla^{\boldsymbol{A}}(F) : F \in \mathscr{C}\}$, for every $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.
- The *relativization* of $\nabla^{\boldsymbol{A}}$ (to the logic $\mathcal{S}$) is the map $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$, defined by $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) := \bigcap\{\nabla^{\boldsymbol{A}}(F') : F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq F'\} = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}\big)$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.
- The map $\nabla^{\boldsymbol{A}^{-1}} \colon \mathrm{Con}\boldsymbol{A} \to \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ is defined by $\nabla^{\boldsymbol{A}^{-1}}(\theta) := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \theta \subseteq \nabla^{\boldsymbol{A}}(G)\}$, for every $\theta \in \mathrm{Con}\boldsymbol{A}$.

The following elementary properties are immediate consequences of the definitions.

Lemma 3.2. *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$.*

1. *$\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$ is an $\mathcal{S}$-operator.*
2. *$\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \nabla^{\boldsymbol{A}}(F)$ for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*
3. *$\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$ is order preserving.*
4. *$\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \nabla^{\boldsymbol{A}}(F)$ for every $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $F \in \mathscr{C}$.*
5. *$\nabla^{\boldsymbol{A}}$ is order preserving if and only if $\nabla^{\boldsymbol{A}} = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$.*

The Leibniz and the Suszko operators are the primary examples of $\mathcal{S}$-operators. The Suszko operator, which is the relativization of the Leibniz operator, is the primary order preserving example. The lifting of the Leibniz operator is the Tarski operator.

Note that $\nabla^{\boldsymbol{A}^{-1}}(\theta) = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \nabla^{\boldsymbol{A}}(G) \in [\theta, A \times A]\}$, which justifies the notation, though $\nabla^{\boldsymbol{A}^{-1}}$ is not, of course, the set-theoretic inverse of $\nabla^{\boldsymbol{A}}$.

Proposition 3.3. *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$. The maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ establish a Galois connection between $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ and $\mathrm{Con}\boldsymbol{A}$ (both ordered under the subset relation).*

Proof. Let $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}A$ and $\theta \in \mathrm{Con}\boldsymbol{A}$. Suppose that $\theta \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. If $F \in \mathscr{C}$, then $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \nabla^{\boldsymbol{A}}(F)$, and hence $\theta \subseteq \nabla^{\boldsymbol{A}}(F)$, that is, $F \in \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Thus, $\mathscr{C} \subseteq \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Conversely, suppose that $\mathscr{C} \subseteq \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Then, $\theta \subseteq \nabla^{\boldsymbol{A}}(G)$, for every $G \in \mathscr{C}$. Thus, $\theta \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. ⊣

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Several consequences follow from Proposition 3.3, by applying the results in Proposition 2.3.

COROLLARY 3.4. *Let* ∇^{A} *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$.

1. *The maps* $\widetilde{\nabla}^{A}$ *and* $\nabla^{A^{-1}}$ *are order reversing.*
2. *The map* $\nabla^{A^{-1}} \circ \widetilde{\nabla}^{A}$ *is a closure operator over* $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, *i.e., is a closure on* $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$.
3. *The map* $\widetilde{\nabla}^{A} \circ \nabla^{A^{-1}}$ *is a closure on* $\mathrm{Con}\boldsymbol{A}$.
4. *The set of fixed points of* $\nabla^{A^{-1}} \circ \widetilde{\nabla}^{A}$ *is* $\mathrm{Ran}(\nabla^{A^{-1}})$.
5. *The set of fixed points of* $\widetilde{\nabla}^{A} \circ \nabla^{A^{-1}}$ *is* $\mathrm{Ran}(\widetilde{\nabla}^{A})$.
6. *The maps* $\widetilde{\nabla}^{A}$ *and* $\nabla^{A^{-1}}$ *restrict to mutually inverse dual order isomorphisms between the set of fixed points of* $\nabla^{A^{-1}} \circ \widetilde{\nabla}^{A}$ *and the set of fixed points of* $\widetilde{\nabla}^{A} \circ \nabla^{A^{-1}}$.

In the following, we shall be interested in characterizing the fixed points of both closures $\nabla^{A^{-1}} \circ \widetilde{\nabla}^{A}$ and $\widetilde{\nabla}^{A} \circ \nabla^{A^{-1}}$; therefore, they receive a proper name.

DEFINITION 3.5. Let ∇^{A} be an $\mathcal{S}$-operator on $\boldsymbol{A}$. A family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is ∇^{A}*-full* if $\mathscr{C} = \nabla^{A^{-1}}\big(\widetilde{\nabla}^{A}(\mathscr{C})\big)$, i.e., if $\mathscr{C} \in \mathrm{Ran}(\nabla^{A^{-1}})$. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is ∇^{A}*-full* if $\theta = \widetilde{\nabla}^{A}\big(\nabla^{A^{-1}}(\theta)\big)$, i.e., if $\theta \in \mathrm{Ran}(\widetilde{\nabla}^{A})$.

Thus, the maps $\widetilde{\nabla}^{A}$ and $\nabla^{A^{-1}}$ restrict to mutually inverse dual order isomorphisms between the sets of all ∇^{A}-full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ and of ∇^{A}-full congruences of $\boldsymbol{A}$. A useful characterization of these ∇^{A}-full objects, which is also a consequence of the Galois connection, is the following.

PROPOSITION 3.6. *Let* ∇^{A} *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$.

1. $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *is* ∇^{A}*-full if and only if it is the largest* $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *such that* $\widetilde{\nabla}^{A}(\mathscr{D}) = \widetilde{\nabla}^{A}(\mathscr{C})$.
2. $\theta \in \mathrm{Con}\boldsymbol{A}$ *is* ∇^{A}*-full if and only if it is the largest* $\theta' \in \mathrm{Con}\boldsymbol{A}$ *such that* $\nabla^{A^{-1}}(\theta') = \nabla^{A^{-1}}(\theta)$.

From this it follows that the closure system $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is always ∇^{A}-full, for any $\mathcal{S}$-operator ∇^{A} on $\boldsymbol{A}$.

3.2. The Leibniz operator as an $\mathcal{S}$-operator. The Leibniz operator is an $\mathcal{S}$-operator defined in a particular way, so that besides the general properties obtained in Section 3.1, it enjoys some further ones, which combined with the previous ones yield some better descriptions of the notions there introduced. As already remarked, the lifting of the Leibniz operator $\boldsymbol{\Omega}^{A}$ is the familiar Tarski operator $\widetilde{\boldsymbol{\Omega}}^{A}$. As to the map $\boldsymbol{\Omega}^{A^{-1}}$, observe that if $\theta \in \mathrm{Con}\boldsymbol{A}$, then

$$
\begin{aligned}
\boldsymbol{\Omega}^{A^{-1}}(\theta) &= \big\{F \in \mathcal{F}i_{\mathcal{S}}(\mathbf{A}) : \theta \subseteq \boldsymbol{\Omega}^{A}(F)\big\} \\
&= \big\{F \in \mathcal{F}i_{\mathcal{S}}(\mathbf{A}) : \theta \text{ is compatible with } F\big\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}.
\end{aligned}
$$

PROPOSITION 3.7. *For every* $\theta \in \mathrm{Con}\boldsymbol{A}$, $\boldsymbol{\Omega}^{A^{-1}}(\theta) = \pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)$ *and* $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) = \pi\big(\boldsymbol{\Omega}^{A^{-1}}(\theta)\big)$. *Moreover, the extended mappings* $\pi\colon \mathscr{P}(A) \to \mathscr{P}(A/\theta)$

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and $\pi^{-1}\colon \mathscr{P}(A/\theta) \to \mathscr{P}(A)$ *restrict to order isomorphisms between the sets* $\boldsymbol{\Omega}^{A^{-1}}(\theta)$ *and* $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$.

Proof. Let $F \in \boldsymbol{\Omega}^{A^{-1}}(\theta)$. This means that θ is compatible with F, and hence that $\pi^{-1}\big(\pi(F)\big) = F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since π is surjective, by Lemma 2.4.2 $\pi(F) \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. So, $F \in \pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)$. Conversely, let $G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. It follows by Lemma 2.4.1 that $\pi^{-1}(G) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Moreover, again by the surjectivity of π, $\pi\big(\pi^{-1}(G)\big) = G$. So, $\pi^{-1}\big(\pi\big(\pi^{-1}(G)\big)\big) = \pi^{-1}(G)$, which tells us that θ is compatible with $\pi^{-1}(G)$. Thus, $\pi^{-1}(G) \in \boldsymbol{\Omega}^{A^{-1}}(\theta)$. This proves the first equality, and the second follows from it by surjectivity of π. As to the second part of the statement, observe that we have just seen that both π and π^{-1} are into (actually, onto) the respective co-domains. Moreover $\big(\pi \restriction \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big) \circ \big(\pi^{-1} \restriction \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) = \mathrm{Id}_{\mathcal{F}i_{\mathcal{S}}(A/\theta)}$, because π is surjective, and $\big(\pi^{-1} \restriction \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) \circ \big(\pi\restriction\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big) = \mathrm{Id}_{\mathcal{F}i_{\mathcal{S}}A}$, by of $\boldsymbol{\Omega}^{A^{-1}}(\theta)$. So, they are mutually inverse bijections. Since they are both order preserving, they are in fact order isomorphisms. ⊣

Since order isomorphisms send least elements to least elements, it is easy to see the following.

Corollary 3.8. *For every* $\theta \in \mathrm{Con}\boldsymbol{A}$*, if* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*, then* F *is the least element of* $\boldsymbol{\Omega}^{A^{-1}}(\theta)$ *if and only if* F/θ *is the least element of* $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$.

Moreover, from the first equality in Proposition 3.7 and the characterization in Proposition 2.7(iv), we also obtain the following.

Corollary 3.9. *For every* $\theta \in \mathrm{Con}\boldsymbol{A}$*, the set* $\boldsymbol{\Omega}^{A^{-1}}(\theta)$ *is full, and hence a closure system.*

Now we characterize the $\boldsymbol{\Omega}^{A}$-full sets of $\mathcal{S}$-filters and the $\boldsymbol{\Omega}^{A}$-full congruences.

Proposition 3.10. *A set* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *is* $\boldsymbol{\Omega}^{A}$*-full if and only if it is full.*

Proof. Observe that in general $\boldsymbol{\Omega}^{A^{-1}}\big(\widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C})\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}(A) : \widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{A}(G)\big\}$. But by definition, $\mathscr{C}$ is $\boldsymbol{\Omega}^{A}$-full when it equals the left-hand side of the equality, and it is full when it equals the right-hand side; this proves the statement. ⊣

Proposition 3.11. *A congruence* $\theta \in \mathrm{Con}\boldsymbol{A}$ *is* $\boldsymbol{\Omega}^{A}$*-full if and only if* $\theta \in \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.

Proof. If $\theta \in \mathrm{Con}\boldsymbol{A}$, we know by Corollary 3.9 that the set $\boldsymbol{\Omega}^{A^{-1}}(\theta)$ is full. So, by Lemma 2.8.3, $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{A}\big(\boldsymbol{\Omega}^{A^{-1}}(\theta)\big) \in \mathsf{Alg}\mathcal{S}$. But if θ is $\boldsymbol{\Omega}^{A}$-full, then $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{A}\big(\boldsymbol{\Omega}^{A^{-1}}(\theta)\big) = \boldsymbol{A}/\theta$. Thus, $\theta \in \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. Conversely, if $\boldsymbol{A}/\theta \in \mathsf{Alg}\mathcal{S}$, by the equivalent definition (8) of this class, $\widetilde{\boldsymbol{\Omega}}^{A/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) = \mathrm{Id}_{A/\theta}$. Then,

$$\begin{aligned}\theta = \mathrm{Ker}(\pi) = \pi^{-1}(\mathrm{Id}_{A/\theta}) &= \pi^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{A/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)\big)\\ &= \widetilde{\boldsymbol{\Omega}}^{A}\big(\pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)\big) = \widetilde{\boldsymbol{\Omega}}^{A}\big(\boldsymbol{\Omega}^{A^{-1}}(\theta)\big),\end{aligned}$$

where we have used Propositions 2.5 and 3.7 in the last two steps. This proves that θ is $\boldsymbol{\Omega}^{A}$-full. ⊣

The two preceding results allow us to instantiate Proposition 3.3 and Corollary 3.4 for the Leibniz operator in a particularly suggestive form.

COROLLARY 3.12. *The maps* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}$ *and* $\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}$ *establish a Galois connection between* $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ *and* $\mathrm{Con}\boldsymbol{A}$ *and restrict to mutually inverse dual order isomorphisms between the poset of all full g-models of* $\mathcal{S}$ *on* $\boldsymbol{A}$ *and the poset* $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.

The second part of this statement is the well-known Isomorphism Theorem [18, Theorem 2.30]; we now see it arises naturally as a by-product of the Galois connection established starting from the Leibniz operator. Finally, from Propositions 3.6 and 3.10 the following characterization of the full g-models follows immediately.

PROPOSITION 3.13. *A subset* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *is full if and only if* $\mathscr{C}$ *is the largest* $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *such that* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D})$.

3.3. $\nabla^{\boldsymbol{A}}$**-classes and** $\nabla^{\boldsymbol{A}}$**-filters.** The notions of $\nabla^{\boldsymbol{A}}$-class and $\nabla^{\boldsymbol{A}}$-filter we consider now are introduced for arbitrary $\mathcal{S}$-operators, although we will use them more extensively for the special kinds of operators considered in the next sections.

DEFINITION 3.14. Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The $\nabla^{\boldsymbol{A}}$-*class of* F is the set

$$[\![F]\!]^{\nabla^{\boldsymbol{A}}} := \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\nabla^{\boldsymbol{A}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

A justification of the usage of the term “class” in this context will be found in the discussion after Proposition 5.1. Let us start by collecting some basic properties concerning $\nabla^{\boldsymbol{A}}$-classes. As a particular case of Corollary 3.9, taking $\theta = \nabla^{\boldsymbol{A}}(F)$, we immediately have the following.

PROPOSITION 3.15. *Let* $\nabla^{\boldsymbol{A}}$ *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$ *and* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. *The* $\nabla^{\boldsymbol{A}}$*-class* $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ *is full. As a consequence, it is a closure system, and* $[\![F]\!]^{\nabla^{\boldsymbol{A}}} = \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^{\nabla^{\boldsymbol{A}}})\big)$.

Thus, given that every closure system is closed under intersections, it makes sense to consider the smallest element in each class.

DEFINITION 3.16. Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The least element of the $\nabla^{\boldsymbol{A}}$-class of F will be denoted by $F^{\nabla^{\boldsymbol{A}}}$; i.e., $F^{\nabla^{\boldsymbol{A}}} := \bigcap [\![F]\!]^{\nabla^{\boldsymbol{A}}}$. We say that F is a $\nabla^{\boldsymbol{A}}$-*filter* if $F = F^{\nabla^{\boldsymbol{A}}}$. The set of all $\nabla^{\boldsymbol{A}}$-filters of $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_{\mathcal{S}}^{\nabla^{\boldsymbol{A}}}\boldsymbol{A}$.

Notice that if $\mathcal{S}$ is a logic without theorems, then for any $\boldsymbol{A}$ the only $\nabla^{\boldsymbol{A}}$-filter of $\boldsymbol{A}$ is the empty filter, because then $\varnothing \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and since $\boldsymbol{\Omega}^{\boldsymbol{A}}(\varnothing) = A \times A$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\varnothing \in [\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and hence necessarily $\varnothing = \bigcap [\![F]\!]^{\nabla^{\boldsymbol{A}}}$. It is therefore clear that the interesting applications of the notions of $\nabla^{\boldsymbol{A}}$-class and $\nabla^{\boldsymbol{A}}$-filter will concern only logics with theorems; however, technically we will not need to assume this in any result.

PROPOSITION 3.17. *Every* $\mathcal{S}$*-operator* $\nabla^{\boldsymbol{A}}$ *on* $\boldsymbol{A}$ *is order reflecting, and therefore injective, on* $\mathcal{F}i_{\mathcal{S}}^{\nabla^{\boldsymbol{A}}}\boldsymbol{A}$.

PROOF. Let $F, G \in \mathcal{F}i_{\mathcal{S}}^{\nabla^{\boldsymbol{A}}}\boldsymbol{A}$ be such that $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$. Then $[\![G]\!]^{\nabla^{\boldsymbol{A}}} \subseteq [\![F]\!]^{\nabla^{\boldsymbol{A}}}$. Thus, $F = \bigcap [\![F]\!]^{\nabla^{\boldsymbol{A}}} \subseteq \bigcap [\![G]\!]^{\nabla^{\boldsymbol{A}}} = G$. ⊣

We now collect a few other basic properties of the notions just introduced.

LEMMA 3.18. *Let* ∇^{A} *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$*. For every* $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

1. $[\![F]\!]^{\nabla^{A}} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F^{\nabla^{A}}}$*; and*
2. *if* $[\![F]\!]^{\nabla^{A}} \subseteq [\![G]\!]^{\nabla^{A}}$*, then* $G^{\nabla^{A}} \subseteq F^{\nabla^{A}}$.

If moreover ∇^{A} *is order preserving, then*

3. *if* $F \subseteq G$*, then* $[\![G]\!]^{\nabla^{A}} \subseteq [\![F]\!]^{\nabla^{A}}$ *and* $F^{\nabla^{A}} \subseteq G^{\nabla^{A}}$*; and*
4. *every* ∇^{A}*-full g-model of* $\mathcal{S}$ *is an up-set of* $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

PROOF. 1. This is a straightforward consequence of the fact that $F^{\nabla^{A}} = \bigcap [\![F]\!]^{\nabla^{A}}$. 2. If $[\![F]\!]^{\nabla^{A}} \subseteq [\![G]\!]^{\nabla^{A}}$, then $G^{\nabla^{A}} = \bigcap [\![G]\!]^{\nabla^{A}} \subseteq \bigcap [\![F]\!]^{\nabla^{A}} = F^{\nabla^{A}}$. 3. If $F \subseteq G$, then $\nabla^{A}(F) \subseteq \nabla^{A}(G)$ by order preservation, and therefore $[\![G]\!]^{\nabla^{A}} \subseteq [\![F]\!]^{\nabla^{A}}$. Moreover, $F^{\nabla^{A}} \subseteq G^{\nabla^{A}}$ by 2. Finally, 4 is a straightforward consequence of the definition of ∇^{A}-full g-model, taking order preservation into account. ⊣

§4. $\mathcal{S}$-compatibility operators and coherent families. In this section we will be interested in a special kind of $\mathcal{S}$-operators, namely those satisfying a compatibility property with respect to the $\mathcal{S}$-filters of the underlying algebra. The general framework developed here will be instantiated for the Leibniz and the Suszko operators in Section 5.

The starting notion of this section was first introduced in [6, p. 199], with a name very similar to the one we now choose.

DEFINITION 4.1. An $\mathcal{S}$*-compatibility operator* on $\boldsymbol{A}$ is an $\mathcal{S}$-operator ∇^{A} on $\boldsymbol{A}$ such that for each $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the congruence $\nabla^{A}(F)$ is compatible with F, that is, satisfies $\nabla^{A}(F) \subseteq \boldsymbol{\Omega}^{A}(F)$.

An alternative formulation of the definition is that an $\mathcal{S}$-operator ∇^{A} on $\boldsymbol{A}$ is an $\mathcal{S}$-compatibility operator if and only if $F \in [\![F]\!]^{\nabla^{A}}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The largest $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$ is obviously $\boldsymbol{\Omega}^{A}$, and the least $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$ is the one sending every $\mathcal{S}$-filter to the identity relation Id_{A} on $\boldsymbol{A}$. Another well-known example of an $\mathcal{S}$-compatibility operator is the Suszko operator relative to $\mathcal{S}$, which turns out to be the largest order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$; this follows trivially from Lemma 3.2.5.

For future reference it will be useful to record here a few elementary general properties, which enhance those of Lemma 3.18.

LEMMA 4.2. *Let* ∇^{A} *be an* $\mathcal{S}$*-compatibility operator on* $\boldsymbol{A}$*. For every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

1. $F \in [\![F]\!]^{\nabla^{A}}$*; and*
2. $F^{\nabla^{A}} \subseteq F$.

If moreover ∇^{A} *is order preserving, then*

3. $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F} \subseteq [\![F]\!]^{\nabla^{A}}$;
4. $[\![F]\!]^{\nabla^{A}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ *if and only if* $F = F^{\nabla^{A}}$*, i.e., if and only if* F *is a* ∇^{A}*-filter; and*
5. $[\![F]\!]^{\nabla^{A}} \subseteq [\![F^{\nabla^{A}}]\!]^{\nabla^{A}}$.

PROOF. 1 and 2 are direct consequences of the definition of $[\![F]\!]^{\nabla^{A}}$ and the notion of $\mathcal{S}$-compatibility. 3. If $F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$, then $\nabla^{A}(F) \subseteq \nabla^{A}(F') \subseteq \boldsymbol{\Omega}^{A}(F')$, and

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therefore $F' \in [\![F]\!]^{\nabla^A}$. 4. Suppose that $[\![F]\!]^{\nabla^A} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Then, $F^{\nabla^A} = \bigcap [\![F]\!]^{\nabla^A} = \bigcap (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = F$. Conversely, suppose that $F = F^{\nabla^A}$. It follows by Lemma 3.18.1 that $[\![F]\!]^{\nabla^A} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and by 3 that $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\nabla^A}$, thus establishing the equality. 5 follows from 2 and Lemma 3.18.3. ⊣

The fact that $\nabla^{\boldsymbol{A}}(F)$ is compatible with F allows to rewrite Corollary 3.8, for congruences of the form $\nabla^{\boldsymbol{A}}(F)$ with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, as follows.

Corollary 4.3. *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then F is a $\nabla^{\boldsymbol{A}}$-filter of $\boldsymbol{A}$ if and only if $F/\nabla^{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$.*

The following straightforward consequence of the definitions will turn out to have some significance later on.

Corollary 4.4. *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. The following conditions are equivalent.*

(i) *Every $\mathcal{S}$-filter is a $\nabla^{\boldsymbol{A}}$-filter of $\boldsymbol{A}$.*
(ii) *For every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, then $F \subseteq G$.*

4.1. The General Correspondence Theorem. We now move towards the main theorem of this section — the General Correspondence Theorem 4.17. This result, the notions it involves, and subsequent ones, concern the behaviour of operators on different algebras. By *a family of* ($\mathcal{S}$*-compatibility*) *operators* we understand a (proper) class $\{\nabla^{\boldsymbol{A}} : \boldsymbol{A}$ an algebra$\}$ such that for each $\boldsymbol{A}$, $\nabla^{\boldsymbol{A}}$ is an ($\mathcal{S}$-compatibility) operator on $\boldsymbol{A}$. Such a family will be denoted collectively by ∇, generalizing the usage of the notations for the Leibniz and Suszko families introduced in Section 2.

Definition 4.5. Let $\nabla^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{B}}$ be $\mathcal{S}$-compatibility operators on $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. The pair $\langle \nabla^{\boldsymbol{A}}, \nabla^{\boldsymbol{B}} \rangle$ *commutes with inverse images by* (*surjective*) *homomorphisms* if for every (surjective) $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$,

$$\nabla^{\boldsymbol{A}}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{\boldsymbol{B}}(G)\big).$$

A family ∇ of $\mathcal{S}$-compatibility operators *commutes with inverse images by* (*surjective*) *homomorphisms* if for all algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ the pair $\langle \nabla^{\boldsymbol{A}}, \nabla^{\boldsymbol{B}} \rangle$ commutes with inverse images by (surjective) homomorphisms in the above sense.

Note that this definition packs two into one (the surjective case and the general case). The difference is significant: Proposition 2.5.1 tells us that the Leibniz operator always commutes with inverse images by surjective homomorphisms, while in general it does not commute with inverse images by arbitrary homomorphisms: for instance, inside the class of protoalgebraic logics, this happens only for equivalential logics (Definition 2.9). Anyway, an important fact is that the more restricted commutativity property is an exclusive feature of the Leibniz operator.

Theorem 4.6. *If ∇ is a family of $\mathcal{S}$-compatibility operators that commutes with inverse images by surjective homomorphisms, then ∇ is the Leibniz operator.*

Proof. Let $\boldsymbol{A}$ be an algebra, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and let $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ be the canonical map, which is surjective. Since $\ker(\pi) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with F, we have that $F = \pi^{-1}\big(\pi(F)\big)$ and $\pi(F) = F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big)$. Now, by $\mathcal{S}$-compatibility of ∇,

$$\nabla^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}\big(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}\big(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) = \mathrm{Id}_{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}.$$

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Hence, $\nabla^{A/\boldsymbol{\Omega}^{A}(F)}\big(F/\boldsymbol{\Omega}^{A}(F)\big) = \boldsymbol{\Omega}^{A/\boldsymbol{\Omega}^{A}(F)}\big(F/\boldsymbol{\Omega}^{A}(F)\big)$. Using the previous facts and that both the Leibniz operator and ∇ commute with inverse images by surjective homomorphisms, we have that

$$\nabla^{A}(F) = \nabla^{A}\big(\pi^{-1}\big(\pi(F)\big)\big) = \pi^{-1}\big(\nabla^{A/\boldsymbol{\Omega}^{A}(F)}\big(F/\boldsymbol{\Omega}^{A}(F)\big)\big)$$
$$= \pi^{-1}\big(\boldsymbol{\Omega}^{A/\boldsymbol{\Omega}^{A}(F)}\big(F/\boldsymbol{\Omega}^{A}(F)\big)\big) = \boldsymbol{\Omega}^{A}\big(\pi^{-1}\big(\pi(F)\big)\big) = \boldsymbol{\Omega}^{A}(F).$$

Since $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ are arbitrary, we conclude that $\nabla = \boldsymbol{\Omega}$. ⊣

It is hence clear that the Suszko operator does not commute, in general, in any of the two senses. In order to find a commutativity property that makes a unified treatment of the properties of the two operators possible, we need to introduce the following technical notion.

Definition 4.7. Let ∇^{A} be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. An $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is *∇^{A}-compatible with F* if $\mathrm{Ker}(h) \subseteq \nabla^{A}(F)$; and it is *$\nabla^{A}$-compatible with $\mathscr{C}$* if it is ∇^{A}-compatible with every member of $\mathscr{C}$.

This is a generalization of two well-known notions. First, $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is $\boldsymbol{\Omega}^{A}$-compatible with F in the sense of Definition 4.7 if and only if the congruence $\mathrm{Ker}(h)$ is compatible with F, and if and only if the matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, h(F)\rangle$ is strict. Second, it turns out that h is a *deductive* matrix homomorphism between $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, h(F)\rangle$ in the sense of [6] if and only if h is $\widetilde{\boldsymbol{\Omega}}{}^{A}_{\mathcal{S}}$-compatible with F, viewed as an algebraic homomorphism, in the sense of Definition 4.7. This is because the property of being "deductive" means that for every $a, b \in A$, $h(a) = h(b)$ implies $\mathrm{Fg}^{A}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{A}_{\mathcal{S}}(F, b)$; but this condition can be written as $\mathrm{Ker}(h) \subseteq \boldsymbol{\Lambda}^{A}_{\mathcal{S}}(F)$, which we know is equivalent to $\mathrm{Ker}(h) \subseteq \widetilde{\boldsymbol{\Omega}}{}^{A}_{\mathcal{S}}(F)$, and this is just to say that h is $\widetilde{\boldsymbol{\Omega}}{}^{A}_{\mathcal{S}}$-compatible with F, in the sense of Definition 4.7. Deductive homomorphisms have been used in [6] to obtain a version of the Correspondence Theorem, which we will find in Theorem 5.15 as an instance of our General Correspondence Theorem 4.17, which is obtained under the generalized assumption of ∇^{A}-compatibility now introduced.

Observe that for each $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\nabla^{A}(F)$ is always ∇^{A}-compatible with F. Moreover, since ∇^{A} is an $\mathcal{S}$-compatibility operator, if h is ∇^{A}-compatible with F then it is also $\boldsymbol{\Omega}^{A}$-compatible with F, that is, $\mathrm{Ker}(h)$ will be compatible with F; then, by Lemmas 2.2.1 and 2.2.2, $F = h^{-1}\big(h(F)\big)$ and $\nabla^{A}(F) = h^{-1}\big(h\big(\nabla^{A}(F)\big)\big)$.

Definition 4.8. A family ∇ of $\mathcal{S}$-compatibility operators is *coherent* if for every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, if h is ∇^{A}-compatible with $h^{-1}(G)$, then $\nabla^{A}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{B}(G)\big)$.

By Proposition 2.5.1, the Leibniz operator is a coherent family. In Proposition 4.19 we will show that coherence is preserved under relativization, and as a consequence the Suszko operator is also a coherent family. This confirms that this restricted commutativity property is common to both operators.

Observe that a family ∇ of $\mathcal{S}$-compatibility operators is coherent if and only if for every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, h is ∇^{A}-compatible with $h^{-1}(G)$ if and only if $\nabla^{A}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{B}(G)\big)$. The only difference with Definition 4.8 is the backwards implication in the property asserted for $h^{-1}(G)$; but this implication always holds, because if $\nabla^{A}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{B}(G)\big)$ then $\mathrm{Ker}(h) = h^{-1}(\mathrm{Id}_B) \subseteq h^{-1}\big(\nabla^{B}(G)\big) = \nabla^{A}\big(h^{-1}(G)\big)$, which means that h is ∇^{A}-compatible with $h^{-1}(G)$.

It is possible, and also practical, to re-state Definition 4.8 in terms of commutativity with images by surjective homomorphisms instead of inverse images.

LEMMA 4.9. *A family ∇ of $\mathcal{S}$-compatibility operators is coherent if and only if for every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is ∇^{A}-compatible with F, then $h\big(\nabla^{A}(F)\big) = \nabla^{B}\big(h(F)\big)$.*

PROOF. Suppose ∇ is coherent, and let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and ∇^{A}-compatible with F. Hence, $F = h^{-1}(h(F))$ and $h(F) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. It follows by coherence that $\nabla^{A}(F) = \nabla^{A}\big(h^{-1}\big(h(F)\big)\big) = h^{-1}\big(\nabla^{B}\big(h(F)\big)\big)$, and hence that $h(\nabla^{A}(F)) = \nabla^{B}(h(F))$ because h is surjective. Conversely, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and ∇^{A}-compatible with $h^{-1}(G)$. Since $h^{-1}(G) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it follows by the assumption and the surjectivity of h that

$$h\big(\nabla^{A}\big(h^{-1}(G)\big)\big) = \nabla^{B}\big(h\big(h^{-1}(G)\big)\big) = \nabla^{B}(G).$$

Applying the property in Lemma 2.2.2 to the ∇^{A}-compatibility of h with $h^{-1}(G)$, we obtain

$$\nabla^{A}\big(h^{-1}(G)\big) = h^{-1}\big(h\big(\nabla^{A}(h^{-1}(G))\big)\big) = h^{-1}\big(\nabla^{B}(G)\big),$$

which shows that ∇ is coherent. ⊣

Isomorphisms are both surjective and ∇^{A}-compatible with any $\mathcal{S}$-filter and for any operator ∇^{A}, as their kernel is the identity.

COROLLARY 4.10. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism, then for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ we have $h\big(\nabla^{A}(F)\big) = \nabla^{B}\big(h(F)\big)$ and $\nabla^{A}\big(h^{-1}(G)\big) = h^{-1}\big(\nabla^{B}(G)\big)$.*

We now see that for coherent families of $\mathcal{S}$-operators, the ∇-full objects defined in Section 3.1 can be given finer characterizations. We first prove the following technical result.

PROPOSITION 4.11. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$,*

$$\nabla^{A^{-1}}\big(\mathrm{Ker}(h)\big) = \big\{F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}) : h^{-1}\big(\nabla^{B}(h(F))\big) = \nabla^{A}\big(F\big)\big\}.$$

PROOF. Let $F \in \nabla^{A^{-1}}(\mathrm{Ker}(h))$. Thus, $\mathrm{Ker}(h) \subseteq \nabla^{A}(F) \subseteq \boldsymbol{\Omega}^{A}(F)$. Therefore, $F = h^{-1}(h(F))$ and hence $h(F) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Since ∇ is a coherent family, $h^{-1}(\nabla^{B}(h(F)) = \nabla^{A}(F)$. Conversely, if $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A})$ is such that $h^{-1}(\nabla^{B}(h(F)) = \nabla^{A}(F)$, since $\mathrm{Ker}(h) \subseteq h^{-1}(\nabla^{B}(h(F))$ always holds, it follows that $\mathrm{Ker}(h) \subseteq \nabla^{A}(F)$ and therefore that $F \in \nabla^{A^{-1}}(\mathrm{Ker}(h))$. ⊣

As a particular case we have:

COROLLARY 4.12. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then for every $\theta \in \mathrm{Con}\boldsymbol{A}$,*

$$\begin{aligned}\nabla^{A^{-1}}(\theta) &= \big\{F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}) : \pi^{-1}\big(\nabla^{A/\theta}(\pi(F))\big) = \nabla^{A}(F)\big\}\\ &= \pi^{-1}\big(\big\{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\big(\nabla^{A/\theta}(G)\big) = \nabla^{A}\big(\pi^{-1}(G)\big)\big\}\big).\end{aligned}$$

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Proof. For the first equality we apply Proposition 4.11 to the quotient homomorphism $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$. To obtain the second note that the inclusion from left to right is clear. The other inclusion follows from the fact that if $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A})$ is such that $\pi^{-1}\big(\nabla^{\boldsymbol{A}/\theta}(\pi(F))\big) = \nabla^{\boldsymbol{A}}(F)$, then $\mathrm{Ker}(\pi)$ is compatible with F and therefore $\pi(F) \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. ⊣

An interesting corollary to Proposition 4.11 is the following characterization of $\nabla^{\boldsymbol{A}}$-full models for coherent families ∇ of $\mathcal{S}$-compatibility operators.

Corollary 4.13. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then $\mathscr{C}$ is a $\nabla^{\boldsymbol{A}}$-full model of $\mathcal{S}$ if and only if $\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}) : h^{-1}\big(\nabla^{\boldsymbol{B}}(h(F))\big) = \nabla^{\boldsymbol{A}}\big(F\big)\big\}$ for some surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$.*

Proof. Suppose that $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a $\nabla^{\boldsymbol{A}}$-full model of $\mathcal{S}$, i.e., $\mathscr{C} = {\nabla^{\boldsymbol{A}}}^{-1}\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big)$. Let $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ and let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ the quotient homomorphism. Then $\mathscr{C} = {\nabla^{\boldsymbol{A}}}^{-1}(\mathrm{Ker}(\pi))$. Thus from Proposition 4.11 we obtain $\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}) : \pi^{-1}\big(\nabla^{\boldsymbol{B}}(\pi(F))\big) = \nabla^{\boldsymbol{A}}\big(F\big)\big\}$. Assume now that $\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}) : h^{-1}\big(\nabla^{\boldsymbol{B}}(h(F))\big) = \nabla^{\boldsymbol{A}}\big(F\big)\big\}$ for some surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$. Then again by Proposition 4.11 we have $\mathscr{C} = {\nabla^{\boldsymbol{A}}}^{-1}(\mathrm{Ker}(h))$. Thus $\mathscr{C} \in \mathrm{Ran}({\nabla^{\boldsymbol{A}}}^{-1})$ and hence it is $\nabla^{\boldsymbol{A}}$-full. ⊣

Considering the proof above and Corollary 4.12, a slightly different corollary can be stated as follows:

Corollary 4.14. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators, and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then $\mathscr{C}$ is $\nabla^{\boldsymbol{A}}$-full if and only if $\mathscr{C} = \pi^{-1}\big(\big\{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\big(\nabla^{\boldsymbol{A}/\theta}(G)\big) = \nabla^{\boldsymbol{A}}\big(\pi^{-1}(G)\big)\big\}\big)$ for some $\theta \in \mathrm{Con}\boldsymbol{A}$, which can be taken to be $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$.*

Corollary 4.14 is particularly interesting in view of Propositions 3.7 and 3.10. There we saw that $\boldsymbol{\Omega}^{\boldsymbol{A}}$-full models of $\mathcal{S}$ are families of $\mathcal{S}$-filters of the form $\pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)$ for some $\theta \in \mathrm{Con}\boldsymbol{A}$. But from $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ it follows that ${\nabla^{\boldsymbol{A}}}^{-1}(\theta) \subseteq {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\theta) = \pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big)$, therefore ${\nabla^{\boldsymbol{A}}}^{-1}(\theta)$ must be a family of the form $\pi^{-1}(\mathscr{D})$ for some $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. Corollary 4.14 determines one such family, which we will see has other interesting properties. We first need the following technical result, which is parallel to Definition 4.8 and Lemma 4.9.

Lemma 4.15. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators, and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective.*

1. *For any $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, if h is $\nabla^{\boldsymbol{A}}$-compatible with $h^{-1}(\mathscr{D})$, then $\widetilde{\nabla}^{\boldsymbol{A}}\big(h^{-1}(\mathscr{D})\big) = h^{-1}\big(\widetilde{\nabla}^{\boldsymbol{B}}(\mathscr{D})\big)$.*
2. *For any $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\nabla^{\boldsymbol{A}}$-compatible with $\mathscr{C}$, then $h\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big) = \widetilde{\nabla}^{\boldsymbol{B}}\big(h(\mathscr{C})\big)$.*

Proof. 1. Assume that h is $\nabla^{\boldsymbol{A}}$-compatible with $h^{-1}(\mathscr{D})$, i.e., that $\mathrm{Ker}(h) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}\big(h^{-1}(\mathscr{D})\big)$. For each $G \in \mathscr{D}$, $\widetilde{\nabla}^{\boldsymbol{A}}\big(h^{-1}(\mathscr{D})\big) \subseteq \nabla^{\boldsymbol{A}}\big(h^{-1}(G)\big)$, and hence h is

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$\nabla^{\boldsymbol{A}}$-compatible with $h^{-1}(G)$. So, by coherence,

$$\widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}(\mathscr{D})) = \bigcap_{G\in\mathscr{D}} \nabla^{\boldsymbol{A}}\big(h^{-1}(G)\big) = \bigcap_{G\in\mathscr{D}} h^{-1}\big(\nabla^{\boldsymbol{B}}(G)\big)$$
$$= h^{-1}\Big(\bigcap_{G\in\mathscr{D}} \nabla^{\boldsymbol{B}}(G)\Big) = h^{-1}\big(\widetilde{\nabla}^{\boldsymbol{B}}(\mathscr{D})\big).$$

2. Assume now that h is $\nabla^{\boldsymbol{A}}$-compatible with $\mathscr{C}$. Thus, if $F\in\mathscr{C}$, then h is $\nabla^{\boldsymbol{A}}$-compatible with F, which implies that $h^{-1}\big(h(F)\big) = F$. Therefore, $h^{-1}\big(h(\mathscr{C})\big) = \mathscr{C}$, so that we can say that h is $\nabla^{\boldsymbol{A}}$-compatible with $h^{-1}\big(h(\mathscr{C})\big)$. Moreover, since $\operatorname{Ker}(h)\subseteq\nabla^{\boldsymbol{A}}(F)$ for each $F\in\mathscr{C}$, we also have that $\operatorname{Ker}(h)\subseteq\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$, which by Lemma 2.2.2 implies that $h^{-1}\big(h\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big)\big) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. Then we can apply point 1 to find that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\nabla}^{\boldsymbol{A}}\big(h^{-1}\big(h(\mathscr{C})\big)\big) = h^{-1}\big(\widetilde{\nabla}^{\boldsymbol{B}}\big(h(\mathscr{C})\big)\big)$ and then by surjectivity of h we conclude that $h\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big) = h\big(h^{-1}\big(\widetilde{\nabla}^{\boldsymbol{B}}\big(h(\mathscr{C})\big)\big)\big) = \widetilde{\nabla}^{\boldsymbol{B}}\big(h(\mathscr{C})\big)$. ⊣

Proposition 4.16. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators and $\theta\in\operatorname{Con}\boldsymbol{A}$. Then θ is $\nabla^{\boldsymbol{A}}$-full if and only if $\widetilde{\nabla}^{\boldsymbol{A}/\theta}\big(\big\{G\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\big(\nabla^{\boldsymbol{A}/\theta}(G)\big) = \nabla^{\boldsymbol{A}}\big(\pi^{-1}(G)\big)\big\}\big) = \operatorname{Id}_{\boldsymbol{A}/\theta}$.*

Proof. To simplify notation, put $\mathscr{D} := \big\{G\in\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\big(\nabla^{\boldsymbol{A}/\theta}(G)\big) = \nabla^{\boldsymbol{A}}\big(\pi^{-1}(G)\big)\big\}$. Then observe that by the comment after Definition 4.8, π is $\nabla^{\boldsymbol{A}}$-compatible with $\mathscr{D}$, and therefore, using Corollary 4.12 and Lemma 4.15.1, θ is $\nabla^{\boldsymbol{A}}$-full if and only if $\theta = \widetilde{\nabla}^{\boldsymbol{A}}\big(\nabla^{\boldsymbol{A}^{-1}}(\theta)\big) = \widetilde{\nabla}^{\boldsymbol{A}}\big(\pi^{-1}(\mathscr{D})\big) = \pi^{-1}\big(\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D})\big)$. This implies, by surjectivity of π, that $\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D}) = \pi\big(\pi^{-1}\big(\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D})\big)\big) = \pi(\theta) = \operatorname{Id}_{\boldsymbol{A}/\theta}$. Conversely, if $\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D}) = \operatorname{Id}_{\boldsymbol{A}/\theta}$, then $\theta = \pi^{-1}(\operatorname{Id}_{\boldsymbol{A}/\theta}) = \pi^{-1}\big(\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D})\big)$, which establishes that θ is $\nabla^{\boldsymbol{A}}$-full by the above consideration. ⊣

Observe that by instantiating the above results to the case of the Leibniz operator, which is a coherent family of $\mathcal{S}$-compatibility operators, we find the result proved directly in Proposition 3.7, namely that $\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta) = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$, because the Leibniz operator commutes with inverse images by all surjective homomorphisms, so that the family $\mathscr{D}$ in the above proof is just the family of all $\mathcal{S}$-filters in the quotient. We also find the result in Proposition 3.11, for the condition that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\theta}(\mathscr{D})$ is the identity amounts to saying that $\theta\in\operatorname{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.

Theorem 4.17 (General Correspondence Theorem). *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective $h\colon \boldsymbol{A}\to\boldsymbol{B}$ and every $F\in\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\nabla^{\boldsymbol{A}}$-compatible with F, then h induces an order isomorphism between $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and $[\![h(F)]\!]^{\nabla^{\boldsymbol{B}}}$, whose inverse is given by h^{-1}.*

Proof. Recall that from the assumption that h is $\nabla^{\boldsymbol{A}}$-compatible with F it follows that $h^{-1}\big(h(F)\big) = F$, and that $h(F)\in\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.

Take first any $F'\in[\![F]\!]^{\nabla^{\boldsymbol{A}}}$. Then $\operatorname{Ker}(h)\subseteq\nabla^{\boldsymbol{A}}(F)\subseteq\boldsymbol{\Omega}^{\boldsymbol{A}}(F')$ and hence by Lemma 2.4.3, $h^{-1}\big(h(F')\big) = F'$ and $h(F')\in\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Moreover, since h is both $\boldsymbol{\Omega}^{\boldsymbol{A}}$-compatible with F' and $\nabla^{\boldsymbol{A}}$-compatible with F and both $\boldsymbol{\Omega}$ and ∇ are coherent, we can apply Lemma 4.9 to both and obtain that $\nabla^{\boldsymbol{B}}\big(h(F)\big) = h\big(\nabla^{\boldsymbol{A}}(F)\big)\subseteq h\big(\boldsymbol{\Omega}^{\boldsymbol{A}}(F')\big) = \boldsymbol{\Omega}^{\boldsymbol{B}}\big(h(F')\big)$. This tells us that $h(F')\in[\![h(F)]\!]^{\nabla^{\boldsymbol{B}}}$.

Now take any $G\in[\![h(F)]\!]^{\nabla^{\boldsymbol{B}}}$, i.e., such that $\nabla^{\boldsymbol{B}}(h(F))\subseteq\boldsymbol{\Omega}^{\boldsymbol{B}}(G)$. We know that $h^{-1}(G)\in\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and that $h\big(h^{-1}(G)\big) = G$. Observe that h is $\nabla^{\boldsymbol{A}}$-compatible with

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$h^{-1}\big(h(F)\big)$, since this is F. Then, by coherence, we have

$$\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{A}}\big(h^{-1}\big(h(F)\big)\big) = h^{-1}\big(\nabla^{\boldsymbol{B}}\big(h(F)\big)\big) \subseteq h^{-1}\big(\boldsymbol{\Omega}^{\boldsymbol{B}}(G)\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(h^{-1}(G)\big).$$

This shows that $h^{-1}(G) \in [\![F]\!]^{\nabla^{\boldsymbol{A}}}$.

Thus, we have established that h induces a bijection between $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and $[\![h(F)]\!]^{\nabla^{\boldsymbol{B}}}$, whose inverse is given by h^{-1}. Since both maps are obviously order preserving, they are in fact order isomorphisms. ⊣

By considering the least elements of the two ∇-classes present in the theorem, we obtain a generalization of Corollary 3.8.

Corollary 4.18. *Under the assumptions of Theorem* 4.17, F *is a* $\nabla^{\boldsymbol{A}}$*-filter of* $\boldsymbol{A}$ *if and only if* $h(F)$ *is a* $\nabla^{\boldsymbol{B}}$*-filter of* $\boldsymbol{B}$.

We now wish to obtain an analogous Correspondence Theorem for the relativized operators $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$. In order to do that, we prove that the notion of coherence is preserved under relativization.

Proposition 4.19. *If* ∇ *is a coherent family of* $\mathcal{S}$*-compatibility operators, then the family* $\widetilde{\nabla}_{\mathcal{S}} = \{\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}} : \boldsymbol{A}$ *an algebra*$\}$ *is also a coherent family of* $\mathcal{S}$*-compatibility operators.*

Proof. First, from the definition of $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$ it immediately follows that if ∇ is a family of $\mathcal{S}$-compatibility operators, also $\widetilde{\nabla}_{\mathcal{S}}$ is. Now, to show the coherence of $\widetilde{\nabla}_{\mathcal{S}}$, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with $h^{-1}(G)$, i.e., such that $\operatorname{Ker}(h) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}\big(h^{-1}(G)\big)$. Consider any $F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}$, i.e., such that $h^{-1}(G) \subseteq F'$. Then $\operatorname{Ker}(h) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}(G)) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F') \subseteq \nabla^{\boldsymbol{A}}(F')$. Hence, h is $\nabla^{\boldsymbol{A}}$-compatible with F', and therefore $F' = h^{-1}(h(F'))$ and $h(F') \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. It follows by hypothesis that

$$\nabla^{\boldsymbol{A}}(F') = \nabla^{\boldsymbol{A}}\big(h^{-1}\big(h(F')\big)\big) = h^{-1}\big(\nabla^{\boldsymbol{B}}\big(h(F')\big)\big). \tag{9}$$

We claim that $h\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}\big) = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$: Let $F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be such that $h^{-1}(G) \subseteq F'$. We have already seen that under the present assumptions, $h(F') \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, and obviously $G = h\big(h^{-1}(G)\big) \subseteq h(F')$. Conversely, let $G' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ be such that $G \subseteq G'$. Then we know that $G' = h\big(h^{-1}(G')\big)$ and $h^{-1}(G') \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and moreover $h^{-1}(G) \subseteq h^{-1}(G')$. This proves the claim. Now, using (9), commutativity of h^{-1} with intersections, and the claim,

$$\begin{aligned}
\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}(G)) &= \bigcap\big\{\nabla^{\boldsymbol{A}}(F') : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}\big\}\\
&= \bigcap\big\{h^{-1}\big(\nabla^{\boldsymbol{B}}\big(h(F')\big)\big) : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}\big\}\\
&= h^{-1}\Big(\bigcap\big\{\nabla^{\boldsymbol{B}}\big(h(F')\big) : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}(G)}\big\}\Big)\\
&= h^{-1}\Big(\bigcap\big\{\nabla^{\boldsymbol{B}}(G') : G' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}\big\}\Big) = h^{-1}\big(\widetilde{\nabla}^{\boldsymbol{B}}_{\mathcal{S}}(G)\big).
\end{aligned}$$

This proves that the family $\widetilde{\nabla}_{\mathcal{S}}$ is coherent. ⊣

In particular, this shows that the Suszko operator is coherent. Finally, as a consequence of the General Correspondence Theorem 4.17 and of Proposition 4.19 we obtain the following.

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THEOREM 4.20. *Let ∇ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F, then h induces an order isomorphism between $[\![F]\!]^{\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}}$ and $[\![h(F)]\!]^{\widetilde{\nabla}^{\boldsymbol{B}}_{\mathcal{S}}}$, whose inverse is given by h^{-1}.*

4.2. Classes of algebras associated with a family of $\mathcal{S}$-operators. We saw in Lemma 2.8 that the classes of algebras usually associated with a logic through the Leibniz and the Suszko operators can be obtained either by considering reduced models, or by a process of reduction. By analogy, we can apply the two procedures to arbitrary families of $\mathcal{S}$-operators, but in principle the classes of algebras resulting from each procedure may be different.

DEFINITION 4.21. Let ∇ be a family of $\mathcal{S}$-operators. The following classes of algebras are associated with it, either directly,

$$\mathsf{Alg}^{\nabla}\mathcal{S} \coloneqq \mathbb{I}\big\{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F) : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\},$$
$$\mathsf{Alg}_{\nabla}\mathcal{S} \coloneqq \mathbb{I}\big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \nabla^{\boldsymbol{A}}(F) = \mathrm{Id}_A\big\},$$

or by applying these definitions to its relativization $\widetilde{\nabla}_{\mathcal{S}}$ (which is again a family of $\mathcal{S}$-operators):

$$\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} \coloneqq \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\},$$
$$\mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} \coloneqq \mathbb{I}\big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_A\big\}.$$

Our goal is to see that for coherent families of $\mathcal{S}$-compatibility operators the two classes of algebras associated with each operator coincide, and that in one of the two definitions the isomorphism operator is superfluous. The key point is to see that the "process of reduction" applied to an arbitrary model of $\mathcal{S}$ always produces a "reduced" model.

LEMMA 4.22. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\theta \in \mathrm{Con}\boldsymbol{A}$, if $\theta \subseteq \nabla^{\boldsymbol{A}}(F)$, then $\nabla^{\boldsymbol{A}/\theta}(F/\theta) = \nabla^{\boldsymbol{A}}(F)/\theta$. In particular $\nabla^{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)}\big(F/\nabla^{\boldsymbol{A}}(F)\big) = \mathrm{Id}_{A/\nabla^{\boldsymbol{A}}(F)}$.*

PROOF. Consider the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\theta$, which is surjective, and is $\nabla^{\boldsymbol{A}}$-compatible with F by the assumption. Then, by coherence and Lemma 4.9, $\nabla^{\boldsymbol{A}/\theta}\big(F/\theta\big) = \nabla^{\boldsymbol{A}/\theta}\big(\pi(F)\big) = \pi\big(\nabla^{\boldsymbol{A}}(F)\big) = \nabla^{\boldsymbol{A}}(F)/\theta$. Taking $\theta \coloneqq \nabla^{\boldsymbol{A}}(F)$, the assumption is trivially satisfied, therefore $\nabla^{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)}\big(F/\nabla^{\boldsymbol{A}}(F)\big) = \nabla^{\boldsymbol{A}}(F)/\nabla^{\boldsymbol{A}}(F) = \mathrm{Id}_{A/\nabla^{\boldsymbol{A}}(F)}$. ⊣

PROPOSITION 4.23. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then $\mathsf{Alg}^{\nabla}\mathcal{S} = \mathsf{Alg}_{\nabla}\mathcal{S}$. Moreover, the class $\{\boldsymbol{A}$: there is $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\nabla^{\boldsymbol{A}}(F) = \mathrm{Id}_A\}$ is closed under isomorphic copies.*

PROOF. The inclusion $\mathsf{Alg}_{\nabla}\mathcal{S} \subseteq \mathsf{Alg}^{\nabla}\mathcal{S}$ holds in general, because $\boldsymbol{A} \cong \boldsymbol{A}/\mathrm{Id}_A$, and the reverse inclusion is a consequence of Lemma 4.22. That the mentioned class is closed under isomorphisms, is a consequence of Corollary 4.10. ⊣

Moreover, we can apply Propositions 4.19 and 4.23 to $\widetilde{\nabla}_{\mathcal{S}}$.

COROLLARY 4.24. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} = \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S}$, and the class $\{\boldsymbol{A}$: there is $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_A\}$ is closed under isomorphic images.*

In the particular cases of the Leibniz operator and the Suszko operator, Proposition 4.23 and Corollary 4.24 yield the equalities $\mathsf{Alg}^{\widetilde{\boldsymbol{\Omega}}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\boldsymbol{\Omega}}}(\mathcal{S}) = \mathsf{Alg}^{*}\mathcal{S}$, and $\mathsf{Alg}^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$, respectively, which are already known (Lemma 2.8).

One can also consider analogous classes for the family $\widetilde{\nabla} = \{\widetilde{\nabla}^{\boldsymbol{A}} : \boldsymbol{A} \text{ an algebra}\}$ of the liftings of the operators $\nabla^{\boldsymbol{A}}$. However, in this case there are several ways to do it; let us choose one for the definition and see that the others yield equivalent results (in one case, under an additional assumption).

Definition 4.25. For any logic $\mathcal{S}$ and any family ∇ of $\mathcal{S}$-operators, we consider the following classes of algebras.

$$\mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S} := \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A} \text{ an algebra}, \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\},$$
$$\mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S} := \mathbb{I}\big\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A\big\}.$$

We are going to see that in fact we get no new classes. However, we first see that, exactly like what happens in the case of the Leibniz operator and its lifting the Tarski operator, each of these classes can be obtained in another way, namely by considering only the ∇-*full* g-models of $\mathcal{S}$, and in the second case also by considering the largest g-model (which is always ∇-full).

Lemma 4.26. *Let ∇ be a family of $\mathcal{S}$-operators. The following hold.*

1. $\mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S} = \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A}$ *an algebra*, $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ ∇-*full*$\big\}$.
2. $\mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S} = \mathbb{I}\big\{\boldsymbol{A}$: *there is a* ∇-*full* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *such that* $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A\big\} = \mathbb{I}\big\{\boldsymbol{A} : \widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A\big\}$.

Proof. 1. The inclusion $\supseteq$ is obvious. To see $\subseteq$, observe that given any $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by the Galois connection (Proposition 3.3 and related results) the congruence $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ is a ∇-full congruence and hence there is some ∇-full $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{D}) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$; therefore, $\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{D}) \in \mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S}$.
2. The first equality is proved by the same argument as point 1. As to the second, $\supseteq$ is again obvious, and $\subseteq$ is a consequence of $\widetilde{\nabla}^{\boldsymbol{A}}$ being order reversing: if $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A$ for some $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then also $\widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A$. ⊣

In general, one of the classes associated with $\widetilde{\nabla}$ is already equal to one of those associated with $\widetilde{\nabla}_{\mathcal{S}}$, and the other one is almost so.

Proposition 4.27. *If ∇ is a family of $\mathcal{S}$-operators, then $\mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S} = \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S}$ and $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} \subseteq \mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S}$.*

Proof. By definition, for each $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}\big)$. From this it follows that $\mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} \subseteq \mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S}$ and that $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} \subseteq \mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S}$. To see the reverse inclusion in the first case, assume that $\boldsymbol{A} \in \mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S}$. By Lemma 4.26.2, $\widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A$. But, if we put $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F_0) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F_0}\big) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A$. Therefore, $\boldsymbol{A} \in \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S}$. ⊣

The proofs of the next two results are completely analogous, modulo Lemma 4.15, to those of Lemma 4.22 and Proposition 4.23, respectively; the last equality in Proposition 4.29 completes Lemma 4.26 and is proved using the first part, and the last equality in Lemma 4.22.2.

Lemma 4.28. *If ∇ is a coherent family of $\mathcal{S}$-compatibility operators, then for every $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\nabla}^{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})}(\mathscr{C}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})) = \mathrm{Id}_{A/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})}$.*

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PROPOSITION 4.29. *If* ∇ *is a coherent family of* $\mathcal{S}$*-compatibility operators, then* $\mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S} = \mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S}$. *Moreover, the class* $\{\boldsymbol{A} :$ *there is* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *such that* $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \mathrm{Id}_A\}$ *is closed under isomorphic images, and* $\mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S} = \mathbb{I}\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) :$ $\boldsymbol{A}$ *an algebra*$\}$.

Merging this with Corollary 4.24 and Proposition 4.27 we find what we announced.

COROLLARY 4.30. *If* ∇ *is a coherent family of* $\mathcal{S}$*-compatibility operators, then* $\mathsf{Alg}^{\widetilde{\nabla}}\mathcal{S} = \mathsf{Alg}_{\widetilde{\nabla}}\mathcal{S} = \mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S} = \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}\mathcal{S}$.

Since, by Proposition 4.19, $\widetilde{\nabla}_{\mathcal{S}}$ is a coherent family of $\mathcal{S}$-compatibility operators when ∇ is, one might try to apply to $\widetilde{\nabla}_{\mathcal{S}}$ everything that has been done for ∇; that is, to consider the relativization of $\widetilde{\nabla}_{\mathcal{S}}$ to $\mathcal{S}$, its lifting to the power sets, the associated classes of algebras, etc. However, since $\widetilde{\nabla}_{\mathcal{S}}$ is order preserving, in view of Lemma 3.2.5 the relativization of $\widetilde{\nabla}_{\mathcal{S}}$ is $\widetilde{\nabla}_{\mathcal{S}}$ itself, and nothing new would be obtained.

§5. Applications to the Leibniz and the Suszko operators.

5.1. The Leibniz operator as an $\mathcal{S}$**-compatibility operator.** In Section 3.2 we looked at the Leibniz operator as an $\mathcal{S}$-operator. In this subsection we shall consider it as an $\mathcal{S}$-compatibility operator, and instantiate the results of Section 4 concerning it and its lifting (the Tarski operator); the results concerning its relativization (the Suszko operator) will be found in Section 5.2. In particular, the $\boldsymbol{\Omega}^{\boldsymbol{A}}$-class of an $\mathcal{S}$-filter and its least element will play an important rôle among the full g-models and the $\mathcal{S}$-filters, respectively.

When instantiating the general notions and constructions of previous sections for $\nabla = \boldsymbol{\Omega}$ we find some familiar ones, which have already well-settled notations and terminology. It will hence be practical to make some changes in order to match them; for instance, we write $(\ )^*$ instead of $(\ )^{\boldsymbol{\Omega}}$, so that the class $\mathsf{Alg}^{\boldsymbol{\Omega}}\mathcal{S}$ becomes the familiar class $\mathsf{Alg}^*\mathcal{S}$ from (5). Recall that, by Definition 3.14, the $\boldsymbol{\Omega}^{\boldsymbol{A}}$-class of F is defined by

$$[\![F]\!]^* := \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\};$$

here we shall call it the *Leibniz class of* F. By Definition 3.16, F^* denotes the least element of the Leibniz class $[\![F]\!]^*$; here we shall call this element *the Leibniz filter of* F. We say that F is a *Leibniz filter* if $F = F^*$, and we denote the set of all Leibniz filters of $\boldsymbol{A}$ by $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$.

The first observation worth mentioning is that every Leibniz class is a closure system. In fact, it is a full closure system; these facts should come with no surprise, given Proposition 3.15, and we collect them together with a couple of related properties in the next proposition for later reference.

PROPOSITION 5.1. *For every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, *the Leibniz class* $[\![F]\!]^*$ *is full, hence it is a closure system on* A; *it is such that* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^*\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$; *and it is the largest* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, *and the only full, such that* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

PROOF. By Proposition 3.15, taking $\nabla^{\boldsymbol{A}} = \boldsymbol{\Omega}^{\boldsymbol{A}}$, it follows that $[\![F]\!]^* = \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^*)\big)$, and that $[\![F]\!]^*$ is full. By Lemma 4.2.1, $F \in [\![F]\!]^*$, which together with the previous equality implies that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^*) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Moreover,

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by Corollary 3.4.3, the map $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}} \circ \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}$ is a closure on Con$\boldsymbol{A}$, so that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big(\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\boldsymbol{\Omega}^{\boldsymbol{A}}(F))\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\llbracket F \rrbracket^*)$. This establishes the second equality. The fact that $\llbracket F \rrbracket^*$ is full implies, by Proposition 3.6, that it is the largest $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big(\llbracket F \rrbracket^*\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Finally, by Corollary 3.12, the Tarski operator is injective on full g-models, and this proves the uniqueness in the last part of the statement. ⊣

We postpone the question of which full g-models are given by Leibniz classes to Proposition 6.1. For the time being, let us turn our attention to the least elements in these classes.

The term "Leibniz filter" was first used in [17, Definition 1], within the scope of protoalgebraic logics (recall, logics for which the Leibniz operator is order preserving), to denote the least elements of the sets of the form

$$[F] \coloneqq \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\},$$

for $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Observe that these sets are the equivalence classes of the kernel of the Leibniz operator, which is an equivalence relation; this somehow explains why the term "class" has been used later on for the related notion of the "Leibniz class" $\llbracket F \rrbracket^*$. Let us clarify the relations between the old and the new notions. Clearly $[F] \subseteq \llbracket F \rrbracket^*$. Our present definition of Leibniz filter generalizes the former one, as we next prove, in the sense that both notions coincide for protoalgebraic logics, and furthermore, for arbitrary logics, every Leibniz filter according to Definition 3.16 is a Leibniz filter according to [17].

Lemma 5.2. *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. $F^* \subseteq \bigcap [F] \subseteq F$;
2. *if $F = F^*$, then $F = \bigcap [F]$; and*
3. *if $\mathcal{S}$ is protoalgebraic, then $F = F^*$ (i.e., F is a Leibniz filter) if and only if $F = \bigcap [F]$.*

Proof. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $F \in [F]$, it holds $\bigcap [F] \subseteq F$. Moreover, it is clear that $[F] \subseteq \llbracket F \rrbracket^*$. Therefore, $F^* = \bigcap \llbracket F \rrbracket^* \subseteq \bigcap [F]$. From (i), (ii) follows immediately. Now to prove (iii) assume that $\mathcal{S}$ is protoalgebraic and $F = \bigcap [F]$. Then, since $F^* \subseteq F$ it follows by order preservation of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F^*) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, and since $F^* \in \llbracket F \rrbracket^*$, it must also hold $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. So, $F^* \in [F]$, and hence $F = \bigcap [F] \subseteq F^*$. ⊣

We warn the reader that a proposal has been made in [19, p. 177] of applying the definition of Leibniz filter for protoalgebraic logics from [17] to arbitrary logics; however, this does not yield our present notion, because while in general $[F]$ needs not have a least element, $\llbracket F \rrbracket^*$ always has one.

The first particular properties of Leibniz filters and Leibniz classes follow easily from the fact that the Leibniz operator is the *largest* of all $\mathcal{S}$-compatibility operators.

Lemma 5.3. *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. Then for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. $\llbracket F \rrbracket^* \subseteq \llbracket F \rrbracket^{\nabla^{\boldsymbol{A}}}$;
2. $F^{\nabla^{\boldsymbol{A}}} \subseteq F^*$; *and*
3. *every $\nabla^{\boldsymbol{A}}$-filter is a Leibniz filter.*

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PROOF. 1 follows from the definitions plus the fact that $\nabla^{A}(F) \subseteq \boldsymbol{\Omega}^{A}(F)$; 2 follows from 1 and the definitions, and 3 follows from 2 plus the fact that $F^{*} \subseteq F$ by Lemma 5.2.1. ⊣

The terminological choice of calling F^{*} "the Leibniz filter of F" is justified by the fact that every such filter is indeed a Leibniz filter.

PROPOSITION 5.4. *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, F^{*} is a Leibniz filter of $\boldsymbol{A}$.*

PROOF. The inclusion $(F^{*})^{*} \subseteq F^{*}$ follows by Lemma 5.2.1. As for the converse inclusion, since $F^{*} \in [\![F]\!]^{*}$, it follows $[\![F^{*}]\!]^{*} \subseteq [\![F]\!]^{*}$. Hence, $F^{*} = \bigcap [\![F]\!]^{*} \subseteq \bigcap [\![F^{*}]\!]^{*} = (F^{*})^{*}$. ⊣

Rephrasing Proposition 3.6 of [18] in terms of the present notion, and taking Lemma 5.2.3 into account, we would find that for a protoalgebraic logic $\mathcal{S}$, an $\mathcal{S}$-filter is a Leibniz filter if and only if it is the least element of some full g-model of $\mathcal{S}$. We can now see that this remains true for arbitrary logics if we replace the notion of [17, 18] by the present one.

THEOREM 5.5. *An $\mathcal{S}$-filter F of $\boldsymbol{A}$ is a Leibniz filter if and only if there exists a full g-model $\langle \boldsymbol{A}, \mathscr{C} \rangle$ of $\mathcal{S}$ such that $F = \bigcap \mathscr{C}$.*

PROOF. Suppose $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Leibniz filter. It is, by definition, the least element of its Leibniz class, which we have seen to be full in Proposition 5.1. Conversely, suppose $F = \bigcap \mathscr{C}$ and $\langle \boldsymbol{A}, \mathscr{C} \rangle$ is a full g-model of $\mathcal{S}$. Since $\bigcap \mathscr{C} \in \mathscr{C}$, it holds $\widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{A}(F)$. Hence, $[\![F]\!]^{*} = \boldsymbol{\Omega}^{A^{-1}}(\boldsymbol{\Omega}^{A}(F)) \subseteq \boldsymbol{\Omega}^{A^{-1}}(\widetilde{\boldsymbol{\Omega}}^{A}(\mathscr{C})) = \mathscr{C}$, where the last equality follows by Proposition 3.10. Thus, $F = \bigcap \mathscr{C} \subseteq \bigcap [\![F]\!]^{*} = F^{*}$. Since the converse inclusion always holds, it follows $F = F^{*}$, i.e., F is a Leibniz filter. ⊣

Instantiating Corollary 4.3 for the Leibniz operator we obtain the next proposition; one can see that it generalizes [17, Proposition 10], if one takes the result in Lemma 5.2.3 into account.

PROPOSITION 5.6. *A filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Leibniz filter of $\boldsymbol{A}$ if and only if $F/\boldsymbol{\Omega}^{A}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{A}(F)$.*

Finally, we apply Theorem 4.17 to the Leibniz operator.

THEOREM 5.7 (Correspondence Theorem for Leibniz classes). *For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\boldsymbol{\Omega}^{A}$-compatible with F, then h induces an order isomorphism between $[\![F]\!]^{*}$ and $[\![h(F)]\!]^{*}$, whose inverse is given by h^{-1}. Moreover, for each $G \in [\![F]\!]^{*}$, h induces an order isomorphism between $[G]$ and $[h(G)]$.*

PROOF. By Proposition 2.5.1, the Leibniz operator is a coherent family of $\mathcal{S}$-compatibility operators, therefore we can apply Theorem 4.17 to it, and obtain the first part of the statement. For the second part, take any $G, H \in [\![F]\!]^{*}$; note that from $G \in [\![F]\!]^{*}$ it follows that $[G] \subseteq [\![F]\!]^{*}$. By the established isomorphism, $h^{-1}\big(h(G)\big) = G$ and $h^{-1}\big(h(H)\big) = H$. Now, using Proposition 2.5.1 and the surjectivity of h,

$$\begin{aligned}
\boldsymbol{\Omega}^{A}(H) = \boldsymbol{\Omega}^{A}(G) \ \text{ iff } \ & \boldsymbol{\Omega}^{A}\big(h^{-1}\big(h(H)\big)\big) = \boldsymbol{\Omega}^{A}\big(h^{-1}\big(h(G)\big)\big) \\
\text{iff } \ & h^{-1}\big(\boldsymbol{\Omega}^{B}\big(h(H)\big)\big) = h^{-1}\big(\boldsymbol{\Omega}^{B}\big(h(G)\big)\big) \\
\text{iff } \ & \boldsymbol{\Omega}^{B}\big(h(H)\big) = \boldsymbol{\Omega}^{B}\big(h(G)\big),
\end{aligned}$$

which shows that $H \in [G]$ if and only if $h(F) \in [h(G)]$. Thus, the order isomorphism induced by h between $[\![F]\!]^*$ and $[\![h(F)]\!]^*$ restricts to one between $[G]$ and $[h(G)]$. ⊣

It is not difficult to see that $[\![F]\!]^* = \bigcup_{G \in [\![F]\!]^*} [G]$, that is, the sets $[G]$ divide the Leibniz class $[\![F]\!]^*$ into disjoint "layers" according to the value of the Leibniz operator. Thus, the second part of Theorem 5.7 is telling us that the isomorphism between the two Leibniz classes is the disjoint union of isomorphisms, one for each corresponding pair of "layers".

Theorem 5.7 generalizes and strengthens the well-known Correspondence Theorem for protoalgebraic logics, as formulated in [2, Corollary 7.7], and its strengthening given in [17, Corollary 9]: it extends its scope to arbitrary logics, and it establishes an order isomorphism between larger sets of filters; in Theorem 6.17 we shall check how the theorem in [2] follows from Theorem 5.7.

Corollary 5.8. *Under the assumptions of Theorem* 5.7, *F is a Leibniz filter of $\boldsymbol{A}$ if and only if $h(F)$ is a Leibniz filter of $\boldsymbol{B}$.*

5.2. The Suszko operator as an $\mathcal{S}$-compatibility operator. Now we do for the Suszko operator what we did in Section 5.1 for the Leibniz operator. Since the notation $\mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$ introduced in (6) on page 428 for what would be our $\mathsf{Alg}^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}\mathcal{S}$ is already well settled in abstract algebraic logic, we shall change all superscripts $(\ )^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}$ to $(\ )^{\mathrm{Su}}$ instead. Moreover, recall that, by Definition 3.14, the $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$-class of F is defined by

$$[\![F]\!]^{\mathrm{Su}} := \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\};$$

we shall call it *the Suszko class of F*. By Definition 3.16, F^{Su} denotes the least element of the Suszko class $[\![F]\!]^{\mathrm{Su}}$. We say that F is a *Suszko filter* if $F = F^{\mathrm{Su}}$, and we denote the set of all Suszko filters of $\boldsymbol{A}$ by $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.

In general, by Lemma 5.3, the Suszko filters of a logic are Leibniz filters. As we will see, Suszko filters and Leibniz filters always coincide in protoalgebraic logics, but in these logics there might be logical filters that are not Leibniz. This situation can also be found outside the class of protoalgebraic logics. For instance, in Positive Modal Logic [9, 24], the logical filters of a positive modal algebra are its lattice filters, whereas the Suszko filters and the Leibniz filters are the lattice filters closed under the interpretation of $\Box$. However, in general, the inclusions between the sets of Suszko filters, Leibniz filters, and $\mathcal{S}$-filters may be proper. For instance, in the logic preserving degrees of truth with respect to the class of MV-algebras [4,16], the logical filters of an MV-algebra are the lattice filters, while the Leibniz filters are exactly the implicative filters, and in many MV-algebras the set of Suszko filters is properly included in the set of Leibniz filters. These results will be published elsewhere. There are also logics where the three classes of filters on any algebra coincide; for example, this holds for truth-equational logics, as we will see in Theorem 6.10.

Similarly to Proposition 5.1, one can prove the following.

Proposition 5.9. *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the Suszko class $[\![F]\!]^{\mathrm{Su}}$ is full, hence it is a closure system on A; it is such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)$; and it is the largest $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and the only full, such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)$.*

That the Suszko operator is the largest order preserving $\mathcal{S}$-compatibility operator was proved in [6, Theorem 1.8]. A short proof of this fact can be given in our setting. Indeed, let $\nabla^{\boldsymbol{A}}$ be an order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. By compatibility, it follows that $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ for every $F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A})$, and by Lemma 3.2.5 we have $\nabla^{\boldsymbol{A}} = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$; therefore, $\nabla^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. As straightforward consequences of this fact and of Lemmas 3.18, 4.2, and 5.3 we have the following two lemmas; the first one is completely parallel to Lemma 5.3.

Lemma 5.10. *Let ∇ be an order preserving $\mathcal{S}$-compatibility operator. Then for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. $[\![F]\!]^{\mathrm{Su}} \subseteq [\![F]\!]^{\nabla^{\boldsymbol{A}}}$;
2. $F^{\nabla^{\boldsymbol{A}}} \subseteq F^{\mathrm{Su}}$;
3. *every $\nabla^{\boldsymbol{A}}$-filter is a Suszko filter.*

Lemma 5.11. *Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then,*

1. $F^{\mathrm{Su}} \subseteq F^{*} \subseteq F$;
2. *every Suszko filter is a Leibniz filter*;
3. *if $F \subseteq G$, then $[\![G]\!]^{\mathrm{Su}} \subseteq [\![F]\!]^{\mathrm{Su}}$ and $F^{\mathrm{Su}} \subseteq G^{\mathrm{Su}}$*;
4. $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F} \subseteq [\![F]\!]^{\mathrm{Su}} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F^{\mathrm{Su}}}$;
5. $[\![F]\!]^{\mathrm{Su}} \subseteq [\![F^{\mathrm{Su}}]\!]^{\mathrm{Su}}$; *and*
6. $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ *if and only if $F = F^{\mathrm{Su}}$, i.e., if and only if F is a Suszko filter.*

The converse of the implication in item 2 of Lemma 5.11 is false, as witnessed by Example 6.21. On a related issue, you may have noticed that we are not calling F^{Su} "the Suszko filter of F". The reason is that we don't have for Suszko filters the nice property of Proposition 5.4 for Leibniz filters: in general, for an $\mathcal{S}$-filter F, the $\mathcal{S}$-filter F^{Su} need not be itself a Suszko filter; this is shown in the following example, suggested to us by our colleague Tommaso Moraschini.

Example 5.12. Consider the language $\mathcal{L} = \langle \Box, \Diamond, c_1, c_2, c_3, \top \rangle$, where $\Box$ and $\Diamond$ are unary function symbols and $c_1, c_2, c_3, \top$ are constant symbols. Consider the set $A = \{a, b, c, d, 1\}$ and the $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, a, b, d, 1 \rangle$, where the unary operations $\Box^{\boldsymbol{A}}$ and $\Diamond^{\boldsymbol{A}}$ are given by the table below. Consider also the logic $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ defined by the calculus with axiom and rules displayed below (x is a variable).

	$\Box^{\boldsymbol{A}}$	$\Diamond^{\boldsymbol{A}}$
a	a	c
b	b	1
c	d	d
d	d	1
1	a	d

Axiom: $\top$
Rule 1: $c_1, c_2 \vdash_{\mathcal{S}} x$
Rule 2: $c_2, c_3 \vdash_{\mathcal{S}} x$

Fact 1. Clearly, the proper $\mathcal{S}$-filters of $\boldsymbol{A}$ are the subsets containing 1, not containing a, b simultaneously, and not containing b, d simultaneously. In particular, the set $F := \{1, b, c\}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$.

Fact 2. $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F} = \{F, A\}$, because the only proper subsets of A containing F are $\{1, a, b, c\}$ and $\{1, b, c, d\}$, but neither is an $\mathcal{S}$-filter of $\boldsymbol{A}$ by the observation in Fact 1.

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FACT 3. $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \big\{\{1,c\},\{a,d\},\{b\}\big\}$, where for simplicity a congruence is described by its partition. One can check by hand that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \big\{\{1,c\},\{a,d\},\{b\}\big\}$. The other equality follows by Fact 2, which implies that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

FACT 4. $F^{\mathrm{Su}} = \{1,c\}$. To see this, first observe that $[\![F]\!]^{\mathrm{Su}} = [\![F]\!]^{*} = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$, which is a direct consequence of Fact 3. But to say that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ is to say that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with G or, by Lemma 2.1, that G is a union of blocks of $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Using the description of $\mathcal{S}$-filters in Fact 1 and the description of the blocks of $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ in Fact 3, we conclude that $[\![F]\!]^{\mathrm{Su}} = \big\{\{1,c\}, F, \{1,a,c,d\}, A\big\}$. From this it follows that $F^{\mathrm{Su}} = \{1,c\}$, as claimed.

FACT 5. $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F^{\mathrm{Su}}) = \mathrm{Id}_A$. This is because F and $\{1,a,c\}$ are two $\mathcal{S}$-filters of $\boldsymbol{A}$, which contain $F^{\mathrm{Su}} = \{1,c\}$, and it is easy to check that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\{1,a,c\}\big) = \mathrm{Id}_A$, using just compatibility arguments and the fact that for any congruence $\equiv$ of this algebra, $1 \equiv c$ if and only if $a \equiv d$.

FACT 6. $(F^{\mathrm{Su}})^{\mathrm{Su}} = \{1\}$. It follows by Fact 5 that $[\![F^{\mathrm{Su}}]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Thus, $(F^{\mathrm{Su}})^{\mathrm{Su}} = \min [\![F^{\mathrm{Su}}]\!]^{\mathrm{Su}} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{1\}$.

We conclude that $F^{\mathrm{Su}} \neq (F^{\mathrm{Su}})^{\mathrm{Su}}$. That is, F^{Su} is not a Suszko filter of $\boldsymbol{A}$. ⊣

We have seen in Theorem 5.5 that Leibniz filters are precisely the least elements of full g-models. Since every Suszko filter is a Leibniz filter, in particular, they are also least elements of full g-models. Now we characterize them as the least elements of the full g-models that are up-sets; but the proof actually shows that there is only one candidate for such a g-model, namely the principal up-set of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ determined by the filter itself.

THEOREM 5.13. *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the following conditions are equivalent.*

(i) *F is a Suszko filter of $\boldsymbol{A}$.*
(ii) *$\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\rangle$ is a full g-model of $\mathcal{S}$.*
(iii) *$F = \bigcap \mathscr{C}$, for some full up-set $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

The principal up-set $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is the only $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ that satisfies (iii).

PROOF. (i)⇒(ii): This follows from Lemma 5.11.6, which tells us that $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, and Proposition 5.9, which tells us that $[\![F]\!]^{\mathrm{Su}}$ is always full.

(ii)⇒(iii): This is because $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is an up-set and $F = \bigcap(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$.

(iii)⇒(i): On the one hand, observe that $F \in \mathscr{C}$ because $\mathscr{C}$, being full, is a closure system. This implies that $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq \mathscr{C}$, because $\mathscr{C}$ is an up-set. On the other hand, since $F = \bigcap \mathscr{C}$ by assumption, clearly $\mathscr{C} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. That is, $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, which proves the final assertion. Moreover, since $\mathscr{C}$ is full, $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$. But, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. So, $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = [\![F]\!]^{\mathrm{Su}}$, and therefore $F = \bigcap [\![F]\!]^{\mathrm{Su}}$ is a Suszko filter. ⊣

Note that this result does not exclude the possibility that a Suszko filter is at the same time the least element of another full g-model, provided this one is not an up-set. Actually, every Suszko filter is a Leibniz filter, and hence it is also the least element of its Leibniz class, which is a full g-model.

We now instantiate Corollary 4.3 for the Suszko operator.

Proposition 5.14. *A filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Suszko filter of $\boldsymbol{A}$ if and only if $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$.*

We have already seen that from Proposition 4.19 it follows that the Suszko operator $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}} := \{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \boldsymbol{A}$ an algebra$\}$ is a coherent family of $\mathcal{S}$-compatible operators. Therefore, Theorem 4.20 gives the following.

Theorem 5.15 (Correspondence Theorem for Suszko classes). *For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if h is $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F, then h induces an order isomorphism between $[\![F]\!]^{\mathrm{Su}}$ and $[\![h(F)]\!]^{\mathrm{Su}}$, whose inverse is given by h^{-1}.*

We can now check that Theorem 5.15 strengthens Corollary 2.7 of [6], which states (in the present terminology) that h is an isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{h(F)}$ under the assumption that h is a surjective and "deductive" matrix homomorphism between $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, h(F)\rangle$. We have already seen, after Definition 4.7, that the property of being deductive amounts to saying that h is $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F, viewed as an algebraic homomorphism. Thus, compared with [6, Corollary 2.7], Theorem 5.15 extends the isomorphism to the whole Suszko classes $[\![F]\!]^{\mathrm{Su}}$ and $[\![h(F)]\!]^{\mathrm{Su}}$, which contain $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{h(F)}$ respectively, by Lemma 5.11.4.

Corollary 5.16. *Under the assumptions of Theorem 5.15, F is a Suszko filter of $\boldsymbol{A}$ if and only if $h(F)$ is a Suszko filter of $\boldsymbol{B}$.*

§6. Applications to the Leibniz hierarchy. In this last section we apply the results on the Leibniz and Suszko operators established in Section 5 to obtain several characterizations of the main classes of logics belonging to the Leibniz hierarchy.

6.1. Characterizations in terms of Leibniz and Suszko filters, and in terms of full g-models. We start our discussion by addressing the issue, postponed after Proposition 5.1, of which full g-models have the form of a Leibniz class.

Proposition 6.1. *Let $\langle \boldsymbol{A}, \mathscr{C}\rangle$ be a full g-model of $\mathcal{S}$. The following conditions are equivalent*:

(i) $\mathscr{C} = [\![F]\!]^*$, *for some* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.
(ii) $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*\mathcal{S}$.

Proof. Suppose $\mathscr{C} = [\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^*) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by Proposition 5.1, and therefore $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*\mathcal{S}$. Conversely, suppose $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*\mathcal{S}$, and put $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$. By the assumption, there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ such that $\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \mathrm{Id}_B$. This implies that $[\![G]\!]^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Now, let $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ be the canonical projection. Since $\mathscr{C}$ is full by assumption, $\mathscr{C} = \pi^{-1}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})$. Thus, $\mathscr{C} = \pi^{-1}\big([\![G]\!]^*\big)$. Finally, $\mathrm{Ker}(\pi) = \pi^{-1}(\mathrm{Id}_B) = \pi^{-1}\big(\boldsymbol{\Omega}^{\boldsymbol{B}}(G)\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\pi^{-1}(G)\big)$, therefore π is $\boldsymbol{\Omega}^{\boldsymbol{A}}$-compatible with $\pi^{-1}(G)$. Now we can apply the Correspondence Theorem 5.7 for Leibniz classes and conclude that $\mathscr{C} = \pi^{-1}\big([\![G]\!]^*\big) = [\![\pi^{-1}(G)]\!]^*$. That is, $F := \pi^{-1}(G) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ witnesses the desired property. ⊣

In general, it does not happen that every full g-model of a logic $\mathcal{S}$ will be of the form $[\![F]\!]^*$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, because there are logics $\mathcal{S}$ for which $\mathsf{Alg}^*\mathcal{S}$ is properly included in $\mathsf{Alg}\mathcal{S}$. That it does happen turns out to be equivalent to an interesting property.

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PROPOSITION 6.2. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$.

(ii) *For every $\boldsymbol{A}$, the family of full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ is $\{[\![F]\!]^* : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\}$.*

(iii) *For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ there is $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$.*

PROOF. (i)⇒(ii): If $\mathscr{C}$ is full, then $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}\mathcal{S}$. From the assumption it follows that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*\mathcal{S}$, and from Proposition 6.1 that $\mathscr{C} = [\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

(ii)⇒(iii): The assumption tells us that every full g-model of $\mathcal{S}$ is of the form of some Leibniz class. In particular, since Suszko classes are full, we have that for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$. But this implies that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}\big([\![G]\!]^*\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, as desired.

(iii)⇒(i): Let $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. Since $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$, there is $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Id}_A$. It follows by hypothesis that there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \mathrm{Id}_A$. Thus, $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}$. This establishes that $\mathsf{Alg}\mathcal{S} \subseteq \mathsf{Alg}^*\mathcal{S}$; the converse inclusion always holds. ⊣

The class of logics $\mathcal{S}$ such that $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$ is larger than it may seem at first sight. It is shown in [18] that it includes all protoalgebraic logics, but also all logics $\mathcal{S}$ such that $\mathsf{Alg}^*\mathcal{S}$ is a quasivariety, and some of these are known to be nonprotoalgebraic, such as the weak relevance logic WR [18, Example 5.4.1]. Nevertheless, we will often use the fact that protoalgebraicity implies $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$; for the sake of completeness, we will prove it here as a consequence of the following fact, originally proved in [6], which is fundamental to our paper.

PROPOSITION 6.3. *A logic $\mathcal{S}$ is protoalgebraic if and only if the Leibniz and the Suszko operators coincide, that is, if and only if for all $\boldsymbol{A}$ and all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.*

PROOF. Recall that we adopted as a definition that $\mathcal{S}$ is protoalgebraic if the Leibniz operator is order preserving on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ for every $\boldsymbol{A}$. Now, by Lemma 5, the Leibniz operator is order preserving if and only if it coincides with the Suszko operator. Therefore, when $\mathcal{S}$ is protoalgebraic both operators coincide. The converse implication follows immediately because the Suszko operator is always order preserving. ⊣

As a consequence, when dealing with protoalgebraic logics, all pairs of notions associated with each of the operators coincide, such as those of Leibniz and Suszko classes, those of Leibniz and Suszko filters, and the associated classes of algebras; in particular, Proposition 6.3 directly implies that $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$. From this, using Proposition 6.2, we obtain the following characterization of the full g-models of protoalgebraic logics in terms of Leibniz classes:

COROLLARY 6.4. *If a logic $\mathcal{S}$ is protoalgebraic, then $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$ and the full g-models of $\mathcal{S}$ are the g-models of $\mathcal{S}$ the form $[\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and some algebra $\boldsymbol{A}$.*

As observed above, the converses of these implications do not hold in general, but in Proposition 6.11 we will see that they do under stronger assumptions. It is

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nevertheless possible to characterize the protoalgebraicity of a logic by the form of its full g-models:

THEOREM 6.5. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is protoalgebraic.*
(ii) *Every full g-model of $\mathcal{S}$ is an up-set; that is, every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some $\mathcal{S}$-filter F of some algebra $\boldsymbol{A}$.*
(iii) *Every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some Suszko $\mathcal{S}$-filter F of some algebra $\boldsymbol{A}$.*
(iv) *$[\![F]\!]^* = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F^*}$, for every $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every algebra $\boldsymbol{A}$.*

PROOF. (i)⇒(ii): Let $\langle \boldsymbol{A}, \mathscr{C}\rangle$ be a full g-model of $\mathcal{S}$. So, $\mathscr{C} = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$. Since by the assumption the Leibniz operator is order preserving, it trivially follows that $\mathscr{C}$ is an up-set. Since $\mathscr{C}$ is a closure system, it is in fact of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some $\mathcal{S}$-filter F of $\boldsymbol{A}$, namely its intersection.

(ii)⇒(iii) is a direct consequence of Theorem 5.13.

(iii)⇒(iv): In general, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $[\![F]\!]^*$ is a full g-model of $\mathcal{S}$. Therefore by assumption $[\![F]\!]^* = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G$, where G is the minimum of $[\![F]\!]^*$; but this is F^* by definition.

(iv)⇒(i): Let $\boldsymbol{A}$ be an algebra and let $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Then, $F^* \subseteq F \subseteq G$. It follows by hypothesis that $G \in [\![F]\!]^*$. So, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Thus, the Leibniz operator is order preserving on every $\boldsymbol{A}$, and this shows that $\mathcal{S}$ is protoalgebraic. ⊣

Notice that we can replace Suszko filter by Leibniz filter in condition (iii). The preceding result extends [18, Theorem 3.4], which proves only the equivalence between items (i) and (ii). Moreover, this characterization of protoalgebraic logics as those whose full g-models are up-sets implies the following enhancement of Proposition 6.3: the coincidence of two more Leibniz- and Suszko-related notions also characterizes protoalgebraicity:

PROPOSITION 6.6. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is protoalgebraic.*
(ii) *The full g-models of $\mathcal{S}$ coincide with its Suszko-full g-models.*
(iii) *$[\![F]\!]^* = [\![F]\!]^{\mathrm{Su}}$ for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.*

PROOF. The implications from (i) to (ii) and to (iii) are a direct consequence of Proposition 6.3. Now assume (ii). Since every Suszko-full g-model is always an up-set (Lemma 4.2.4), the condition implies that the full g-models of $\mathcal{S}$ are all up-sets, and by Theorem 6.5 this implies that $\mathcal{S}$ is protoalgebraic. Finally, assume (iii) and consider any $\boldsymbol{A}$ and any $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Then by Lemma 5.11.4, $G \in [\![F]\!]^{\mathrm{Su}} = [\![F]\!]^*$, which implies that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. This shows that the Leibniz operator is order preserving on every $\boldsymbol{A}$, which implies that $\mathcal{S}$ is protoalgebraic. ⊣

By contrast, the coincidence of the following Leibniz- and Suszko-related notions does not characterize protoalgebraicity.

- $F^* = F^{\mathrm{Su}}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.
- F is a Suszko filter if and only if F is a Leibniz filter, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.

The reason is that these two properties hold (vacuously) in all truth-equational logics, because, as we will soon see in Theorem 6.10, in them all filters are Suszko filters, and hence also Leibniz filters.

In order to find a characterization of truth-equational logics in a spirit similar (but dual) to Theorem 6.5, we first need to take a closer look at Suszko classes, specially at those associated with Suszko filters. Note that, given $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the inclusion $[\![F]\!]^{\mathrm{Su}} \supseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ always holds, by Lemma 5.11.4. So, by Lemma 5.11.6, an $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Suszko filter when the converse inclusion holds; that is, when it satisfies

$$\text{for all } G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, \text{ if } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G), \text{ then } F \subseteq G. \tag{10}$$

Now, it is not difficult to see that requiring this condition for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is equivalent to the property that the Leibniz operator on $\boldsymbol{A}$ is completely order reflecting; this was first observed in [25, p. 108]:

Lemma 6.7. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ if and only if condition* (10) *holds for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

Proof. Suppose the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting. Let $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = \bigcap\big\{\boldsymbol{\Omega}^{\boldsymbol{A}}(F') : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big\}$, it follows by the assumption that $F = \bigcap(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq G$. Conversely, suppose condition (10) holds, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $\{F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : i \in I\}$ and $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be such that $\bigcap_{i \in I} \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap_{i \in I} F_i) \subseteq \bigcap_{i \in I} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_i) \subseteq \bigcap_{i \in I} \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. It follows by (10) that $\bigcap_{i \in I} F_i \subseteq G$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. ⊣

This property, together with Corollary 4.4, instantiated to the case of the Suszko operator, implies the following.

Proposition 6.8. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ if and only if every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter.*

Independently, Corollary 4.4, instantiated for the Leibniz operator, gives the following.

Proposition 6.9. *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is order reflecting on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ if and only if every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Leibniz filter.*

Now, taking Definition 2.9 into account, we get the following.

Theorem 6.10. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is truth-equational.*
(ii) *For every algebra $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter.*
(iii) *For every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter.*

Proof. (i)⇒(ii): It follows immediately by Proposition 6.8, given Definition 2.9.

(ii)⇒(iii): Obvious.

(iii)⇒(i): Let $\boldsymbol{A}$ be an arbitrary algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ be the canonical map. Consider $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. Notice that $\pi(F) \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$, by Lemma 2.4.3. Since $F_0 \subseteq \pi(F)$, using that the Suszko operator is order preserving and Lemma 4.22 we obtain that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(F_0\big) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(\pi(F)\big) = \mathrm{Id}_{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}$. Hence, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(F_0\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(\pi(F)\big)$.

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Now, by hypothesis, both F_0 and $\pi(F)$ are Suszko filters of $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, because $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathsf{Alg}^{\mathrm{Su}}\mathcal{S} = \mathsf{Alg}\mathcal{S}$. Since by Proposition 3.17 the Suszko operator is always injective over Suszko filters, it follows that $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = F_0 = \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. By Proposition 5.14, this establishes that F is a Suszko filter. Finally, again by Proposition 6.8 and Definition 2.9, $\mathcal{S}$ is truth-equational. ⊣

Now we are able to show that under the assumption of truth-equationality, the converse to the implications in Corollary 6.4 holds:

PROPOSITION 6.11. *Let $\mathcal{S}$ be a truth-equational logic. If $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$, then $\mathcal{S}$ is protoalgebraic.*

PROOF. If $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$, then by Proposition 6.2 every full g-model of $\mathcal{S}$ is of the form $[\![G]\!]^*$, for some $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and some algebra $\boldsymbol{A}$. In particular, so are Suszko classes. Take any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for an arbitrary $\boldsymbol{A}$. Then, $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$, for some $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Hence, $F^{\mathrm{Su}} = G^*$. But, since $\mathcal{S}$ is truth-equational by hypothesis, by Theorem 6.10 every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter, and in general every Suszko filter is a Leibniz filter, by Lemma 5.11.2. Therefore, $F = F^{\mathrm{Su}} = G^* = G$. Thus, $[\![F]\!]^{\mathrm{Su}} = [\![F]\!]^*$. Since this has been proved for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and all $\boldsymbol{A}$, this implies protoalgebraicity by Proposition 6.6. ⊣

COROLLARY 6.12. *A logic $\mathcal{S}$ is weakly algebraizable if and only if it is truth-equational and $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$.*

We can now obtain a characterization of truth-equational logics in terms of the form of their full g-models. Observe how the next condition (ii) is in some sense symmetrical to condition (ii) of Theorem 6.5.

THEOREM 6.13. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is truth-equational.*
(ii) *$\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is a full g-model of $\mathcal{S}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.*
(iii) *$[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.*

PROOF. It follows immediately by applying the characterizations of the notion of Suszko filter in Lemma 5.11 and in Theorem 5.13 to Theorem 6.10. ⊣

Putting Theorems 6.5 and 6.13 together we see that weakly algebraizable logics can also be characterized by the form of their full g-models:

COROLLARY 6.14. *A logic $\mathcal{S}$ is weakly algebraizable if and only if the full g-models of $\mathcal{S}$ are exactly all the g-matrices of the form $\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ for any algebra $\boldsymbol{A}$ and any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

By combining several of the previous results we obtain an interesting reformulation.

PROPOSITION 6.15. *A logic $\mathcal{S}$ is weakly algebraizable if and only if all its full g-models are Suszko classes and all its filters are Suszko filters.*

PROOF. If $\mathcal{S}$ is weakly algebraizable, then it is protoalgebraic, therefore by Theorem 6.5 every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. But $\mathcal{S}$ is also truth-equational, so that by Theorem 6.13 $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = [\![F]\!]^{\mathrm{Su}}$. Thus, every full g-model of $\mathcal{S}$ is a Suszko class. And by Theorem 6.10 every $\mathcal{S}$-filter is a Suszko filter. Conversely, assume that the two properties hold. By the second one, $\mathcal{S}$ is truth-equational, and by Theorem 6.13 this implies that any Suszko class is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Thus, by the first property, every full g-model

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of $\mathcal{S}$ is of this form, and by Theorem 6.5 this implies that $\mathcal{S}$ is protoalgebraic. Thus, it is weakly algebraizable. ⊣

Theorem 6.13 also allows us to prove one more characterization of truth-equational logics:

Corollary 6.16. *A logic $\mathcal{S}$ is truth-equational if and only if $\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

Proof. Let $\boldsymbol{A}$ be an algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and consider the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. By Proposition 3.7 and the definition of Suszko class we know that

$$\pi^{-1}\big(\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)\big) = {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = [\![F]\!]^{\mathrm{Su}}.$$

Now assume that $\mathcal{S}$ is truth-equational. From Theorem 6.13 and surjectivity of π it follows that $\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \pi\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Conversely, if $\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \pi\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$, then $\pi([\![F]\!]^{\mathrm{Su}}) = \pi\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$. Having in mind that $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}(F)$ is compatible with every $\mathcal{S}$-filter in $[\![F]\!]^{\mathrm{Su}}$, and moreover $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\mathrm{Su}}$, it follows that $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Thus, $\mathcal{S}$ is truth-equational, again by Theorem 6.13. ⊣

To end this section we consider characterizations in terms of correspondence theorems formulated for matrix homomorphisms (rather than for algebraic homomorphisms, as it has been up to now). The first result, with an almost equivalent wording, was obtained Blok and Pigozzi in [2, Corollary 7.7]. The present formulation can be found as Theorem 2.7 of [22] and as Theorem 1.1.8 of [5]; our interest here is just to see how it follows from previous results.

Theorem 6.17 (Correspondence Theorem for protoalgebraic logics). *A logic $\mathcal{S}$ is protoalgebraic if and only if every strict surjective matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between $\mathcal{S}$-models induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^G$, whose inverse is given by h^{-1}.*

Proof. Assume $\mathcal{S}$ is protoalgebraic. If $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is strict and surjective, then $F = h^{-1}(G)$ and $G = h\big(h^{-1}(G)\big) = h(F)$, so that $F = h^{-1}\big(h(F)\big)$. This means that, viewed as an algebraic homomorphism, h is $\boldsymbol{\Omega}^{\boldsymbol{A}}$-compatible with F. Therefore, we can apply Theorem 5.7 to obtain that h induces an order isomorphism between $[\![F]\!]^*$ and $[\![G]\!]^*$, with inverse given by h^{-1}. This isomorphism restricts to an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^G$, because by protoalgebraicity and Lemma 5.11.4, these up-sets are contained in $[\![F]\!]^*$ and $[\![G]\!]^*$, respectively, and F and G correspond to each other under h and h^{-1}. The converse implication would be proved as in [5], i.e., by showing that the stated condition easily implies that the Leibniz operator is order preserving. ⊣

A new characterization of truth-equational logics can be given which is similar in spirit to the preceding one, though with two modifications, one in the class of homomorphisms it applies to, and the other in the set of filters it involves. It arises from the following observation. As we already mentioned, Theorem 5.15 implies Corollary 2.7 of [6], which in turn implies that every strict surjective matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between $\mathcal{S}$-models that is $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^G$, whose inverse is given by h^{-1}. Notice that, when applied to truth-equational logics, one can write

$(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G^{\mathrm{Su}}}$ instead of $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$ in this property, because for these logics $G^{\mathrm{Su}} = G$ when $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Interestingly, it turns out that, reformulated in this way, the property does indeed characterize truth-equational logics.

THEOREM 6.18 (Correspondence Theorem for truth-equational logics). *A logic $\mathcal{S}$ is truth-equational if and only if every strict surjective matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between $\mathcal{S}$-models that is $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G^{\mathrm{Su}}}$, whose inverse is given by h^{-1}.*

PROOF. If $\mathcal{S}$ is truth-equational, the desired conclusion follows by the discussion above. For the converse, we assume the stated property, and prove that every $\mathcal{S}$-filter is a Suszko filter, which by Theorem 6.10 shows that $\mathcal{S}$ is truth-equational. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and put $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Then, the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ is a strict and surjective matrix homomorphism between the $\mathcal{S}$-models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, F/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\rangle$, and it is clearly $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F. Therefore, by the assumption, π induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{\pi(F)^{\mathrm{Su}}}$, with inverse given by π^{-1}. But the Suszko operator is a coherent family of $\mathcal{S}$-compatible operators, therefore by Lemma 4.9, $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{B}}_{\mathcal{S}}\big(\pi(F)\big) = \pi\big(\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \mathrm{Id}_{B}$. This implies that $[\![\pi(F)]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and hence that $\pi(F)^{\mathrm{Su}} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{\pi(F)^{\mathrm{Su}}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. But on the other hand we can apply the General Correspondence Theorem 5.15 to π, and we find that it induces an order isomorphism between $[\![F]\!]^{\mathrm{Su}}$ and $[\![\pi(F)]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, with inverse given by π^{-1} as well. Thus, necessarily $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$, which tells us that F is a Suszko filter. ⊣

Finally, by just extending the scope of the order isomorphism in the last result to all strict and surjective matrix homomorphisms, we reach weakly algebraizable logics.

THEOREM 6.19. *A logic $\mathcal{S}$ is weakly algebraizable if and only if every strict surjective matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ between $\mathcal{S}$-models induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G^{\mathrm{Su}}}$, whose inverse is given by h^{-1}.*

PROOF. If $\mathcal{S}$ is weakly algebraizable, in particular it is protoalgebraic and by Theorem 6.17 we obtain the order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G}$; but since $\mathcal{S}$ is also truth-equational, every $\mathcal{S}$-filter is a Suszko filter and hence $G^{\mathrm{Su}} = G$, which produces the desired result. Conversely, assume the stated property, and remark that in particular it holds for all h that are $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with F. Therefore, by Theorem 6.18, $\mathcal{S}$ is truth-equational. But then all $\mathcal{S}$-filters will be Suszko, so that $G^{\mathrm{Su}} = G$, and the assumed condition establishes, for all the h described, an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G}$. Thus we can apply Theorem 6.17 and conclude that $\mathcal{S}$ is protoalgebraic as well. That is, $\mathcal{S}$ is weakly algebraizable. ⊣

6.2. Characterizations in terms of the Leibniz operator. One of the well-known general properties of the Leibniz operator is that for every $\boldsymbol{A}$ its range is $\mathrm{Con}_{\mathsf{Alg}^{*}\mathcal{S}}\boldsymbol{A}$. Moreover, by Proposition 3.17, we know that it is also always order reflecting, and hence injective, on the set of Leibniz filters of $\boldsymbol{A}$, that is, on $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$. If we assume protoalgebraicity, then $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is order preserving on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, hence in particular on $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$, and we find the result of [17, Theorem 3]:

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Proposition 6.20. *If* $\mathcal{S}$ *is protoalgebraic, then for every* $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ *is an order isomorphism.*

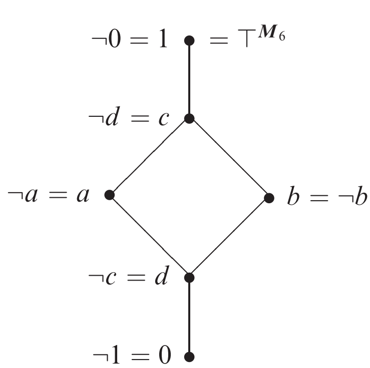
The converse of this result does not hold. A counterexample is the Łukasiewicz logic that preserves degrees of truth studied in [4, 16]; the details of why this is the case will be presented elsewhere. On the other hand, we know that the property need not hold for arbitrary logics, because the Leibniz operator need not be order preserving even on the Leibniz filters, as the next example shows.

Example 6.21. Consider Dunn–Belnap's four-valued logic $\mathcal{B}$ over the signature $\mathcal{L} = \langle \wedge, \vee, \neg, \top \rangle$ [11]; note that we are adding the constant $\top$ to the signature (which will be a theorem), whereas this logic is usually presented without it. Consider the 6-element De Morgan lattice $\boldsymbol{M}_6$, with universe $M_6 = \{0, a, b, c, d, 1\}$, sometimes called "the crystal lattice", and whose structure is described in Figure 2. By direct inspection of the table it is clear that the Leibniz $\mathcal{B}$-filters of $\boldsymbol{M}_6$ are $\{1\}$, $\{1, c\}$, and M_6. Now, $\boldsymbol{\Omega}^{\boldsymbol{M}_6}(\{1\}) = \theta_1$ and $\boldsymbol{\Omega}^{\boldsymbol{M}_6}(\{1, c\}) = \theta_2$, but θ_1 and θ_2 are not comparable. Thus, the Leibniz operator is not order preserving on the Leibniz filters of this algebra. Moreover, it is easy to see that the Suszko filters are here $\{1\}$ and M_6. Thus, this example also shows that not every Leibniz filter is a Suszko filter; the converse implication does indeed hold, as seen in Lemma 5.11.

Now we consider whether it is possible to obtain some version of Proposition 6.20 with Suszko filters rather than with Leibniz filters. We start by relating two conditions already seen to hold in protoalgebraic logics:

Lemma 6.22. *If* $\boldsymbol{\Omega}^{\boldsymbol{A}}\colon \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ *is an order isomorphism, for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, *then* $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$.

Proof. Let $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$. Consider the $\mathcal{S}$-filter $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It is clearly the smallest Leibniz filter. Since we are assuming that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is order preserving on Leibniz filters, it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F_0) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ for every $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. So, $[\![F]\!]^* \subseteq [\![F_0]\!]^*$, for every $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. Now, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be arbitrary. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) \in \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$, it follows by the assumption (surjectivity) that there exists some



$F \in \mathcal{F}i_{\mathcal{B}}(\boldsymbol{M}_6)$	$\theta = \boldsymbol{\Omega}^{\boldsymbol{M}_6}(F)$	blocks of θ
M_6	$M_6 \times M_6$	M_6
$\{1\}$	θ_1	$\{1\}$ $\{0\}$
$\{1, a, b, c, d\}$		$\{a, b, c, d\}$
$\{1, c\}$	θ_2	$\{1, c\}$
$\{1, c, a\}$		$\{a\}$ $\{b\}$
$\{1, c, b\}$		$\{d, 0\}$

Figure 2. The algebra $\boldsymbol{M}_6$, its $\mathcal{B}$-filters, and their Leibniz congruences.

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$F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$; so, $G \in [\![G]\!]^* = [\![F]\!]^* \subseteq [\![F_0]\!]^*$. Thus, $[\![F_0]\!]^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It follows by Proposition 6.1 that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \in \mathsf{Alg}^*\mathcal{S}$. But since $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \mathrm{Id}_A$ and $\boldsymbol{A} \cong \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$. Therefore, $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{S}$. This shows that $\mathsf{Alg}\mathcal{S} \subseteq \mathsf{Alg}^*\mathcal{S}$. The converse inclusion always holds. ⊣

It is not hard to show that if the condition in the lemma (that $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ is an order isomorphism) holds for every for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, as stated there, then it holds for every algebra $\boldsymbol{A}$. We leave the details to the interested reader.

The proof of Lemma 6.22 works, *mutatis mutandis*, for Suszko filters, since $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Therefore:

Lemma 6.23. *If* $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ *is an order isomorphism, for every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, *then* $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$.

With this lemma at hand, we are now able to prove the refinement of Proposition 6.20 we are looking for.

Theorem 6.24. *A logic* $\mathcal{S}$ *is protoalgebraic if and only if the Leibniz operator restricted to the Suszko filters* $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ *is an order isomorphism, for every* $\boldsymbol{A}$.

Proof. The direct implication is just a rephrasing of Proposition 6.20, because under protoalgebraicity the Leibniz filters and the Suszko filters coincide. Now assume the stated condition. We will prove separately that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$ and that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; this will imply that the Leibniz and the Suszko operators coincide, which is equivalent to protoalgebraicity by Proposition 6.3. So, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. To prove the first equality note that since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, it follows by Lemma 6.23 and the surjectivity of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ that there exists $G \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. This implies that $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$, and hence that $F^{\mathrm{Su}} = G^* = G$, because every Suszko filter is a Leibniz filter. Thus, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$. As to the second equality, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$, it follows from the assumption that there exists $H \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(H)$. Then, $[\![F]\!]^* = [\![H]\!]^*$, and hence $F^* = H^* = H$, again because every Suszko filter is a Leibniz filter. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Moreover, $F^* = H$ is a Suszko filter. That is, $(F^*)^{\mathrm{Su}} = F^*$. Now, since $F^* \subseteq F$, it holds $[\![F]\!]^{\mathrm{Su}} \subseteq [\![F^*]\!]^{\mathrm{Su}}$, and therefore $(F^*)^{\mathrm{Su}} \subseteq F^{\mathrm{Su}}$. So, $F^* \subseteq F^{\mathrm{Su}}$. The converse inclusion always holds, by Lemma 5.11.1. Thus, $F^* = F^{\mathrm{Su}}$, which implies that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$. ⊣

The preceding result is a weakened extension to protoalgebraic logics of several characterizations in the same spirit existing in the literature for more restricted classes of logics; see [7, Theorem 4.8] and [23]. From it, taking Definition 2.9 into account, we readily obtain one such characterization for equivalential logics.

Corollary 6.25. *A logic* $\mathcal{S}$ *is equivalential if and only if the Leibniz operator commutes with inverse images by homomorphisms and for every* $\boldsymbol{A}$, *the operator* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *restricts to an order isomorphism between* $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ *and* $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$.

Theorem 6.24 provides an alternative proof of the known fact, which is somehow implicit in the way we chose to define the classes in the Leibniz hierarchy, that a logic $\mathcal{S}$ is weakly-algebraizable if and only if for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$ [7, Theorem 4.8]. Indeed, by Definition 2.9, $\mathcal{S}$ is weakly algebraizable if and only if it is protoalgebraic and truth-equational.

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By Theorem 6.10, this second condition is equivalent to every $\mathcal{S}$-filter being a Suszko filter, and this turns the characterization of Theorem 6.24 into the mentioned one. The converse is less interesting, as the condition directly implies that for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is both order preserving and completely order reflecting on the set of all filters (because it is an order isomorphism between two posets which are actually complete lattices, with intersection as meet), and this means that $\mathcal{S}$ is both protoalgebraic and truth-equational, that is, weakly algebraizable. If we add the condition that the Leibniz operator commutes with inverse images by homomorphisms, then by Corollary 6.25, we obtain the known characterization [21, Corollary 3.14] that a logic $\mathcal{S}$ is algebraizable if and only if the Leibniz operator commutes with inverse images by homomorphisms and for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$.

6.3. Characterizations in terms of the Suszko operator. Let us start with the characterization of truth-equationally in terms of the Suszko operator formulated in [25, Theorem 28]; from it other characterizations will follow without effort within the framework we have settled.

THEOREM 6.26. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is truth-equational.*
(ii) *The Suszko operator is globally injective, that is, $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is injective on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ for every $\boldsymbol{A}$.*
(iii) *$\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is injective on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$.*

PROOF. (i)⇒(ii): Observe that by Proposition 3.17 the Suszko operator is always injective on Suszko filters. If $\mathcal{S}$ is truth-equational, then by Theorem 6.10 every $\mathcal{S}$-filter is a Suszko filter, so the Suszko operator is globally injective.

(ii)⇒(iii): Obvious.

(iii)⇒(i): Let $\boldsymbol{A}$ be an arbitrary algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Consider $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. Notice that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathsf{Alg}^{\mathrm{Su}}\mathcal{S} = \mathsf{Alg}\mathcal{S}$. Moreover, by Lemma 2.4.3, $F/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. So $F_0 \subseteq F/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, and since the Suszko operator is order preserving, and using Lemma 4.22, we obtain that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}(F_0) \subseteq \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(F/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \mathrm{Id}_{A/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)}$. It follows by hypothesis that $F/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = F_0 = \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. By Proposition 5.14, this establishes that F is a Suszko filter. We have shown that every $\mathcal{S}$-filter is a Suszko filter, which implies that $\mathcal{S}$ is truth-equational, again by Theorem 6.10. ⊣

Next, let us apply Proposition 3.17 to the Suszko operator:

PROPOSITION 6.27. *The Suszko operator restricted to Suszko filters $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ is an order embedding.*

PROOF. By its own definition, the Suszko operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is order preserving on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, hence so is its restriction to $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. Moreover, by Proposition 3.17, the Suszko operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is order reflecting on $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. Finally, $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ is into $\mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$ and $\mathsf{Alg}^{\mathrm{Su}}\mathcal{S} = \mathsf{Alg}\mathcal{S}$, by Lemma 2.8. ⊣

So, it is natural to ask under what assumptions is the operator $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ also surjective. It turns out that demanding surjectivity is equivalent to protoalgebraicity, as the next result shows. Moreover, the equivalence is related to yet another fundamental characterization of protoalgebraic logics as those where

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the Suszko operator (viewed as a family of $\mathcal{S}$-operators) satisfies a property typical of the Leibniz operator, namely commutativity with inverse images by surjective homomorphisms.

THEOREM 6.28. *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

(i) *$\mathcal{S}$ is protoalgebraic.*
(ii) *The Suszko operator commutes with inverse images by surjective homomorphisms.*
(iii) *The Suszko operator restricted to Suszko filters $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$ is surjective, for every $\boldsymbol{A}$.*

PROOF. (i)$\Leftrightarrow$(ii): By Proposition 6.3, if $\mathcal{S}$ is protoalgebraic, then the Suszko and the Leibniz operators coincide, and by Proposition 2.5.1 the Leibniz operator always commutes with inverse images by surjective homomorphisms. Conversely, by Theorem 4.6, the assumption implies that the Suszko operator coincides with the Leibniz operator, which by Proposition 6.3 again implies that $\mathcal{S}$ is protoalgebraic.

(i)$\Rightarrow$(iii): By Proposition 6.3 and Corollary 6.4, if $\mathcal{S}$ is protoalgebraic, then $\boldsymbol{\Omega}^{\boldsymbol{A}} = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}$ and $\mathsf{Alg}\mathcal{S} = \mathsf{Alg}^*\mathcal{S}$. Therefore $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A} = \mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$, and then Theorem 6.24 in particular establishes (iii).

(iii)$\Rightarrow$(i): We prove that for every $\boldsymbol{A}$, every full g-model on $\boldsymbol{A}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. This last condition implies protoalgebraicity, by Theorem 6.5. So, let $\mathscr{C}$ be a full g-model of $\mathcal{S}$ on some $\boldsymbol{A}$. Then, $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}(\mathscr{C}) \in \mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. It follows by hypothesis that there exists a Suszko filter $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}{}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$. But since F is a Suszko filter, by Theorem 5.13 $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is full, and then the Isomorphism Theorem (Corollary 3.12) implies that $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, as wished. ⊣

We then obtain a first set of characterizations of the main classes of the Leibniz hierarchy by properties of the Suszko operator, which are set-theoretic rather than order-theoretic.

THEOREM 6.29. *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}$ is protoalgebraic if and only if the Suszko operator commutes with inverse images by surjective homomorphisms.*
2. *$\mathcal{S}$ is equivalential if and only if the Suszko operator commutes with inverse images by homomorphisms.*
3. *$\mathcal{S}$ is truth-equational if and only if the Suszko operator is globally injective.*
4. *$\mathcal{S}$ is weakly algebraizable if and only if the Suszko operator is globally injective and commutes with inverse images by surjective homomorphisms.*
5. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator is globally injective and commutes with inverse images by homomorphisms.*

PROOF. 1. This is the equivalence between the first two conditions of Theorem 6.28.

2. Suppose $\mathcal{S}$ is equivalential. Then, by Definition 2.9, $\mathcal{S}$ is protoalgebraic and the Leibniz operator commutes with inverse images by homomorphisms. But protoalgebraicity implies the coincidence of the Suszko and Leibniz operators. Thus the Suszko operator commutes with inverse images by homomorphisms. Conversely, suppose that the Suszko operator commutes with inverse images by homomorphisms. In particular, it does commute with inverse images by surjective

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homomorphisms. By Theorem 4.6, this implies that it is the Leibniz operator. Hence, the Leibniz operator commutes with inverse images by homomorphisms. Thus $\mathcal{S}$ is equivalential.

3 is Theorem 6.26, and 4 and 5 follow from the previous points, given the definitions of the classes of weakly algebraizable and of algebraizable logics. ⊣

It is interesting to notice that characterizations analogous to those in points 2 and 5 hold for the Leibniz operator if we require that it is order preserving, a property that the Suszko operator has for free; see Theorems 3.13.2 and 3.13.5 of [20]. The final set of characterizations involves the order-theoretic behaviour of the Suszko operator either with respect to the Suszko filters or with respect to arbitrary filters:

THEOREM 6.30. *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}$ is protoalgebraic if and only if for each $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ restricts to an order isomorphism between $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.*
2. *$\mathcal{S}$ is truth-equational if and only if for each $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order embedding of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.*
3. *$\mathcal{S}$ is weakly algebraizable if and only if for each $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.*
4. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator commutes with inverse images by homomorphisms and for each $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$.*

PROOF. 1. Suppose $\mathcal{S}$ is protoalgebraic. Then, by Proposition 6.27 and Theorem 6.28 we get the desired result. Conversely, suppose that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism between $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. In particular it is surjective over the Suszko filters. Then, again by Theorem 6.28, $\mathcal{S}$ is protoalgebraic.

2. Observe that by Proposition 6.27 the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is always an order embedding of $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. But truth-equationality implies that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, by Proposition 6.9, so that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is actually an order embedding of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$. Conversely, that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism means in particular that it is injective, for every $\boldsymbol{A}$, and by Theorem 6.26 this implies that $\mathcal{S}$ is truth-equational.

3. Suppose $\mathcal{S}$ is weakly algebraizable. By Theorems 6.10 and 6.24, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. But $\mathcal{S}$ is also protoalgebraic, therefore the Suszko operator and the Leibniz operator coincide, and $\mathsf{Alg}^*\mathcal{S} = \mathsf{Alg}\mathcal{S}$, and we obtain the desired isomorphism. Conversely, suppose that the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. In particular, it is injective on $\mathcal{S}$-filters, and it follows from Theorem 6.26 that $\mathcal{S}$ is truth-equational. Moreover, every $\mathcal{S}$-filter is a Suszko filter, by Proposition 6.9, so that the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is actually surjective over the Suszko filters. It follows from Theorem 6.28 that $\mathcal{S}$ is protoalgebraic. In sum, $\mathcal{S}$ is weakly algebraizable.

4 follows from point 3 and Theorem 6.29.2. ⊣

It is not known whether some of the characterizations in the previous results can be limited to consider the Suszko operator on the formula algebra and restrict the commutativity to substitutions, as happens with similar characterizations regarding the Leibniz operator. However, we tend to be pessimistic in our case, because the Suszko operator has been seen to behave in a different way; for instance, Raftery [25, Example 1] shows that its injectivity on theories does not imply it is injective on filters of arbitrary algebras.

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