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ALTERNATING CIRCULAR SUMS OF BINOMIAL COEFFICIENTS

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Abstract

By combining the generating function approach with the Lagrange expansion formula, we evaluate, in closed form, two multiple alternating sums of binomial coefficients, which can be regarded as alternating counterparts of the circular sum evaluation discovered by Carlitz ['The characteristic polynomial of a certain matrix of binomial coefficients', *Fibonacci Quart.* **3**(2) (1965), 81–89].

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1. Introduction and outline

In mathematics and applied science, Fibonacci and Lucas numbers are well known. They are defined by the same recurrence relations

$$F_n = F_{n-1} + F_{n-2} \quad \text{and} \quad L_n = L_{n-1} + L_{n-2}$$

but with different initial conditions

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad L_0 = 2,\ L_1 = 1.$$

According to these recurrence relations, it is not hard to check that both definitions for Fibonacci and Lucas numbers can be extended to negative integer indices by

$$F_{-n} = (-1)^n F_n \quad \text{and} \quad L_{-n} = (-1)^n L_n \quad \text{for } n \in \mathbb{N}.$$

In 1965, Carlitz [2] discovered an elegant identity for circular sums:

$$\sum_{0\le k_1,k_2,\dots,k_m\le n} \binom{n-k_1}{k_m} \prod_{i=1}^{m-1} \binom{n-k_{i+1}}{k_i} = \frac{F_{mn+m}}{F_m}. \tag{1.1}$$

There are three different proofs that can be found in [1, 5, 8]. Denote an m-tuple of integers by $\mathbf{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}_0^m$. We define two component-related sums by

$$|\mathbf{k}| = \sum_{i=1}^{m} k_i \quad \text{and} \quad \|\mathbf{k}\| = \sum_{i=1}^{m} ik_i.$$

Recently, the author [5] found the following alternating counterpart:

$$\sum_{0\le k_1,k_2,\ldots,k_m\le n} (-1)^{|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i} = (-1)^{n\lfloor m/3\rfloor}\begin{cases} n+1 & \text{if } m\equiv_3 0;\\ u_n & \text{if } m\equiv_3 1;\\ v_n & \text{if } m\equiv_3 2;\end{cases}$$

where u_n and v_n are two periodic sequences (see A010892 and A049347 recorded in [9]) and related by $u_{2n+1} = v_n$ and given explicitly by

$$u_n = \begin{cases} 1 & \text{if } n\equiv_6 0,\ 1;\\ -1 & \text{if } n\equiv_6 3,\ 4;\\ 0 & \text{if } n\equiv_6 2,\ 5;\end{cases} \quad \text{and} \quad v_n = \begin{cases} 1 & \text{if } n\equiv_3 0;\\ -1 & \text{if } n\equiv_3 1;\\ 0 & \text{if } n\equiv_3 2.\end{cases}$$

To reduce lengthy expressions, the following abbreviations will be used throughout the paper. For a real number x, the greatest integer not exceeding it will be denoted by $\lfloor x\rfloor$. We shall use χ for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. For three integers i, j, m with $m > 0$, the notation $i \equiv_m j$ stands for 'i is congruent to j modulo m'.

The aim of the present paper is to evaluate two further alternating circular sums of binomial coefficients by making use of the generating function approach:

$$\Phi_m(n) = \sum_{0\le k_1,k_2,\ldots,k_m\le n} (-1)^{\|\mathbf{k}\|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}; \tag{1.2}$$

$$\Psi_m(n) = \sum_{0\le k_1,k_2,\ldots,k_m\le n} (-1)^{\|\mathbf{k}\|-|\mathbf{k}|}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i}. \tag{1.3}$$

To fulfil this task, the Lagrange expansion formula [7] will be crucial. We give it in the following form (see Comtet [6, Section 3.8] and Chu [3, 4]). For a formal power series $\varphi(x)$ subject to the condition $\varphi(0) \neq 0$, the functional equation $y = x/\varphi(x)$ determines x as an implicit function of y. Then for another formal power series $F(x)$ in the variable x, the following expansions hold for both composite series:

$$F(x(y)) = F(0) + \sum_{n=1}^{\infty} \frac{y^n}{n}[x^{n-1}]\{F'(x)\varphi^n(x)\},$$

$$\frac{F(x(y))}{1 - x\varphi'(x)/\varphi(x)} = \sum_{n=0}^{\infty} y^n [x^n]\{F(x)\varphi^n(x)\}; \tag{1.4}$$

where $[x^k]\phi(x)$ denotes the coefficient of x^k in the formal power series $\phi(x)$. □

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2. The first alternating sum $\Phi_m(n)$

Observe that $\Phi_m(n)$ defined in (1.2) can be rewritten as

$$\Phi_m(n) = \sum_{0\le k_1,k_2,\ldots,k_m\le n} (-1)^{\sum k_{2i-1}} \binom{n-k_1}{k_m} \prod_{i=1}^{m-1} \binom{n-k_{i+1}}{k_i},$$

where $\sum k_{2i-1}$ denotes the sum of the odd indexed components of $\mathbf{k}$. Then we have the following theorem.

THEOREM 2.1 (generating functions: $m, n \in \mathbb{N}$). *We have*

$$\Phi_m(n) = \begin{cases} [y^n]\dfrac{1}{1 - yL_\lambda + (-1)^\lambda y^2} & \text{if } m = 2\lambda; \\[2ex] [y^n]\dfrac{1}{1 - yF_{\lambda-2} - (-1)^\lambda y^2} & \text{if } m = 2\lambda - 1. \end{cases}$$

Recall the Binet formulae:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \quad \text{where } \alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

When m is even with $m = 2\lambda$, we can rewrite the generating function and then decompose it into partial fractions:

$$\frac{1}{1 - yL_\lambda + (-1)^\lambda y^2} = \frac{1}{1 - y(\alpha^\lambda + \beta^\lambda) + y^2(-1)^\lambda} = \frac{1}{\alpha^\lambda - \beta^\lambda}\left\{\frac{\alpha^\lambda}{1 - y\alpha^\lambda} - \frac{\beta^\lambda}{1 - y\beta^\lambda}\right\}.$$

By extracting the coefficient of x^n across the above equation, we find the following counterpart of Carlitz' formula (1.1).

PROPOSITION 2.2 (explicit formula: $n, \lambda \in \mathbb{N}$). *We have*

$$\Phi_{2\lambda}(n) = \frac{F_{n\lambda+\lambda}}{F_\lambda}.$$

However, when m is odd with $m = 2\lambda - 1$, there exists no such closed formula. By expanding the corresponding generating function into power series

$$\frac{1}{1 - yF_{\lambda-2} - (-1)^\lambda y^2} = \sum_{i=0}^{\infty} \{yF_{\lambda-2} + (-1)^\lambda y^2\}^i = \sum_{i=0}^{\infty} \sum_{k=0}^{i} (-1)^{k\lambda} \binom{i}{k} y^{i+k} F_{\lambda-2}^{i-k}$$

and then extracting the coefficient of y^n, we get the binomial sum expression.

PROPOSITION 2.3 (explicit formula: $n, \lambda \in \mathbb{N}$).

$$\Phi_{2\lambda-1}(n) = \sum_{k=0}^{n} (-1)^{k\lambda} \binom{n-k}{k} F_{\lambda-2}^{n-2k}.$$

The first five values are highlighted in the following corollary.

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COROLLARY 2.4 (explicit formula: $n \in \mathbb{N}$). *We have*

$$\Phi_1(n) = [y^n]\frac{1}{1-y+y^2} = (-1)^{\lfloor (n+1)/3 \rfloor}\chi(n \not\equiv_3 2);$$

$$\Phi_3(n) = [y^n]\frac{1}{1-y^2} = \frac{1+(-1)^n}{2};$$

$$\Phi_5(n) = [y^n]\frac{1}{1-y+y^2} = (-1)^{\lfloor (n+1)/3 \rfloor}\chi(n \not\equiv_3 2);$$

$$\Phi_7(n) = [y^n]\frac{1}{1-y-y^2} = F_{n+1};$$

$$\Phi_9(n) = [y^n]\frac{1}{1-2y+y^2} = n+1.$$

PROOF OF THEOREM 2.1. First, it is routine to verify the relations

$$(-1)^{k_1}\binom{n-k_2}{k_1} = [x^{k_1}](1-x)^{n-k_2},$$
$$\binom{n-k_1}{k_m} = [x^{n-k_1}]\frac{x^{k_m}}{(1-x)^{1+k_m}}.$$

They can be used to express the binomial sum with respect to k_1 in $\Phi_m(n)$ as

$$\Phi_m^1(n) = \sum_{k_1=0}^{n}(-1)^{k_1}\binom{n-k_2}{k_1}\binom{n-k_1}{k_m}$$
$$= [x^n]\frac{x^{k_m}(1-x)^{n-k_2}}{(1-x)^{1+k_m}}.$$

We proceed next with the binomial sum with respect to k_2 in $\Phi_m(n)$:

$$\Phi_m^2(n) = \sum_{k_2=0}^{n}[x^n]\frac{x^{k_m}(1-x)^{n-k_2}}{(1-x)^{1+k_m}}\binom{n-k_3}{k_2}$$
$$= [x^n]\frac{x^{k_m}(1-x)^n}{(1-x)^{1+k_m}} \times \left\{\frac{2-x}{1-x}\right\}^{n-k_3}$$
$$= [x^n]\frac{x^{k_m}(2-x)^n}{(1-x)^{1+k_m}} \times \left\{\frac{1-x}{2-x}\right\}^{k_3}.$$

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Repeat this process for the binomial sum with respect to k_3 in $\Phi_m(n)$:

$$\begin{aligned}\Phi_m^3(n) &= \sum_{k_3=0}^{n}[x^n]\frac{x^{k_m}(2-x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{1-x}{2-x}\right\}^{k_3}\binom{n-k_4}{k_3}(-1)^{k_3}\\ &= [x^n]\frac{x^{k_m}(2-x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{1}{2-x}\right\}^{n-k_4} = [x^n]\frac{x^{k_m}(2-x)^{k_4}}{(1-x)^{1+k_m}}\\ &= [x^n]\frac{x^{k_m}(F_2-xF_0)^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_3-xF_1}{F_2-xF_0}\right\}^{k_4},\end{aligned}$$

and then the binomial sum with respect to k_4 in $\Phi_m(n)$:

$$\begin{aligned}\Phi_m^4(n) &= \sum_{k_4=0}^{n}[x^n]\frac{x^{k_m}(2-x)^{k_4}}{(1-x)^{1+k_m}}\binom{n-k_5}{k_4} = [x^n]\frac{x^{k_m}(3-x)^{n-k_5}}{(1-x)^{1+k_m}}\\ &= [x^n]\frac{x^{k_m}(F_4-xF_2)^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_2-xF_0}{F_4-xF_2}\right\}^{k_5}.\end{aligned}$$

By means of the induction principle, we can show that for $1 \le j \le m/2$, the binomial sums with respect to k_{2j-1} and k_{2j-2} in $\Phi_m(n)$ are respectively given by

$$\begin{aligned}\Phi_m^{2j-1}(n) &= \sum_{k_{2j-1}=0}^{n}\Phi_m^{2j-2}(n)\binom{n-k_{2j}}{k_{2j-1}}(-1)^{k_{2j-1}}\\ &= [x^n]\frac{x^{k_m}(F_j-xF_{j-2})^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_{j+1}-xF_{j-1}}{F_j-xF_{j-2}}\right\}^{k_{2j}},\\ \Phi_m^{2j-2}(n) &= \sum_{k_{2j-2}=0}^{n}\Phi_m^{2j-3}(n)\binom{n-k_{2j-1}}{k_{2j-2}}\\ &= [x^n]\frac{x^{k_m}(F_{j+1}-xF_{j-1})^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_{j-1}-xF_{j-3}}{F_{j+1}-xF_{j-1}}\right\}^{k_{2j}}.\end{aligned}$$

According to the parity of m, we can determine $\Phi_m(n)$ separately as follows. For even $m = 2\lambda$, we can evaluate

$$\begin{aligned}\Phi_m(n) &= \sum_{k_m=0}^{n}\Phi_m^{m-1}(n)\\ &= \sum_{k_m=0}^{n}[x^n]\frac{x^{k_m}(F_\lambda-xF_{\lambda-2})^n}{(1-x)^{1+k_m}}\left\{\frac{F_{\lambda+1}-xF_{\lambda-1}}{F_\lambda-xF_{\lambda-2}}\right\}^{k_m}\end{aligned}$$

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$$= [x^n]\frac{(F_\lambda - xF_{\lambda-2})^n}{1-x}\sum_{k_m=0}^{\infty}\Big\{\frac{x(F_{\lambda+1}-xF_{\lambda-1})}{(1-x)(F_\lambda - xF_{\lambda-2})}\Big\}^{k_m}$$

$$= [x^n]\frac{(F_\lambda - xF_{\lambda-2})^{n+1}}{(1-x)(F_\lambda - xF_{\lambda-2}) - x(F_{\lambda+1}-xF_{\lambda-1})},$$

which simplifies to

$$\Phi_m(n) = [x^n]\frac{(F_\lambda - xF_{\lambda-2})^{n+1}}{F_\lambda(1-3x+x^2)}.$$

By means of the Lagrange expansion formula, these expressions can be simplified further. In fact, specialise first in (1.4) by

$$F(x) = \frac{F_\lambda - xF_{\lambda-2}}{F_\lambda(1-3x+x^2)} \quad \text{and} \quad \varphi(x) = F_\lambda - xF_{\lambda-2}.$$

Then we can deduce that

$$y = x/\varphi(x) \quad \Longleftrightarrow \quad x = \frac{yF_\lambda}{1+yF_{\lambda-2}},$$

together with the generating function

$$\begin{aligned}\frac{F(x(y))}{1-x\varphi'(x)/\varphi(x)} &= \frac{F_\lambda - xF_{\lambda-2}}{F_\lambda(1-3x+x^2)}\Big/\Big\{1+\frac{xF_{\lambda-2}}{F_\lambda - xF_{\lambda-2}}\Big\}\\ &= \frac{(F_\lambda - xF_{\lambda-2})^2}{F_\lambda^2(1-3x+x^2)}, \quad \text{where } x = \frac{yF_\lambda}{1+yF_{\lambda-2}}\\ &= \frac{1}{1+2yF_{\lambda-2}-3yF_\lambda + y^2F_\lambda^2 + y^2F_{\lambda-2}^2 - 3y^2F_\lambda F_{\lambda-2}}.\end{aligned}$$

After some routine simplification, we find the elegant formula

$$\Phi_m(n) = [y^n]\frac{1}{1-yL_\lambda + (-1)^\lambda y^2}, \quad \text{where } m = 2\lambda.$$

Analogously for odd $m = 2\lambda - 1$, we can evaluate

$$\begin{aligned}\Phi_m(n) &= \sum_{k_m=0}^{n}\Phi_m^{m-1}(n)\\ &= \sum_{k_m=0}^{n}[x^n]\frac{x^{k_m}(F_{\lambda+1}-xF_{\lambda-1})^n}{(1-x)^{1+k_m}}\Big\{-\frac{F_{\lambda-1}-xF_{\lambda-3}}{F_{\lambda+1}-xF_{\lambda-1}}\Big\}^{k_m}\end{aligned}$$

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$$= [x^n]\frac{(F_{\lambda+1} - xF_{\lambda-1})^n}{1-x}\sum_{k_m=0}^{\infty}\left\{\frac{-x(F_{\lambda-1} - xF_{\lambda-3})}{(1-x)(F_{\lambda+1} - xF_{\lambda-1})}\right\}^{k_m}$$

$$= [x^n]\frac{(F_{\lambda+1} - xF_{\lambda-1})^{n+1}}{(1-x)(F_{\lambda+1} - xF_{\lambda-1}) + x(F_{\lambda-1} - xF_{\lambda-3})},$$

which simplifies to

$$\Phi_m(n) = [x^n]\frac{(F_{\lambda+1} - xF_{\lambda-1})^{n+1}}{F_{\lambda+1}(1-x) + x^2F_{\lambda-2}}.$$

By means of the Lagrange expansion formula, these expressions can be simplified further. In fact, specialise now in (1.4) by

$$F(x) = \frac{F_{\lambda+1} - xF_{\lambda-1}}{F_{\lambda+1}(1-x) + x^2F_{\lambda-2}} \quad \text{and} \quad \varphi(x) = F_{\lambda+1} - xF_{\lambda-1}.$$

Then we can deduce that

$$y = x/\varphi(x) \quad \Longleftrightarrow \quad x = \frac{yF_{\lambda+1}}{1 + yF_{\lambda-1}},$$

together with the generating function below

$$\frac{F(x(y))}{1 - x\varphi'(x)/\varphi(x)} = \frac{F_{\lambda+1} - xF_{\lambda-1}}{F_{\lambda+1}(1-x) + x^2F_{\lambda-2}}\Big/\left\{1 + \frac{xF_{\lambda-1}}{F_{\lambda+1} - xF_{\lambda-1}}\right\}$$

$$= \frac{(F_{\lambda+1} - xF_{\lambda-1})^2}{F_{\lambda+1}^2(1-x) + x^2F_{\lambda+1}F_{\lambda-2}}, \quad \text{where } x = \frac{yF_{\lambda+1}}{1 + yF_{\lambda-1}}$$

$$= \frac{1}{1 + 2yF_{\lambda-1} - yF_{\lambda+1} + y^2F_{\lambda-1}^2 + y^2F_{\lambda+1}F_{\lambda-2} - y^2F_{\lambda+1}F_{\lambda-1}}.$$

After some routine simplification, we confirm another elegant formula,

$$\Phi_m(n) = [y^n]\frac{1}{1 - yF_{\lambda-2} - (-1)^{\lambda}y^2}, \quad \text{where } m = 2\lambda - 1.$$

Hence, both statements of Theorem 2.1 are proved. □

3. The second alternating sum $\Psi_m(n)$

Analogously, we can reformulate the circular sum $\Psi_m(n)$ as

$$\Psi_m(n) = \sum_{0 \le k_1, k_2, \ldots, k_m \le n} (-1)^{\sum k_{2i}}\binom{n-k_1}{k_m}\prod_{i=1}^{m-1}\binom{n-k_{i+1}}{k_i},$$

where $\sum k_{2i}$ stands for the sum of the even indexed components of $\mathbf{k}$. We have the following theorem.

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THEOREM 3.1 (generating functions).

$$\Psi_m(n) = \begin{cases} [y^n]\dfrac{1}{1 - yL_\lambda + (-1)^\lambda y^2} & \textit{if } m = 2\lambda; \\ [y^n]\dfrac{1}{1 - yF_{\lambda+1} + (-1)^\lambda y^2} & \textit{if } m = 2\lambda - 1. \end{cases}$$

REMARK 3.2. Comparing the generating functions in Theorems 2.1 and 3.1, we can see that $\Phi_m(n) = \Psi_m(n)$ for all $n \in \mathbb{N}$ when m is even. This is not surprising because in this case, both sums $\Psi_m(n)$ and $\Phi_m(n)$, defined respectively in (1.3) and (1.2), are equivalent if we reverse the order for the components of $\mathbf{k}$. When m is odd, we have instead $\Psi_m(n) = \Phi_{m+6}(n)$ for all $n \in \mathbb{N}$.

According to this remark, we have the following summation formulae, where $\Psi_5(n)$, $\Psi_7(n)$ and $\Psi_9(n)$ are integer sequences A006190, A004254 and A041025, respectively, recorded in [9]:

$$\Psi_1(n) = [y^n]\frac{1}{1 - y - y^2} = F_{n+1};$$

$$\Psi_3(n) = [y^n]\frac{1}{(1 - y)^2} = n + 1;$$

$$\Psi_5(n) = [y^n]\frac{1}{1 - 3y - y^2} = \sum_{k=0}^{n}\binom{n-k}{k}3^{n-2k};$$

$$\Psi_7(n) = [y^n]\frac{1}{1 - 5y + y^2} = \sum_{k=0}^{n}\binom{n+k+1}{2k+1}3^{k};$$

$$\Psi_9(n) = [y^n]\frac{1}{1 - 8y - y^2} = \sum_{k=0}^{n}\binom{n-k}{k}8^{n-2k}.$$

PROOF OF THEOREM 3.1. We only need to prove the case when m is odd with $m = 2\lambda - 1$. According to the binomial relations

$$\binom{n-k_2}{k_1} = [x^{k_1}](1+x)^{n-k_2},$$

$$\binom{n-k_1}{k_m} = [x^{n-k_1}]\frac{x^{k_m}}{(1-x)^{1+k_m}},$$

we can express the binomial sum with respect to k_1 in $\Psi_m(n)$ as

$$\Psi_m^1(n) = \sum_{k_1=0}^{n}\binom{n-k_2}{k_1}\binom{n-k_1}{k_m}$$

$$= [x^n]\frac{x^{k_m}(1+x)^{n-k_2}}{(1-x)^{1+k_m}}.$$

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We proceed next with the binomial sum with respect to k_2 in $\Psi_m(n)$:

$$\begin{aligned}\Psi_m^2(n) &= \sum_{k_2=0}^{n}[x^n]\frac{x^{k_m}(1+x)^{n-k_2}}{(1-x)^{1+k_m}}\binom{n-k_3}{k_2}(-1)^{k_2}\\ &= [x^n]\frac{x^{k_m}(1+x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{x}{1+x}\right\}^{n-k_3}\\ &= [x^n]\frac{x^{k_m+n}}{(1-x)^{1+k_m}}\times\left\{\frac{1+x}{x}\right\}^{k_3}.\end{aligned}$$

Repeat this process for the binomial sums with respect to k_3 in $\Psi_m(n)$:

$$\begin{aligned}\Psi_m^3(n) &= \sum_{k_3=0}^{n}[x^n]\frac{x^{k_m+n}}{(1-x)^{1+k_m}}\times\left\{\frac{1+x}{x}\right\}^{k_3}\binom{n-k_4}{k_3}\\ &= [x^n]\frac{x^{k_m}(1+2x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{x}{1+2x}\right\}^{k_4}\\ &= [x^n]\frac{x^{k_m}(F_2+xF_3)^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_0+xF_1}{F_2+xF_3}\right\}^{k_4},\end{aligned}$$

and then with respect to k_4 in $\Psi_m(n)$:

$$\begin{aligned}\Psi_m^4(n) &= \sum_{k_4=0}^{n}[x^n]\frac{x^{k_m}(1+2x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{x}{1+2x}\right\}^{k_4}\binom{n-k_5}{k_4}(-1)^{k_4}\\ &= [x^n]\frac{x^{k_m}(1+x)^n}{(1-x)^{1+k_m}}\times\left\{\frac{1+2x}{1+x}\right\}^{k_5}\\ &= [x^n]\frac{x^{k_m}(F_1+xF_2)^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_2+xF_3}{F_1+xF_2}\right\}^{k_5}.\end{aligned}$$

By means of the induction principle, we can show that for $1 \le j \le m/2$, the binomial sums with respect to k_{2j-1} and k_{2j-2} in $\Psi_m(n)$ are respectively given by

$$\begin{aligned}\Psi_m^{2j-1}(n) &= \sum_{k_{2j-1}=0}^{n}\Psi_m^{2j-2}(n)\binom{n-k_{2j}}{k_{2j-1}}\\ &= [x^n]\frac{x^{k_m}(F_j+xF_{j+1})^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_{j-2}+xF_{j-1}}{F_j+xF_{j+1}}\right\}^{k_{2j}},\\ \Psi_m^{2j-2}(n) &= \sum_{k_{2j-2}=0}^{n}\Psi_m^{2j-3}(n)\binom{n-k_{2j-1}}{k_{2j-2}}(-1)^{k_{2j-2}}\\ &= [x^n]\frac{x^{k_m}(F_{j-2}+xF_{j-1})^n}{(1-x)^{1+k_m}}\times\left\{\frac{F_{j-1}+xF_j}{F_{j-2}+xF_{j-1}}\right\}^{k_{2j}}.\end{aligned}$$

Since $m = 2\lambda - 1$ is odd, we can further evaluate

$$\begin{aligned}\Psi_m(n) &= \sum_{k_m=0}^{n} \Psi_m^{m-1}(n) = \sum_{k_{2\lambda-1}=0}^{n} \Psi_{2\lambda-1}^{2\lambda-2}(n)\\ &= \sum_{k_m=0}^{n} [x^n] \frac{x^{k_m}(F_{\lambda-2}+xF_{\lambda-1})^n}{(1-x)^{1+k_m}} \Big\{ \frac{F_{\lambda-1}+xF_\lambda}{F_{\lambda-2}+xF_{\lambda-1}} \Big\}^{k_m}\\ &= [x^n] \frac{(F_{\lambda-2}+xF_{\lambda-1})^n}{1-x} \sum_{k_m=0}^{\infty} \Big\{ \frac{x(F_{\lambda-1}+xF_\lambda)}{(1-x)(F_{\lambda-2}+xF_{\lambda-1})} \Big\}^{k_m}\\ &= [x^n] \frac{(F_{\lambda-2}+xF_{\lambda-1})^{n+1}}{(1-x)(F_{\lambda-2}+xF_{\lambda-1}) - x(F_{\lambda-1}+xF_\lambda)},\end{aligned}$$

which simplifies to

$$\Psi_m(n) = [x^n] \frac{(F_{\lambda-2}+xF_{\lambda-1})^{n+1}}{F_{\lambda-2}(1-x) - x^2 F_{\lambda+1}}, \quad m \neq 3.$$

REMARK 3.3. When $m = 3$, the triple sum $\Psi_3(n)$ results in

$$\Psi_3(n) = \sum_{k_3=0}^{n} [x^n] \frac{x^{k_3+n}}{(1-x)^{1+k_3}} \times \Big\{ \frac{1+x}{x} \Big\}^{k_3} = [x^0] \frac{(\frac{1+x}{1-x})^{n+1} - 1}{2x} = n+1.$$

By means of the Lagrange expansion formula, we can further simplify the generating function for $\Psi_m(n)$. In fact, specialise in (1.4) by

$$F(x) = \frac{F_{\lambda-2}+xF_{\lambda-1}}{F_{\lambda-2}(1-x) - x^2 F_{\lambda+1}} \quad \text{and} \quad \varphi(x) = F_{\lambda-2} + xF_{\lambda-1}.$$

Then we can deduce that

$$y = x/\varphi(x) \iff x = \frac{yF_{\lambda-2}}{1 - yF_{\lambda-1}},$$

together with the generating function

$$\begin{aligned}\frac{F(x(y))}{1 - x\varphi'(x)/\varphi(x)} &= \frac{F_{\lambda-2}+xF_{\lambda-1}}{F_{\lambda-2}(1-x) - x^2 F_{\lambda+1}} \Big/ \Big\{ 1 - \frac{xF_{\lambda-1}}{F_{\lambda-2}+xF_{\lambda-1}} \Big\}\\ &= \frac{(F_{\lambda-2}+xF_{\lambda-1})^2}{F_{\lambda-2}^2(1-x) - x^2 F_{\lambda+1}F_{\lambda-2}}, \quad \text{where } x = \frac{yF_{\lambda-2}}{1-yF_{\lambda-1}}\\ &= \frac{1}{1 - yF_{\lambda-2} - 2yF_{\lambda-1} + y^2F_{\lambda-1}^2 + y^2F_{\lambda-1}F_{\lambda-2} - y^2F_{\lambda+1}F_{\lambda-2}}.\end{aligned}$$

After some routine simplification, we arrive at

$$\Psi_m(n) = [y^n] \frac{1}{1 - yF_{\lambda+1} + (-1)^\lambda y^2}, \quad \text{where } m = 2\lambda - 1,$$

which is also valid for $m = 3$. This completes the proof of Theorem 3.1. □

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4. Open problems

For Carlitz' identity (1.1), there is a combinatorial proof by Benjamin and Rouse [1] through the domino tiling. It would be interesting to construct a similar proof for the identity displayed in Proposition 2.2. Another intriguing problem is to find a combinatorial interpretation of the identity $\Psi_m(n) = \Phi_{m+6}(n)$ for all $n \in \mathbb{N}$ and odd $m \in \mathbb{N}$.

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