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GRAPH IMMERSIONS, INVERSE MONOIDS AND DECK TRANSFORMATIONS

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Abstract

If $f : \tilde{\Gamma} \to \Gamma$ is a covering map between connected graphs, and H is the subgroup of $\pi_1(\Gamma, v)$ used to construct the cover, then it is well known that the group of deck transformations of the cover is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(\Gamma, v)$. We show that an entirely analogous result holds for immersions between connected graphs, where the subgroup H is replaced by the closed inverse submonoid of the inverse monoid $L(\Gamma, v)$ used to construct the immersion. We observe a relationship between group actions on graphs and deck transformations of graph immersions. We also show that a graph immersion $f : \tilde{\Gamma} \to \Gamma$ may be extended to a cover $g : \tilde{\Delta} \to \Gamma$ in such a way that all deck transformations of f are restrictions of deck transformations of g.

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1. Introduction

It is well known that group theory provides a powerful algebraic tool for studying covering spaces of topological spaces. For example, under mild conditions on a connected topological space $\mathcal{X}$, the connected covers of $\mathcal{X}$ may be classified via subgroups of the fundamental group of $\mathcal{X}$. This may be used to study deck transformations of covering spaces and actions of groups on topological spaces. However, the study of *immersions* between connected topological spaces seems to require somewhat different algebraic tools, even for graphs (one-dimensional CW-complexes).

In his paper [16], Stallings made use of immersions between finite graphs to study finitely generated subgroups of free groups. Here by an immersion between graphs we mean a locally injective graph morphism, that is, a graph morphism that is injective on star sets. Subsequently, Margolis and Meakin [7] showed how the theory of *inverse monoids* may be used to classify immersions between connected graphs. These results have been extended by Meakin and Szakács [9, 10] to classify immersions between higher-dimensional cell complexes.

In the present paper we extend the ideas of [7] to show how inverse monoids may be used to study deck transformations of immersions between connected graphs. By a deck transformation we mean a graph automorphism that respects the immersion.

In Section 2 of the paper we introduce the terminology needed to describe covers and immersions of graphs. We then summarize some of the classical algebraic and topological ideas involving the classification of covers and the theory of deck transformations of connected covers of graphs.

Section 3 summarizes some of the basic theory of inverse monoids that will be needed subsequently. We then describe the use of closed inverse submonoids of free inverse monoids to classify immersions between connected graphs. We prove an apparently new result constructing the group of right ω-cosets of a closed inverse submonoid of an inverse monoid in its normalizer.

In Section 4 we provide a calculation of the group of deck transformations of a connected immersion between graphs. If H is the closed inverse submonoid of the free inverse monoid that is used to construct the immersion, then the group of deck transformations of the immersion is the group of right ω-cosets of H in its normalizer (Theorem 4.4).

In Section 5 we describe how the results of earlier sections of the paper specialize in the case that the graph immersion is actually a cover of connected graphs. In Section 6 we make an observation relating graph immersions to actions of groups on graphs. In Section 7 we prove that an immersion between graphs may be extended to a covering map between graphs in such a way that deck transformations of the immersion are restrictions of deck transformations of the cover.

2. Covers and immersions of graphs

By a *graph* $\Gamma = (\Gamma^0, \Gamma^1)$ we mean a graph in the sense of Serre [15]. Here Γ^0 is the set of vertices and Γ^1 is the set of edges of Γ. Thus every directed edge $e : v \to w$ comes equipped with an inverse edge $e^{-1} : w \to v$ such that $(e^{-1})^{-1} = e$ and $e^{-1} \neq e$. The initial vertex of e is denoted by $\alpha(e)$ and the terminal vertex of e is denoted by $\omega(e)$: thus $\alpha(e^{-1}) = \omega(e)$ and $\omega(e^{-1}) = \alpha(e)$. For each edge e we designate one of the edges in the set $\{e, e^{-1}\}$ as being *positively oriented*, and its inverse edge as being *negatively oriented*. We normally only indicate the positively oriented edges in a sketch of a graph. A *path* in the graph Γ is a finite string $p = e_1e_2 \cdots e_n$ where $\omega(e_i) = \alpha(e_{i+1})$ for $i = 1, \ldots, n-1$: here the edges e_i may be either positively or negatively oriented. We denote the initial vertex of p by $\alpha(p)$ and the terminal vertex by $\omega(p)$: that is, if $p = e_1e_2 \cdots e_n$, then $\alpha(p) = \alpha(e_1)$ and $\omega(p) = \omega(e_n)$. The *inverse* of the path $p = e_1e_2 \cdots e_n$ is the path $p^{-1} = e_n^{-1} \cdots e_2^{-1}e_1^{-1}$. The path p is a *circuit* if $\alpha(p) = \omega(p)$. A *tree* is a connected graph in which every circuit $e_1e_2 \cdots e_n$ contains a subpath of the form ee^{-1} for some edge e. Thus the Cayley graph $\Gamma(X)$ of the free group $FG(X)$ with respect to a set X of free generators is a tree.

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The *free category* on a graph Γ is the category $FC(\Gamma)$ whose objects are the vertices of Γ and whose morphisms are the paths in Γ. The product $p.q$ of paths p and q is defined in $FC(\Gamma)$ if and only if $\omega(p) = \alpha(q)$ and in that case $p.q = pq$, the concatenation of the path p followed by the path q. We say that a path p_1 is an *initial segment* of a path p if there is a path p_2 such that $p = p_1p_2$, and in this case p_2 is a *terminal segment* of p.

A *morphism* from the graph Γ to the graph Γ' is a pair of functions $f : \Gamma \to \Gamma'$ that takes vertices to vertices and edges to edges, and preserves incidence and orientation of edges. (Here we abuse notation slightly by using the same symbol f to denote the corresponding function that takes vertices to vertices and the function that takes edges to edges.) If $v \in V(\Gamma)$, let $\mathrm{star}(\Gamma, v) = \{e \in E(\Gamma) : \alpha(e) = v\}$. A morphism $f : \Gamma \to \Gamma'$ induces a map $f_v : \mathrm{star}(\Gamma, v) \to \mathrm{star}(\Gamma', f(v))$ between star sets in the obvious way. Following Stallings [16], we say that a graph morphism f is a *cover* if each f_v is a bijection and that f is an *immersion* if each f_v is an injection.

In his paper [16], Stallings made use of immersions between graphs to study subgroups of free groups. Since then, Stallings foldings and Stallings graphs have been used extensively to study subgroups of free groups. See, for example, the survey paper by Kapovich and Myasnikov [4] or the paper by Birget *et al.* [1] for just some of the relevant literature.

It is clear from the definition of a graph that it is possible to label the edges of a graph with labels of positively oriented edges coming from some set X so that no two positively oriented edges with the same initial or terminal vertex are assigned the same label. The labeling of positively oriented edges may be extended to a labeling of all edges in the graph in such a way that if e is a positively oriented edge labeled by $x \in X$ then e^{-1} is labeled by $x^{-1} \in X^{-1}$, where X^{-1} is a set disjoint from X and in one-to-one correspondence with X via the map $x \to x^{-1}$. We denote the label on an edge e by $\ell(e)$. Thus $\ell(e) \in X$ if e is a positively oriented edge, and in general $\ell(e) \in X \cup X^{-1}$. The labeling of edges in Γ extends to a labeling of paths in Γ in the obvious way via $\ell(pq) = \ell(p)\ell(q)$ if pq is defined. Thus if p is a path in Γ then $\ell(p) \in (X \cup X^{-1})^*$.

For example, the Cayley graph $\Gamma(G, X)$ of a group G relative to a set X of generators is obviously labeled over $X \cup X^{-1}$: its vertices are the elements of G and there is a directed edge labeled by x from g to gx for each $g \in G$ and each $x \in X \cup X^{-1}$. The bouquet of circles B_X has one vertex and one positively oriented edge labeled by x for each $x \in X$: of course B_X also has negatively oriented edges labeled by elements $x^{-1} \in X^{-1}$ for each $x \in X$.

If Γ is labeled over $X \cup X^{-1}$ as above, then the associated natural map f_Γ from Γ to B_X that preserves edge labeling is a graph immersion. If $f : \Gamma \to \Gamma'$ is a graph immersion and the edges of Γ' are labeled over $X \cup X^{-1}$ as above, then this labeling induces a labeling of the edges of Γ in a natural way so that f preserves edge labeling and $f_{\Gamma'} \circ f = f_\Gamma$.

While the essential results in this paper may be formulated without resort to labeling the edges of our graphs, it is more convenient to do so. Hence we adopt the convention

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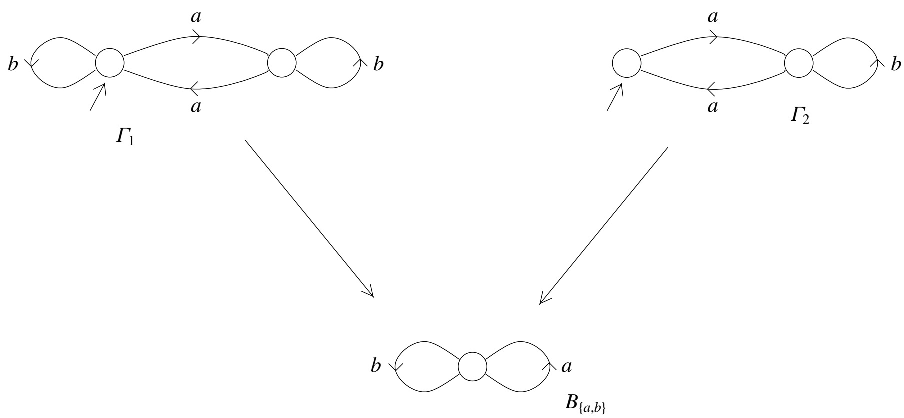

Figure 1.

that *all graphs that we will consider in this paper will be edge-labeled as described above and all immersions will preserve edge labeling*.

A simple example illustrating these ideas is provided in Figure 1: the natural map from the graph Γ_1 to $B_{\{a,b\}}$ is a cover, while the natural map from Γ_2 to $B_{\{a,b\}}$ is an immersion that is not a cover.

We now list some straightforward propositions that will be used in the sequel.

Proposition 2.1. *An immersion $f : \Gamma \to \Gamma'$ between connected edge-labeled graphs is uniquely determined by the image $f(v)$ of any vertex v in Γ. More precisely, if v is a vertex of Γ and v' is a vertex of Γ', then there is at most one graph immersion $f : \Gamma \to \Gamma'$ such that $f(v) = v'$. Such an immersion exists if and only if, for every path p in Γ with $\alpha(p) = v$, there is a path p' in Γ' with $\alpha(p') = v'$ and $\ell(p') = \ell(p)$ and such that p' is a circuit if p is a circuit.*

Proof. Suppose that there is an immersion f from Γ to Γ' such that $f(v) = v'$. Let v_1 be any vertex of Γ and p any path in Γ from v to v_1. Since f preserves edge labels, it maps paths in Γ to paths in Γ' with the same label, so there must be a path p' in Γ' with $\alpha(p') = v'$ and $\ell(p') = \ell(p)$. Furthermore, the path p' is unique since edge labeling is consistent with an immersion into B_X. It follows that we must have $f(v_1) = \omega(p')$. If p is a circuit, then $\omega(p) = v$ so $\omega(p') = f(\omega(p)) = f(v) = v'$, and so p' is a circuit. Also, if e is an edge in Γ with $\alpha(e) = v_1$, then we must have an edge e' in Γ' with $\alpha(e') = \omega(p')$ and $f(e) = e'$. The uniqueness of such an edge follows since $\ell(e') = \ell(e)$. So if there is an immersion from Γ to Γ' that maps v to v', there is only one such immersion.

Suppose conversely that, for every path p in Γ with $\alpha(p) = v$, there is a path p' in Γ' with $\alpha(p') = v'$ and $\ell(p') = \ell(p)$ and such that p' is a circuit if p is a circuit. Let v_1 be a vertex in Γ and p a path from v to v_1 in Γ. Then there is a (necessarily unique) path p' in Γ' with $\alpha(p') = v'$ and $\ell(p') = \ell(p)$. Define $f(v_1) = \omega(p')$. If p_1 is another path in Γ from v to v_1, then, as above, there is a unique path p'_1 in Γ' with $\alpha(p'_1) = v'$ and

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$\ell(p_1') = \ell(p_1)$. Since p_1p^{-1} is a circuit in Γ from v to v it follows by hypothesis that there is a circuit q' in Γ' at v' with $\ell(q') = \ell(p_1p^{-1}) = \ell(p_1)\ell(p^{-1}) = \ell(p_1')\ell(p'^{-1})$. So p_1' is an initial segment of q' and p'^{-1} is a terminal segment of q', and hence $p_1'p'^{-1}$ is a circuit in Γ' from v' to v' with $\ell(p_1'p'^{-1}) = \ell(q')$. It follows that $\omega(p_1') = \omega(p')$, so f is well defined on vertices. If e is an edge in Γ with $\alpha(e) = v_1$, then pe is a path in Γ starting at v so there is some path s' in Γ' starting at v' with $\ell(s') = \ell(pe) = \ell(p)\ell(e) = \ell(p')\ell(e)$. By uniqueness of a path starting at v' with this label and the fact that p' is a path starting at v' with $\ell(p') = \ell(p)$, it follows that there must be a (unique) edge e' in Γ' with $\alpha(e') = \omega(p')$ and $\ell(e') = \ell(e)$. Then define $f(e) = e'$. This is well defined by uniqueness of the edge e'. So f is a well-defined morphism from Γ to Γ' which is clearly an immersion since it preserves edge labeling. □

An *isomorphism* from the edge-labeled graph Γ onto the edge-labeled graph Γ' is a (label-preserving) graph morphism that is a bijection from vertices of Γ to vertices of Γ' and also a bijection from edges of Γ to edges of Γ'. Such an isomorphism must restrict to isomorphisms between the connected components of Γ and the connected components of Γ' and, by Proposition 2.1, it is determined by the image of any vertex in Γ if Γ is connected. The following fact is an easy consequence of Proposition 2.1.

Proposition 2.2. *Let Γ and Γ' be (edge-labeled) connected graphs, v a vertex in Γ and v' a vertex in Γ'. Then a bijective map $f : \Gamma \to \Gamma'$ such that $f(v) = v'$ is an isomorphism from Γ onto Γ' if and only if there is a label-preserving bijection $p \leftrightarrow p'$ between the paths p in Γ starting at v and the paths p' in Γ' starting at v' such that p is a circuit at v if and only if p' is a circuit at v'.*

Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion of connected graphs, v a vertex in Γ and $\tilde{v} \in f^{-1}(v)$. Then we say that a path p in Γ starting at v *lifts* to $\tilde{v}$ if there is a path $\tilde{p}$ in $\tilde{\Gamma}$ starting at $\tilde{v}$ such that $f(\tilde{p}) = p$. In this case it is clear that the lifted path $\tilde{p}$ is uniquely determined by $\tilde{v}$ and p since $\ell(\tilde{p}) = \ell(p)$. We refer to the proposition below as the "path lifting" proposition.

Proposition 2.3. *Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion between connected graphs. Then f is a cover if and only if, for each vertex v in Γ and each vertex $\tilde{v}$ in $f^{-1}(v)$, every path p in Γ starting at v lifts to a (unique) path in $\tilde{\Gamma}$ starting at $\tilde{v}$.*

Proof. If f is a cover, then the fact that every path p lifts to all preimages of v is easy to prove and may be viewed as a very special case of a much more general path lifting property for covers of topological spaces (see, for example, Proposition 1.30 of Hatcher's book [3]). Conversely, suppose that every path in Γ starting at v lifts to a path in $\tilde{\Gamma}$ starting at $\tilde{v}$. It follows that if e is an edge in Γ with $\alpha(e) = v$, then there is an edge $\tilde{e}$ in $\tilde{\Gamma}$ with $\alpha(\tilde{e}) = \tilde{v}$ and $f(\tilde{e}) = e$. So the map $f_{\tilde{v}}$ from $\mathrm{star}(\tilde{\Gamma}, \tilde{v})$ to $\mathrm{star}(\Gamma, v)$ is surjective. Since it is also injective by hypothesis, it is a bijection, and so the immersion f is a cover. □

The path lifting proposition above does not hold for immersions that are not covers in general, but it is easy to see that maximal initial segments of paths in Γ lift uniquely to paths in $\tilde{\Gamma}$, as described in the following proposition.

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Proposition 2.4. *Let* $f : \tilde{\Gamma} \to \Gamma$ *be an immersion between connected graphs, let* v *be a vertex of* Γ *and let* p *be a path in* Γ *with* $\alpha(p) = v$. *Then for every vertex* $\tilde{v} \in f^{-1}(v)$ *there is a unique (possibly empty) maximal initial segment* p_1 *of* p *that lifts to a path at* $\tilde{v}$. *Furthermore, the lift of* p_1 *at* $\tilde{v}$ *is unique.*

Proof. Since an immersion is locally injective on star sets, it is clear that an edge e of Γ starting at v lifts to at most one edge $\tilde{e}$ in $\tilde{\Gamma}$ starting at $\tilde{v}$. The result then follows by an easy inductive argument. □

We briefly summarize some of the most basic facts linking group theory and covers of graphs. Recall (see Stallings [16]) that two paths p and q in a graph Γ are said to be *homotopy equivalent* (written $p \sim q$) if and only if it is possible to pass from p to q by a finite sequence of insertions or deletions of paths of the form ee^{-1} for various edges e of Γ. Clearly $\alpha(p) = \alpha(q)$ and $\omega(p) = \omega(q)$ if $p \sim q$. We denote the equivalence class (homotopy class) of a path p in Γ by $[p]$. The *fundamental groupoid* $\pi_1(\Gamma)$ is a groupoid whose objects are the vertices of Γ and whose morphisms are the homotopy classes $[p]$. We regard $[p]$ as a morphism from $\alpha(p)$ to $\omega(p)$. The multiplication in the groupoid is defined by $[p][q] = [pq]$ if $\omega(p) = \alpha(q)$ and is undefined otherwise.

For each vertex v of Γ the set $\pi_1(\Gamma, v) = \{[p] : p \text{ is a circuit from } v \text{ to } v \text{ in } \Gamma\}$ is a group with respect to the multiplication in $\pi_1(\Gamma)$, called the *fundamental group* of Γ based at v. The following fact is classical and can be found in many sources, for example [16].

Proposition 2.5. *Let* Γ *be a connected graph and* v, w *vertices of* Γ. *Then:*

(a) $\pi_1(\Gamma, v)$ *is a free group whose rank is the number of positively oriented edges of* Γ *that are not in a spanning tree for* Γ.

(b) *If* p *is a path in* Γ *from* v *to* w, *then the map* $[q] \to [p][q][p^{-1}]$ *for* $[q] \in \pi_1(\Gamma, w)$ *defines an isomorphism from* $\pi_1(\Gamma, w)$ *onto* $\pi_1(\Gamma, v)$.

It is well known (see, for example, Hatcher's book [3]) that under suitable conditions on a connected topological space $\mathcal{X}$, connected covers of $\mathcal{X}$ may be classified via conjugacy classes of subgroups of the fundamental group of $\mathcal{X}$. The following version of this result for graph covers may be found in many sources, for example [3] or [16].

Theorem 2.6.

(a) *Let* $f : \tilde{\Gamma} \to \Gamma$ *be a cover of connected graphs, let* v *be a vertex of* Γ *and* $\tilde{v} \in f^{-1}(v)$. *Then* f *induces an embedding of* $\pi_1(\tilde{\Gamma}, \tilde{v})$ *into* $\pi_1(\Gamma, v)$. *If* $\tilde{v_1}$ *is another vertex in* $f^{-1}(v)$, *then the groups* $f(\pi_1(\tilde{\Gamma}, \tilde{v}))$ *and* $f(\pi_1(\tilde{\Gamma}, \tilde{v_1}))$ *are conjugate subgroups of* $\pi_1(\Gamma, v)$.

(b) *Conversely, let* Γ *be a connected graph,* v *a vertex in* Γ *and* $H \leq \pi_1(\Gamma, v)$. *Then there exist a unique (up to labeled graph isomorphism) connected graph* $\tilde{\Gamma}$, *a unique (up to equivalence) covering map* $f : \tilde{\Gamma} \to \Gamma$ *and a vertex* $\tilde{v} \in \tilde{\Gamma}$ *such that* $f(\tilde{v}) = v$ *and* $f(\pi_1(\tilde{\Gamma}, \tilde{v})) = H$.

If $f : \tilde{\Gamma} \to \Gamma$ is an immersion of connected graphs, a labeled graph automorphism γ of $\tilde{\Gamma}$ is called a *deck transformation* of $\tilde{\Gamma}$ if $f = f \circ \gamma$, that is, $f(\tilde{v}) = f(\gamma(\tilde{v}))$ for all vertices $\tilde{v}$ in $\tilde{\Gamma}$. The deck transformations of $\tilde{\Gamma}$ form a group $G(\tilde{\Gamma})$ with respect to composition of automorphisms.

A graph cover $f : \tilde{\Gamma} \to \Gamma$ is called a *normal* cover if, for every vertex v in Γ and every pair of vertices $\tilde{v}_1, \tilde{v}_2 \in f^{-1}(v)$, there is a deck transformation that takes $\tilde{v}_1$ to $\tilde{v}_2$; equivalently [3, Proposition 1.39], f is a normal cover if and only if the subgroup $H = f(\pi_1(\tilde{\Gamma}, \tilde{v}))$ of $\pi_1(\Gamma, f(v))$ that defines the cover is a normal subgroup of $\pi_1(\Gamma, f(v))$. The *universal cover* of Γ is the cover $\tilde{\Gamma}$ corresponding to the trivial subgroup of $\pi_1(\Gamma, f(v))$: a cover of Γ is isomorphic to the universal cover if and only if it is a tree.

If $f : \tilde{\Gamma} \to \Gamma$ is a *cover* of connected graphs, then there is a well-known connection between the group $G(\tilde{\Gamma})$ of deck transformations and the fundamental group of Γ. The following result is a special case of a more general standard result in topology (see, for example, [3, Proposition 1.39]).

THEOREM 2.7. *Let $f : \tilde{\Gamma} \to \Gamma$ be a cover of connected graphs, let $\tilde{v}$ be a vertex of $\tilde{\Gamma}$ with $f(\tilde{v}) = v$ and let H be the subgroup $H = f(\pi_1(\tilde{\Gamma}, \tilde{v}))$ of $\pi_1(\Gamma, v)$. Then $G(\tilde{\Gamma}) \cong N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(\Gamma, v)$. In particular, if $\tilde{\Gamma}$ is the universal cover of Γ, then $G(\tilde{\Gamma}) \cong \pi_1(\Gamma, v)$.*

While group theory provides a powerful algebraic tool for classifying and studying *covers* of graphs (or topological spaces in general), it appears that groups do not provide an adequate algebraic tool to classify *immersions* between graphs. For example, let $\Gamma(a)$ denote the Cayley graph of $\mathbb{Z} = Gp\langle a : \emptyset\rangle$ with respect to the generating set $\{a\}$. The vertices of $\Gamma(a)$ may be identified with the integers and there is a directed edge labeled by a from n to $n + 1$ for each integer n. Then $\Gamma(a)$ is the universal cover of the circle $B_{\{a\}}$. Any connected subgraph of $\Gamma(a)$ immerses into $B_{\{a\}}$ but all such graphs have trivial fundamental groups, so they cannot be distinguished by subgroups of $\mathbb{Z}$. We need a different algebraic tool to classify immersions and to encode the fact that paths only sometimes lift under immersions (Proposition 2.4). In subsequent sections of this paper, we show that the theory of *inverse monoids* provides a useful algebraic tool to study graph immersions and, in particular, to obtain analogues of Theorems 2.6 and 2.7.

3. Inverse monoids and their closed inverse submonoids

An *inverse monoid* is a monoid M such that for every element $a \in M$ there is a *unique* element a^{-1} in M such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$

It is clear that the elements aa^{-1} and $a^{-1}a$ of an inverse monoid are *idempotents* of M. In general, $aa^{-1} \neq a^{-1}a$ and neither of these idempotents is necessarily equal to the identity 1 of M. We denote the set of idempotents of an inverse monoid M

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by $E(M)$. An important elementary fact about inverse monoids is that their *idempotents commute*, that is, $ef = fe$ for all $e, f \in M$. In fact inverse monoids may be characterized alternatively as (von Neumann) regular monoids whose idempotents commute.

Inverse monoids provide an appropriate algebraic tool for studying *partial symmetry* of mathematical objects in much the same way as groups are used to study symmetry. We refer the reader to the book by Lawson [5] for an exposition of this point of view and for much basic information about inverse monoids.

A standard example of an inverse monoid is the *symmetric inverse monoid* $SIM(X)$ on a set X. This is the set of bijections between subsets of X with respect to the usual composition of partial maps. The inverse of a bijection $f \in SIM(X)$ with domain A and range B is the inverse map f^{-1} with domain B and range A: the idempotents of $SIM(X)$ are the identity maps on subsets of X (including the empty map 0 with domain the empty subset). The analogue for inverse monoids of Cayley's theorem for groups is the *Wagner–Preston theorem*, namely, every inverse monoid embeds in a suitable symmetric inverse monoid (see [5] for a proof of this theorem).

The *natural partial order* on an inverse monoid M is defined by $a \leq b$ if and only if $a = eb$ for some idempotent $e \in E(M)$ (equivalently $a = bf$ for some idempotent $f \in E(M)$, or equivalently $a = aa^{-1}b$, or equivalently $a = ba^{-1}a$). This extends the natural partial order on $E(M)$ defined by $e \leq f$ if and only if $e = ef = fe$. With respect to this partial order, $E(M)$ forms a *lower semilattice* with meet operation $e \wedge f = ef$ for all $e, f \in E(M)$. For the symmetric inverse monoid $SIM(X)$, the natural partial order corresponds to restriction of a partial one-to-one map to a subset of its domain. The semilattice of idempotents of $SIM(X)$ is of course the Boolean lattice of subsets of X with respect to inclusion.

For each subset N of an inverse monoid M, we denote by N^ω the set of all elements $m \in M$ such that $m \geq n$ for some $n \in N$. The subset N of M is called *closed* if $N = N^\omega$. Closed inverse submonoids of an inverse monoid M arise naturally in the representation theory of M by partial injections on a set, developed by Schein [14]. An inverse monoid M acts by injective partial functions on a set Q if there is a homomorphism from M to $SIM(Q)$. Denote by qm the image of q under the action of m if q is in the domain of the action by m. (Here we are considering M as acting on the right on Q.)

If an inverse monoid M acts on Q by injective partial functions, then for every $q \in Q$, $\mathrm{Stab}(q) = \{m \in M : qm = q\}$ is a closed inverse submonoid of M. Conversely, given a closed inverse submonoid H of M, we can construct a transitive representation of M as follows. A subset of M of the form $(Hm)^\omega$ where $mm^{-1} \in H$ is called a *right ω-coset* of H. Let X_H denote the set of right ω-cosets of H. If $(Hn)^\omega \in X_H$ and $m \in M$, define an action by $(Hn)^\omega m = (Hnm)^\omega$ if $(Hnm)^\omega \in X_H$ and undefined otherwise. This defines a transitive action of M (on the right) on X_H. Conversely, if M acts transitively on Q, then this action is equivalent in the obvious sense to the action of M on the right ω-cosets of $\mathrm{Stab}(q)$ in M for any $q \in Q$. See [14] or [12] for details. Dually, *left ω-cosets* of H in M (sets of the form $(mH)^\omega$ where $m^{-1}m \in H$) arise in connection with left actions of M by partial one-to-one maps on some set.

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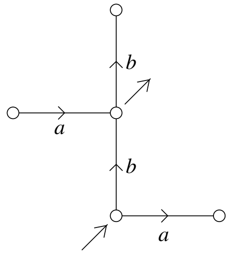

Figure 2.

If M is generated as an inverse monoid by a set X and H is a closed inverse submonoid of M then the graph of right ω-cosets of H in M is the graph with vertices the set X_H of right ω-cosets of H in M and with an edge labeled by $x \in X \cup X^{-1}$ from $(Hm)^\omega$ to $(Hmx)^\omega$ if $mm^{-1}, mxx^{-1}m^{-1} \in H$.

It is well known that free inverse monoids exist. We will denote the free inverse monoid on a set X by $FIM(X)$. The structure of free inverse monoids on one generator was determined by Gluskin [2]. The structure of free inverse monoids in general was determined much later independently by Scheiblich [13] and Munn [11]. Scheiblich's description for elements of $FIM(X)$ is in terms of rooted Schreier subsets of the free group $FG(X)$, while Munn's description is in terms of birooted edge-labeled trees. Scheiblich's description provides an important example of a McAlister triple, in the spirit of the McAlister P-theorem [8], while Munn's description lends itself most directly to a solution to the word problem for $FIM(X)$. It is not difficult to see the equivalence of the two descriptions.

A variation on Scheiblich's approach is provided in Lawson's book [5]. The version below is a slight variation on Munn's approach, the essential difference being that for some purposes it is somewhat more convenient to regard Munn's birooted trees as subtrees of the Cayley graph of the free group $FG(X)$, with the initial root identified with the vertex 1 in the Cayley graph.

Denote by $\Gamma(X)$ the Cayley graph of the free group $FG(X)$ with respect to the usual presentation, $FG(X) = Gp\langle X : \emptyset\rangle$. Thus $\Gamma(X)$ is an infinite tree whose vertices correspond to the elements of $FG(X)$ (in reduced form) and with a directed edge labeled by $x \in X$ from g to gx (and an inverse edge labeled by x^{-1} from gx to g). For each word $w \in (X \cup X^{-1})^*$, denote by $MT(w)$ the finite subtree of the tree $\Gamma(X)$ obtained by reading the word w as the label of a path in $\Gamma(X)$, starting at 1. Thus, for example, if $w = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$, then $MT(w)$ is the tree pictured in Figure 2.

One may view $MT(w)$ as a birooted tree, with initial root 1 and terminal root $r(w)$, the reduced form of the word w in the usual group-theoretic sense. Munn's solution [11] to the word problem in $FIM(X)$ may be stated in the following form.

Theorem 3.1. *If $u, v \in (X \cup X^{-1})^*$, then $u = v$ in $FIM(X)$ if and only if $MT(u) = MT(v)$ and $r(u) = r(v)$.*

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Thus elements of $FIM(X)$ may be viewed as pairs $(MT(w), r(w))$ (or as birooted edge-labeled trees, which was the way that Munn described his results). Multiplication in $FIM(X)$ is performed as follows. If $u, v \in (X \cup X^{-1})^*$, then $MT(uv) = MT(u) \cup r(u).MT(v)$ (just translate $MT(v)$ so that its initial root coincides with the terminal root of $MT(u)$ and take the union of $MT(u)$ and the translated copy of $MT(v)$: the terminal root is of course $r(uv)$). The idempotents of $FIM(X)$ are the *Dyck words*, that is the words w whose reduced form is 1: two Dyck words represent the same idempotent of $FIM(X)$ if and only if they have the same Munn tree. Munn's approach has been greatly extended by Stephen [17] to a general theory of presentations of inverse monoids by generators and relations.

In their paper [7], Margolis and Meakin studied closed inverse submonoids of free inverse monoids and showed how they could be used to classify immersions between connected graphs. In particular, they showed that closed inverse submonoids of free inverse monoids have surprisingly nice finiteness properties, and they may be constructed from free actions of groups on trees. A closed inverse submonoid of a free inverse monoid is not necessarily a free inverse monoid, but it admits an idempotent pure morphism onto a free inverse monoid. We briefly recall some of the results from the paper [7] that are relevant for our current purposes.

Recall that an *inverse category* C is a category with the property that for each morphism p there is a unique morphism p^{-1} such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$.

The *loop monoid* of such a category at the vertex (object) v is the set $L(C, v) = \{p : p$ is a morphism in C from v to $v\}$. Then $L(C, v)$ is an inverse monoid. Let Γ be a (connected) graph. Recall from [7] that the *free inverse category* $FIC(\Gamma)$ on Γ is the quotient of the free category $FC(\Gamma)$ on Γ by the category congruence $\sim_i$ induced by all relations of the form $p = pp^{-1}p$ and $pp^{-1}qq^{-1} = qq^{-1}pp^{-1}$ if $\alpha(p) = \alpha(q)$ for paths p, q in Γ. Denote the $\sim_i$-class of p by $[[p]]$. Then $FIC(\Gamma)$ is an inverse category. Of course if $\Gamma = B_X$ then $FIC(\Gamma) = FIM(X)$.

The *loop monoid* of $FIC(\Gamma)$ at the vertex v of Γ is

$$L(\Gamma, v) = \{[[p]] : p \text{ is a circuit in } \Gamma \text{ based at } v\}.$$

If Γ is labeled over $X \cup X^{-1}$ consistent with an immersion into B_X it follows that if $p \sim_i q$ for paths p, q in Γ, then $\ell(p)$ and $\ell(q)$ are equal in $FIM(X)$. So we will *view labels of paths in* Γ *as elements of* $FIM(X)$ throughout the sequel. We note that $FIM(X)$ acts on Γ^0 in a natural way (namely, $w \in FIM(X)$ acts on v_1 and takes v_1 to v_2 if there is a path labeled by w from v_1 to v_2). Then $L(\Gamma, v)$ is the stabilizer of v under this action, so *each loop monoid is a closed inverse submonoid of* $FIM(X)$. See [7, Proposition 4.3] for details.

Recall that two closed inverse submonoids H and K of an inverse monoid M are said to be *conjugate* (written $H \approx K$) if there exists $m \in M$ such that $mHm^{-1} \subseteq K$ and $m^{-1}Km \subseteq H$. Conjugation is an equivalence relation on the set of closed inverse submonoids of M. However, we caution that, unlike the situation in group theory, conjugate closed inverse submonoids of inverse monoids are not necessarily isomorphic. For example, the subsets $\{1, aa^{-1}, a^2a^{-2}\}$ and $\{1, aa^{-1}, a^{-1}a, aa^{-2}a\}$ of

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$FIM(\{a\})$ are conjugate closed inverse submonoids of $FIM(\{a\})$ that are clearly not isomorphic.

Immersions of connected graphs over B_X are classified via conjugacy classes of closed inverse submonoids of $FIM(X)$ as indicated in the following theorem [7, Theorem 4.4].

THEOREM 3.2. *Let Γ be a connected graph with edges labeled over $X \cup X^{-1}$ consistent with an immersion into B_X. Then each loop monoid is a closed inverse submonoid of $FIM(X)$ and the set of all loop monoids $L(\Gamma, v)$ for v a vertex of Γ is a conjugacy class of the set of closed inverse submonoids of $FIM(X)$. Conversely, if H is any closed inverse submonoid of a free inverse monoid $FIM(X)$ then there exist some graph Γ and an immersion $f : \Gamma \to B_X$ such that H is a loop monoid of $FIC(\Gamma)$. Furthermore, Γ is unique (up to graph isomorphism) and f is unique (up to equivalence).*

The results of this theorem can be extended somewhat to obtain a classification of immersions over arbitrary graphs, as indicated in the following theorem [7, Theorem 4.5]. This theorem may be viewed as the analogue for graph immersions of the classification theorem for graph covers (Theorem 2.6 above).

THEOREM 3.3. *Let $f : \Delta \to \Gamma$ be an immersion of connected graphs where Δ and Γ are edge-labeled over $X \cup X^{-1}$ consistent with immersions into B_X. If v is a vertex of Γ and $v_1 \in f^{-1}(v)$ then f induces an embedding of $L(\Delta, v_1)$ into $L(\Gamma, v)$. Conversely, let Γ be a graph edge-labeled over $X \cup X^{-1}$ as usual and let H be a closed inverse submonoid of $FIM(X)$ so that $H \subseteq L(\Gamma, v)$ for some vertex v in Γ. Then there exist a graph Δ, an immersion $f : \Delta \to \Gamma$ and a vertex v_1 in Δ such that $f(v_1) = v$ and $f(L(\Delta, v_1)) = H$. Furthermore, Δ is unique (up to graph isomorphism) and f is unique (up to equivalence). If H, K are two closed inverse submonoids of $FIM(X)$ with $H, K \subseteq L(\Gamma, v)$, then the corresponding immersions $f : \Delta \to \Gamma$ and $g : \Delta' \to \Gamma$ are equivalent if and only if $H \approx K$ in $FIM(X)$.*

REMARK 3.4. From the proof of Theorem 3.3 in [7], it follows that the graph Δ constructed in this theorem is the graph of right ω-cosets of H in $FIM(X)$.

Theorems 3.2 and 3.3 have been extended to classify immersions between two-dimensional CW-complexes and more generally between finite-dimensional Δ-complexes in the papers by Meakin and Szakács [9, 10]. In these more general cases it is necessary to construct closed inverse submonoids of certain inverse monoids presented by generators and relations associated with the complexes.

We close this section with a result about normalizers of closed inverse submonoids of inverse monoids. Part (a) of Proposition 3.5 below was observed in a paper by Lawson, Margolis and Steinberg [6, Lemma 2.10], but as far as we know the other parts of the proposition are new. This result will be needed in the construction of the group of deck transformations of a graph immersion later in this paper.

Let H be a closed inverse submonoid of an inverse monoid M. Define the *normalizer* $N(H)$ of H in M to be the set

$$N(H) = \{a \in M : aHa^{-1}, a^{-1}Ha \subseteq H\},$$

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Proposition 3.5. *Let H be a closed inverse submonoid of an inverse monoid M. Then:*

(a) *$N(H)$ is a closed inverse submonoid of M and H is a full inverse submonoid of $N(H)$ (that is, H contains all of the idempotents of $N(H)$);*

(b) *if $a \in N(H)$ then the right ω-coset $(Ha)^{\omega}$ and the left ω-coset $(aH)^{\omega}$ both exist and $(Ha)^{\omega} = (aH)^{\omega}$;*

(c) *the relation ρ_H on $N(H)$ defined by $a \, \rho_H \, c$ if and only if $(Ha)^{\omega} = (Hc)^{\omega}$ is a congruence on $N(H)$ and $N(H)/\rho_H$ is a group with operation $(Ha)^{\omega}.(Hb)^{\omega} = (Hab)^{\omega}$.*

Proof. (a) It is clear that $a^{-1} \in N(H)$ if $a \in N(H)$ and also that $N(H)$ contains the identity of M. Also if $a, b \in N(H)$ then $(ab)^{-1}H(ab) = b^{-1}a^{-1}Hab \subseteq b^{-1}Hb \subseteq H$, and similarly $(ab)H(ab)^{-1} \subseteq H$. Thus $ab \in N(H)$, so $N(H)$ is an inverse submonoid of $FIM(X)$. If $a \in N(H)$ and $b \geq a$ then $a = be$ for some idempotent $e \in M$. Hence if $h \in H$ then $beheb^{-1} = aha^{-1} \in H$. Hence $bhb^{-1} \in H$ since H is closed and $bhb^{-1} \geq beheb^{-1}$. Thus $bHb^{-1} \subseteq H$; similarly, $b^{-1}Hb \subseteq H$. Hence $b \in N(H)$ and so $N(H)$ is a closed inverse submonoid of M. It is clear that $H \subseteq N(H)$. If e is an idempotent of $N(H)$, then $e = e1e^{-1} \in H$ since $1 \in H$, so H is a full inverse submonoid of $N(H)$.

(b) If $a \in N(H)$ then aa^{-1}, $a^{-1}a \in H$ since H has an identity, so both left and right ω-cosets exist. Note that if $a \in N(H)$ then $aHa^{-1} \subseteq H$ and so $a^{-1}aHa^{-1}a \subseteq a^{-1}Ha \subseteq H$, so via conjugation by a we get $aHa^{-1} = aa^{-1}aHa^{-1}aa^{-1} \subseteq aa^{-1}Haa^{-1} \subseteq aHa^{-1}$. It follows that $aHa^{-1} = aa^{-1}Haa^{-1}$ and so $aHa^{-1}a = aa^{-1}Ha$.

Let $x \in (Ha)^{\omega}$, so $x \geq ha$, for some $h \in H$. Hence $x \geq aa^{-1}ha = ah_1a^{-1}a$ for some $h_1 \in H$ by the observation above. Since $h_2 = h_1a^{-1}a \in H$ and $x \geq ah_2$, it follows that $x \in (aH)^{\omega}$. Hence $(Ha)^{\omega} \subseteq (aH)^{\omega}$. A similar argument shows the converse inclusion.

(c) If $a \in N(H)$ then, by part (b), $(Ha)^{\omega}$ exists and clearly $a \in (Ha)^{\omega}$. This shows that ρ_H is a reflexive relation on $N(H)$. Since ρ_H is obviously symmetric and transitive, it is an equivalence relation on $N(H)$. It is well known that $(Ha)^{\omega}$=$(Hc)^{\omega}$ if and only if $ac^{-1} \in H$ (see, for example, [12, Lemma IV.4.5]). Suppose $a \, \rho_H \, c$ and $b \, \rho_H \, d$, so that $(Ha)^{\omega}$=$(Hc)^{\omega}$ and $(Hb)^{\omega}$=$(Hd)^{\omega}$. Then $ac^{-1}, bd^{-1} \in H$. We have $abd^{-1}c^{-1} \in aHc^{-1}$, so $abd^{-1}c^{-1} = ahc^{-1}$, for some $h \in H$. But $ah \in (aH)^{\omega} = (Ha)^{\omega}$ since $a \in N(H)$, so $ah \geq h_1a$ for some $h_1 \in H$. Hence $abd^{-1}c^{-1} \geq h_1ac^{-1} \in H$ and so $abd^{-1}c^{-1} \in H$. Hence $(Hab)^{\omega} = (Hcd)^{\omega}$, that is, ρ_H is a congruence on $N(H)$.

The operation of congruence classes in $N(H)/\rho_H$ is of course $(Ha)^{\omega}.(Hb)^{\omega} = (Hab)^{\omega}$. Clearly $H = (H1)^{\omega}$ is a right ω-coset. It follows that, by definition, $(H1)^{\omega}.(Ha)^{\omega} = (Ha)^{\omega} = (Ha)^{\omega}.(H1)^{\omega}$, so H acts as an identity. Also, if $a \in N(H)$, then $a^{-1} \in N(H)$ and $(Ha)^{\omega}.(Ha^{-1})^{\omega} = (Haa^{-1})^{\omega} = H$ since $aa^{-1} \in H$. Similarly, $(Ha^{-1})^{\omega}.(Ha)^{\omega} = H$, so $N(H)/\rho_H$ is a group. □

We denote the group $N(H)/\rho_H$ by $N(H)/H$ and refer to it as the *group of ω-cosets of H in $N(H)$*.

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4. The group of deck transformations

Throughout this section $f : \tilde{\Gamma} \to \Gamma$ is an immersion of connected graphs, v_0 is a fixed basepoint in Γ, $\tilde{v}_0 \in f^{-1}(v_0)$ is a fixed basepoint in $\tilde{\Gamma}$ and $H = f(L(\tilde{\Gamma}, \tilde{v}_0))$, a closed inverse submonoid of $L(\Gamma, v_0)$.

A *partial isomorphism* of $\tilde{\Gamma}$ is a (labeled graph) isomorphism $\phi : \Delta_1 \to \Delta_2$ between subgraphs Δ_1 and Δ_2 of $\tilde{\Gamma}$ that respects the immersion, that is, $f(\tilde{v}) = f(\phi(\tilde{v}))$ for all vertices $\tilde{v}$ in Δ_1. It follows that $f(\tilde{e}) = f(\phi(\tilde{e}))$ for all edges $\tilde{e}$ in Δ_1. It is convenient to also consider the empty map from $\emptyset$ to $\emptyset$ as a partial isomorphism: this is denoted by 0. We denote the set of partial isomorphisms of $\tilde{\Gamma}$ by $PI(\tilde{\Gamma})$. Partial isomorphisms may be composed in the usual way. Namely, if $\phi_1 : \Delta_1 \to \Delta_2$ and $\phi_2 : \Delta_3 \to \Delta_4$ are partial isomorphisms, then $\phi_2 \circ \phi_1$ is the corresponding partial isomorphism from $\phi_1^{-1}(\Delta_2 \cap \Delta_3)$ to $\phi_2(\Delta_2 \cap \Delta_3)$ defined by $(\phi_2 \circ \phi_1)(\tilde{e}) = \phi_2(\phi_1(\tilde{e}))$ for all vertices and edges $\tilde{e} \in \phi_1^{-1}(\Delta_2 \cap \Delta_3)$.

Proposition 4.1. *With respect to the multiplication above, $PI(\tilde{\Gamma})$ is an inverse monoid. The idempotents of $PI(\tilde{\Gamma})$ are the identity automorphisms on subgraphs of $\tilde{\Gamma}$ and the corresponding maximal subgroups are isomorphic to the group of deck transformations of the subgraph. In particular, the group of units of $PI(\tilde{\Gamma})$ is the group $G(\tilde{\Gamma})$ of deck transformations of $\tilde{\Gamma}$. The natural partial order on $\tilde{\Gamma}$ is defined by $\phi_1 \leq \phi_2$ if the domain of ϕ_1 is a subgraph of the domain of ϕ_2 and ϕ_1 is the restriction of ϕ_2 to the domain of ϕ_1.*

Proof. The proof that $PI(\tilde{\Gamma})$ is an inverse semigroup that has the stated properties is a standard routine argument similar to the proof of the corresponding properties for $SIM(X)$. For example, the identity of $PI(\tilde{\Gamma})$ is clearly the identity automorphism of $\tilde{\Gamma}$. A partial isomorphism $\phi : \Delta_1 \to \Delta_2$ is in the group of units of $PI(\tilde{\Gamma})$ if and only if $\Delta_1 = \Delta_2 = \tilde{\Gamma}$, so it follows that the group of units is $G(\tilde{\Gamma})$. □

If Δ is a connected subgraph of $\tilde{\Gamma}$ and $\tilde{v}$ is a vertex of Δ, then we define $H(\Delta, \tilde{v}) = f(L(\Delta, \tilde{v}))$. By Theorem 3.3 this is a closed inverse submonoid of $FIM(X)$ that is contained in $L(\Gamma, v)$ where $v = f(\tilde{v})$. In particular, if $\tilde{v}_0 \in \Delta$, then $H(\Delta, \tilde{v}_0) \leq H = H(\tilde{\Gamma}, \tilde{v}_0)$.

Lemma 4.2. *Let Δ_1 and Δ_2 be connected subgraphs of $\tilde{\Gamma}$ containing vertices $\tilde{v}_1$ and $\tilde{v}_2$, respectively. Then there is a partial isomorphism from Δ_1 onto Δ_2 that maps $\tilde{v}_1$ onto $\tilde{v}_2$ if and only if $H(\Delta_1, \tilde{v}_1) = H(\Delta_2, \tilde{v}_2)$.*

Proof. By Proposition 2.2 a partial isomorphism ϕ from Δ_1 onto Δ_2 that maps $\tilde{v}_1$ onto $\tilde{v}_2$ induces a label-preserving bijection from paths in Δ_1 that start at $\tilde{v}_1$ to paths in Δ_2 that start at $\tilde{v}_2$ and maps circuits to circuits. This bijection also maps $\sim_i$-equivalent paths to $\sim_i$-equivalent paths, so it induces an isomorphism from $L(\Delta_1, \tilde{v}_1)$ onto $L(\Delta_2, \tilde{v}_2)$. Also $f(\tilde{v}_1) = f(\tilde{v}_2)$. Denote this vertex of Γ by v. Since $L(\Delta_i, \tilde{v}_i) \cong H(\Delta_i, \tilde{v}_i)$ by Theorem 3.3, this means that there is a labeled graph isomorphism from $H(\Delta_1, \tilde{v}_1)$ onto $H(\Delta_2, \tilde{v}_2)$ that fixes v. But this implies that $H(\Delta_1, \tilde{v}_1) = H(\Delta_2, \tilde{v}_2)$ since the labeling of edges in Γ is consistent with an immersion into B_X.

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Conversely, suppose that $H(\Delta_1, \tilde{v}_1) = H(\Delta_2, \tilde{v}_2)$. Let $\tilde{q}_1$ be a path in Δ_1 starting at $\tilde{v}_1$ and let q be its projection into Γ. Then $\tilde{q}_1\tilde{q}_1^{-1} \in L(\Delta_1, \tilde{v}_1)$, so $qq^{-1} \in H(\Delta_1, \tilde{v}_1) = H(\Delta_2, \tilde{v}_2)$, so qq^{-1} lifts to a path $\tilde{q}_2\tilde{q}_2^{-1}$ with $\tilde{q}_2\tilde{q}_2^{-1} \in L(\Delta_2, \tilde{v}_2)$. Hence q lifts to the path $\tilde{q}_2$ in Δ_2 starting at $\tilde{v}_2$. Clearly $\ell(\tilde{q}_1) = \ell(\tilde{q}_2) = \ell(q)$. Dually, if $\tilde{q}_2$ is a path in Δ_2 starting at $\tilde{v}_2$ then there is a corresponding path $\tilde{q}_1$ in Δ_1 starting at $\tilde{q}_1$. The correspondence $\tilde{q}_1 \leftrightarrow \tilde{q}_2$ is one-to-one by equality of labels on these paths. Since it also maps circuits to circuits, it induces an isomorphism from Δ_1 onto Δ_2 that maps $\tilde{v}_1$ onto $\tilde{v}_2$ by Proposition 2.2. □

If v is a vertex in the connected graph Γ and H is a closed inverse submonoid of $L(\Gamma, v)$, then we denote by $N(H, v)$ the normalizer of H in $L(\Gamma, v)$, that is, $N(H, v) = \{p \in L(\Gamma, v) : pHp^{-1}, p^{-1}Hp \subseteq H\}$.

Lemma 4.3. *Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion between connected graphs, v_0 a vertex of Γ, $\tilde{v}_0 \in f^{-1}(v_0)$ and $H = f(L(\tilde{\Gamma}, \tilde{v}_0))$. Then:*

(a) *If there is a deck transformation that maps $\tilde{v}_0$ to $\tilde{v}_1$, then any path $\tilde{p}$ from $\tilde{v}_0$ to $\tilde{v}_1$ projects onto a path $p \in N(H, v_0)$.*
(b) *Each path $p \in N(H, v_0)$ lifts to a path $\tilde{p}$ in $\tilde{\Gamma}$ starting at $\tilde{v}_0$.*
(c) *Each path $p \in N(H, v_0)$ determines a deck transformation ϕ_p of $\tilde{\Gamma}$ that maps $\tilde{v}_0$ to $\omega(\tilde{p}) \in f^{-1}(v_0)$. Furthermore, if q is another element of $N(H, \tilde{v}_0)$, then $\phi_p = \phi_q$ if and only if $\omega(\tilde{p}) = \omega(\tilde{q})$.*

Proof. (a) Suppose there is a deck transformation that maps $\tilde{v}_0$ to $\tilde{v}_1$ and $\tilde{p}$ is a path from $\tilde{v}_0$ to $\tilde{v}_1$. Then $\tilde{p}L(\tilde{\Gamma}, \tilde{v}_1)\tilde{p}^{-1} \subseteq L(\tilde{\Gamma}, \tilde{v}_0)$ and $\tilde{p}^{-1}L(\tilde{\Gamma}, \tilde{v}_0)\tilde{p} \subseteq L(\tilde{\Gamma}, \tilde{v}_1)$. Since $f(L(\tilde{\Gamma}, \tilde{v}_1)) = f(L(\tilde{\Gamma}, \tilde{v}_0)) = H$ by Lemma 4.2, this implies that $pHp^{-1} \subseteq H$ and $p^{-1}Hp \subseteq H$, so $p \in N(H, v_0)$.

(b) Let $p \in N(H, v_0)$. Since $pp^{-1} \in H$ and every circuit in H based at v_0 lifts to a circuit in $\tilde{\Gamma}$ based at $\tilde{v}_0$, it follows that pp^{-1} lifts to a circuit which has the same label as pp^{-1} based at $\tilde{v}_0$. By uniqueness of a path starting at $\tilde{v}_0$ with a given label, this circuit must be of the form $\tilde{p}\tilde{p}^{-1}$ for some lift $\tilde{p}$ of the path p starting at $\tilde{v}_0$.

(c) Let p and $\tilde{p}$ be as in part (b) above, denote $\omega(\tilde{p})$ by $\tilde{v}_1$ and let $H_1 = f(L(\tilde{\Gamma}, \tilde{v}_1))$. If $h \in H$, then $php^{-1} \in H$ since $p \in N(H, v_0)$. This lifts to a circuit $\tilde{p}\tilde{h}_1\tilde{p}^{-1}$ based at $\tilde{v}_0$ since every circuit in H lifts to a circuit in $\tilde{\Gamma}$ based at $\tilde{v}_0$. This forces $\tilde{h}_1$ to be a circuit in $\tilde{\Gamma}$ based at $\tilde{v}_1$ and with the same label as h, and so $h = f(\tilde{h}_1) \in H_1$. Hence $H \subseteq H_1$.

Conversely, suppose that $h \in H_1$. Then h lifts to a circuit $\tilde{h}_1$ in $\tilde{\Gamma}$ based at $\tilde{v}_1$. So $\tilde{p}\tilde{h}_1\tilde{p}^{-1} \in L(\tilde{\Gamma}, \tilde{v}_0)$, and so $php^{-1} \in H$. It follows that $p^{-1}php^{-1}p \in H$ since $p \in N(H, v_0)$, so this lifts to a circuit of the form $\tilde{p}^{-1}\tilde{p}\tilde{h}\tilde{p}^{-1}\tilde{p}$ based at $\tilde{v}_0$ for some path $\tilde{h}$ with the same label as h. This path $\tilde{h}$ must be a circuit based at $\tilde{v}_0$, and so $f(\tilde{h}) = h \in H$. Hence $H_1 \subseteq H$ and so $H_1 = H$. By Lemma 4.2 this implies that there is a deck transformation ϕ_p of $\tilde{\Gamma}$ that maps $\tilde{v}_0$ onto $\tilde{v}_1$. The last statement of part (c) of the lemma is immediate since automorphisms of an edge-labeled graph are determined by where they send a point. □

The following theorem provides an analogue for graph immersions of Theorem 2.7.

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THEOREM 4.4. *Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion of connected graphs with $\tilde{v}_0 \in f^{-1}(v_0)$ and $H = f(L(\tilde{\Gamma}, \tilde{v}_0))$. Then $G(\tilde{\Gamma}) \cong N(H, v_0)/H$.*

PROOF. Define $r : N(H, v_0) \to G(\tilde{\Gamma})$ by $r(p) = \phi_p$ in the notation of Lemma 4.3. If $p, q \in N(H, v_0)$, then by Proposition 2.2 the deck transformation ϕ_p maps $\omega(\tilde{q})$ to $\omega(\tilde{q}_1)$ where $\tilde{q}_1$ is a path starting at $\omega(\tilde{p})$ with $\ell(\tilde{q}_1) = \ell(\tilde{q})$. So $\phi_{pq}(\tilde{v}_0) = \omega(\tilde{p}\tilde{q}_1) = \omega(\tilde{q}_1) = \phi_p(\omega(\tilde{q})) = \phi_p(\phi_q(\tilde{v}_0))$, and so $\phi_{pq} = \phi_p \circ \phi_q$. Hence r is a homomorphism. It is surjective by Lemma 4.3. Now $\tilde{\Gamma}$ is identified with the graph of right ω-cosets of H in $FIM(X)$, and under this identification a vertex $\tilde{v}$ of $\tilde{\Gamma}$ is identified with the right ω-coset $(Hp)^\omega$ where p lifts to a path $\tilde{p}$ in $\tilde{\Gamma}$ starting at $\tilde{v}_0$ and ending at $\tilde{v}$. By Lemma 4.3, $\phi_p = \phi_q$ if and only if $\omega(\tilde{p}) = \omega(\tilde{q})$, so the kernel of the map r coincides with the equivalence relation ρ_H that identifies p and q if $(Hp)^\omega = (Hq)^\omega$. It follows that $G(\tilde{\Gamma}) \cong N(H, v_0)/H$. □

5. Covers

In this section we characterize covers of graphs in terms of the concepts introduced earlier. A result related to part (a) of the following theorem was obtained by Meakin and Szakács [9] in the more general context of immersions between 2-complexes.

THEOREM 5.1. *Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion of connected graphs. Choose basepoints $v_0 \in \Gamma$ and $\tilde{v}_0 \in \tilde{\Gamma}$ such that $\tilde{v}_0 \in f^{-1}(v_0)$ and let $H = f(L(\tilde{\Gamma}, \tilde{v}_0))$. Then:*

(a) *$\tilde{\Gamma}$ is a cover of Γ if and only if H is a full inverse submonoid of $L(\Gamma, v_0)$.*
(b) *$\tilde{\Gamma}$ is a normal cover of Γ if and only if $N(H, v_0) = L(\Gamma, v_0)$.*
(c) *$\tilde{\Gamma}$ is the universal cover of Γ if and only if H consists of the idempotents in $L(\Gamma, v_0)$, that is, $H = \{p \in L(\Gamma, v_0) : \ell(p)$ is a Dyck word$\}$.*

PROOF. (a) Suppose that $\tilde{\Gamma}$ is a cover of graphs. Then every path p in Γ that starts at v_0 lifts to a (unique) path $\tilde{p}$ starting at $\tilde{v}_0$ by Proposition 2.3. In particular, if e is an idempotent of $L(\Gamma, v_0)$, then $\ell(e)$ is a Dyck word since the only idempotents in $FIC(\tilde{\Gamma})$ are Dyck words, so $\ell(\tilde{e})$ is a path starting at $\tilde{v}_0$ whose label is a Dyck word. This forces $\tilde{e}$ to be a circuit at $\tilde{v}_0$, so $e = f(\tilde{e}) \in H$. Hence H is full in $L(\Gamma, v_0)$. Conversely, suppose that H is full in $L(\Gamma, v_0)$. Let v_1 be any vertex in Γ and $\tilde{v}_1$ any vertex in $f^{-1}(v_1)$ and let p be a path in Γ starting at v_1. There is a path $\tilde{q}$ in $\tilde{\Gamma}$ from $\tilde{v}_0$ to $\tilde{v}_1$ and the projection of this path is a path q in Γ from v_0 to v_1. Then $qpp^{-1}q^{-1}$ is an idempotent in $L(\Gamma, v_0)$ so it is in H and hence it lifts to a path $\tilde{q}_1\tilde{p}_1\tilde{p}_1^{-1}\tilde{q}_1^{-1}$ in $\tilde{\Gamma}$ starting at $\tilde{v}_0$. Since $\ell(\tilde{q}_1) = \ell(q) = \ell(\tilde{q})$ we must have $\tilde{q}_1 = \tilde{q}$. Hence $\omega(\tilde{q}_1) = \tilde{v}_1$, and so $\tilde{p}_1$ is a lift of p that starts at $\tilde{v}_1$. Hence all paths in Γ lift everywhere, so $\tilde{\Gamma}$ is a cover of Γ by Proposition 2.3.

(b) Suppose that $\tilde{\Gamma}$ is a normal cover of Γ and $v_0 \in \Gamma$. Then for all vertices $\tilde{v}_0, \tilde{v}_1 \in f^{-1}(v_0)$ there is a (unique) deck transformation ϕ mapping $\tilde{v}_0$ onto $\tilde{v}_1$. If $p \in L(\Gamma, v_0)$ then pp^{-1} is an idempotent in $L(\Gamma, v_0)$ so $pp^{-1} \in H$ by part (a). Hence pp^{-1} lifts to a path $\tilde{p}\tilde{p}^{-1}$ at $\tilde{v}_0$ and so p lifts to the path $\tilde{p}$ from $\tilde{v}_0$ to some vertex $\tilde{v}_1$. Since there is a deck transformation ϕ that maps $\tilde{v}_0$ onto $\tilde{v}_1$, part (a) of Lemma 4.3 implies that $p \in N(H, v_0)$. Hence $L(\Gamma, v_0) \subseteq N(H, v_0)$ and so $L(\Gamma, v_0) = N(H, v_0)$.

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Conversely, suppose that $L(\Gamma, v_0) = N(H, v_0)$ and e is an idempotent in $L(\Gamma, v_0)$. Then $e \in N(H, v_0)$. But H is full in $N(H, v_0)$ by Proposition 3.5, so $e \in H$. Hence $\tilde{\Gamma}$ is a cover of Γ by part (a). Now let $\tilde{v}_0, \tilde{v}_1 \in f^{-1}(v_0)$. There is some path $\tilde{r}$ from $\tilde{v}_0$ to $\tilde{v}_1$ that projects to a path $r \in L(\Gamma, v_0) = N(\Gamma, v_0)$. So by Lemma 4.3 there is a deck transformation that maps $\tilde{v}_0$ to $\tilde{v}_1$, and so $\tilde{\Gamma}$ is a normal cover of Γ.

(c) Suppose that $\tilde{\Gamma}$ is the universal cover of Γ. Then $\tilde{\Gamma}$ is a tree, so the label of any circuit $\tilde{p}$ in $\tilde{\Gamma}$ based at any point is a Dyck word. Such circuits project onto circuits in Γ whose label is also a Dyck word, so H consists just of idempotents in $L(\Gamma, v_0)$. On the other hand, if e is an idempotent in $L(\Gamma, v_0)$ then $\ell(e)$ is a Dyck word (since any idempotent in $FIC(\Gamma)$ is a path whose label is a Dyck word). Since e lifts to a path $\tilde{e}$ based at $\tilde{v}_0$ and since $\ell(\tilde{e})$ is a Dyck word, $\tilde{e}$ is a circuit based at $\tilde{v}_0$, so $f(\tilde{e}) = e \in H$. Hence H consists exactly of all the Dyck words in $L(\Gamma, v_0)$. Conversely, if H consists of these idempotents, then H is full in $L(\Gamma, v_0)$, so $\tilde{\Gamma}$ is a cover of Γ by part (a). By definition of H, any circuit in $\tilde{\Gamma}$ based at $\tilde{v}_0$ projects onto a circuit in H based at v_0, so its label must be a Dyck word. It follows that $\tilde{\Gamma}$ is a tree. Hence $\tilde{\Gamma}$ is the universal cover of Γ. □

6. Actions of groups on graphs

By an *action* of a group G on a graph Γ we mean a homomorphism $\phi : G \to Aut(\Gamma)$ of G into the group of (labeled graph) automorphisms of Γ. Here we adopt the convention that the action is a left action and denote the image under $\phi(g)$ of the vertex v (or edge e) by $g.v$ (respectively, $g.e$). If $f : \tilde{\Gamma} \to \Gamma$ is an immersion between connected graphs then there is an obvious action of the group $G(\tilde{\Gamma})$ on $\tilde{\Gamma}$. The following fact is well known for *covers* of connected graphs, but also holds for immersions (with essentially the same proof).

Proposition 6.1. *If $f : \tilde{\Gamma} \to \Gamma$ is an immersion of connected graphs with group $G = G(\tilde{\Gamma})$ of deck transformations, then the quotient map $g : \tilde{\Gamma} \to \tilde{\Gamma}/G$ is a normal covering and G is the group of deck transformations of this cover.*

Proof. If γ is a deck transformation of the immersion f then, by Proposition 2.1, γ is uniquely determined by the image of any vertex in Γ, and it follows that if v is a vertex and e is a edge of Γ and γ is not the identity automorphism of Γ, then $\gamma(v) \neq v$ and $\gamma(e) \neq e$. From this it follows that the action of G on $\tilde{\Gamma}$ satisfies condition $(*)$ on page 72 of Hatcher [3] and so the result follows by [3, Proposition 1.40]. □

Example 6.2. The graph $\tilde{\Gamma}$ in Figure 3 has four vertices v_1, v_2, v_3 and v_4 and six positively labeled edges. It immerses in the obvious way into the bouquet $B_{\{a,b\}}$ of two circles via the map f that preserves edge labels.

Let e_1 be the directed edge from v_1 to v_4 with label a; e_2 the directed edge from v_2 to v_3 with label a; e_3 the directed edge from v_3 to v_2 with label a; e_4 the directed edge from v_4 to v_1 with label a; e_5 the directed edge from v_1 to v_2 with label b; and e_6 the directed edge from v_4 to v_3 with label b.

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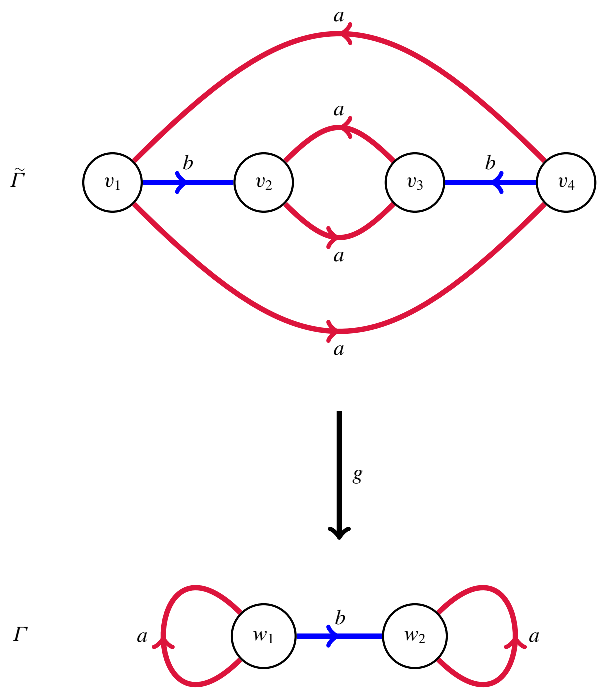

FIGURE 3.

The stabilizer of the vertex v_1 under the action by $FIM(a, b)$ is the closed inverse submonoid H of $FIM(a, b)$ generated by the elements $ba^2b^{-1}, aba^{-1}b^{-1}$ and $bab^{-1}a$. The graph $\tilde{\Gamma}$ is the graph of right ω-cosets of H in $FIM(a, b)$. The distinct right ω-cosets of H are of course $H, (Hb)^\omega, (Hba)^\omega$ and $(Ha)^\omega$, and they may be identified with the four vertices v_1, v_2, v_3 and v_4, respectively.

The map that interchanges e_1 and e_4, interchanges e_2 and e_3, and interchanges e_5 and e_6 defines a deck transformation γ of the immersion f that interchanges v_1 with v_4 and v_2 with v_3. This is the only nontrivial deck transformation, so the group $G = G(\tilde{\Gamma})$ is isomorphic to the cyclic group $\mathbb{Z}_2$ of order 2. Note that the only right ω-coset of H that lies in $N(H)$ is $(Ha)^\omega$, so $N(H)/H = \{H, (Ha)^\omega\}$ is isomorphic to $\mathbb{Z}_2$ in accord with Theorem 4.4.

The group $G = \mathbb{Z}_2$ acts on $\tilde{\Gamma}$ in the obvious way and the quotient graph $\Gamma = \tilde{\Gamma}/G$ is the graph with two vertices w_1, w_2 and one positively oriented edge from w_1 to w_2 with label b, as depicted in Figure 3. Clearly the map g from $\tilde{\Gamma}$ to Γ that preserves edge labels is a normal cover of $\Gamma = \tilde{\Gamma}/G$, in accord with Proposition 6.1. □

7. Extending immersions to covers

THEOREM 7.1. *Let $f : \tilde{\Gamma} \to \Gamma$ be an immersion of connected graphs. Then there is a graph cover $g : \tilde{\Delta} \to \Gamma$ such that:*

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(a) $\tilde{\Gamma}$ *is a subgraph of* $\tilde{\Delta}$ *and* f *is the restriction of* g *to* $\tilde{\Gamma}$*; and*
(b) *any deck transformation of* $\tilde{\Gamma}$ *is the restriction of some deck transformation of* $\tilde{\Delta}$.

PROOF. (a) If $f : \tilde{\Gamma} \to \Gamma$ is a cover of graphs there is nothing to prove, so assume that this is not the case. Let the edges of Γ and $\tilde{\Gamma}$ be labeled over a set $X \cup X^{-1}$ consistent with an immersion into B_X as usual. Then there is a vertex $\tilde{v}$ of $\tilde{\Gamma}$ for which there is an edge e in Γ with $f(\tilde{v}) = \alpha(e)$ such that e does not lift to any edge in $\tilde{\Gamma}$ starting at $\tilde{v}$. If $\ell(e) = x \in X \cup X^{-1}$ then there is no edge with label x starting at $\tilde{v}$. We refer to such a vertex as an *incomplete vertex* of $\tilde{\Gamma}$ and say that $\tilde{v}$ is *missing an edge labeled by* x. Enlarge the graph $\tilde{\Gamma}$ by adding a new vertex $\tilde{v}_x$ and a new edge $\tilde{e}_x$ from $\tilde{v}$ to $\tilde{v}_x$ for each incomplete vertex $\tilde{v}$ of $\tilde{\Gamma}$ that is missing an edge labeled by x. Since distinct edges starting at $f(\tilde{v})$ have distinct labels, the new vertices and edges that we added to $\tilde{\Gamma}$ are all distinct. Clearly the graph $\tilde{\Delta}_1$ immerses into Γ via the map that preserves edge labeling. Then apply the same process to Δ_1, adding new edges as necessary at any incomplete vertices, to form the graph Δ_2. Continue in this fashion to build a sequence of graphs $\tilde{\Gamma} \subseteq \tilde{\Delta}_1 \subseteq \tilde{\Delta}_2 \subseteq \cdots$ by adding new vertices and edges to the previous graph at any incomplete vertices. Let $\tilde{\Delta}$ be the union of the graphs $\tilde{\Delta}_i$ as i ranges from 1 to ∞. The graph $\tilde{\Delta}$ is obtained from $\tilde{\Gamma}$ by adding (possibly infinite) trees to incomplete vertices of $\tilde{\Gamma}$. Then the map $g : \tilde{\Delta} \to \Gamma$ that extends f and maps paths in $\tilde{\Delta} \setminus \tilde{\Gamma}$ to their obvious images in Γ is a covering map since $\tilde{\Delta}$ has no incomplete vertices.

(b) Suppose now that γ is a deck transformation of the immersion $f : \tilde{\Gamma} \to \Gamma$ that takes a vertex $\tilde{v}$ to a vertex $\tilde{v}_1$. Any path $\tilde{p}$ in $\tilde{\Delta}$ starting at $\tilde{v}$ factors in the form $\tilde{p} = \tilde{p}_1\tilde{q}_1\tilde{p}_2\tilde{q}_2 \cdots \tilde{p}_n\tilde{q}_n$, where $\tilde{p}_1\tilde{p}_2 \cdots \tilde{p}_n$ is a path in $\tilde{\Gamma}$ and the $\tilde{q}_i$ are circuits in the forest $\tilde{\Delta} \setminus \tilde{\Gamma}$ based at $\omega(\tilde{p}_i)$ for $i = 1, \ldots, n-1$ and $\tilde{q}_n$ is a path in $\tilde{\Delta} \setminus \tilde{\Gamma}$ starting at $\omega(\tilde{p}_n)$. By Proposition 2.2, there is a path $\tilde{p}'_1$ in $\tilde{\Gamma}$ starting at $\tilde{v}_1$ with $\ell(\tilde{p}'_1) = \ell(\tilde{p}_1)$. Also by the same proposition, there is an edge $\tilde{e}$ in $\tilde{\Gamma}$ starting at $\omega(\tilde{p}_1)$ with label x if and only if there is an edge $\tilde{e}'$ in $\tilde{\Gamma}$ starting at $\omega(\tilde{p}')$ with the same label. It follows that there is a path of the form $\tilde{q}'_1$ in $\tilde{\Delta} \setminus \tilde{\Gamma}$ with $\ell(\tilde{q}'_1) = \ell(\tilde{q}_1)$, $\alpha(\tilde{q}'_1) = \omega(\tilde{p}'_1)$, and $\tilde{q}'_i$ is a circuit if and only if $\tilde{q}_i$ is a circuit. Continuing in this fashion, we see that there is a path $\tilde{p}' = \tilde{p}'_1\tilde{q}'_1\tilde{p}'_2\tilde{q}'_2 \cdots \tilde{p}'_n\tilde{q}'_n$ with $\ell(\tilde{p}') = \ell(\tilde{p})$. Furthermore, $\tilde{p}'$ is a circuit if and only if $\tilde{p}$ is a circuit, since $\omega(\tilde{p}') = \gamma(\omega(\tilde{p}))$. Hence by Proposition 2.2 there is a deck transformation $\tilde{\gamma}$ of $\tilde{\Delta}$ that takes $\tilde{v}$ to $\tilde{v}_1$. The bijection $\tilde{\gamma}$ extends γ and maps a path in $\tilde{\Delta} \setminus \tilde{\Gamma}$ starting at an incomplete vertex $\tilde{w}$ to the path with the same label starting at $\gamma(\tilde{w})$. □

We remark that it is not clear that it is possible to extend *finite* immersions between graphs to *finite* covers in such a way that deck transformations of the immersion are restrictions of deck transformations of the cover.

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