THE PRINCIPAL IDEAL THEOREM IN PRIME NOETHERIAN RINGS

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1. Introduction. In the study of certain prime Noetherian rings it is natural to consider the set C of elements which are regular modulo all height-1 prime ideals of R. For R commutative, this set C is simply the set of units. In general this is not the case, though with certain additional conditions we can state non-commutative versions of the Principal Ideal Theorem.

We start by giving some equivalent forms of the classical principal ideal theorem of commutative algebra. For an ideal I of a ring R we use $C(I)$ to denote the set of elements of R which are regular (i.e. not zero-divisors) modulo I. For general background information and terminology we refer to Chatters–Hajarnavis [3].

Standard notation: $C = \bigcap C(P)$, when P ranges over all the height-1 primes of R.

THEOREM 1.1. *Let R be a commutative Noetherian integral domain. Then the following statements are true*:

(a) (Classical Principal Ideal Theorem) *if a is a non-zero non-unit element of R and P is a prime ideal minimal over a then* $\mathrm{height}(P) = 1$;

(b) *the elements of C are units*;

(c) *every non-zero prime ideal of R is the union of the height-1 prime ideals it contains*;

(d) *if P is a non-zero prime ideal of R then $C(P) = \bigcap C(Q)$, where the intersection ranges over those height-1 prime ideals $Q \subseteq P$*;

(d′) *in the notation of* (d), $\bigcap C(Q) \subseteq C(P)$.

Proof. See, for example, Theorem 142, Commutative Rings, Irving Kaplansky.

It is trivial to show that the five statements are all equivalent for commutative Noetherian rings, and that they suggest non-commutative generalizations which are likely to be inequivalent and not generally true. Some non-commutative versions of (a) have been proved, but they require a to satisfy a special condition such as $aR = Ra$, or the prime P is chosen minimal with respect to containing all the ideals contained in aR (see, for example, Jategaonkar [7], [9] or Chatters–Hajarnavis [3, Chapter 3]).

We shall prove that statement (b) is true for all prime Noetherian PI rings and we shall also prove some results similar to (d′). The reason for concentrating on (d′) rather than (d) is that if P and Q are prime ideals of a non-commutative ring then it is possible to have $Q \subseteq P$ and $C(P) \nsubseteq C(Q)$. Statement (c) is often false even in very well-behaved non-commutative Noetherian rings, as is illustrated by the next two examples.

EXAMPLE 1.2. Let $S = F[x_1, \ldots, x_n]$, the commutative polynomial ring in n indeterminates $x_1, x_2, \ldots, x_n$ over a field F, and set $R = M_n(S)$. Then R is a prime Noetherian P.I. ring which is a maximal order and module-finite over its centre.

If $P = x_1 S + \ldots + x_n S$ then $M_n(P)$ is a prime ideal of R of height n. Let $a = \text{diag}(x_1, \ldots, x_n) \in R$. Then $RaR = M_n(P)$. Thus $M_n(P)$ is the only prime ideal of R containing a, and hence (a) and (c) of Theorem 2.1 are false. The height-1 primes of R are of the form $M_n(Q)$ where Q is a height-1 prime ideal of S, and an easy argument using determinants shows that (b), (d) and (d') of Theorem 1.1 are true.

EXAMPLE 1.3. Let R be the universal enveloping algebra of the 2-dimensional non-Abelian solvable Lie algebra over the field F of complex numbers. Thus R is the F-algebra with generators x and y subject to $xy - yx = y$. It is well known that R is a Noetherian domain with a unique height-1 prime ideal $yR = Ry$ and the maximal ideals of R all have height 2 and are of the form $(x - a)R + yR$ with $a \in F$. In this case $C = R \setminus Ry$. In particular $x \in C$ and x is not a unit of R, and $RxR = xR + yR$. Hence all the statements in Theorem 1.1 are false for R.

EXAMPLE 1.4. Let R be a prime Noetherian ring with a unique proper ideal P (e.g. Robson [13, Example 7.3]). We have $P = P^2$ and $P = x_1 R + \ldots + x_n R$ for some $x_i \in P$. Then in $M_n(R)$ the only height-1 prime ideal is $M_n(P)$. Now $x_i = \sum_{j=1}^{n} x_j \alpha_{ij}$, for some $\alpha_{ij} \in P$. Thus, setting $A = (\alpha_{ij})$, $A - I_n \in C(M_n(P))$, but clearly $A - I_n$ is not regular in $M_n(R)$.

In the next section, we shall work in the context of a ring R which is prime Noetherian fully bounded and every non-zero ideal of R contains a non-zero central element. The reason for considering this class of rings is that it includes all prime Noetherian rings which either satisfy a polynomial identity or are integral over their centre. This class is also closed under localisations at Ore sets.

2. The elements of C.   We recall our standard notation $C = \bigcap C(P)$, where the intersection ranges over all height-1 prime ideals P of R. In the cases which concern us here it will turn out that all the elements of C are always regular, and this is certainly the case in a prime Noetherian ring with an infinite number of height-1 prime ideals.

The following lemma is known, but is not readily available in the literature, so we include a proof for the readers' convenience.

LEMMA 2.1. *Let R be a fully bounded Noetherian ring with nilpotent radical N, and let $c \in C(N)$. Then cR contains a non-zero ideal of R.*

REMARK. This is a consequence of Krause–Lenagan–Stafford [10, Lemma 3].

Proof. Since R is fully bounded Noetherian, N has weak ideal invariance (Stafford [17]). That is, if K is a right ideal of R, and $|M|$ denotes the Krull dimension of an R-module M, then $|R/K| < |R/N|$ implies that $|N/KN| < |R/N|$.

Now suppose cR contains no non-zero ideal of R. Then $|R/cR| = |R| = |R/N|$ (Jategaonkar [8, Lemma 2.1]). But $c \in C(N)$; so $|R/cR + N| < |R/N|$. So we proceed to

show by induction that $|R/cR + N^k| < |R/N|$ for $k = 1, 2, \ldots$. Suppose that $|R/cR + N^{k-1}| < |R/N|$. Then, by weak ideal invariance, $|N/(cR + N^{k-1})N| < |R/N|$. Hence

$$|cR + N/cR + N^k| \leq |N/cN + N^k| \leq |N/(cR + N^{k-1})N| < |R/N|.$$

Thus combining these we have

$$|R/cR + N^k| = \sup\{|R/cR + N|, |cR + N/cR + N^k|\} < |R/N|.$$

But for some m, $N^m = 0$. Hence $|R/cR| < |R/N|$, a contradiction.

Recall that the bound of a right ideal I of R is the largest two-sided ideal contained in I.

THEOREM 2.2. *Let R be a prime fully bounded Noetherian ring such that every non-zero ideal of R contains a non-zero central element. Then the elements of C are units of R.*

Proof. Suppose first that R contains an infinite number of height-1 prime ideals. Then $c \in C$ implies that c is regular. Now suppose that c is not a unit. Let B be the bound of cR. Thus cR/B contains no non-zero ideal of R/B. Therefore, by Lemma 2.1, there is a prime ideal minimal over B such that $c \notin C(P)$. Then, by the proof of the Principal Ideal Theorem (Chatters–Goldie–Hajarnavis–Lenagan [2]), height $(P) = 1$. Since $c \in C$, this is a contradiction.

Suppose now that R has only finitely many height-1 prime ideals $P_1, \ldots, P_n$. Set $I = P_1 \cap \ldots \cap P_n \neq 0$. We note that $C = C(I)$. Let $c \in C(I)$. Then $c + x$ is regular for some $x \in I$ by Robson [14]. Therefore $c + x$ is a unit of R as above. Hence R/I is Artinian and I is the Jacobson radical of R. Thus R is semilocal and 1-dimensional, and c is a unit of R.

Note that the P.I. case when R has only finitely many height-1 prime ideals is due to Amitsur–Small [1].

Note also that the assumption that R is fully bounded cannot be dropped. For example, if R is the universal enveloping algebra of any non-Abelian nilpotent Lie algebra then not all the elements of C are units even though every ideal of R has a centralising set of generators.

3. The height-1 prime ideals related to a given prime ideal.   We now consider some generalizations of statement (d') of Theorem 1.1.

THEOREM 3.1. *Let R be a prime fully bounded Noetherian ring and suppose that every non-zero ideal of R contains a non-zero element of the centre Z of R. Let P be a non-zero prime ideal of R. Then $C(P) \cap C(0) \supseteq \bigcap C(Q)$, where the intersection ranges over those height-1 prime ideals Q such that $Q \cap Z \subseteq P \cap Z$.*

Proof. Write $P' = P \cap Z$ and let S be the partial quotient ring of R formed by inverting the elements of $Z \setminus P'$.

Let $c \in \bigcap C(Q)$ as above. The height-1 prime ideals of S are of the form QS when Q is a height-1 prime ideal of R with $Q \cap Z \subseteq P'$. Thus c is regular modulo all height-1

prime ideals of S, and hence c is a unit of S by Theorem 2.2. So $1 = cad^{-1}$ for some $a \in R$ and $d \in Z \setminus P'$. That is $d = ca \in C(P) \cap C(0)$. Thus $c \in C(P) \cap C(0)$.

EXAMPLE 3.2. We shall construct a prime Noetherian P.I. ring which is a maximal order and a non-zero prime ideal P and a height-1 prime ideal $Q \subseteq P$ such that $C(P) \cap C(0) \not\subseteq C(Q)$.

Let k be a field of characteristic zero. Let R be the ring generated by 2×2 generic matrices X and Y over k (see, for example, Cohn [4, Section 12.6]). Let $T = T(R)$ be the trace ring of R. Then we know T is a Noetherian P.I. domain and a maximal order (Small–Stafford [15]).

Further, let tr( ) denote the trace of a given matrix and det( ) the determinant. Then we know that $T . (XY - YX)$ is a height-1 prime ideal of T and that $T/T . (XY - YX)$ is a polynomial ring over k generated by the images of $X, Y, \text{tr}(X), \text{tr}(Y)$ (Formanek–Halpin–Li [5]).

Now $\det(X) = X . (\text{tr}(X) - X)$ by the Cayley–Hamilton Theorem. Let

$$P = T . X + T . (XY - YX).$$

Then P is a height-2 prime ideal of T with $\det(X) \in P$ and $\det(X)$ is central. Thus, by Jategaonkar's Principal Ideal Theorem (Jategaonkar [9]), there exists a height-1 prime ideal $Q \subseteq P$ such that $\det(X) \in Q$. But then $X - \text{tr}(X) \in C(P) \cap C(0)$ and $X - \text{tr}(X) \notin C(Q)$.

THEOREM 3.3. *Let R be as in Theorem* 3.1 *and let P be a non-zero localisable prime ideal of R. Then $C(P) = \bigcap C(Q)$, where the intersection ranges over those height-1 prime ideals $Q \subseteq P$.*

Proof. We have that R satisfies the Ore condition with respect to $C(P)$ and it is well known (Smith [16, Lemma 4.1]) that this implies that $C(P) \subseteq C(Q)$ for every prime ideal $Q \subseteq P$. The proof is now very similar to that of Theorem 3.1, except that S is taken to be the localisation of R at P.

We have been unable to answer the following question: If R is the type of ring which we are considering and P is a non-zero prime ideal of R, is it true that $C(P) \supseteq \bigcap C(Q)$, where the intersection ranges over those height-1 prime ideals $Q \subseteq P$?

A positive answer to the above question would imply that if height $(P) \geq 2$ then P contains infinitely many height-1 prime ideals, a result which is known to be true in the P.I. case (Resco–Small–Stafford [12]), but is an open question for fully bounded Noetherian rings. We shall in the next section show that the answer is "Yes" if R is also a maximal order.

4. Prime Noetherian maximal orders.

The questions we are considering can also be answered for prime Noetherian bounded maximal orders. Let R be a prime Noetherian ring with full quotient ring Q. Then R is a maximal order if given $q \in Q$ such that $q . I \subseteq I$ or $I . q \subseteq I$ for some non–zero ideal I of R then $q \in R$. In the commutative case this is equivalent to R being integrally closed. For further details we refer to Maury–Raynaud [11].

If R is a maximal order and I an ideal of R, we set $I^* = \{q \in Q$ such that $qI \subseteq R\}$. Since R is a maximal order $I^* = \{q \in Q$ such that $Iq \subseteq R\}$.

Recall that the bound of cR is the largest two-sided ideal contained in cR.

LEMMA 4.1. *Let R be a prime Noetherian maximal order and let c be a regular element of R such that the bound B of cR is non-zero. Let P be a prime ideal minimal over B. Then $c \notin C(P)$, P is reflexive and* height$(P) = 1$.

Proof. We have $c^{-1}B \subseteq R$ so that $c^{-1} \in B^*$. Thus $c^{-1}B^{**} \subseteq R$, i.e. $B^{**} \subseteq cR$. It follows that $B = B^{**}$. From Hajarnavis–Williams [6, Lemma 3.2], P is reflexive and height$(P) = 1$. Finally, $c \in C(P)$ implies that $cR \cap P = cP \supseteq B$. But then $B \cdot P^* \subseteq cP \cdot P^* = cR$ which implies that $P^* = R$, a contradiction. Hence $c \notin C(P)$.

THEOREM 4.2. *Let R be a prime Noetherian bounded maximal order. Then the elements of C are units of R.*

Proof. Suppose first that R has infinitely many height-1 primes. Then $c \in C$ is regular and hence cR has a bound B. If c is not a unit then pick a minimal prime ideal P over B. By Lemma 4.1, P has height$(P) = 1$ and $c \notin C(P)$, a contradiction.

If R has only finitely many height-1 primes $Q_1, \ldots, Q_n$, set $I = Q_1 \cap \ldots \cap Q_n \neq 0$. We note that $C = C(I)$. Let $c \in C(I)$. Then $c + x$ is regular for some $x \in I$ by Robson [14]. Therefore $c + x$ is a unit of R by Lemma 4.1. Hence R/I is Artinian and I is the Jacobson radical of R. So R is 1-dimensional and semi-local and the result follows.

THEOREM 4.3. *Let R be a prime Noetherian bounded maximal order and let P be a non-zero prime ideal of R. Then $C(P) \supseteq \bigcap C(Q)$, where the intersection ranges over those height-1 primes $Q \subseteq P$.*

Proof. Let $c \in \bigcap C(Q)$ as above. If P contains infinitely many such Q then clearly c is regular. If P contains only finitely many height-1 primes $Q_1, \ldots, Q_n$ then, by Robson [14], we have $c + x$ regular for some $x \in Q_1 \cap \ldots \cap Q_n$. Further, $c + x \in C(P)$ if and only if $c \in C(P)$. Therefore, without loss of generality, we may suppose c regular.

Let B be the bound of cR and suppose that $c \notin C(P)$. Then we must have $B \subseteq P$. Hence, by Lemma 4.1, there exists a height-1 prime Q such that $B \subseteq Q \subseteq P$ and $c \notin C(Q)$, a contradiction.

COROLLARY 4.4. *Let R be a prime Noetherian bounded maximal order and let P be a prime ideal of R with* height$(P) \geq 2$. *Then P contains infinitely many height-1 primes.*

Proof. Suppose that P contains only finitely many height-1 primes $Q_1, \ldots, Q_n$ and let $I = Q_1 \cap \ldots \cap Q_n$. Thus P/I is a non-minimal prime ideal of the semiprime Noetherian ring R/I. Therefore, by Goldie's theorem, there exists $c \in P$ such that $c \in C(I)$. But then $c \in C(P)$ by Theorem 4.3, a contradiction.

Note that in view of Example 1.3, the word "bounded" cannot be deleted from the statements of the results in this section.

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