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Non-real zeros of real differential polynomials

J. K. Langley
School of Mathematical Sciences, The University of Nottingham,
University Park, Nottingham NG7 2RD, UK (jkl@maths.nott.ac.uk)

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The main results of the paper determine all real meromorphic functions f of finite lower order in the plane such that f has finitely many zeros and non-real poles and such that certain combinations of derivatives of f have few non-real zeros.

1. Introduction

This paper concerns non-real zeros of certain combinations of derivatives of real meromorphic functions in the plane, that is, meromorphic functions mapping $\mathbb{R}$ into $\mathbb{R}\cup\{\infty\}$. Research into the non-real zeros of derivatives of real entire functions has a long history. Wiman conjectured around 1911 [1,2] that if f is a real entire function such that f and f'' have only real zeros, then f belongs to the Laguerre–Pólya class LP of entire functions which are locally uniform limits of real polynomials with real zeros. This was proved in [32] for f of finite order and in [6] for infinite order (see also [27] for the case of 'large' infinite order). It was further shown that, for an entire function $f = Ph$, where h is a real entire function with real zeros and P is a real polynomial, the number of non-real zeros of $f^{(k)}$ is 0 for large k if $h \in \mathrm{LP}$ [7,8,17,18], and tends to infinity with k otherwise [5,23]. These results proved a conjecture of Pólya [30].

For real meromorphic functions with poles there are less complete results. All meromorphic functions f in the plane for which all derivatives $f^{(k)}$ ($k \geqslant 0$) have only real zeros were determined by Hinkkanen [14–16], while functions with real poles, for which some of the derivatives have only real zeros, were considered in several papers including [11,12,31]. The following theorem was proved in [25] (see also [24]).

Theorem 1.1 (Langley [25]). *Let f be a real meromorphic function in the plane, not of the form $f = S\mathrm{e}^P$ with S being a rational function and P being a polynomial. Let μ and k be integers with $1 \leqslant \mu < k$. Assume that all but finitely many zeros of f and $f^{(k)}$ are real, and that $f^{(\mu)}$ has finitely many zeros. Then $\mu = 1$ and $k = 2$ and f satisfies*

$$f(z) = \frac{R(z)\mathrm{e}^{\mathrm{i}cz} - 1}{AR(z)\mathrm{e}^{\mathrm{i}cz} - \bar{A}},$$

where $c \in (0,\infty)$, $A \in \mathbb{C}\setminus\mathbb{R}$, and R is a rational function with $|R(x)| = 1$ for all $x \in \mathbb{R}$. Moreover, all but finitely many poles of f are real.

A related result was proved for $k = 2$ and $\mu = 1$ in [12], but with the reality of poles of f as a hypothesis rather than a conclusion. The starting point of the present paper is the analogous problem where f (instead of $f^{(\mu)}$) is assumed to have finitely many zeros. In particular, the following theorem is a combination of results of Hellerstein and Williamson [11] and Rossi [31].

Theorem 1.2 (Hellerstein and Williamson [11]; Rossi [31]). *Let f be a real meromorphic function in the plane with real poles and no zeros, and assume that all zeros of f' are real. If f has infinite order then f'' has infinitely many non-real zeros. The same conclusion holds if f has finite order and infinitely many poles.*

For f of finite order, the assumption in theorem 1.2 that f' has only real zeros is not particularly strong. Indeed, if $g = 1/f$ is a real transcendental meromorphic function of finite lower order in the plane with finitely many poles and non-real zeros, then g' has finitely many non-real zeros and so has f' (see § 2). The following theorem will be proved.

Theorem 1.3. *Let f be a real meromorphic function of finite lower order in the plane, with finitely many zeros and non-real poles, and assume that*

$$N_0(r) = o(T(r, f'/f)) \quad \textit{as } r \to \infty, \tag{1.1}$$

where $N_0(r)$ counts the non-real zeros of f''. Then f satisfies

$$f = S\mathrm{e}^P, \textit{ with } S \textit{ a rational function and } P \textit{ a polynomial.} \tag{1.2}$$

For example, taking $f(z) = \exp(-z^2)$ gives $f''(z) = (4z^2 - 2)\exp(-z^2)$, which has real zeros only. Observe that theorem 1.3 certainly applies if $N_0(r) = O(\log r)$ as $r \to \infty$ because, in this case, either f'/f is a rational function or $N_0(r) = o(T(r, f'/f))$, and both alternatives lead to (1.2). Theorem 1.3 will be deduced from a result concerning non-real zeros of $ff'' - a(f')^2$, for a meromorphic function f with finitely many zeros and non-real poles, and certain real values of a. Langley [21] proved a conjecture of Bergweiler [3] by showing that if f is a meromorphic function in the plane and $ff'' - a(f')^2$ has finitely many zeros, where $a \in \mathbb{C} \setminus \{1\}$ and $1/(a-1)$ is not a positive integer, then f satisfies (1.2). The methods of [3, 21] involved a modified Newton function defined via

$$h = \frac{1}{1-a}, \qquad a = \frac{h-1}{h}, \qquad F(z) = z - h\frac{f(z)}{f'(z)}, \qquad F' = h\left(\frac{ff''}{(f')^2} - a\right). \tag{1.3}$$

Several results have been proved [29] establishing the existence of non-real zeros of $ff'' - a(f')^2$, when f is a real entire function, including the following theorem.

Theorem 1.4 (Nicks [29]). *Let f be a real entire function and let $a < 1$ be a real number. If f and $ff''/(f')^2 - a$ have finitely many non-real zeros then $f \in U^*_{2p}$ for some $p \geqslant 0$, and $ff''/(f')^2 - a$ has at least $2p$ non-real zeros. If $a \leqslant \frac{1}{2}$, f'/f has finite lower order and $ff''/(f')^2 - a$ has finitely many non-real zeros, then $f \in U^*_{2p}$ again for some p.*

The class U^*_{2p} is defined for $p \geqslant 0$ as the set of entire functions $f = Ph$, where $h \in V_{2p} \setminus V_{2p-2}$ and P is a real polynomial with no real zeros. Here $V_{-2} = \emptyset$, while V_{2p} for $p \geqslant 0$ consists of all entire functions $f(z) = g(z)\exp(-az^{2p+2})$, where $a \geqslant 0$ is real and g is a real entire function with real zeros of genus at most $2p+1$ [10, p. 29]. It is well known that $V_0 = \mathrm{LP}$. The significance of the conditions on a in theorem 1.4 lies in the fact that $a < 1$ implies that $h > 0$ in (1.3), so that if z and $f'(z)/f(z)$ have positive imaginary part, then so has $F(z)$, in analogy with the method of [32]. Furthermore, if $a \leqslant \frac{1}{2}$, then $0 < h \leqslant 2$ and zeros of f are attracting or rationally indifferent fixpoints of F: the hypothesis that f'/f has finite lower order then facilitates the application of Hinchliffe's extension to finite lower order [13] of a theorem of [4] concerning singularities of the inverse function. The following result will be proved for meromorphic functions and $a < 1$.

Theorem 1.5. *Let f be a real meromorphic function of finite lower order in the plane, with finitely many zeros and non-real poles. Let $a \in \mathbb{R}$ satisfy $a < 1$, and assume that (1.1) holds, where $N_0(r)$ counts the non-real zeros of $ff''/(f')^2 - a$. Then f satisfies (1.2).*

Theorem 1.3 follows easily from the case $a = 0$ of theorem 1.5. Writing

$$g = \frac{1}{f}, \qquad \frac{gg''}{(g')^2} - a = 2 - a - \frac{ff''}{(f')^2}$$

leads to the following immediate consequence of theorem 1.5, which complements theorem 1.4.

Theorem 1.6. *Let f be a real meromorphic function of finite lower order in the plane, with finitely many poles and non-real zeros. Let $a \in \mathbb{R}$ satisfy $a > 1$, and assume that (1.1) holds, where $N_0(r)$ counts the non-real zeros of $ff''/(f')^2 - a$. Then f satisfies (1.2).*

For example, taking $f(z) = \exp(z^2)$ and $a = 2$ gives

$$\frac{f(z)f''(z)}{f'(z)^2} - 2 = \frac{1 - 2z^2}{2z^2},$$

which has only real zeros. The case $a = 1$ is exceptional: indeed, if $f(z)$ is $\cos z$ or $\sec z$, then $ff''/(f')^2 - 1$ has no zeros at all [3, 28].

The methods of the present paper are also applicable when f'' in theorem 1.3 is replaced by $F = f'' + a_1 f' + a_0 f$ for certain rational functions a_j, albeit with a stronger hypothesis on the frequency of non-real zeros of F.

Theorem 1.7. *Let f be a real meromorphic function of finite lower order in the plane, such that f has finitely many zeros and non-real poles. Let a_1 and a_0 be real rational functions such that $a_1(z)$ and $za_0(z)$ both vanish at infinity, and assume that $F = f'' + a_1 f' + a_0 f$ has finitely many non-real zeros. Then f satisfies (1.2).*

Theorem 1.7 will be proved by showing that the hypotheses imply that f and F have finitely many zeros in the plane, so that the conclusion follows at once from the main result of [19]. Meromorphic functions f in the plane, for which f and $f'' + a_1 f' + a_0 f$ have finitely many zeros, for arbitrary rational functions a_1 and a_0, were classified in [20] by means of representations for f and f'/f. However, these representations in [20] are complicated, and so the investigation of this more general problem in the context of non-real zeros will be left for future research.

It turns out that results in the direction of theorems 1.3, 1.5 and 1.6 may be proved for functions of infinite order, but these require completely different methods and will be presented elsewhere. The case of finite lower order treated here depends on proposition 3.1, which fails for infinite lower order.

2. The Levin–Ostrovskii factorization

Let g be a real transcendental meromorphic function in the plane with finitely many poles and non-real zeros. Then the logarithmic derivative has a Levin–Ostrovskii factorization [26, 27]

$$\frac{g'}{g} = \phi_1 \psi, \tag{2.1}$$

in which ϕ_1 and ψ are real meromorphic functions, such that ϕ_1 has finitely many poles and ψ is constructed as follows. If g has finitely many zeros, set $\psi = 1$. For g with infinitely many zeros, denote by α_p the distinct real zeros of g, ordered so that

$$\cdots < \alpha_{p-1} < \alpha_p < \alpha_{p+1} < \cdots .$$

For $|p| \geqslant p_0$, where p_0 is large, α_p and α_{p+1} are of the same sign, and there is a zero β_p of g' in the interval (α_p, α_{p+1}). Thus, the product

$$\psi(z) = \prod_{|p| \geqslant p_0} \frac{1 - z/\beta_p}{1 - z/\alpha_p}$$

converges by the alternating series test, and satisfies

$$0 < \sum_{|p| \geqslant p_0} \arg \frac{1 - z/\beta_p}{1 - z/\alpha_p} = \sum_{|p| \geqslant p_0} \arg \frac{\beta_p - z}{\alpha_p - z} < \pi \quad \text{for } z \in H = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\},$$

so that $\psi(H) \subseteq H$, which implies in turn that [26, chapter I.6, theorem 8′]

$$\tfrac{1}{5}|\psi(\mathrm{i})| \frac{\sin\theta}{r} < |\psi(r\mathrm{e}^{\mathrm{i}\theta})| < 5|\psi(\mathrm{i})| \frac{r}{\sin\theta} \quad \text{for } r \geqslant 1, \theta \in (0, \pi). \tag{2.2}$$

Regardless of whether or not g has infinitely many zeros, this gives

$$T(r, \phi_1) \leqslant N(r, \phi_1) + m\left(r, \frac{g'}{g}\right) + m\left(r, \frac{1}{\psi}\right) \leqslant m\left(r, \frac{g'}{g}\right) + O(\log r)$$

as $r \to \infty$. It follows at once that if g has finite lower order, then ϕ_1 is a rational function (and g' has finitely many non-real zeros as asserted in § 1).

Assume for the remainder of this section that g has infinitely many zeros. Since the image of H under $\log \psi$ contains no disc of radius greater than $\frac{1}{2}\pi$, Bloch's theorem implies that

$$\left| \frac{\psi'(r\mathrm{e}^{\mathrm{i}\theta})}{\psi(r\mathrm{e}^{\mathrm{i}\theta})} \right| \leqslant \frac{c_0}{r \sin\theta} \quad \text{for } r \geqslant 1,\ \theta \in (0, \pi), \tag{2.3}$$

with c_0 a positive absolute constant. Furthermore, ψ has a representation [26]

$$\left.\begin{aligned} \psi(z) &= Az + B + \sum B_k\left(\frac{1}{A_k - z} - \frac{1}{A_k}\right), \\ \sum \frac{B_k}{A_k^2} &< \infty, \qquad B_k = -\operatorname{Res}(\psi, A_k) > 0, \end{aligned}\right\} \tag{2.4}$$

where $A \geqslant 0$, $B \in \mathbb{R}$ and the A_k are the poles of ψ (all of which lie in $\mathbb{R} \setminus \{0\}$). In particular, this gives $c_1 > 0$ such that

$$\psi'(x) = A + \sum \frac{B_k}{(A_k - x)^2} \geqslant \sum_{|A_k| \leqslant |x|} \frac{B_k}{4x^2} \geqslant \frac{c_1}{x^2} \quad \text{as } |x| \to +\infty,\ x \in \mathbb{R}. \tag{2.5}$$

3. Lower bounds for certain differential polynomials

We have the following proposition.

PROPOSITION 3.1. *Let f be a real meromorphic function of finite lower order in the plane, with finitely many zeros and infinitely many poles, all but finitely many of which are real. Let $L = f'/f$, let b be a positive real number, and let c and d be real rational functions such that $c(z)$ and $zd(z)$ both vanish at infinity. Then there exists $R_1 \in (0, \infty)$ such that*

$$Q(x) = bL(x)^2 + c(x)L(x) + L'(x) + d(x)$$

is positive or infinite for every real x with $|x| \geqslant R_1$.

The case where $b = 1$, $c = d = 0$, $1/f \in V_{2p}$ and f' has only real zeros is treated in [11, theorem 2], but the present approach is simpler and more general. The example

$$f(z) = \exp(\sin z), \quad \frac{f''(z)}{f(z)} = L(z)^2 + L'(z) = \cos^2 z - \sin z, \quad b = 1, \quad c = d = 0,$$

shows that proposition 3.1 is false for infinite lower order.

The proof of proposition 3.1 will occupy the remainder of this section. Let f, b, c, d and Q be as in the hypotheses, and set $g = 1/f$. Then g satisfies the hypotheses of § 2, and has infinitely many zeros. Thus, (2.1) implies the representations

$$\frac{f'}{f} = L = -\frac{g'}{g} = \phi\psi, \qquad Q = bL^2 + cL + L' + d = b\phi^2\psi^2 + c\phi\psi + \phi'\psi + \phi\psi' + d, \tag{3.1}$$

where ψ is as constructed in § 2 and where $\phi = -\phi_1$ is a real rational function. Hence, ϕ satisfies

$$\frac{z\phi'(z)}{\phi(z)} = O(1) \quad \text{as } z \to \infty. \tag{3.2}$$

Since $b > 0$, all but finitely many poles of f are poles of Q. Hence, Q is transcendental, and because $f(z)$ may be replaced by $f(-z)$, it obviously suffices to show that $Q(x)$ is positive or infinite for large x on the positive real axis $\mathbb{R}^+$. The proof will now be divided into a number of cases. In each case, let $x \in \mathbb{R}^+$ be large, but not a pole of f.

Case 1. Suppose that $\phi(\infty) = 0$.

Let ε be small and positive. Then (2.2) and (3.1) imply that

$$L(z) = O(1), \quad \log g(z) = O(|z|) \quad \text{as } |z| \to +\infty \text{ with } \varepsilon \leqslant |\arg z| \leqslant \pi - \varepsilon.$$

Since g has finite lower order and finitely many poles, it follows, using a standard application of the Phragmén–Lindelöf principle, that $\rho(g) \leqslant 1$. Hence, L has a representation

$$L(z) = \frac{f'(z)}{f(z)} = a + \sum_{k=1}^{\infty} \left(\frac{1}{d_k - z} - \frac{1}{d_k} \right) + \frac{R'(z)}{R(z)},$$

where $a \in \mathbb{C}$, the d_k are the poles of f in $\mathbb{R} \setminus \{0\}$, repeated according to multiplicity, and R is a rational function. This then implies that

$$\begin{aligned} Q(x) &= bL(x)^2 + c(x)L(x) + L'(x) + d(x) \\ &= bL(x)^2 + c(x)L(x) + \sum_{k=1}^{\infty} \frac{1}{(d_k - x)^2} + O\left(\frac{1}{x^2}\right). \end{aligned} \tag{3.3}$$

Estimating the sum in (3.3) gives

$$x^2 \sum_{k=1}^{\infty} \frac{1}{(d_k - x)^2} \geqslant x^2 \sum_{|d_k| \leqslant x} \frac{1}{(d_k - x)^2} \geqslant \sum_{|d_k| \leqslant x} \frac{1}{4} \to +\infty \quad \text{and} \quad \frac{1}{x^2} = o(L'(x))$$

as $x \to +\infty$. If $|c(x)L(x)| \leqslant bL(x)^2$, it is then obvious from (3.3) that $Q(x) > 0$, while the contrary case gives $L(x) = O(1/x)$, and hence $c(x)L(x) = o(L'(x))$ and so $Q(x) > 0$ again. This proves proposition 3.1 in case 1.

Case 2. Suppose that $\phi(\infty) \in \mathbb{C} \setminus \{0\}$.

In this case, since f has infinitely many real poles and the residues of ψ are negative, $\phi(\infty)$ must be real and positive by (3.1), and $\phi(x)\psi'(x) > 0$ by (2.5). Furthermore, by (2.4), (3.1) and the fact that

$$B_k \phi(A_k) = -\operatorname{Res}(L, A_k) \geqslant 1$$

for large k, the estimate (2.5) may be replaced by

$$\lim_{x \to +\infty} x^2 \psi'(x) = +\infty, \tag{3.4}$$

so that

$$d(x) = o(\phi(x)\psi'(x)) \quad \text{as } x \to +\infty. \tag{3.5}$$

It now follows from (3.1) that $Q(x) > 0$, unless

$$b\phi(x)^2\psi(x)^2 < |(c(x)\phi(x) + \phi'(x))\psi(x)|, \tag{3.6}$$

in which case $0 \neq \psi(x) = O(x^{-1})$ using (3.2). But in this case,

$$(c(x)\phi(x) + \phi'(x))\psi(x) = O(x^{-2}) = o(\psi'(x)) = o(\phi(x)\psi'(x)),$$

using (3.2) and (3.4), and (3.1) and (3.5) give $Q(x) > 0$ again. This disposes of case 2.

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CASE 3. Suppose that $\phi(\infty) = \infty$ and f has infinitely many poles on $\mathbb{R}^+$.

Again it follows from (2.4), (3.1) and a consideration of residues that $\phi(x) > 0$, and hence that $\phi(x)\psi'(x) > 0$, using (2.5). Again (3.5) is satisfied, which then forces $Q(x) > 0$, unless (3.6) holds. But (3.6) implies, this time in view of (3.2) and the fact that $\phi(\infty) = \infty$, that

$$0 \neq b|\psi(x)| < \frac{|c(x)\phi(x) + \phi'(x)|}{\phi(x)^2}, \quad \psi(x) = O(x^{-2})$$

and, using (2.5),

$$(c(x)\phi(x) + \phi'(x))\psi(x) = O(x^{-3})\phi(x) = o(\phi(x)\psi'(x)),$$

which gives $Q(x) > 0$ in (3.1) as in case 2.

CASE 4. Suppose that $\phi(\infty) = \infty$ and f has finitely many poles on $\mathbb{R}^+$.

In this case, let ε be small and positive. Then the function $h(z) = 1/(z\psi(z))$ is bounded on the rays $\arg z = \pm\varepsilon$, by (2.2). But ψ has finitely many positive poles, and hence finitely many positive zeros, by construction. Since ψ has finite lower order it follows, using the Phragmén–Lindelöf principle, that $h(z)$ is bounded as $z \to \infty$ with $|\arg z| \leqslant \varepsilon$. Similar considerations, starting from (2.3), show that $z\psi'(z)/\psi(z)$ is also bounded as $z \to \infty$ with $|\arg z| \leqslant \varepsilon$. On recalling (3.2) it now follows that

$$\frac{1}{\phi(x)\psi(x)} = O(1)$$

and

$$c(x)\phi(x)\psi(x) + \phi'(x)\psi(x) + \phi(x)\psi'(x) + d(x) = o(|\phi(x)\psi(x)|),$$

which implies, using (3.1), that $Q(x) > 0$.

This completes the proof of proposition 3.1.

4. A consequence of a result of Eremenko

The proofs of theorems 1.3, 1.5 and 1.6 depend on the following result from [22].

THEOREM 4.1 (Langley [22]). *Suppose that the function F is transcendental and meromorphic of finite lower order in the plane, with*

$$N_1(r, F) = N(r, F) - \bar{N}(r, F) + N(r, 1/F') = o(T(r, F)) \tag{4.1}$$

as $r \to \infty$. Then F has a sequence of fixpoints $z_k \to \infty$ with $F'(z_k) \to \infty$.

The function $N_1(r, F)$ counts the multiple points of F [10, chapter 2]. Theorem 4.1 was proved in [22] using Eremenko's characterization [9] of transcendental meromorphic functions of finite lower order in the plane that satisfy (4.1).

5. Proof of theorem 1.5

Assume that f and a are as in the hypotheses of theorem 1.5, but that f is not of the form (1.2). Hence, $L = f'/f$ is transcendental and f has infinitely many poles, all but finitely many of which are real. Define h and F by (1.3), and write

$$H = (a-1)L^2 - L' = L^2\left(a - \frac{ff''}{(f')^2}\right) = \frac{-L^2F'}{h}. \tag{5.1}$$

If z_0 is a pole of f'/f with residue m, then

$$F(z_0) = z_0, \qquad F'(z_0) = 1 - \frac{h}{m}, \tag{5.2}$$

and so F' cannot have a zero at a pole of f, since $h > 0$. Because $a - 1$ is negative, proposition 3.1 implies that $H(x)$ is negative or infinite for all $x \in \mathbb{R}$ with $|x|$ large. Since all but finitely many poles of L are poles of f, it now follows from (5.1) that F' has finitely many real zeros, and so

$$N(r, 1/F') = o(T(r, L)) \quad \text{as } r \to \infty, \tag{5.3}$$

using (1.1) and the definition of $N_0(r)$.

Observe next that a multiple pole of F can only arise from a multiple zero of $L = f'/f$. But $g = 1/f$ satisfies the conditions of § 2, and so $g'/g = -L$ has a Levin–Ostrovskii factorization (2.1), in which ϕ_1 is a rational function, since g has finite lower order, and all zeros of ψ are simple by construction. This implies that all but finitely many zeros of L are simple. These considerations and (5.3) now yield

$$N_1(r, F) = o(T(r, L)) = o(T(r, F)).$$

Thus theorem 4.1 implies that F has a sequence of fixpoints $z_k \to \infty$ satisfying $F'(z_k) \to \infty$, which is impossible by (5.2). This contradiction proves theorem 1.5 and, as observed in § 1, theorems 1.3 and 1.6 both follow from theorem 1.5.

6. Proof of theorem 1.7

Assume that f, a_1 and a_0 are as in the hypotheses of theorem 1.7, but that f is not of the form (1.2). Hence $L = f'/f$ is transcendental, and f has infinitely many poles, all but finitely many of which are real. Proposition 3.1 with $b = 1$, $c = a_1$ and $d = a_0$ implies that $f''/f + a_1 f'/f + a_0$ has finitely many real zeros. Thus, f and $f'' + a_1 f' + a_0 f$ have finitely many zeros in the plane and, by the main theorem of [19], the function L is rational. This contradiction completes the proof of theorem 1.7.

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