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Rank-one convexity implies polyconvexity for isotropic, objective and isochoric elastic energies in the two-dimensional case

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We show that, in the two-dimensional case, every objective, isotropic and isochoric energy function that is rank-one convex on $\mathrm{GL}^+(2)$ is already polyconvex on $\mathrm{GL}^+(2)$. Thus, we answer in the negative Morrey's conjecture in the subclass of isochoric nonlinear energies, since polyconvexity implies quasi-convexity. Our methods are based on different representation formulae for objective and isotropic functions in general, as well as for isochoric functions in particular. We also state criteria for these convexity conditions in terms of the deviatoric part of the logarithmic strain tensor.

Keywords: rank-one convexity; polyconvexity; quasi-convexity;
Morrey's conjecture; isochoric energies; nonlinear elasticity

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1. Introduction

We consider different convexity properties of a real-valued function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ on the group $\mathrm{GL}^+(2) = \{X \in \mathbb{R}^{2\times 2} \mid \det X > 0\}$ of invertible 2×2-matrices with positive determinant. Our work is mainly motivated by the theory of nonlinear hyperelasticity, where $W(\nabla\varphi)$ is interpreted as the energy density of a deformation $\varphi\colon \Omega \to \mathbb{R}^2$; here, $\Omega \subset \mathbb{R}^2$ corresponds to a planar elastic body in its reference configuration. The elastic energy W is assumed to be *objective* as well as *isotropic*, i.e. assumed to satisfy the equality

$$W(Q_1 F Q_2) = W(F) \quad \text{for all } F \in \mathrm{GL}^+(2) \text{ and all } Q_1, Q_2 \in \mathrm{SO}(2),$$

where $\mathrm{SO}(2) = \{X \in \mathbb{R}^{2\times 2} \mid X^{\mathrm{T}} X = \mathbb{1}, \det X = 1\}$ denotes the special orthogonal group.

Different notions of convexity play an important role in elasticity theory. Here, we focus on the concepts of *rank-one convexity*, *polyconvexity* and *quasi-convexity*. Following a definition by Ball [7, definition 3.2], we say that W is *rank-one convex* on $\mathrm{GL}^+(2)$ if it is convex on all closed line segments in $\mathrm{GL}^+(2)$ with end points differing by a matrix of rank 1, i.e.

$$W(F + (1-\theta)\xi \otimes \eta) \leqslant \theta W(F) + (1-\theta) W(F + \xi \otimes \eta)$$

for all $F \in \mathrm{GL}^+(2)$, $\theta \in [0,1]$ and all $\xi, \eta \in \mathbb{R}^2$ with $F + t\xi \otimes \eta \in \mathrm{GL}^+(2)$ for all $t \in [0,1]$, where $\xi \otimes \eta$ denotes the dyadic product. For sufficiently regular functions $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, rank-one convexity is equivalent to *Legendre–Hadamard ellipticity* (see [29]) on $\mathrm{GL}^+(2)$:

$$D_F^2 W(F)(\xi \otimes \eta, \xi \otimes \eta) \geqslant 0 \quad \text{for all } \xi, \eta \in \mathbb{R}^2 \setminus \{0\},\ F \in \mathrm{GL}^+(2).$$

The rank-one convexity is connected with the study of wave propagation [2, 20, 68, 79] or hyperbolicity of the dynamic problem, and plays an important role in the existence and uniqueness theory for linear elastostatics and elastodynamics [32,34,59,73] (cf. [33,45]). It also ensures the correct spatial and temporal behaviour of the solution to the boundary-value problems for a large class of materials [19, 21, 35, 36]. Important criteria for the rank-one convexity of functions were established by Knowles and Sternberg [44] as well as by Šilhavý [71] and Dacorogna [26].

The notion of *polyconvexity* was introduced into the context of nonlinear elasticity theory by Ball [6, 7] (see [6, 27, 65]). In the two-dimensional case, a function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ is called polyconvex if and only if it is expressible in the form

$$W(F) = P(F, \det F), \quad P\colon \mathbb{R}^{2\times 2} \times \mathbb{R} \cong \mathbb{R}^5 \to \mathbb{R} \cup \{+\infty\},$$

where $P(\cdot,\cdot)$ is convex. Since the polyconvexity of an energy W already implies the weak lower semi-continuity of the corresponding energy functional, it is of fundamental importance to the direct methods in the calculus of variations. In particular, this implication is still valid for functions W defined only on $\mathrm{GL}^+(2)$, which do not satisfy polynomial growth conditions; this is generally the case in nonlinear elasticity.

Lastly, a function W is called *quasi-convex* at $\bar{F} \in \mathrm{GL}^+(n)$ if the condition

$$\int_\Omega W(\bar{F} + \nabla\vartheta)\mathrm{d}x \geqslant \int_\Omega W(\bar{F})\mathrm{d}x = W(\bar{F}) \cdot |\Omega| \quad \text{for every bounded open set } \Omega \subset \mathbb{R}^n \tag{1.1}$$

holds for all $\vartheta \in C_0^\infty(\Omega)$ such that $\det(\bar{F} + \nabla\vartheta) > 0$. Note carefully that there are alternative definitions of quasi-convexity for functions on $\mathrm{GL}^+(n)$ (see [11]). Although quasi-convexity of an energy function W is sufficient for the weak lower semi-continuity of the corresponding energy functional if $W\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous and satisfies suitable growth conditions [18, 74], it is generally not sufficient in the case of energy functions defined only on $\mathrm{GL}^+(n)$.

It is well known that the implications

$$\text{polyconvexity} \quad \Longrightarrow \quad \text{quasi-convexity} \quad \Longrightarrow \quad \text{rank-one convexity}$$

hold for arbitrary dimension n. However, it is also known that rank-one convexity does not imply polyconvexity in general (see the Alibert–Dacorogna–Marcellini example [1]; cf. [27, p. 221] and [4]), and that for $n > 2$ rank-one convexity does not imply quasi-convexity [12, 27, 64, 75].

The question of whether rank-one convexity implies quasi-convexity in the two-dimensional case is considered to be one of the major open problems in the calculus of variations [9, 10, 24, 60, 61]. Morrey conjectured in 1952 that the two are not equivalent [3, 42, 43, 48, 51, 63], i.e. that there exists a function $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ that is rank-one convex but not quasi-convex. A number of possible candidates have already been proposed: for example [77], the function $W^\#\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with

$$\begin{aligned} W^\#(F) &= \begin{cases} -4\det F, & \sqrt{\|F\|^2 - 2\det F} + \sqrt{\|F\|^2 + 2\det F} \leqslant 1, \\ 2\sqrt{\|F\|^2 - 2\det F} - 1, & \text{otherwise}, \end{cases} \\ &= \begin{cases} -4\lambda_{\min}\lambda_{\max}, & \lambda_{\max} \leqslant \frac{1}{2}, \\ 2(\lambda_{\max} - \lambda_{\min}) - 1, & \text{otherwise} \end{cases} \end{aligned} \tag{1.2}$$

(where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and the largest singular value of F, respectively) is known to be rank-one convex,[1] but it is not known whether this function is quasi-convex at $F = 0$.

There are, however, a number of special cases for which the two convexity conditions are, in fact, equivalent: for example, every quasi-convex quadratic form is polyconvex [47, 70, 75, 76] and, as Müller [49] has shown, rank-one convexity implies quasi-convexity in dimension two on diagonal matrices [17, 22, 23]. Moreover, Ball

[1] This follows from the convexity of the function

$$\lambda_{\max} \pm \lambda_{\min} = \sqrt{\|F\|^2 \pm 2\det F} = \sqrt{(F_{11} \pm F_{22})^2 (F_{21} \mp F_{12})^2}$$

(see [23, lemma 2.2]). In [23, remark 1] it is also noted that any SO(2)-invariant polyconvex function can be written as the supremum of linear combinations of the functions $\varphi_c^\pm = \lambda_{\max} \pm \lambda_{\min} - (\lambda_{\max}\lambda_{\min}/c)$, for $c \in \mathbb{R} \setminus \{0\}$, $\varphi_0^\pm = -\lambda_{\max}\lambda_{\min}$, by writing it first as the supremum of polyaffine functions and then exploiting SO(2) invariance. Thus, the individual branches of $W^\#$ are polyconvex.

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and Murat [12] showed that every energy function $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ of the form $W(F) = \|F\|^\alpha + h(\det F)$ with a function $h\colon \mathbb{R} \to \mathbb{R}$ and $1 \leqslant \alpha < 2$ is polyconvex if and only if it is rank-one convex. Iwaniec *et al*. even conjectured that "continuous rank-one convex functions $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ are quasi-convex" [3, conjecture 1.1] in general[2] (whereas Pedregal found "some evidence in favour" [63] of the hypothesis that the two conditions are not equivalent [62]).

In this spirit, we present another condition under which rank-one convexity implies polyconvexity (and thus quasi-convexity), thereby further complicating the search for a counterexample: we show that any function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ that is isotropic and objective (i.e. bi-SO(2)-invariant) as well as *isochoric* is rank-one convex if and only if it is polyconvex. A function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ is called isochoric[3] if

$$W(aF) = W(F) \quad \text{for all } a \in \mathbb{R}^+ := (0, \infty).$$

Note carefully that we explicitly consider functions that are defined only on $\mathrm{GL}^+(2)$, and not on all of $\mathbb{R}^{2\times 2}$. Such a function W can equivalently be expressed as a (discontinuous) function $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ with $W(F) = +\infty$ for all $F \notin \mathrm{GL}^+(2)$. In many fields, these energy functions are more suitable for applications than finite-valued functions on $\mathbb{R}^{2\times 2}$. In the theory of nonlinear hyperelasticity, for example, the requirement $W(F) \to \infty$ as $\det F \to 0$ is commonly assumed to hold. The left and right SO(2) invariance is also motivated by applications in nonlinear elasticity and corresponds to the requirements of objectivity and isotropy, respectively.[4] While Morrey's conjecture is usually stated for finite-valued functions on all of $\mathbb{R}^{2\times 2}$ only, energy functions on $\mathrm{GL}^+(2)$ have long been a valuable source of inspiring examples; indeed, for $n > 2$, an early example of a non-continuous function mapping $\mathbb{R}^{n\times n}$ to $\mathbb{R} \cup \{+\infty\}$ which is rank-one convex but not quasi-convex was given by Ball [8], even before Šverák [75] found a continuous finite-valued counterexample. Additional conditions for rank-one convexity of objective and isotropic energy functions on $\mathrm{GL}^+(2)$ have also been considered by Šilhavý [72], Parry and Šilhavý [61], Aubert [5] and Davies [28].

Note also that a function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ is isotropic, objective and isochoric if and only if W is (left and right) *conformally invariant*, i.e. $W(AFB) = W(F)$ for all $A, B \in \mathrm{CSO}(2)$, where

$$\mathrm{CSO}(2) = \mathbb{R}^+ \cdot \mathrm{SO}(2) = \{aQ \in \mathrm{GL}^+(2) \mid a \in \mathbb{R}^+,\ Q \in \mathrm{SO}(2)\}$$

[2] Interestingly, the related (but not equivalent) question of whether isotropic rank-one convex sets in $\mathbb{R}^{2\times 2}$ are already quasi-convex has a positive answer [38, 46].

[3] In elasticity theory, isochoric energy functions measure only the *change of form* of an elastic body, not the *change of size*. For more general elastic energy functions $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, an additive *isochoric–volumetric split* [50] of the form

$$W(F) = W^{\mathrm{iso}}(F) + W^{\mathrm{vol}}(\det F) = W^{\mathrm{iso}}\left(\frac{F}{(\det F)^{1/2}}\right) + W^{\mathrm{vol}}(\det F)$$

into an isochoric part $W^{\mathrm{iso}}\colon \mathrm{GL}^+(2) \to \mathbb{R}$ and a volumetric part $W^{\mathrm{vol}}\colon \mathbb{R}^+ \to \mathbb{R}$ is sometimes assumed (see § 5.2).

[4] If functions on $\mathbb{R}^{2\times 2}$ are considered, then the isotropy requirement is often assumed to be right O(2) invariance, whereas right SO(2) invariance is the natural isotropy condition for functions on $\mathrm{GL}^+(2)$.

denotes the *conformal special orthogonal group*. In the literature, one also encounters the concept of *conformal energies* [78], which are functions W such that $W(F)$ vanishes if and only if $F \in \mathrm{CSO}(2)$, e.g. $W(F) = \|F\|^2 - 2\det F$. However, as this example shows, such energies are generally not isochoric (or conformally invariant).

The idea of finding new isochoric functions that are rank-one convex arose from the search for a function of the isotropic invariants $\|\mathrm{dev}_2 \log U\|^2$ and $[\mathrm{tr}(\log U)]^2$ of the logarithmic strain tensor $\log U$ that is rank-one convex or polyconvex (see [25, 54, 54, 55, 58, 69]), since the commonly used quadratic Hencky energy

$$W_{\mathrm{H}}(F) = W_{\mathrm{H}}^{\mathrm{iso}}\left(\frac{F}{(\det F)^{1/2}}\right) + W_{\mathrm{H}}^{\mathrm{vol}}(\det F) = \mu\|\mathrm{dev}_2 \log U\|^2 + \tfrac{1}{2}\kappa[\mathrm{tr}(\log U)]^2$$

is not rank-one convex even in $\mathrm{SL}(2) := \{X \in \mathrm{GL}^+(2) \,|\, \det X = 1\}$ (see [56]). Here, $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{1}{3}(2\mu + 3\lambda) > 0$ is the infinitesimal bulk modulus, λ is the first Lamé constant, $F = \nabla\varphi$ is the gradient of deformation, $U = \sqrt{F^{\mathrm{T}}F}$ is the right stretch tensor and $\log U$ denotes the principal matrix logarithm of U. For $X \in \mathbb{R}^{2\times 2}$, we denote by $\|X\|$ the Frobenius tensor norm, $\mathrm{tr}(X)$ is the trace of X, $\mathrm{dev}_2 X = X - \frac{1}{2}\mathrm{tr}(X) \cdot \mathbb{1}$ is the deviatoric part of X and $\mathbb{1}$ denotes the identity tensor on $\mathbb{R}^{2\times 2}$.

Promising candidates for an appropriate polyconvex formulation in terms of $\|\mathrm{dev}_2 \log U\|^2$ and $[\mathrm{tr}(\log U)]^2$ are the exponentiated Hencky energies previously considered in the series of papers [37, 53, 56, 57]:

$$\begin{aligned} W_{\mathrm{eH}}(F) &= W_{\mathrm{eH}}^{\mathrm{iso}}\left(\frac{F}{(\det F)^{1/2}}\right) + W_{\mathrm{eH}}^{\mathrm{vol}}(\det F) \\ &= \frac{\mu}{k}\exp(k\|\mathrm{dev}_2 \log U\|^2) + \frac{\kappa}{2\hat{k}}\exp(\hat{k}[(\log\det U)]^2), \end{aligned} \tag{1.3}$$

where k, $\hat{k}$ are additional dimensionless parameters.

2. Preliminaries

In order to establish our main result, i.e. that rank-one convexity and polyconvexity are equivalent for isochoric energy functions, we first need to recall some conditions for these convexity properties. In the following, we shall assume $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ to be an objective, isotropic function. It is well known that such a function can be expressed in terms of the singular values of F: there exists a uniquely determined function $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$W(F) = g(\lambda_1, \lambda_2) \tag{2.1}$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Note that the isotropy of W also implies the symmetry condition $g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_1)$.

2.1. A sufficient condition for polyconvexity

A proof of the following lemmas can be found in [37].

Lemma 2.1. *If* $Y\colon [1,\infty) \to \mathbb{R}$ *is non-decreasing and convex and* $Z\colon \mathrm{GL}^+(2) \to [1,\infty)$ *is polyconvex, then* $Y \circ Z$ *is polyconvex.*

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Lemma 2.2. *The function* $Z\colon \mathrm{GL}^+(2) \to [1,\infty)$ *with* $Z(F) = \|F\|_{\mathrm{op}}^2/\det F$*, where* $\|F\|_{\mathrm{op}} = \max\{\lambda_1, \lambda_2\}$ *denotes the spectral norm of* $F \in \mathrm{GL}^+(2)$ *with singular values* λ_1*,* λ_2*, is polyconvex on* $\mathrm{GL}^+(2)$*. Note that the function* Z *can be expressed as* $Z(F) = g(\lambda_1, \lambda_2)$ *with* $g(\lambda_1\lambda_2) = \max\{\lambda_1^2, \lambda_2^2\}/\lambda_1\lambda_2$*.*

These two lemmas immediately imply the next proposition [37], which will play a key role in showing that isochoric, rank-one convex energies are already polyconvex.

Proposition 2.3. *If, for given* $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$*, there exists a non-decreasing and convex function* $h\colon [1,\infty) \to \mathbb{R}$ *such that* $W = h \circ Z$*, where* $Z(F) = \|F\|_{\mathrm{op}}^2/\det F$*, then* W *is polyconvex.*

2.2. A necessary condition for rank-one convexity

We prove the following well-known necessary condition for rank-one convexity.

Lemma 2.4. *Let* $W : \mathrm{GL}^+(2) \to \mathbb{R}$ *be objective, isotropic and rank-one convex, and let* $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *denote the representation of* W *in terms of singular values. Then* g *is separately convex, i.e. the mapping* $\lambda_1 \mapsto g(\lambda_1, \lambda_2)$ *is convex for fixed* $\lambda_2 \in \mathbb{R}^+$ *and the mapping* $\lambda_2 \mapsto g(\lambda_1, \lambda_2)$ *is convex for fixed* $\lambda_1 \in \mathbb{R}^+$*.*

Proof. For $a, b \in \mathbb{R}$, we define

$$\mathrm{diag}(a,b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Let $\lambda_2 \in \mathbb{R}^+$ be fixed. Since the matrix $\mathrm{diag}(1,0)$ has rank 1, the rank-one convexity of W implies that the mapping

$$t \mapsto W(\mathrm{diag}(1,\lambda_2) + t\,\mathrm{diag}(1,0)) = W(\mathrm{diag}(1+t,\lambda_2)) = g(1+t,\lambda_2), \quad t \in (-1,\infty),$$

is convex. Therefore, the function g is convex in the first component and, for symmetry reasons, convex in the second component. □

Note that for an energy function W of class C^2 the separate convexity of g is equivalent to the *tension-extension inequalities (TE inequalities)*

$$\frac{\partial^2 g}{\partial \lambda_1^2} \geqslant 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial \lambda_2^2} \geqslant 0 \quad \text{for } \lambda_1, \lambda_2 \in \mathbb{R}^+.$$

3. The equivalence of rank-one convexity and polyconvexity for isochoric energy functions

3.1. The main result

We now focus on isochoric functions W on $\mathrm{GL}^+(2)$, i.e. functions that satisfy $W(aF) = W(F)$ for all $F \in \mathrm{GL}^+(2)$ and all $a > 0$. These functions can be uniquely represented in terms of the ratio λ_1/λ_2 of the singular values of F.

Lemma 3.1. *Let* $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$*,* $F \mapsto W(F)$ *be an objective, isotropic function that is additionally isochoric, i.e. it satisfies* $W(aF) = W(F)$ *for all* $F \in \mathrm{GL}^+(2)$ *and all* $a > 0$*. Then there exists a unique function* $h\colon \mathbb{R}^+ \to \mathbb{R}$ *with* $h(t) = h(1/t)$ *such that* $W(F) = h(\lambda_1/\lambda_2)$ *for all* $F \in \mathrm{GL}^+(2)$ *with singular values* $\lambda_1, \lambda_2 \in \mathbb{R}^+$*.*

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Remark 3.2. Note that lemma 3.1 explicitly requires W to be defined on $\mathrm{GL}^+(2)$ only: for functions on all of $\mathrm{GL}(2)$, the isotropy requirement must be extended from right $\mathrm{SO}(2)$ invariance to right $\mathrm{O}(2)$ invariance in order to ensure a representation in terms of the singular values; if singular matrices are included in the domain of W, then h is not well defined in the form stated in the lemma.

Proof. Since W is objective and isotropic, there exists a function $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ with $W(F) = g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_1)$ for all $F \in \mathrm{GL}^+(2)$, where λ_1, λ_2 are the singular values of F. Then

$$W(F) = W\left(\frac{F}{\sqrt{\det F}}\right) = g\left(\frac{\lambda_1}{\sqrt{\lambda_1\lambda_2}}, \frac{\lambda_2}{\sqrt{\lambda_1\lambda_2}}\right) = g\left(\sqrt{\frac{\lambda_1}{\lambda_2}}, \sqrt{\frac{\lambda_2}{\lambda_1}}\right).$$

Hence, for $h\colon \mathbb{R}^+ \to \mathbb{R}$ with $h(t) := g(\sqrt{t}, 1/\sqrt{t})$ we find

$$h\left(\frac{\lambda_1}{\lambda_2}\right) = g\left(\sqrt{\frac{\lambda_1}{\lambda_2}}, \frac{1}{\sqrt{\lambda_1/\lambda_2}}\right) = g\left(\sqrt{\frac{\lambda_1}{\lambda_2}}, \sqrt{\frac{\lambda_2}{\lambda_1}}\right) = W(F),$$

and the symmetry of g (which follows from the isotropy of W) implies

$$h(t) = g\left(\sqrt{t}, \frac{1}{\sqrt{t}}\right) = g\left(\frac{1}{\sqrt{t}}, \sqrt{t}\right) = g\left(\sqrt{\frac{1}{t}}, \sqrt{\frac{1}{1/t}}\right) = h\left(\frac{1}{t}\right).$$

Finally, the uniqueness of h follows directly from the equality $h(t) = W(\mathrm{diag}(t, 1))$. □

We are now ready to prove our main result.

Theorem 3.3. *Let $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ be an objective, isotropic and isochoric function, and let $h\colon \mathbb{R}^+ \to \mathbb{R}$, $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ denote uniquely determined functions with*

$$W(F) = g(\lambda_1, \lambda_2) = h\left(\frac{\lambda_1}{\lambda_2}\right) = h\left(\frac{\lambda_2}{\lambda_1}\right)$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Then the following are equivalent:

(i) *W is polyconvex;*

(ii) *W is rank-one convex;*

(iii) *g is separately convex;*

(iv) *h is convex on $\mathbb{R}^+$;*

(v) *h is convex and non-decreasing on $[1, \infty)$.*

Proof. The implication (i) ⇒ (ii) is known to hold in general, whereas the implication (ii) ⇒ (iii) is stated in lemma 2.4.

(iii) ⇒ (iv). If g is separately convex, then the mapping

$$\lambda_1 \mapsto g(\lambda_1, 1) = h(\lambda_1)$$

is convex; thus, h is convex on $\mathbb{R}^+$.

(iv) ⇒ (v). Assume that h is convex on $\mathbb{R}^+$. Then, of course, h is also convex on $[1,\infty)$, and it remains to show the monotonicity of h. Let $1 \leqslant t_1 < t_2$. Then $1/t_2 < 1 \leqslant t_1 < t_2$, i.e. t_1 lies in the convex hull of $1/t_2$ and t_2. But then $t_1 = s/t_2 + (1-s)t_2$ for some $s \in (0,1)$, and thus the convexity of h on $\mathbb{R}^+$ implies that

$$h(t_1) = h\left(s\frac{1}{t_2} + (1-s)t_2\right) \leqslant s\,h\left(\frac{1}{t_2}\right) + (1-s)h(t_2) = s\,h(t_2) + (1-s)\,h(t_2) = h(t_2).$$

Hence, h is non-decreasing on $[1,\infty)$.

(iv) ⇒ (v). Assume that h is convex and non-decreasing on $[1,\infty)$. Then we can apply proposition 2.3: since the mapping

$$F \mapsto \frac{\|F\|_{\mathrm{op}}^2}{\det F} = \frac{\max\{\lambda_1^2, \lambda_2^2\}}{\lambda_1\lambda_2} \in [1,\infty)$$

is polyconvex [37] and h is convex and non-decreasing on $[1,\infty)$, the mapping

$$F \mapsto h\left(\frac{\max\{\lambda_1^2, \lambda_2^2\}}{\lambda_1\lambda_2}\right) = h\left(\frac{\lambda_1}{\lambda_2}\right) = W(F)$$

is polyconvex as well. □

If the function h is continuously differentiable, then the criteria in theorem 3.3 can be simplified even further.

Corollary 3.4. *Let $W : \mathrm{GL}^+(2) \to \mathbb{R}$ be an objective, isotropic and isochoric function, and let $h \colon \mathbb{R}^+ \to \mathbb{R}$ denote the uniquely determined function with $W(F) = h(\lambda_1/\lambda_2)$ for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. If $h \in C^1(\mathbb{R}^+)$, then W is polyconvex if and only if h is convex on $[1,\infty)$.*

Proof. We need only to show that the stated criterion is sufficient for the polyconvexity of W. Assume therefore that h is convex on $[1,\infty)$. Taking the derivative on both sides of the equality $h(t) = h(1/t)$, which holds for all $t \in \mathbb{R}^+$, yields

$$h'(t) = -\frac{1}{t^2}\,h'\left(\frac{1}{t}\right).$$

In particular, $h'(1) = -h'(1)$ and thus $h'(1) = 0$. Since the convexity of h implies the monotonicity of h' on $[1,\infty)$, we find $h'(t) \geqslant 0$ for all $t \in [1,\infty)$. This means that h is non-decreasing on $[1,\infty)$, and applying theorem 3.3(v) yields the polyconvexity of W. □

4. Criteria for rank-one convexity and polyconvexity in terms of different energy representations

4.1. Energy functions in terms of the logarithmic strain

We shall now assume that the function W is of class C^2. While the criterion $h''(t) \geqslant 0$ for all $t \in [1,\infty)$ in corollary 3.4 is easy to state, isochoric elastic energy functions in nonlinear hyperelasticity are typically not immediately given in terms of the quantity λ_1/λ_2. We therefore consider different representations of such functions in our search for easily verifiable polyconvexity criteria.

LEMMA 4.1. *Let* $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ *be objective, isotropic and isochoric. Then there exist unique functions* $f, \tilde{f}\colon [0,\infty) \to \mathbb{R}$ *such that*

(i) $W(F) = f\Big(\log^2 \dfrac{\lambda_1}{\lambda_2}\Big)$,

(ii) $W(F) = \tilde{f}(\|\mathrm{dev}_2 \log U\|^2)$

for all $F \in \mathrm{GL}^+(2)$, *where* λ_1, λ_2 *denote the singular values of* F, $U = \sqrt{F^{\mathrm{T}}F}$ *is the positive definite symmetric polar factor in the right polar decomposition of* F, $\mathrm{dev}_2 X = X - \frac{1}{2}\mathrm{tr}(X)\cdot \mathbb{1}$ *is the deviatoric part of* $X \in \mathbb{R}^{2\times 2}$, $\log$ *denotes the principal matrix logarithm on* $\mathrm{PSym}(2)$ *and* $\|\cdot\|$ *is the Frobenius matrix norm.*

Proof.
(i) Let us first recall from lemma 3.1 that there exists a unique function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $W(F) = h(\lambda_1/\lambda_2)$ for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Let $f(\theta) = h(\exp(\sqrt{\theta}))$ for $\theta > 0$. Since

$$\sqrt{\log^2 \frac{\lambda_1}{\lambda_2}} = \left|\log \frac{\lambda_1}{\lambda_2}\right| = \log \frac{\max\{\lambda_1,\lambda_2\}}{\min\{\lambda_1,\lambda_2\}},$$

we find

$$\begin{aligned} f\Big(\log^2 \frac{\lambda_1}{\lambda_2}\Big) &= h\Big(\exp\Big(\sqrt{\log^2 \frac{\lambda_1}{\lambda_2}}\Big)\Big) \\ &= h\Big(\exp\Big(\log \frac{\max\{\lambda_1,\lambda_2\}}{\min\{\lambda_1,\lambda_2\}}\Big)\Big) \\ &= h\Big(\frac{\max\{\lambda_1,\lambda_2\}}{\min\{\lambda_1,\lambda_2\}}\Big) \\ &= h\Big(\frac{\lambda_1}{\lambda_2}\Big) \\ &= W(F) \end{aligned}$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. To show the uniqueness of f, we simply note that

$$f(\theta) = f\Big(\log^2\Big(\frac{\mathrm{e}^{\sqrt{\theta}}}{1}\Big)\Big) = W(\mathrm{diag}(\mathrm{e}^{\sqrt{\theta}}, 1))$$

for all $\theta > 0$.

(ii) It was shown in [56] that

$$\|\mathrm{dev}_2 \log U\|^2 = \tfrac{1}{2}\log^2 \frac{\lambda_1}{\lambda_2}.$$

The equality $W(F) = \tilde{f}(\|\mathrm{dev}_2 \log U\|^2)$ is therefore satisfied for all $F \in \mathrm{GL}^+(2)$ if and only if $\tilde{f}(t) = f(2t)$, where f is given by (i). □

Note carefully that, for $n > 2$, not every objective, isotropic and isochoric energy $W\colon \mathrm{GL}^+(n) \to \mathbb{R}$ can be written in terms of $\|\mathrm{dev}_n \log U\|^2$ in the way lemma 4.1 states for $n = 2$. However, there always exists a function $\widehat{W}\colon \mathrm{Sym}(n) \to \mathbb{R}$ such that $W(F) = \widehat{W}(\mathrm{dev}_n \log U)$ for all $F \in \mathrm{GL}^+(n)$ with $U = \sqrt{F^{\mathrm{T}}F}$.

We can now state theorem 3.3 in terms of the functions f, $\tilde{f}$ as defined in lemma 4.1.

PROPOSITION 4.2. *Let $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ be an objective, isotropic and isochoric function and let $f, \tilde{f}\colon [0,\infty) \to \mathbb{R}$ denote the uniquely determined functions with*

$$W(F) = \tilde{f}(\|\mathrm{dev}_2 \log U\|^2) = f\left(\log^2 \frac{\lambda_1}{\lambda_2}\right)$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. If $f, \tilde{f} \in C^2([0,\infty))$, then the following are equivalent:

(i) *W is polyconvex,*

(ii) *W is rank-one convex,*

(iii) $2\theta\, f''(\theta) + (1 - \sqrt{\theta})f'(\theta) \geqslant 0$ *for all* $\theta \in (0,\infty)$,

(iv) $2\eta\, \tilde{f}''(\eta) + (1 - \sqrt{2\eta})\tilde{f}'(\eta) \geqslant 0$ *for all* $\eta \in (0,\infty)$.

Proof. For $h\colon \mathbb{R}^+ \to \mathbb{R}$ with $h(t) = f(\log^2 t)$ we find

$$h\left(\frac{\lambda_1}{\lambda_2}\right) = f\left(\log^2 \frac{\lambda_1}{\lambda_2}\right) = W(F)$$

for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. If $f \in C^2([0,\infty))$, then $h \in C^2(\mathbb{R}^+)$. Thus, we can apply corollary 3.4 to find that W is polyconvex (and, equivalently, rank-one convex) if and only if h is convex on $[1,\infty)$. Since h'' is continuous on $\mathbb{R}^+$, this convexity of h is equivalent to $h''(t) \geqslant 0$ for all $t \in (1,\infty)$. We compute

$$h'(t) = 2f'(\log^2 t)\,\frac{\log t}{t} \tag{4.1}$$

as well as

$$\begin{aligned} h''(t) &= 4f''(\log^2 t)\frac{\log^2 t}{t^2} - 2f'(\log^2 t)\frac{\log t}{t^2} + 2f'(\log^2 t)\frac{1}{t^2} \\ &= \frac{2}{t^2}(2(\log^2 t)\, f''(\log^2 t) + (1 - \log t)\, f'(\log^2 t)). \end{aligned}$$

Writing $t > 1$ as $t = \mathrm{e}^{\sqrt{\theta}}$ with $\theta > 0$ we find

$$h''(t) = \frac{2}{\mathrm{e}^{2\sqrt{\theta}}}(2\theta f''(\theta) + (1 - \sqrt{\theta})f'(\theta)).$$

Since the mapping $\theta \to \mathrm{e}^{\sqrt{\theta}}$ is bijective from $(0,\infty)$ to $(1,\infty)$, the condition

$$h''(t) \geqslant 0 \quad \text{for all } t \in (1,\infty) \tag{4.2}$$

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is therefore equivalent to

$$2\theta f''(\theta) + (1 - \sqrt{\theta}) f'(\theta) \geqslant 0 \quad \text{for all } \theta \in (0, \infty), \tag{4.3}$$

which is exactly criterion (iii).

It remains to show that (iii) and (iv) are equivalent. Since $\tilde{f}(\eta) = f(2\eta)$ (see lemma 4.1), we find

$$2\eta \tilde{f}''(\eta) + (1 - \sqrt{2\eta}\, \tilde{f}'(\eta)) = 2[2(2\eta) f''(2\eta) + (1 - \sqrt{(2\eta)}) f'(2\eta)].$$

Thus, (iv) is satisfied for all $\eta \in \mathbb{R}^+$ if and only if (iii) is satisfied for all $\theta = 2\eta \in \mathbb{R}^+$. □

In addition to proposition 4.2(iii), the polyconvexity of W also implies the monotonicity of f.

Corollary 4.3. *Under the assumptions of proposition 4.2, if W is polyconvex (or, equivalently, rank-one convex), then $f'(\theta) \geqslant 0$ for all $\theta > 0$.*

Proof. According to theorem 3.3, the polyconvexity of W implies that $h = f \circ \log^2$ is non-decreasing on $[1, \infty)$. Then

$$0 \leqslant h'(t) = 2 f'(\log^2 t) \frac{\log t}{t}$$

for all $t > 1$, and thus $0 \leqslant f'(\log^2 t)$ for all $t > 1$, which immediately implies $f'(\theta) \geqslant 0$ for all $\theta > 0$. □

4.2. Energy functions in terms of the distortion function

We now consider the representation of an isochoric energy $W(F)$ in terms of

$$\mathbb{K}(F) = \frac{1}{2} \frac{\|F\|^2}{\det F},$$

where $\|\cdot\|$ denotes the Frobenius matrix norm; the mapping $\mathbb{K}$ is also known as the (planar) *distortion function* [40] or *outer distortion* [41, eq. (14)]. Note that $\mathbb{K} \geqslant 1$ and that, for $F \in \mathrm{GL}^+(2)$, $\mathbb{K}(F) = 1$ if and only if F is conformal, i.e. if $F = a \cdot R$ with $a \in \mathbb{R}^+$ and $R \in \mathrm{SO}(2)$. In the two-dimensional case, every objective, isotropic and isochoric (i.e. conformally invariant) energy can be written in terms of $\mathbb{K}$.

Lemma 4.4. *Let $W \colon \mathrm{GL}^+(2) \to \mathbb{R}$ be objective, isotropic and isochoric. Then there exists a unique function $z \colon [1, \infty) \to \mathbb{R}$ with*

$$W(F) = z(\mathbb{K}(F)) = z\left(\frac{1}{2} \frac{\|F\|^2}{\det F}\right)$$

for all $F \in \mathrm{GL}^+(2)$.

Proof. It can easily be seen that the function $p \colon [1, \infty) \to [1, \infty)$ with

$$p(t) = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

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is bijective, and that its inverse is given by

$$q(s) = p^{-1}(s) = y + \sqrt{y^2 - 1}.$$

Then

$$q\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) = t \quad \text{for all } t \in [1, \infty),$$

while for $t \in (0, 1)$ we find

$$q\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) = q\left(\frac{1}{2}\left(\overbrace{\frac{1}{t}}^{>1} + \frac{1}{1/t}\right)\right) = \frac{1}{t}.$$

Therefore,

$$q\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) = \max\left\{t, \frac{1}{t}\right\} \quad \text{for all } t \in \mathbb{R}^+ = (0, \infty).$$

According to lemma 3.1, there exists a unique function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $W(F) = h(\lambda_1/\lambda_2) = h(\lambda_2/\lambda_1)$ for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2. Then the function $z := h \circ q$ has the desired property: since

$$\frac{1}{2}\frac{\|F\|^2}{\det F} = \frac{1}{2}\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2} = \frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right),$$

we find

$$\begin{aligned} z\left(\frac{1}{2}\frac{\|F\|^2}{\det F}\right) &= h\left(q\left(\frac{1}{2}\frac{\|F\|^2}{\det F}\right)\right) \\ &= h\left(q\left(\frac{1}{2}\left(\frac{\lambda_1}{\lambda_2} + \frac{1}{\lambda_1/\lambda_2}\right)\right)\right) \\ &= h\left(\max\left\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\right\}\right) \\ &= W(F). \end{aligned}$$

The uniqueness follows directly from the observation that

$$z(r) = W(\mathrm{diag}(r + \sqrt{r^2 - 1}, 1))$$

for all $r \in [1, \infty)$. □

By means of this representation formula, we can easily show that every objective, isotropic and isochoric function on $\mathrm{GL}^+(2)$ satisfies the *tension–compression symmetry* condition $W(F^{-1}) = W(F)$: since

$$F^{-1} = \frac{1}{\det F}\begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix} \quad \text{for } F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

we find

$$\begin{aligned} \mathbb{K}(F^{-1}) = \frac{1}{2}\frac{\|F^{-1}\|^2}{\det(F^{-1})} &= \frac{\det F}{2}\left\|\frac{1}{\det F}\begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix}\right\|^2 \\ &= \frac{1}{2\det F}\left\|\begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix}\right\|^2 = \frac{1}{2}\frac{\|F\|^2}{\det F} = \mathbb{K}(F) \end{aligned}$$

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and thus $W(F) = z(\mathbb{K}(F)) = z(\mathbb{K}(F^{-1})) = W(F^{-1})$ for all $F \in \mathrm{GL}^+(2)$. Note that this implication is restricted to the two-dimensional case: isochoric energy functions on $\mathrm{GL}^+(n)$ are generally not tension–compression symmetric for $n > 2$.

Criteria for the polyconvexity of W can now be established in terms of the function z corresponding to W.

PROPOSITION 4.5. *Let $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ be an objective, isotropic and isochoric function and let $z\colon [1,\infty) \to \mathbb{R}$ denote the uniquely determined function with*

$$W(F) = z(\mathbb{K}(F)) = z\left(\frac{1}{2}\frac{\|F\|^2}{\det F}\right)$$

for all $F \in \mathrm{GL}^+(2)$. If $z \in C^2([0,\infty))$, then the following are equivalent:

(i) *W is polyconvex;*

(ii) *W is rank-one convex;*

(iii) $(r^2-1)(r+\sqrt{r^2-1})z''(r) + z'(r) \geqslant 0$ *for all $r \in (1,\infty)$.*

Proof. As indicated in the proof of lemma 4.4, the unique function h with $W(F) = h(\lambda_1/\lambda_2) = h(\lambda_2/\lambda_1)$ for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2 is given by

$$h(t) = z\left(\frac{t}{2} + \frac{1}{2t}\right) \quad \text{for all } t \geqslant 1.$$

By corollary 3.4, we only need to show that condition (iii) is equivalent to the convexity of h on $[1,\infty)$, i.e. equivalent to $h''(t) \geqslant 0$ for all $t > 1$. For $t > 1$, we find

$$h'(t) = \frac{1}{2}\left(1 - \frac{1}{t^2}\right) z'\left(\frac{t}{2} + \frac{1}{2t}\right)$$

and

$$\begin{aligned}
h''(t) &= \frac{1}{4}\left(1-\frac{1}{t^2}\right)^2 z''\left(\frac{t}{2}+\frac{1}{2t}\right) + \frac{1}{t^3} z'\left(\frac{t}{2}+\frac{1}{2t}\right)\\
&= \frac{1}{t^3}\left[\frac{1}{4}t\left(t-\frac{1}{t}\right)^2 z''\left(\frac{t}{2}+\frac{1}{2t}\right) + z'\left(\frac{t}{2}+\frac{1}{2t}\right)\right]\\
&= \frac{1}{t^3}\left[t\left(\left(\frac{t}{2}+\frac{1}{2t}\right)^2 - 1\right) z''\left(\frac{t}{2}+\frac{1}{2t}\right) + z'\left(\frac{t}{2}+\frac{1}{2t}\right)\right].
\end{aligned}$$

Thus,

$$h''(t) \geqslant 0 \quad \Longleftrightarrow \quad 0 \leqslant t\left(\left(\frac{t}{2}+\frac{1}{2t}\right)^2 - 1\right) z''\left(\frac{t}{2}+\frac{1}{2t}\right) + z'\left(\frac{t}{2}+\frac{1}{2t}\right).$$

Recall from the proof of lemma 4.4 that the mapping $r \mapsto q(r) = r + \sqrt{r^2-1}$ bijectively maps $(1,\infty)$ onto itself and that

$$\frac{q(r)}{2} + \frac{1}{2q(r)} = r \quad \text{for all } r > 1.$$

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Therefore, by writing $t = q(r)$, we find that the inequality $h''(t) \geqslant 0$ holds for all $t > 1$ if and only if

$$\begin{aligned}
0 &\leqslant q(r)\left(\left(\frac{q(r)}{2} + \frac{1}{2q(r)}\right)^2 - 1\right) z''\left(\frac{q(r)}{2} + \frac{1}{2q(r)}\right) + z'\left(\frac{q(r)}{2} + \frac{1}{2q(r)}\right) \\
&= q(r)(r^2 - 1)\, z''(r) + z'(r) \\
&= (r + \sqrt{r^2 - 1})(r^2 - 1)\, z''(r) + z'(r) \quad \text{for all } r > 1.
\end{aligned}$$

□

An example of the application of proposition 4.5 can be found in Appendix B.

5. Applications

5.1. The quadratic and the exponentiated isochoric Hencky energy

Proposition 4.2 can be applied directly to isochoric energy functions given in terms of $\|\operatorname{dev}_2 \log U\|^2$.

Corollary 5.1.

(i) *The isochoric Hencky energy* $\|\operatorname{dev}_2 \log U\|^2 = \frac{1}{2}\log^2(\lambda_1/\lambda_2)$ *is not polyconvex and not rank-one convex on* $\mathrm{GL}^+(2)$.

(ii) *The exponentiated isochoric Hencky energy*

$$\exp(k\|\operatorname{dev}_2 \log U\|^2) = \exp\left(k\left\|\log \frac{U}{\det U^{1/2}}\right\|^2\right) = \exp\left(\frac{k}{2}\log^2 \frac{\lambda_1}{\lambda_2}\right)$$

is rank-one convex (and therefore polyconvex) on $\mathrm{GL}^+(2)$ *if and only if* $k \geqslant \frac{1}{4}$.

Proof.
(i) In the case of the isochoric Hencky energy

$$W(F) = \|\operatorname{dev}_2 \log U\|^2 = \frac{1}{2}\log^2 \frac{\lambda_1}{\lambda_2},$$

the function $\tilde{f}$ is defined by $\tilde{f}(\eta) = \eta$. This function does not satisfy proposition 4.2(iv); since

$$2\eta\tilde{f}''(\eta) + (1 - \sqrt{2\eta})\tilde{f}'(\eta) = 1 - \sqrt{2\eta}, \tag{5.1}$$

the inequality is not satisfied for $\eta > \frac{1}{2}$.

(ii) For the exponentiated isochoric Hencky energy

$$W(F) = \exp(k\|\operatorname{dev}_2 \log U\|^2) = \exp\left(k\left\|\log \frac{U}{\det U^{1/2}}\right\|^2\right) = \exp\left(\frac{k}{2}\log^2 \frac{\lambda_1}{\lambda_2}\right),$$

the functions $f, \tilde{f} \colon [0,\infty) \to \mathbb{R}$ are given by $f(\theta) = \mathrm{e}^{k\theta/2}$ and $\tilde{f}(\eta) = \mathrm{e}^{k\eta}$. We find

$$2\eta\tilde{f}''(\eta) + (1 - \sqrt{2\eta})\tilde{f}'(\eta) = 2\eta k^2 \mathrm{e}^{k\eta} + (1 - \sqrt{2\eta})k\mathrm{e}^{k\eta}.$$

Thus, proposition 4.2(iv) is equivalent to

$$k \geqslant \frac{\sqrt{2\eta}-1}{2\eta} \quad \text{for all } \eta > 0.$$

This inequality is satisfied if and only if $k \geqslant \frac{1}{4}$. Therefore, the requirement $k \geqslant \frac{1}{4}$ is necessary and sufficient for the rank-one convexity as well as for the polyconvexity of the isochoric exponentiated Hencky energy $\exp(k\|\text{dev}_2 \log U\|^2)$. □

5.2. The isochoric-volumetric split

Our results can also be applied to non-isochoric energy functions possessing an additive *isochoric–volumetric split*,[5] i.e. energy functions W of the form

$$W\colon \text{GL}^+(2) \to \mathbb{R},$$
$$W(F) = W^{\text{iso}}(F) + W^{\text{vol}}(\det F) = W^{\text{iso}}\left(\frac{F}{(\det F)^{1/2}}\right) + W^{\text{vol}}(\det F)$$

with an isochoric function $W^{\text{iso}}\colon \text{GL}^+(2) \to \mathbb{R}$ and a function $W^{\text{vol}}\colon \mathbb{R}^+ \to \mathbb{R}$. In this case, theorem 3.3 and propositions 4.2 and 4.5 provide sufficient criteria for the polyconvexity of W: if W^{vol} is convex on $\mathbb{R}^+$, then the polyconvexity of W^{iso} is sufficient for W to be polyconvex as well. For example, since the mapping $t \mapsto (\kappa/2\hat{k})\exp(\hat{k}[(\log t)]^2)$ is convex on $\mathbb{R}^+$ for $\hat{k} \geqslant \frac{1}{8}$, it follows from corollary 5.1 that the exponentiated Hencky energy $W_{\text{eH}}\colon \text{GL}^+(2) \to \mathbb{R}$ with

$$\begin{aligned} W_{\text{eH}}(F) &= W_{\text{eH}}^{\text{iso}}(F) + W_{\text{eH}}^{\text{vol}}(\det F) \\ &= \frac{\mu}{k}\exp(k\|\text{dev}_2 \log U\|^2) + \frac{\kappa}{2\hat{k}}\exp(\hat{k}[(\log\det U)]^2) \end{aligned}$$

is polyconvex for $k \geqslant \frac{1}{4}$ and $\hat{k} \geqslant \frac{1}{8}$, as indicated in § 1.

5.3. Growth conditions for polyconvex isochoric energies

By integrating the polyconvexity criteria given in proposition 4.2, we obtain an exponential growth condition for the function f that is necessarily satisfied if W is rank-one convex (i.e. polyconvex).

Corollary 5.2. *Let*

$$W\colon \text{GL}^+(2) \to \mathbb{R} \quad \textit{with } W(F) = \tilde{f}(\|\text{dev}_2 \log U\|^2) = f\left(\log^2 \frac{\lambda_1}{\lambda_2}\right)$$

be a polyconvex energy function with $f \in C^2([0,\infty))$. If $f'(\theta) \neq 0$ for all $\theta > 0$, then the function f satisfies the inequality

$$f(\theta) \geqslant (\mathrm{e}^{\sqrt{\theta}} - 1)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}} f'(\varepsilon) + f(0) \quad \textit{for all } \theta, \varepsilon > 0. \tag{5.2}$$

[5] In nonlinear elasticity theory, the assumption that an elastic energy function takes this specific form is due to the physically plausible requirement that the mean pressure should depend only on the determinant of the deformation gradient F, i.e. that there exists a function $\mathcal{F}\colon \mathbb{R}^+ \to \mathbb{R}$ such that $(1/n)\operatorname{tr}\sigma = \mathcal{F}(\det F)$, where σ denotes the Cauchy stress tensor (cf. [16, 66, 67]).

Proof. According to proposition 4.2 and corollary 4.3, if the energy W is polyconvex, then

$$2\theta f''(\theta) + (1-\sqrt{\theta})f'(\theta) \geqslant 0 \quad \text{and} \quad f'(\theta) \geqslant 0 \quad \text{for all } \theta > 0. \tag{5.3}$$

Under our assumption $f'(\theta) \neq 0$, we therefore find $f'(\theta) > 0$ for all $\theta > 0$ and deduce

$$\frac{f''(\theta)}{f'(\theta)} \geqslant \frac{\sqrt{\theta}-1}{2\theta} \quad \text{for all } \theta > 0. \tag{5.4}$$

By integration from $\varepsilon > 0$ to θ, it follows that

$$\log f'(\theta) \geqslant \log f'(\varepsilon) + \tfrac{1}{2}(2\sqrt{\theta} - \log\theta) - \tfrac{1}{2}(2\sqrt{\varepsilon} - \log\varepsilon) \quad \text{for all } \theta, \varepsilon > 0. \tag{5.5}$$

Thus, we obtain

$$\begin{aligned} f'(\theta) &\geqslant \exp(\log f'(\varepsilon) + \tfrac{1}{2}(2\sqrt{\theta} - \log\theta) - \tfrac{1}{2}(2\sqrt{\varepsilon} - \log\varepsilon)) \\ &= f'(\varepsilon)\exp(-\sqrt{\varepsilon} + \tfrac{1}{2}\log\varepsilon)\exp(\sqrt{\theta} - \tfrac{1}{2}\log\theta) \\ &= f'(\varepsilon)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}}\frac{\mathrm{e}^{\sqrt{\theta}}}{\sqrt{\theta}} \end{aligned} \tag{5.6}$$

for all $\theta, \varepsilon > 0$. By another integration on the interval $[\delta, \theta]$, $\delta > 0$, we obtain

$$f(\theta) \geqslant f'(\varepsilon)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}}\mathrm{e}^{\sqrt{\theta}} + f(\delta) - f'(\varepsilon)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}}\mathrm{e}^{\sqrt{\delta}} \quad \text{for all } \theta, \varepsilon, \delta > 0. \tag{5.7}$$

Taking the limit case $\delta \to 0$ and using the continuity of the function f, we finally obtain

$$f(\theta) \geqslant f'(\varepsilon)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}}\mathrm{e}^{\sqrt{\theta}} + f(0) - f'(\varepsilon)\frac{\sqrt{\varepsilon}}{\mathrm{e}^{\sqrt{\varepsilon}}} \quad \text{for all } \theta, \varepsilon > 0, \tag{5.8}$$

and the proof is complete. □

REMARK 5.3. Since $f'(\theta) \geqslant 0$ for all $\theta \geqslant 0$, a necessary condition is that

$$f(\theta) \geqslant C_1 \mathrm{e}^{\sqrt{\theta}} + C_2 \quad \text{for all } \theta > 0, \tag{5.9}$$

for $C_1 = (1/\mathrm{e})f'(1) > 0$ and $C_2 = f(0) - (1/\mathrm{e})f'(1) \in \mathbb{R}$. In terms of the function h with $W(F) = h(\lambda_1/\lambda_2)$, inequality (5.9) also implies

$$h(t) \geqslant C_1 t + C_2 \quad \text{for all } t > 1,$$

since $h(t) = f(\log^2 t)$.

Sendova and Walton [69] derive similar necessary growth conditions for the three-dimensional case. Growth conditions for polyconvex functions were also considered by Yan [78], who showed that non-constant polyconvex conformal energy functions defined on all of $\mathbb{R}^{n\times n}$ must grow at least with power n.

Appendix A. Additional examples and applications

The criteria given in §§ 3 and 4 can be applied to a number of isochoric energy functions in order to determine whether or not they are polyconvex or, equivalently, rank-one convex.

Corollary A.1. *The following functions* $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ *are rank-one convex and polyconvex:*

(i) $W(F) = \left\| \dfrac{U}{(\det U)^{1/2}} - \left(\dfrac{U}{(\det U)^{1/2}}\right)^{-1} \right\|^2$;

(ii) $W(F) = \exp(\|\mathrm{dev}_2 \log U\|^2) \dfrac{\|F\|^2}{\det F}$;

(iii) $W(F) = \cosh(\|\mathrm{dev} \log U\|^2) = \cosh(\|\mathrm{dev} \log \sqrt{F^{\mathrm{T}}F}\|^2)$.

The following functions $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ *are neither rank-one convex nor polyconvex:*

(iv) $W(F) = \|\mathrm{dev}_2 \log U\|^\beta$ *for* $\beta > 0$;

(v) $W(F) = \exp(\|\mathrm{dev}_2 \log U\|^2 + \sin(\|\mathrm{dev}_2 \log U\|^2))$.

Proof.
(i) The squared Frobenius norm of a symmetric matrix X is the squared sum of its eigenvalues, and thus for $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2, we find

$$W(F) = \left(\frac{\lambda_1}{\sqrt{\lambda_1\lambda_2}} - \frac{\lambda_1^{-1}}{\sqrt{\lambda_1^{-1}\lambda_2^{-1}}}\right)^2 + \left(\frac{\lambda_2}{\sqrt{\lambda_1\lambda_2}} - \frac{\lambda_2^{-1}}{\sqrt{\lambda_1^{-1}\lambda_2^{-1}}}\right)^2 = 2\left(\sqrt{\frac{\lambda_1}{\lambda_2}} - \sqrt{\frac{\lambda_2}{\lambda_1}}\right)^2.$$

Therefore, the function $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $h(t) = h(1/t)$ with $W(F) = h(\lambda_1/\lambda_2)$ for all $F \in \mathrm{GL}^+(2)$ with singular values λ_1, λ_2 is given by

$$h(t) = 2\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 = 2\left(t + \frac{1}{t}\right) - 4,$$

and we find

$$h'(t) = 2\left(1 - \frac{1}{t^2}\right) \quad \text{and} \quad h''(t) = \frac{4}{t^3} \geqslant 0$$

for all $t \in \mathbb{R}^+$. Thus, according to theorem 3.3, W is polyconvex.

(ii) Again, we write $W(F)$ in terms of the singular values λ_1, λ_2 of F:

$$\begin{aligned} W(F) &= \exp(\|\mathrm{dev}_2 \log U\|^2) \frac{\|F\|^2}{\det F} \\ &= \exp\left(\frac{1}{2}\log^2 \frac{\lambda_1}{\lambda_2}\right) \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1\lambda_2} \\ &= \exp\left(\frac{1}{2}\log^2 \frac{\lambda_1}{\lambda_2}\right)\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right). \end{aligned}$$

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Then $W(F) = h(\lambda_1/\lambda_2)$, where

$$h(t) = \exp(\tfrac{1}{2}\log^2 t)\left(t + \frac{1}{t}\right),$$

and we compute

$$h'(t) = \exp(\tfrac{1}{2}\log^2 t)\left[1 - \frac{1}{t^2} + \frac{\log t}{t}\left(t + \frac{1}{t}\right)\right] = \exp(\tfrac{1}{2}\log^2 t)\left[1 - \frac{1}{t^2} + \log t\left(1 + \frac{1}{t^2}\right)\right]$$

as well as

$$\begin{aligned} h''(t) & \\ &= \exp(\tfrac{1}{2}\log^2 t)\left[\frac{\log t}{t}\left(1 - \frac{1}{t^2} + \log t\left(1 + \frac{1}{t^2}\right)\right) + \left(\frac{2}{t^3} + \frac{1}{t}\left(1 + \frac{1}{t^2}\right) - \frac{2\log t}{t^3}\right)\right] \\ &= \exp(\tfrac{1}{2}\log^2 t)\left[\frac{\log t}{t} - \frac{\log t}{t^3} + \left(\frac{1}{t} + \frac{1}{t^3}\right)\log^2 t + \frac{2}{t^3} + \frac{1}{t} + \frac{1}{t^3} - \frac{2\log t}{t^3}\right] \\ &= \exp(\tfrac{1}{2}\log^2 t)\left[\frac{1}{t} + \frac{3}{t^3} + \left(\frac{1}{t} - \frac{3}{t^3}\right)\log t + \left(\frac{1}{t} + \frac{1}{t^3}\right)\log^2(t)\right]. \end{aligned}$$

Therefore, theorem 3.3 states that W is polyconvex if and only if

$$\left(\frac{3}{t^2} - 1\right)\log t \leqslant 1 + \frac{3}{t^2} + \left(1 + \frac{1}{t^2}\right)\log^2 t \tag{A 1}$$

for all $t > 0$. For $t < 1$ or $t > \sqrt{3}$, the left-hand side is negative and the inequality is therefore satisfied. If $1 \leqslant t \leqslant \sqrt{3}$, then $0 \leqslant \log t < 1$ and $3/t^2 - 1 \geqslant 0$; thus,

$$\left(\frac{3}{t^2} - 1\right)\log t \leqslant \frac{3}{t^2} - 1 < 1 + \frac{3}{t^2} + \left(1 + \frac{1}{t^2}\right)\log^2 t.$$

Hence, inequality (A 1) is satisfied in this case as well.

(iii) The function $\tilde{f}\colon [0,\infty) \to \mathbb{R}$ with $W(F) = \tilde{f}(\|\mathrm{dev}_2 \log U\|^2)$ for all $F \in \mathrm{GL}^+(2)$ is given by $\tilde{f}(\eta) = \cosh(\eta)$. For $\eta \in \mathbb{R}^+$ we find

$$\begin{aligned} 2\eta\tilde{f}''(\eta) + (1 - \sqrt{2\eta})\tilde{f}'(\eta) &= 2\eta\cosh(\eta) + (1 - \sqrt{2\eta})\sinh(\eta) \\ &\geqslant (2\eta + 1 - \sqrt{2\eta})\sinh(\eta) \\ &\geqslant 0. \end{aligned}$$

Thus, W is polyconvex according to proposition 4.2.

(iv) Let $\alpha := \frac{1}{2}\beta$. Then $W(F) = \tilde{f}(\|\mathrm{dev}_2 \log U\|^2)$ for $\tilde{f}(\eta) = \eta^\alpha$. Since

$$\begin{aligned} 2\eta\tilde{f}''(\eta) + (1 - \sqrt{2\eta})\tilde{f}'(\eta) &= 2\eta\alpha(\alpha - 1)\eta^{\alpha-2} + (1 - \sqrt{2\eta})\alpha\eta^{\alpha-1} \\ &= \alpha\eta^{\alpha-1}[2\alpha - 1 - \sqrt{2\eta}], \end{aligned}$$

we use proposition 4.2 to find that W is polyconvex if and only if

$$0 \leqslant 2\alpha - 1 - \sqrt{2\eta} \quad \text{for all } \eta \in \mathbb{R}^+,$$

which is obviously not the case for any $\beta = 2\alpha > 0$. This result was also hinted at by Hutchinson and Neale [39].

(v) We apply proposition 4.2 to the function $\tilde{f}$ with $\tilde{f}(\eta) = \exp(\eta + \sin\eta)$. Since

$$\tilde{f}'(\eta) = \exp(\eta+\sin\eta)(1+\cos\eta) \quad \text{and} \quad \tilde{f}''(\eta) = \exp(\eta+\sin\eta)((1+\cos\eta)^2 - \sin\eta),$$

we find

$$2\eta\tilde{f}''(\eta) + (1-\sqrt{2\eta})\tilde{f}'(\eta) = 2\eta\exp(\eta+\sin\eta)((1+\cos\eta)^2 - \sin\eta) + (1-\sqrt{2\eta})\exp(\eta+\sin\eta)(1+\cos\eta).$$

Thus, W is polyconvex if and only if

$$2\eta((1+\cos\eta)^2 - \sin\eta) + (1-\sqrt{2\eta})(1+\cos\eta) \geqslant 0 \quad \text{for all } \eta \in (0,\infty).$$

This inequality is not satisfied for $\eta = \frac{1}{2}\pi$. Note that $\tilde{f}$ is monotone on $\mathbb{R}^+$ with exponential growth, but is not convex. □

Appendix B. On $\mathrm{dist}^2(F/(\det F)^{1/2}, \mathrm{SO}(2))$

For $F \in \mathrm{GL}^+(2)$, we consider the squared distance from $F/(\det F)^{1/2} \in \mathrm{SL}(2)$ to the special orthogonal group $\mathrm{SO}(2)$ with respect to different distance measures. Such distances are closely connected to a number of elastic energy functions, including the isochoric quadratic Hencky energy [58], and they provide an important class of examples for isochoric energy functions on $\mathrm{GL}^+(2)$. In this appendix, we collect some related results which are scattered throughout the literature.

B.1. The Euclidean distance of $F \in \mathbb{R}^{2\times 2}$ to SO(2)

We first consider the Euclidean distance

$$\mathrm{dist}^2_{\mathrm{Euclid}}(F, \mathrm{SO}(2)) := \inf_{R\in\mathrm{SO}(2)} \|F - R\|^2$$

of $F \in \mathbb{R}^{2\times 2}$ to $\mathrm{SO}(2)$, where $\|\cdot\|$ denotes the Frobenius matrix norm. In the two-dimensional case, this distance can be calculated explicitly: since

$$\mathrm{dist}^2_{\mathrm{Euclid}}(F, \mathrm{SO}(2)) = \inf_{R\in\mathrm{SO}(2)} \|F - R\|^2 = \inf_{\alpha\in[-\pi,\pi]} \left\| F - \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \right\|^2,$$

we find

$$\left\| \begin{pmatrix} F_{11} - \cos\alpha & F_{12} - \sin\alpha \\ F_{21} + \sin\alpha & F_{22} - \cos\alpha \end{pmatrix} \right\|^2 = (F_{11} - \cos\alpha)^2 + (F_{12} - \sin\alpha)^2 + (F_{21} + \sin\alpha)^2 + (F_{22} - \cos\alpha)^2.$$

Taking the derivative with respect to α yields the stationarity condition

$$(F_{11} + F_{22})\sin\alpha + (F_{21} - F_{12})\cos\alpha = 0 \iff \left\langle \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \begin{pmatrix} F_{11} + F_{22} \\ F_{21} - F_{12} \end{pmatrix} \right\rangle = 0,$$

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which implies

$$\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix} = \pm\frac{1}{\sqrt{\|F\|^2+2\det F}}\begin{pmatrix}-(F_{21}-F_{12})\\ F_{11}+F_{22}\end{pmatrix}.$$

The minimum is easily seen to be realized by

$$\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix} = \frac{1}{\sqrt{\|F\|^2+2\det F}}\begin{pmatrix}-(F_{21}-F_{12})\\ F_{11}+F_{22}\end{pmatrix},$$

and thus

$$\operatorname{dist}^2_{\mathrm{Euclid}}(F,\mathrm{SO}(2)) = \inf_{R\in\mathrm{SO}(2)}\|F-R\|^2 = \|F\|^2 - 2\sqrt{\|F\|^2+2\det F}+2$$

for arbitrary $F\in\mathbb{R}^{2\times 2}$. Let us recall the Biot energy term

$$W_{\mathrm{Biot}}(F) = \|U-\mathbb{1}\|^2 = \|U\|^2 - 2\operatorname{tr}(U)+2.$$

For $F\in\mathrm{GL}^+(2)$, we find

$$\begin{aligned}&\|U\|^2 - [\operatorname{tr}(U)]^2 + 2\det U = 0\\ &\qquad\qquad\implies \operatorname{tr}(U) = \sqrt{\|U\|^2+2\det U} \overset{F\in\mathrm{GL}^+(2)}{=} \sqrt{\|F\|^2+2\det F}.\end{aligned}$$

Hence,

$$\begin{aligned}W_{\mathrm{Biot}}(F) &= \|F\|^2 - 2\sqrt{\|F\|^2+2\det F}+2\\ &= (\sqrt{\|F\|^2+2\det F}-1)^2 + 1 - 2\det F\\ &\overset{F\in\mathrm{GL}^+(2)}{=} \|U\|^2 - 2\sqrt{\|U\|^2+2\det U}+2\end{aligned}$$

and we observe that

$$\operatorname{dist}^2_{\mathrm{Euclid}}(F,\mathrm{SO}(2)) = \|U-\mathbb{1}\|^2 = W_{\mathrm{Biot}}(F)\quad\text{for all } F\in\mathrm{GL}^+(2), \tag{B 1}$$

while in general

$$\operatorname{dist}^2_{\mathrm{Euclid}}(F,\mathrm{SO}(2)) \geqslant W_{\mathrm{Biot}}(F)$$

for $F\in\mathbb{R}^{2\times 2}$. Note that W_{Biot} is not rank-one convex [13].

B.2. The polyconvexity of $F\mapsto \operatorname{dist}^2_{\mathrm{Euclid}}(F/(\det F)^{1/2},\mathrm{SO}(2))$

In order to show that the mapping

$$F\mapsto \operatorname{dist}^2_{\mathrm{Euclid}}\left(\frac{F}{(\det F)^{1/2}},\mathrm{SO}(2)\right)$$

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is polyconvex on $\mathrm{GL}^+(2)$, we apply (B 1) to $F/(\det F)^{1/2}$ and find

$$\begin{aligned}
&\mathrm{dist}^2_{\mathrm{Euclid}}\left(\frac{F}{(\det F)^{1/2}}, \mathrm{SO}(2)\right)\\
&\quad = \left(\sqrt{\left\|\frac{F}{(\det F)^{1/2}}\right\|^2 + 2\det\left(\frac{F}{(\det F)^{1/2}}\right)} - 1\right)^2 + 1 - 2\det\left(\frac{F}{(\det F)^{1/2}}\right)\\
&\quad = \left(\sqrt{\frac{\|F\|^2}{\det F} + 2} - 1\right)^2 - 1.
\end{aligned}$$

Since the function

$$t \mapsto (\sqrt{t+2} - 1)^2 - 1.$$

is convex and monotone, we only need to prove that the mapping $F \mapsto \|F\|^2/\det F$ is polyconvex. This is shown (in a slightly generalized version) in the following lemma, using the criteria developed in § 4.

Lemma B.1. *Let $\beta > 0$. Then the function*

$$W\colon \mathrm{GL}^+(2) \to \mathbb{R}, \qquad W(F) = \left(\frac{\|F\|^2}{\det F}\right)^\beta$$

is polyconvex (and, equivalently, rank-one convex) if and only if $\beta \geqslant 1$.

Proof. The unique function $z\colon [1, \to \infty) \to \mathbb{R}$ with

$$W(F) = z\left(\frac{1}{2}\frac{\|F\|^2}{\det F}\right) \quad \text{for all } F \in \mathrm{GL}^+(2)$$

is given by $z(r) = 2^\beta r^\beta$. Then

$$z'(r) = 2^\beta \beta r^{\beta-1} \quad \text{and} \quad z''(r) = 2^\beta \beta(\beta-1) r^{\beta-2}.$$

Thus, according to proposition 4.5, the function W is polyconvex if and only if

$$\begin{aligned}
0 &\leqslant (r^2-1)(r + \sqrt{r^2-1}) z''(r) + z'(r)\\
&= 2^\beta \beta r^{\beta-2}[(\beta-1)(r^2-1)(r+\sqrt{r^2-1}) + r] \quad \text{for all } r > 1.
\end{aligned}$$

Since $2^\beta \beta r^{\beta-2} > 0$ for all $\beta > 0$ and $r > 1$, this inequality is equivalent to

$$\begin{aligned}
&0 \leqslant (\beta-1)(r^2-1)(r+\sqrt{r^2-1}) + r\\
&\qquad \iff \beta - 1 \geqslant -\frac{r}{(r^2-1)(r+\sqrt{r^2-1})} \quad \text{for all } r > 1.
\end{aligned}$$

The right-hand side in the above equality is always negative, so the polyconvexity condition is satisfied for all $\beta \geqslant 1$. Furthermore, the right-hand expression tends to 0 as r tends to ∞, and hence the condition cannot be satisfied for $\beta < 1$. □

Note that in the three-dimensional case the mapping $F \mapsto (\|F\|^3/\det F)^\beta$ is polyconvex if and only if $\beta \geqslant \frac{1}{2}$, as shown in [16, proposition 6].

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B.3. The quasi-convex hull of $\mathrm{dist}^2_{\mathrm{Euclid}}(F, \mathrm{SO}(2))$

In contrast to the isochoric function $F \mapsto \mathrm{dist}^2_{\mathrm{Euclid}}(F/(\det F)^{1/2}, \mathrm{SO}(2))$, the squared Euclidean distance of F to $\mathrm{SO}(2)$ is not polyconvex and not even rank-one convex. However, the quasi-convex hull of the function can be computed explicitly using the Brighi–Theorem, adapted to the two-dimensional case.

THEOREM B.2 (Bousselsal and Brighi [14, theorem 3.2]). *Let $q\colon \mathbb{R}^{2\times 2} \to \mathbb{R}_+$ be a non-negative quadratic form. For a function $\varphi\colon \mathbb{R}^+ \to [0,\infty)$, define $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ by*

$$W(F) = \varphi(q(F)).$$

Let $\mu^*, \alpha \in \mathbb{R}$ be such that

$$\mu^* = \inf_{t\in\mathbb{R}_+} \varphi(t) = \varphi(\alpha).$$

Then

$$R[W(F)] = Q[W(F)] = P[W(F)] = C[W(F)] = \mu^* \quad \textit{for all } F \in \mathbb{R}^{2\times 2},\ q(F) \leqslant \alpha,$$

where $R[W(\cdot)]$, $Q[W(\cdot)]$, $P[W(\cdot)]$ and $C[W(\cdot)]$ denote the rank-one convex hull, the quasi-convex hull, the polyconvex hull and the convex hull of W, respectively.

We apply this theorem to $q\colon \mathbb{R}^{2\times 2} \to \mathbb{R}_+$ with

$$q(F) = \|F\|^2 + 2\det F.$$

Note that q is indeed a non-negative quadratic form due to the arithmetic–geometric mean inequality. Consider the function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\varphi(t) = (\sqrt{t} - 1)^2 \quad \Longrightarrow \quad \inf_{t\in\mathbb{R}_+} = 0 = \varphi(1) \quad \Longrightarrow \quad \mu^* = 0, \quad \alpha = 1,$$

and let

$$W(F) = \varphi(q(F)) = (\sqrt{\|F\|^2 + 2\det F} - 1)^2.$$

From theorem B.2 we conclude that

$$R[W(F)] = Q[W(F)] = P[W(F)] = C[W(F)] = 0 \quad \text{for all } F \in \mathbb{R}^{2\times 2},\ q(F) \leqslant 1.$$

Now set

$$\widehat{W}(F) := \begin{cases} 0, & q(F) \leqslant 1, \\ (\sqrt{q(F)} - 1)^2, & q(F) > 1, \end{cases}$$
$$= \begin{cases} 0, & \|F\|^2 + 2\det F \leqslant 1, \\ (\sqrt{\|F\|^2 + 2\det F} - 1)^2, & \|F\|^2 + 2\det F \geqslant 1. \end{cases}$$

Then $\widehat{W}$ is convex (and therefore quasi-convex) as the composition $\widehat{W} = \hat{\varphi} \circ q$ of the (convex) quadratic form q and the non-decreasing convex function $\hat{\varphi}\colon \mathbb{R}^+ \to \mathbb{R}$ with

$$\hat{\varphi}(t) := \begin{cases} 0, & t \leqslant 1, \\ (\sqrt{t} - 1)^2, & t \geqslant 1. \end{cases}$$

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We observe that

$$\widehat{W}(F) = 0 = Q[W(F)] \quad \text{for all } F \in \mathbb{R}^{2\times 2} \text{ with } q(F) \leqslant 1$$

and that

$$\widehat{W}(F) = W(F) \geqslant Q[W(F)] \quad \text{for all } F \in \mathbb{R}^{2\times 2} \text{ with } q(F) > 1.$$

Thus, $\widehat{W}$ is a quasi-convex function with

$$\widehat{W}(F) \geqslant Q[W(F)] \quad \text{and} \quad \widehat{W}(F) \leqslant W(F) \quad \text{for all } F \in \mathbb{R}^{2\times 2}.$$

Hence, $\widehat{W}$ is the quasi-convex hull of W:

$$Q[W(F)] = \widehat{W}(F) = \begin{cases} 0, & \|F\|^2 + 2\det F \leqslant 1, \\ (\sqrt{\|F\|^2 + 2\det F} - 1)^2, & \|F\|^2 + 2\det F > 1. \end{cases}$$

Taking the representation

$$\begin{aligned} \operatorname{dist}^2_{\text{Euclid}}(F, \mathrm{SO}(2)) &= (\sqrt{\|F\|^2 + 2\det F} - 1)^2 + 1 - 2\det F \\ &= W(F) + 1 - 2\det F, \end{aligned}$$

it is easy to see that

$$Q[\operatorname{dist}^2_{\text{Euclid}}(F, \mathrm{SO}(2))] = Q[W(F)] + 1 - 2\det F = \widehat{W}(F) + 1 - 2\det F,$$

since $F \mapsto 1 - 2\det F$ is a null Lagrangian. We therefore find

$$\begin{aligned} &Q[\operatorname{dist}^2_{\text{Euclid}}(F, \mathrm{SO}(2))] \\ &\qquad = \begin{cases} 1 - 2\det F, & \|F\|^2 + 2\det F \leqslant 1, \\ (\sqrt{\|F\|^2 + 2\det F} - 1)^2 + 1 - 2\det F, & \|F\|^2 + 2\det F \geqslant 1, \end{cases} \\ &\qquad = \begin{cases} 1 - 2\det F, & \|F\|^2 + 2\det F \leqslant 1, \\ \operatorname{dist}^2_{\text{Euclid}}(F, \mathrm{SO}(2)), & \|F\|^2 + 2\det F \geqslant 1, \end{cases} \end{aligned}$$

for $F \in \mathbb{R}^{2\times 2}$. The same result has been given by Dolzmann [30, 31] with an alternative proof. The quasi-convex hull of the mapping $F \mapsto \operatorname{dist}^2_{\text{Euclid}}(F, \mathrm{SO}(3))$ is not yet known.

B.4. A comparison of distance functions on $\mathrm{GL}^+(2)$

Let $\operatorname{dist}_{\text{geod}}(F, \mathrm{SO}(2)) = \|\log U\|^2$ denote the *geodesic distance* [54, 55, 58] of F to $\mathrm{SO}(2)$. Then we can list the following convexity properties of (modified) distance functions to SO(2):

$$\operatorname{dist}^2_{\text{Euclid}}(F, \text{SO}(2)) = \|U - \mathbb{1}\|^2 \textit{ is not } \text{rank-one convex [13];}$$

$$\operatorname{dist}^2_{\text{Euclid}}\left(\frac{F}{(\det F)^{1/2}}, \text{SO}(2)\right) = \left\|\frac{U}{(\det U)^{1/2}} - \mathbb{1}\right\|^2 \textit{ is } \text{polyconvex (§ B.2);}$$

$$\operatorname{dist}^2_{\text{geod}}(F, \text{SO}(2)) = \|\log U\|^2 \textit{ is not } \text{rank-one convex [15, 52];}$$

$$\operatorname{dist}^2_{\text{geod}}\left(\frac{F}{(\det F)^{1/2}}, \text{SO}(2)\right) = \|\operatorname{dev}_2 \log U\|^2 \textit{ is not } \text{rank-one convex [52];}$$

$$\exp(\operatorname{dist}^2_{\text{geod}}(F, \text{SO}(2))) = \exp(\|\log U\|^2) \textit{ is not } \text{rank-one convex [56];}$$

$$\exp\left(\operatorname{dist}^2_{\text{geod}}\left(\frac{F}{(\det F)^{1/2}}, \text{SO}(2)\right)\right) = \exp(\|\operatorname{dev}_2 \log U\|^2) \textit{ is } \text{polyconvex [37].}$$

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