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IRREDUCIBLE FAMILIES OF COMPLEX MATRICES CONTAINING A RANK-ONE MATRIX

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Abstract

We show that an irreducible family $\mathcal{S}$ of complex $n \times n$ matrices satisfies Paz's conjecture if it contains a rank-one matrix. We next investigate properties of families of rank-one matrices. If $\mathcal{R}$ is a linearly independent, irreducible family of rank-one matrices then (i) $\mathcal{R}$ has length at most n, (ii) if all pairwise products are nonzero, $\mathcal{R}$ has length 1 or 2, (iii) if $\mathcal{R}$ consists of elementary matrices, its minimum spanning length M is the smallest integer M such that every elementary matrix belongs to the set of words in $\mathcal{R}$ of length at most M. Finally, for any integer k dividing $n-1$, there is an irreducible family of elementary matrices with length $k+1$.

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1. Introduction

If $\mathcal{V}$ is a finite-dimensional vector space over a field $\mathbb{F}$, a family $\mathcal{F}$ of linear transformations on $\mathcal{V}$ is called *irreducible* if the only linear subspaces of $\mathcal{V}$ that are invariant under all of the elements of $\mathcal{F}$ are (0) and $\mathcal{V}$. Burnside's well-known theorem (see [4], [13, Theorem 1.2.2]) states that the only irreducible algebra of linear transformations on a vector space, over an algebraically closed field and of finite dimension greater than one, is the algebra of all linear transformations on the space.

For any family $\mathcal{S}$ of complex $n \times n$ matrices and any natural number k, denote by $\mathcal{V}_k(\mathcal{S})$ the linear span of the words in $\mathcal{S}$ of length at most k, where the word of length zero (the 'empty word') is taken to be the identity. If $\mathcal{S}$ is finite and irreducible, thanks to Burnside's theorem, writing $\mathcal{V}_k$ for $\mathcal{V}_k(\mathcal{S})$,

$$\mathbb{C}I = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_{L-1} \subset \mathcal{V}_L = M_n(\mathbb{C}),$$

for some natural number L, called the *length* of $\mathcal{S}$.

For such a family $\mathcal{S}$ (different from $\{0\}$ if $n = 1$) and for any positive integer k, let $\mathcal{V}'_k(\mathcal{S})$, or simply $\mathcal{V}'_k$, be the linear span of the words in $\mathcal{S}$ of positive length at most k. Again thanks to Burnside's theorem,

$$\mathcal{V}'_1 \subset \mathcal{V}'_2 \subset \cdots \subset \mathcal{V}'_{M-1} \subset \mathcal{V}'_M = M_n(\mathbb{C}),$$

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for some positive integer M, called the *minimum spanning length (msl)* of $\mathcal{S}$. The idea of the msl of a finite irreducible family $\mathcal{S}$ first arose in [5].

Clearly $\mathcal{V}_k = \mathcal{V}'_k + \mathbb{C}I$, for every $k \in \mathbb{Z}^+$. It is clear that $L \le M \le L + 1$, that is, $L = M$ or $M - 1$. If $I \in \mathcal{V}'_L$ then $\mathcal{V}'_L = \mathcal{V}_L = M_n(\mathbb{C})$, so $M \le L$ and $M = L$. If $I \notin \mathcal{V}'_L$, then $\mathcal{V}'_L \neq M_n(\mathbb{C})$, so $L < M$ and $L = M + 1$. Summarising: the length L of $\mathcal{S}$ is equal to the msl M of $\mathcal{S}$ or the msl plus one according as to whether $I \in \mathcal{V}'_L$ or not.

We will investigate below the ideas of 'irreducibility', 'length' and 'minimum spanning length' as they apply to families of $n \times n$ complex matrices which contain, or are entirely made up of, rank-one matrices. Notice that, as far as these ideas are concerned, we may restrict our investigation to linearly independent families. Indeed, if $\mathcal{S}$ is any family of $n \times n$ complex matrices and $\mathcal{S}_0$ is any subset of $\mathcal{S}$ which is a basis for the linear span of $\mathcal{S}$, then $\mathcal{S}$ is irreducible if and only if $\mathcal{S}_0$ is, and then the length of $\mathcal{S}$ and the length of $\mathcal{S}_0$ are equal. This is because, for every $k \in \mathbb{N}$, the linear span of the words in $\mathcal{S}$ of length at most k equals the linear span of the words in $\mathcal{S}_0$ of length at most k. That $\mathcal{S}$ and $\mathcal{S}_0$ have the same minimum spanning lengths follows similarly.

Every rank-one matrix $R \in M_n(\mathbb{C})$ is of the form $R = e \otimes f$, where $e, f \in \mathbb{C}^n$ are nonzero vectors, and where $Rx = (x|e)f$, for all $x \in \mathbb{C}^n$, and '$(\cdot|\cdot)$' denotes the usual inner product. If $A, B \in M_n(\mathbb{C})$ we have $ARB = A(e \otimes f)B = (B^*e) \otimes (Af)$, where '$*$' denotes adjoint.

For $1 \le i, j \le n$, let $E_{i,j}$ denote the usual $n \times n$ elementary matrix. Then $E_{i,j} = e_j \otimes e_i$. Notice that $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$, where $\delta_{j,k}$ is the Kronecker delta function.

2. Irreducible families containing a rank-one matrix

In [12] Paz conjectures that the length L of an irreducible finite family $\mathcal{S}$ of (complex) $n \times n$ matrices satisfies $L \le 2n - 2$. Much work has been done on this conjecture (see references [1–11, 13, 14]), but it is still unresolved. We show that such an $\mathcal{S}$ will satisfy Paz's conjecture if it contains at least one rank-one matrix.

Theorem 2.1. *If $\mathcal{S} = \{A_1, A_2, \ldots, A_k\} \bigcup \{R\}$ is an irreducible family of $n \times n$ matrices and R has rank one, then the set*

$$\bigcup \{\mathcal{V}_{j-1}(\mathcal{A})R\mathcal{V}_{i-1}(\mathcal{A}) : 1 \le i, j \le n, (i, j) \neq (n, n)\} \cup \mathcal{V}_{n-1}(\mathcal{S})$$

spans $M_n(\mathbb{C})$, where $\mathcal{V}_u(\mathcal{A})$ denotes the linear span of the words of length at most u in $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$, including the empty word. Consequently, the length of $\mathcal{S}$ is at most $2n - 2$.

Proof. Let $R = e \otimes f$, where $e, f \in \mathbb{C}^n$, and, for every $u \in \mathbb{N}$, let $\mathcal{V}_u = \mathcal{V}_u(\mathcal{S})$. If $x \in \mathbb{C}^n$ is any nonzero vector, $\mathcal{V}_{n-1}x = \mathbb{C}^n$ since

$$\langle x \rangle = \mathcal{V}_0 x \subset \mathcal{V}_1 x \subset \mathcal{V}_2 x \subset \cdots \subset \mathcal{V}_{r-1} x \subset \mathcal{V}_r x = \mathcal{V}_{r+1} x = \mathbb{C}^n$$

for some $r \le n - 1$. If $\mathcal{G}$ denotes the unital algebra generated by $\mathcal{A}$ then $\mathcal{G}f$ is invariant under each element of $\mathcal{A}$ and R, so $\mathcal{G}f = \mathbb{C}^n$. Since

$$\langle f \rangle = \mathcal{V}_0(\mathcal{A})f \subset \mathcal{V}_1(\mathcal{A})f \subset \mathcal{V}_2(\mathcal{A})f \subset \cdots \subset \mathcal{V}_{s-1}(\mathcal{A})f \subset \mathcal{V}_s(\mathcal{A})f = \mathcal{V}_{s+1}(\mathcal{A})f$$

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for some s, it follows that $\mathcal{V}_s(\mathcal{A})f = \mathcal{G}f = \mathbb{C}^n$ and that the dimension of $\mathcal{V}_s(\mathcal{A})$ is at least $s+1$. Thus $n \geq s+1$ and so $\mathcal{V}_{n-1}(\mathcal{A})f = \mathbb{C}^n$. Applying this result to $\mathcal{S}^* = \{A_1^*, A_2^*, \dots, A_k^*\} \cup \{R^*\}$ yields $\mathcal{V}_{n-1}(\mathcal{A}^*)e = \mathbb{C}^n$. Thus the set of matrices

$$\mathcal{V}_{n-1}(\mathcal{A})R\mathcal{V}_{n-1}(\mathcal{A}) = \{U^*e \otimes Vf : U, V \in \mathcal{V}_{n-1}(\mathcal{A})\}$$

contains every $n \times n$ rank-one matrix and so spans $M_n(\mathbb{C})$.

Choose a basis $\{f_1, f_2, \dots, f_n\}$ such that $f_j \in \mathcal{V}_{j-1}(\mathcal{A})f$, for $j = 1, 2, \dots, n$. Similarly, choose a basis $\{e_1, e_2, \dots, e_n\}$ such that $e_i \in \mathcal{V}_{i-1}(\mathcal{A}^*)e$, for $i = 1, 2, \dots, n$. Then $\{e_i \otimes f_j : 1 \leq i, j \leq n\}$ is a basis for $M_n(\mathbb{C})$ for which $e_i \otimes f_j \in \mathcal{V}_{i+j-1}$. All of these basis elements belong to $\bigcup\{\mathcal{V}_{j-1}(\mathcal{A})R\mathcal{V}_{i-1}(\mathcal{A}) : 1 \leq i, j \leq n, (i, j) \neq (n, n)\}$ with the possible exception of $e_n \otimes f_n$. We complete the proof by showing that

$$e_n \otimes f_n \in \operatorname{span}\Big(\bigcup\{\mathcal{V}_{j-1}(\mathcal{A})R\mathcal{V}_{i-1}(\mathcal{A}) : 1 \leq i, j \leq n, (i, j) \neq (n, n)\} \cup \mathcal{V}_{n-1}\Big).$$

If $T \in M_n(\mathbb{C})$ is any matrix, it can be written as $T = \sum_{i,j=1}^n t_{i,j}(e_i \otimes f_j)$ for some scalars $\{t_{i,j} : 1 \leq i, j \leq n\}$. If $\{e_1^*, e_2^*, \dots, e_n^*\}$ is the basis dual to $\{e_1, e_2, \dots, e_n\}$ and $\{f_1^*, f_2^*, \dots, f_n^*\}$ is the basis dual to $\{f_1, f_2, \dots, f_n\}$, then $t_{i,j} = (Te_i^* | f_j^*)$, for all i and j. Now $\mathcal{V}_{n-1}e_n^* = \mathbb{C}^n$, so there exists $W \in \mathcal{V}_{n-1}$ such that $(We_n^* | f_n^*) = 1$. Then

$$W - (e_n \otimes f_n) \in \operatorname{span}\Big(\bigcup\{\mathcal{V}_{j-1}(\mathcal{A})R\mathcal{V}_{i-1}(\mathcal{A}) : 1 \leq i, j \leq n, (i, j) \neq (n, n)\}\Big).$$

Thus,

$$e_n \otimes f_n \in \operatorname{span}\Big(\bigcup\{\mathcal{V}_{j-1}(\mathcal{A})R\mathcal{V}_{i-1}(\mathcal{A}) : 1 \leq i, j \leq n, (i, j) \neq (n, n)\} \cup \mathcal{V}_{n-1}\Big)$$

as required. This completes the proof. □

Remark 2.2. The inequality in the statement of the preceding theorem is sharp, that is, the length of $\mathcal{S} = \{A_1, A_2, \dots, A_k\} \cup \{R\}$ can be $2n-2$. If we take $\mathcal{A} = \{J\}$, the upper triangular Jordan matrix, and $R = J^{*n-1}$, the length of $\mathcal{S}$ is $2n-2$. This was observed in [5].

3. Irreducible families of rank-one matrices

In the remainder of the paper, we will be concerned only with irreducible families consisting entirely of rank-one matrices. We first consider the case where all pairwise products are nonzero.

Theorem 3.1. *If $n \geq 2$ and $\mathcal{R} = \{R_1, R_2, \dots, R_k\}$ is a linearly independent, irreducible family of rank-one matrices satisfying $R_iR_j \neq 0$, for $1 \leq i, j \leq k$, then $n \leq k \leq n^2$, where both inequalities are sharp, and the length of $\mathcal{R}$ is 1 or 2. Indeed,*

(i) *if $k < n^2 - 1$, the length of $\mathcal{R}$ is 2;*
(ii) *for $k = n^2$, the length of $\mathcal{R}$ is 1;*
(iii) *for $k = n^2 - 1$, the length of $\mathcal{R}$ can be 1 or 2 and both possibilities actually occur.*

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PROOF. It is clear, by linear independence, that $k \leq n^2$. Since, by irreducibility, the linear span of $\bigcup\{\text{range}(R_i) : 1 \leq i \leq k\}$ is $\mathbb{C}^n$, we have $k \geq n$. Note that, for $1 \leq i, j \leq k$ and every matrix X, there exists a scalar λ such that $R_i X R_j = \lambda R_i R_j$. It follows that the length of $\mathcal{R}$ is 1 or 2. If $k < n^2 - 1$, the length cannot be 1 so it is 2. If $m = n^2$, then clearly length$(\mathcal{R}) = 1$. The following examples (I) and (II) show that k can equal n or n^2, and examples (III) and (IV) show that the length of $\mathcal{R}$ can be 1 and it can be 2 when $k = n^2 - 1$.

(I) (*k can be n^2*) First, observe that the vector $e = (1, 1, 1, \ldots, 1)$ is a cyclic vector for $A = \text{diag}(1, 2, 3, \ldots, n)$. Let $f_j = A^{j-1}e$, $1 \leq j \leq n$. Put $R_{i,j} = e_i \otimes f_j$, where $\{e_i\}_1^n$ are the standard basis vectors. Then $R_{i,j}R_{s,t} = (f_t|e_i)(e_s \otimes f_j) \neq 0$, since every component of f_j is nonzero. The set $\mathcal{R} = \{R_{i,j} : 1 \leq i, j \leq n\}$ is linearly independent since $\{e_i\}_1^n$ and $\{f_j\}_1^n$ are linearly independent. It is also irreducible. For, let M be a nonzero (common) invariant subspace of $\mathcal{R}$. Let $0 \neq x \in M$. Then $(x|e_{i_0}) \neq 0$ for some i_0. Thus $R_{i_0 j}x = (x|e_{i_0})f_j \in M$, for every $1 \leq j \leq n$. So $M = \mathbb{C}^n$.

(II) (*k can be n*) For $1 \leq j \leq n$, let $f_j = \sum_{i=1}^{j} e_i$, where $\{e_i\}_1^n$ are the standard basis vectors. Put $R_j = f_j \otimes f_j$, for $1 \leq j \leq n$. Since $(f_j|f_i) = \min\{i, j\} \neq 0$ for every i, j, it follows that $R_i R_j = (f_j|f_i)(f_j \otimes f_i) \neq 0$, for every i, j. Let $\{f_j^*\}_1^n$ be the basis dual to $\{f_j\}_1^n$. If $\sum_{m=1}^n \lambda_m R_m = 0$, where $\{\lambda_m\}_1^n$ are scalars then, for every j, $(\sum_{m=1}^n \lambda_m R_m)f_j^* = 0 = \lambda_j f_j$, so $\lambda_j = 0$. Thus $\mathcal{R}$ is linearly independent. Irreducibility is proved as follows. Clearly $\{f_j : 1 \leq j \leq n\}$ spans $\mathbb{C}^n$. Let M be a nonzero (common) invariant subspace of $\mathcal{R}$. Let $0 \neq x \in M$. Then $(x|f_{j_0}) \neq 0$ for some j_0. Thus $R_{j_0}x = (x|f_{j_0})f_{j_0} \in M$. So $f_{j_0} \in M$ and therefore $R_j f_{j_0} = (f_{j_0}|f_j)f_j \in M$ for $1 \leq j \leq n$. Thus $M = \mathbb{C}^n$.

(III) (*$k = n^2 - 1$ and length equal to* 1) Let $\mathcal{R}$ be as in (I). We claim that $e_1 \notin \text{span}\{f_2, f_3, \ldots, f_n\}$. Suppose that $e_1 = \sum_{j=1}^{n-1} \alpha_j A^j e$. Then $e_1 = A(\sum_{j=1}^{n-1} \alpha_j A^{j-1}e)$ and, since the unique solution to $Ax = e_1$ is $x = e_1$, it follows that $e_1 = \sum_{j=1}^{n-1} \alpha_j A^{j-1}e$. Comparing coefficients in the latter to those in $e_1 = \sum_{j=1}^{n-1} \alpha_j A^j e$ gives $\alpha_{n-1} = 0 = \alpha_1$ and $\alpha_j = \alpha_{j-1}$ for $2 \leq j \leq n-1$. Hence $\alpha_j = 0$ for every j. This contradicts $e_1 \neq 0$.

It now follows that $I \notin \text{span}(\{R_{i,j} : (i, j) \neq (1, 1)\})$, since the first column of I does not belong to the span of the first columns of those $R_{i,j}$ with $(i, j) \neq (1, 1)$. Since $\mathcal{R}$ is linearly independent, so is $\mathcal{R}_1 = \{R_{i,j} : (i, j) \neq (1, 1)\}$. It follows that $\mathcal{R}_1 \cup \{I\}$ is a basis for $M_n(\mathbb{C})$ and that the length of $\mathcal{R}_1$ is 1.

(IV) (*$k = n^2 - 1$ and length equal to* 2) Let $\mathcal{R}$ and $\{f_j\}_1^n$ be as in (I). Since $e_1 \notin \text{span}\{f_2, f_3, \ldots, f_n\}$ we have $e_1 + f_2 \notin \text{span}\{f_2, f_3, \ldots, f_n\}$. Let $g_1 = f_2 + e_1$, and let $g_j = f_j$ if $j \neq 1$. Then $\{g_j\}_1^n$ is a basis for $\mathbb{C}^n$. Define $\mathcal{R}_2 = \{e_i \otimes g_j : 1 \leq i, j \leq n\}$. Clearly $e_i \otimes e_i \in \text{span}(\mathcal{R}_2 \backslash \{e_1 \otimes g_3\})$, for every i, so $I \in \text{span}(\mathcal{R}_2 \backslash \{e_1 \otimes g_3\})$. It follows that the length of $\mathcal{R}_2$ is 2. □

The simplest families of rank-one matrices are those that arise as subsets of the set of elementary matrices $\{E_{i,j} : 1 \leq i, j \leq n\}$, where $E_{i,j} = e_j \otimes e_i$ and $\{e_i\}_1^n$ is the standard basis for $\mathbb{C}^n$. All such families are linearly independent, but not necessarily irreducible. We use these families to provide simple examples of what the length of an irreducible, linearly independent family of rank-one matrices can be. First we consider the question of irreducibility. In the following, we interpret $\text{span}(\bigcup\{\text{range}(R) : R \in \emptyset\})$ as $\{0\}$ and $\bigcap\{\ker(R) : R \in \emptyset\}$ as $\mathbb{C}^n$.

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THEOREM 3.2. *A family $\mathcal{R}$ of rank-one $n \times n$ matrices is irreducible if and only if*

(i) $\operatorname{span}(\bigcup\{\operatorname{range} R : R \in \mathcal{R}\}) = \mathbb{C}^n$ *and* $\bigcap\{\ker R : R \in \mathcal{R}\} = \{0\}$;

(ii) *if $\mathcal{E}$ is a proper, nonempty subset of $\mathcal{R}$ such that* $\operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\}) \neq \mathbb{C}^n$, *there exist $R \in \mathcal{E}$ and $S \in \mathcal{R}\backslash\mathcal{E}$ such that* $\operatorname{range} S \nsubseteq \operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\})$ *and* $SR \neq 0$.

PROOF. If R is a rank-one matrix, the subspace M of $\mathbb{C}^n$ is invariant under R if and only if $\operatorname{range} R \subseteq M$ or $M \subseteq \ker R$.

First, let $\mathcal{R}$ be irreducible. Since $\operatorname{span}(\bigcup\{\operatorname{range} R : R \in \mathcal{R}\})$ is invariant under $\mathcal{R}$ and is not $\{0\}$, it must be $\mathbb{C}^n$. Also, since $\bigcap\{\ker R : R \in \mathcal{R}\}$ is invariant under $\mathcal{R}$ and is not $\mathbb{C}^n$, it must be $\{0\}$.

Let $\mathcal{E}$ be a subset of $\mathcal{R}$ which is neither $\emptyset$ nor $\mathcal{R}$ with $\operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\}) \neq \mathbb{C}^n$. Suppose that $SR = 0$ for every $R \in \mathcal{E}$ and every $S \in \mathcal{R}\backslash\mathcal{E}$ satisfying the condition $\operatorname{range} S \nsubseteq \operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\})$. The subspace $M = \operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\})$ is invariant under $\mathcal{R}$, as is readily verified. Since $M \neq \{0\}$ or $\mathbb{C}^n$, this contradicts the irreducibility of $\mathcal{R}$.

Conversely, suppose that conditions (i) and (ii) are satisfied. Suppose that $\mathcal{R}$ has a proper, nontrivial invariant subspace, M, say. Let $\mathcal{E} = \{E \in \mathcal{R} : \operatorname{range} E \subseteq M\}$. By condition (i), $\mathcal{E}$ is a proper and nonempty subset of $\mathcal{R}$. Let $R \in \mathcal{E}$ and $S \notin \mathcal{E}$. Then $\operatorname{range} S \nsubseteq M$ so $M \subseteq \ker S$. But $\operatorname{range} R \subseteq M$ so $\operatorname{range} R \subseteq M \subseteq \ker S$ and so $SR = 0$. This contradicts condition (ii). This completes the proof. □

COROLLARY 3.3. *Let $\{e_i\}_1^n$ and $\{f_j\}_1^n$ be bases for $\mathbb{C}^n$. The set of rank-one matrices $\{e_i \otimes f_j : 1 \leq i, j \leq n\}$ is linearly independent. A nonempty subset $\mathcal{D}$ of this set of rank-one matrices is irreducible if and only if*

(i) $\operatorname{span}(\bigcup\{f_j : e_i \otimes f_j \in \mathcal{D}$ *for some* $i\}) = \operatorname{span}(\bigcup\{e_i : e_i \otimes f_j \in \mathcal{D}$ *for some* $j\}) = \mathbb{C}^n$, *and*

(ii) *for every proper, nonempty subset $\mathcal{E}$ of $\mathcal{D}$ with* $\operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\}) \neq \mathbb{C}^n$ *there exist $R \in \mathcal{E}$ and $S \in \mathcal{D}\backslash\mathcal{E}$ such that* $\operatorname{range} S \nsubseteq \operatorname{span}(\bigcup\{\operatorname{range} E : E \in \mathcal{E}\})$ *and* $SR \neq 0$.

PROOF. Let $\{e_i^*\}_1^n$ be the basis dual to $\{e_i\}_1^n$. Suppose $\sum_{k,j=1}^n \lambda_{k,j}(e_k \otimes f_j) = 0$, where the $\lambda_{k,j}$ are scalars. Applying the left-hand side to e_i^* gives $\sum_{j=1}^n \lambda_{i,j} f_j = 0$. Hence $\lambda_{i,j} = 0$, for every i and j. This proves linear independence. The remainder of the proof follows from Theorem 3.2. □

Applying Corollary 3.3 to families of elementary matrices gives the following result.

COROLLARY 3.4. *Let $\mathcal{D}$ be a nonempty subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. The family of elementary matrices $\{E_{i,j} : (i, j) \in \mathcal{D}\}$ is irreducible if and only if*

(i) *for every i with $1 \leq i \leq n$, there exists j such that $(i, j) \in \mathcal{D}$ (that is, every row intersects $\mathcal{D}$), and*

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(ii) *for every* j *with* $1 \le j \le n$*, there exists* i *such that* $(i, j) \in \mathcal{D}$ *(that is, every column intersects* $\mathcal{D}$*), and*

(iii) *for every proper, nonempty subset* $\mathcal{E}$ *of* $\mathcal{D}$*, satisfying* $\mathrm{span}(\bigcup\{e_i : (i, j) \in \mathcal{E}\}) \neq \mathbb{C}^n$ *there exist* i, j, k *such that* $(i, j) \in \mathcal{E}$*,* $(k, i) \in \mathcal{D} \backslash \mathcal{E}$ *and* $(k, m) \notin \mathcal{E}$ *for* $1 \le m \le n$*.*

We next turn to the question of length for linearly independent, irreducible families of rank-one matrices. In the following theorem we use two elementary facts about rank-one matrices:

(a) if $R, X, Y \in M_n(\mathbb{C})$ and $XRY = 0$ where R has rank one, then $XR = 0$ or $RY = 0$;

(b) if $E = e \otimes f$ and $M = \mathrm{span}(\{f_1, f_2, \ldots, f_t\})$, then E does not leave M invariant if and only if $f \notin \mathrm{span}(\{f_1, f_2, \ldots, f_t\})$ and $Ef_s \neq 0$, for some s.

Theorem 3.5. *Let* $n \ge 2$ *and let* $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$ *be a linearly independent, irreducible family of* $n \times n$ *rank-one matrices given by* $R_i = e_i \otimes f_i$ *for* $1 \le i \le k$*. Then for* $1 \le j \le k$*, there exist integers* $i_1, i_2, \ldots, i_{n-1}$ *in* $\{1, 2, \ldots, k\}$ *such that* $\{f_j, f_{i_1}, f_{i_2}, \ldots, f_{i_{n-1}}\}$ *is a basis for* $\mathbb{C}^n$ *and, for each* s *with* $1 \le s \le n-1$*, there exists a nonzero word in* $\mathcal{R}$*, of length at most* $s+1$*, beginning with* R_{i_s} *and ending with* R_j*. Consequently, the length of* $\mathcal{R}$ *is at most* n*.*

Proof. Let $1 \le j \le k$. By irreducibility (taking $\mathcal{E} = \{R_j\}$ in Theorem 3.2) there exists i_1 with $1 \le i_1 \le k$ such that $\{f_j, f_{i_1}\}$ is linearly independent and $R_{i_1}R_j \neq 0$. If $n = 2$, the theorem is proved. Otherwise, if $n \ge 3$ (taking $\mathcal{E} = \{R_{i_1}, R_j\}$), there exists i_2 with $1 \le i_2 \le k$ such that $\{f_j, f_{i_1}, f_{i_2}\}$ is linearly independent and $R_{i_2}R_{i_1} \neq 0$ or $R_{i_2}R_j \neq 0$. If $R_{i_2}R_{i_1} \neq 0$, then $R_{i_2}R_{i_1}R_j \neq 0$ since $R_{i_1}R_j \neq 0$. If $n = 3$ the proof is complete.

Continuing in this way, after m steps, we construct a linearly independent set $\{f_j, f_{i_1}, f_{i_2}, \ldots, f_{i_m}\}$ and, for every s with $1 \le s \le m$, there is a nonzero word in $\mathcal{R}$ beginning with R_{i_s} and ending with R_j of elements of $\mathcal{R}$ and of length less than or equal to $s+1$. If $m+1 < n$ we can continue to the next step. The process stops when we have a basis $\{f_j, f_{i_1}, f_{i_2}, \ldots, f_{i_{n-2}}, f_{i_{n-1}}\}$ of $\mathbb{C}^n$ and, for $1 \le s \le n-1$, a nonzero word in $\mathcal{R}$ beginning with R_{i_s} and ending with R_j of length less than or equal to $s+1$.

Let $1 \le j \le n$ and continue the notation as in the preceding paragraph. For any $n \times n$ matrix X and for $1 \le s \le n-1$, we have $R_{i_s}XR_j = \gamma(e_j \otimes f_{i_s})$ for some scalar γ. If $R_{i_s}XR_j \neq 0$ then $\gamma \neq 0$. It follows that $e_j \otimes f_{i_s} \in \mathcal{V}_n$, for $s = 1, 2, \ldots, n-1$. Since $e_j \otimes f_j \in \mathcal{V}_n$ we get $e_j \otimes f \in \mathcal{V}_n$, for every $f \in \mathbb{C}^n$. Since $\bigcap\{\ker R_j : 1 \le j \le k\} = \{0\}$, it follows that $\{e_1, e_2, \ldots, e_k\}$ spans $\mathbb{C}^n$, so $e \otimes f \in \mathcal{V}_n$ for all vectors $e, f \in \mathbb{C}^n$. Thus $\mathcal{V}_n = M_n(\mathbb{C})$, so the length of $\mathcal{R}$ is at most n. □

Theorem 3.6. *Let* $n \ge 2$ *and let* $\{e_i\}_1^n, \{f_j\}_1^n$ *be bases for* $\mathbb{C}^n$*. Let* $\mathcal{D}$ *be an irreducible subset of* $\{e_i \otimes f_j : 1 \le i, j \le n\}$*. There is a positive integer* $k \le n$ *with the property that every rank-one matrix* $e_i \otimes f_j, 1 \le i, j \le n$*, belongs to the set of scalar multiples of words in* $\mathcal{D}$ *of length at most* k*.*

The msl M *of* $\mathcal{D}$ *is equal to the least such positive integer* k *(and the length* L *of* $\mathcal{D}$ *is* M *or* $M-1$ *according as* $I \in \mathcal{V}'_L$ *or not).*

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Proof. Using the preceding theorem, we find that n will serve as a k. Denote the least such k by K. We have $\mathcal{V}'_K = M_n(\mathbb{C})$, so $M \le K$. We next show that $\mathcal{V}'_{K-1} \ne M_n(\mathbb{C})$. Since $K-1$ will not serve as a k, there exists (i, j) such that there is a nonzero word W of length K with $W \in \text{span}(e_i \otimes f_j)$, but $V \notin \text{span}(e_i \otimes f_j)$ for every nonzero word V of length at most $K-1$. If $\{e_s^*\}_1^n, \{f_q^*\}_1^n$ are the dual bases, since every word in $\mathcal{D}$ of length at most $K-1$ is of the form $\gamma(e_p \otimes f_q)$ for some scalar γ and some p, q, it follows that $(Ve_i^*|f_j^*) = 0$, for every word of length at most $K-1$. Hence $W \notin \mathcal{V}'_{K-1}$ (since $(We_i^*|f_j^*) \ne 0$), so $K = M$. □

Examples below show some of the values that the length of an irreducible, linearly independent family of rank-one matrices can be. Attention is restricted to families of elementary matrices. Recall that $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$, where $\delta_{j,k}$ is the Kronecker delta function. For $1 \le s, t \le n-1$, define the sets Δ_s, Δ_{-t} by

$$\Delta_s = \{E_{i,i+s} : 1 \le i \le n-s\} \quad \text{and} \quad \Delta_{-t} = \{E_{i,i-t} : t+1 \le i \le n\}.$$

(So Δ_s is the set of elementary matrices in sth diagonal positions and Δ_{-t} is the set of elementary matrices in $-t$th diagonal positions.) Also, if $\mathcal{E}$ and $\mathcal{F}$ are families of complex matrices we define their #-product ('hash product'), $\mathcal{E}\#\mathcal{F}$, to be the set of all products $\{XY : X \in \mathcal{E}, Y \in \mathcal{F}\} \cup \{YX : X \in \mathcal{E}, Y \in \mathcal{F}\}$.

It is clear that, if $1 \le |s|, |t| \le n-1$ then $\Delta_s\#\Delta_t \subseteq \Delta_{s+t} \cup \{0\}$. In more detail, we have, and will frequently use, the following facts. Let $1 \le s, t \le n-1$. Then

(P) $\Delta_s\#\Delta_t = \Delta_{s+t} \cup \{0\}$, if $s+t \le n-1$;
(Q) $\Delta_s\#\Delta_t = \{0\}$, if $s+t > n-1$;
(R) $\Delta_{-s}\#\Delta_{-t} = \Delta_{-(s+t)} \cup \{0\}$, if $s+t \le n-1$;
(S) $\Delta_s\#\Delta_{-t} = \Delta_{s-t} \cup \{0\}$, if $s+t \le n$.

Example 3.7. In the following, all the examples are on the space $\mathbb{C}^n$, where $n \ge 2$ and 'a word of length m' means 'the matrix which arises as a word of length m'.

(1) *Length* 2. The length of $\{E_{i,j} : 1 \le i, j \le n\}\backslash\Delta_0$ is 2, since $\Delta_1\#\Delta_{-1} = \Delta_0 \cup \{0\}$.

(2) *Length n*. Let $\mathcal{E}_1 = \Delta_{n-1} \cup \Delta_{-1}$. This set of elementary matrices is obviously linearly independent. It is also irreducible since it contains $J^* = \sum\{E_{i,j} : E_{i,j} \in \Delta_{-1}\}$, the lower triangular Jordan matrix. It is well known that J^* and $E_{1,n} = J^{n-1}$ have no common nontrivial invariant subspaces. The set of words of length one in $\mathcal{E}_1$ is

$$\mathcal{W}_1 = \mathcal{E}_1 = \{E_{2,1}, E_{3,2}, E_{4,3}, \ldots, E_{n,n-1}\} \cup \{E_{1,n}\}.$$

If $n \ge 3$ the set of words of length two is

$$\mathcal{W}_2 = \{E_{2,n,}, E_{3,1}, E_{4,2}, \ldots, E_{n,n-2}, E_{1,n-1}\} \cup \{0\} = \Delta_{n-2} \cup \Delta_{-2} \cup \{0\}.$$

It is easily proved my induction that, given $n \ge 3$, the set of words of length m is given by

$$\mathcal{W}_m = \Delta_{n-m} \cup \Delta_{-m} \cup \{0\} \quad \text{for } 2 \le m \le n-1.$$

Thus, $\mathcal{V}_{n-1} = \text{span}(\{E_{i,j} : 1 \le i, j \le n, (i, j) \notin \Delta_0\}) + \mathbb{C}I$ for $n \ge 2$. Now $\Delta_0 \subseteq \mathcal{W}_n$ since $\Delta_1\#\Delta_{-1} = \Delta_0 \cup \{0\}$, so the length of $\mathcal{E}_1$ is n.

(3) *Length* $n-1$. Let $\mathcal{D} = \Delta_1 \cup \Delta_{-1}$. This set of elementary matrices is obviously linearly independent. It is also irreducible since the upper triangular Jordan matrix $J = \sum\{E_{i,j} : E_{i,j} \in \Delta_1\}$ has no reducing subspaces. Here the set of words of length one is $\mathcal{W}_1 = \Delta_1 \cup \Delta_{-1}$ and, for words of length 2 and 3, $\mathcal{W}_2 = \Delta_2 \cup \Delta_{-2} \cup \Delta_0 \cup \{0\}$ if $n \geq 3$ and $\mathcal{W}_3 = \Delta_3 \cup \Delta_{-3} \cup \Delta_1 \cup \Delta_{-1} \cup \{0\}$ if $n \geq 4$. We prove by induction that, if $n \geq 3$

$$\bigcup_{u=1}^{v} \mathcal{W}_u = \Big(\bigcup_{u=1}^{v} \Delta_u \cup \Delta_{-u}\Big) \cup \Delta_0 \cup \{0\} \quad \text{for } 2 \leq v \leq n-1. \tag{3.1}$$

We have observed that (3.1) is true for $v = 2$, so there is no more to be proved if $n = 3$. Assume that $n \geq 4$ and that (3.1) is true for all integers v satisfying $2 \leq v \leq m$, where $2 \leq m \leq n-2$. We shall show that (3.1) is true for $v = m+1$. Since

$$\bigcup_{u=1}^{m+1} \mathcal{W}_u = \Big(\bigcup_{u=1}^{m} \mathcal{W}_u\Big) \cup \Big(\mathcal{W}_{m+1} \backslash \bigcup_{u=1}^{m} \mathcal{W}_u\Big)$$

and

$$\mathcal{W}_m = \Big(\mathcal{W}_m \backslash \bigcup_{u=1}^{m-1} \mathcal{W}_u\Big) \cup \Big(\mathcal{W}_m \cap \bigcup_{u=1}^{m-1} \mathcal{W}_u\Big) = (\Delta_m \cup \Delta_{-m}) \cup \Big(\mathcal{W}_m \cap \bigcup_{u=1}^{m-1} \mathcal{W}_u\Big),$$

it follows that

$$\mathcal{W}_{m+1} \backslash \bigcup_{u=1}^{m} \mathcal{W}_u = (\mathcal{W}_1 \# \mathcal{W}_m) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u = (\mathcal{W}_1 \# (\Delta_m \cup \Delta_{-m})) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u.$$

The induction proof is completed by showing that

$$(\mathcal{W}_1 \# (\Delta_m \cup \Delta_{-m})) \backslash \Big(\bigcup_{u=1}^{m} \mathcal{W}_u\Big) = \Delta_{m+1} \cup \Delta_{-(m+1)}.$$

Noticing that

$$\begin{aligned} \Delta_1 \# \Delta_m &= \Delta_{m+1} \cup \{0\}, \quad & \Delta_1 \# \Delta_{-m} &= \Delta_{-m+1} \cup \{0\}, \\ \Delta_{-1} \# \Delta_m &= \Delta_{m-1} \cup \{0\}, \quad & \Delta_{-1} \# \Delta_{-m} &= \Delta_{-(m+1)} \cup \{0\} \end{aligned}$$

gives

$$\begin{aligned} (\mathcal{W}_1 \# (\Delta_m \cup \Delta_{-m})) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u &= ((\Delta_1 \# \Delta_{-1}) \# (\Delta_m \cup \Delta_{-m})) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u \\ &= \Delta_{m+1} \cup \Delta_{-(m+1)}. \end{aligned}$$

Since (3.1) is true for $2 \leq v \leq n-1$, it follows that $\mathcal{V}_{n-2} \neq M_n(\mathbb{C})$ and $\mathcal{V}_{n-1} = M_n(\mathbb{C})$. Hence $\mathcal{D}$ has length $n-1$. If $n = 2$, we have $\mathcal{W}_2 = \Delta_0 \cup \{0\}$, so $\mathcal{D}$ has length $n = 2$.

THEOREM 3.8. *Let $n \geq 2$ and $n-1 = pk$ where $p, k \in \mathbb{Z}^+$. The set of elementary matrices $\mathcal{E}_p = (\bigcup_{s=n-p}^{n-1} \Delta_s) \cup (\bigcup_{t=1}^{p} \Delta_{-t})$ is linearly independent, irreducible and has length $k+1$.*

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Proof. The set $\mathcal{E}_p$ is clearly linearly independent. Because of an earlier example it is also irreducible since it contains Δ_{-1} and Δ_{n-1} and we may suppose that $p \geq 2$. Moreover, by an earlier example, we can also suppose that $k \geq 2$ (so $n \geq 5$).

Let $\mathcal{P} = \bigcup_{s=n-p}^{n-1} \Delta_s$ and let $\mathcal{N} = \bigcup_{t=1}^{p} \Delta_{-t}$, so that $\mathcal{E}_p = \mathcal{P} \cup \mathcal{N}$. The set of words in $\mathcal{E}_p$ of length one is $\mathcal{W}_1 = \mathcal{P} \cup \mathcal{N}$. The set of words of length two is

$$\mathcal{W}_2 = \mathcal{W}_1 \# \mathcal{W}_1 = (\mathcal{P}\#\mathcal{P}) \cup (\mathcal{N}\#\mathcal{N}) \cup (\mathcal{P}\#\mathcal{N}).$$

We simplify this as follows.

(i) $\mathcal{P}\#\mathcal{P} = (\bigcup_{s=n-p}^{n-1} \Delta_s)\#(\bigcup_{s'=n-p}^{n-1} \Delta_{s'}) = \{0\}$, since $s + s' > n-1$ if $s, s' \in [n-p, n-1]$ (because $2p \leq n-1$).

(ii) $\mathcal{N}\#\mathcal{N} = (\bigcup_{t=1}^{p} \Delta_{-t})\#(\bigcup_{t'=1}^{p} \Delta_{-t'}) = (\bigcup_{t''=2}^{2p} \Delta_{-t''}) \cup \{0\}$, since $t + t' \leq n-1$ if $t, t' \in [1, p]$.

(iii) $\mathcal{P}\#\mathcal{N}$. We claim that $\mathcal{P}\#\mathcal{N} = (\bigcup_{s=n-p}^{n-1} \Delta_s)\#(\bigcup_{t=1}^{p} \Delta_{-t}) = (\bigcup_{s'=n-2p}^{n-2} \Delta_{s'}) \cup \{0\}$. Clearly $\mathcal{P}\#\mathcal{N} \subseteq \bigcup_{s'=n-2p}^{n-2} \Delta_{s'} \cup \{0\}$, since $s - t \in [n-2p, n-2]$ if $s \in [n-p, n-1]$ and $t \in [1, p]$. We show that $\Delta_{s'} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N}$, for $s' \in [n-2p, n-2]$.

First, let $t \in [1, p-1]$ and put $s = n-1-t$. Then $s \in [n-p, n-1]$ and $s + t \leq n$, and so $\Delta_s\#\Delta_{-t} = \Delta_{s-t} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N}$. This shows that

$$\Delta_{s'} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N} \quad \text{if } s' \in \{n-3, n-5, \ldots, n-2p+1\}.$$

Secondly, let $t \in [1, p]$ and put $s = n-t$. Then $s \in [n-p, n-1]$ and $s + t \leq n$. Thus $\Delta_s\#\Delta_{-t} = \Delta_{s-t} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N}$, so that $\Delta_{s'} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N}$ if $s' \in \{n-2, n-4, \ldots, n-2p\}$. It now follows that $\Delta_{s'} \cup \{0\} \subseteq \mathcal{P}\#\mathcal{N}$ if $s' \in [n-2p, n-2]$.

Thus $\mathcal{W}_2 = (\bigcup_{t''=2}^{2p} \Delta_{-t''}) \cup (\bigcup_{s'=n-2p}^{n-2} \Delta_{s'}) \cup \{0\}$. For $1 \leq m \leq k$, define

$$\mathcal{P}_m = \bigcup_{s_m=n-mp}^{n-(m-1)p-1} \Delta_{s_m} \quad \text{and} \quad \mathcal{N}_m = \bigcup_{t_m=(m-1)p+1}^{mp} \Delta_{-t_m}.$$

Then $\mathcal{W}_1 = \mathcal{P}_1 \cup \mathcal{N}_1$ and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{P}_1 \cup \mathcal{N}_1 \cup \mathcal{P}_2 \cup \mathcal{N}_2 \cup \{0\}$. Moreover, the sets $\{\mathcal{P}_m \cup \mathcal{N}_m : 1 \leq m \leq k\}$ are pairwise disjoint with union $\{E_{i,j} : 1 \leq i, j \leq n\} \backslash \Delta_0$.

We prove by induction that

$$\bigcup_{u=1}^{v} \mathcal{W}_u = \bigcup_{u=1}^{v} (\mathcal{P}_u \cup \mathcal{N}_u) \cup \{0\} \quad \text{for } 2 \leq v \leq k,$$

where $\mathcal{W}_u$ denotes the words of length u in $\mathcal{E}_p$. We have already seen that the result is true for $v = 2$. Let $2 \leq m \leq k-1$ and suppose that the result is true for all v with $2 \leq v \leq m$. We show that it is true for $m+1$.

Observe that

$$\bigcup_{u=1}^{m+1} \mathcal{W}_u = \Big(\bigcup_{u=1}^{m} \mathcal{W}_u\Big) \cup \Big(\mathcal{W}_{m+1} \backslash \bigcup_{u=1}^{m} \mathcal{W}_u\Big)$$

and

$$\mathcal{W}_{m+1} \backslash \bigcup_{u=1}^{m} \mathcal{W}_u = (\mathcal{W}_1 \# (\mathcal{P}_m \cup \mathcal{N}_m)) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u = ((\mathcal{P}_1 \cup \mathcal{N}_1) \# (\mathcal{P}_m \cup \mathcal{N}_m)) \backslash \bigcup_{u=1}^{m} \mathcal{W}_u.$$

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We show that

$$((\mathcal{P}_1 \cup \mathcal{N}_1)\#(\mathcal{P}_m \cup \mathcal{N}_m))\setminus \bigcup_{u=1}^{m} \mathcal{W}_u = \mathcal{P}_{m+1} \cup \mathcal{N}_{m+1}.$$

We simplify $(\mathcal{P}_1 \cup \mathcal{N}_1)\#(\mathcal{P}_m \cup \mathcal{N}_m) = (\mathcal{P}_1\#\mathcal{P}_m) \cup (\mathcal{P}_1\#\mathcal{N}_m) \cup (\mathcal{N}_1\#\mathcal{P}_m) \cup (\mathcal{N}_1\#\mathcal{N}_m)$.

(1) $\mathcal{P}_1\#\mathcal{P}_m = (\bigcup_{s_1=n-p}^{n-1} \Delta_{s_1})\#(\bigcup_{s_m=n-mp}^{n-(m-1)p-1} \Delta_{s_m}) = \{0\}$, since $s_1 + s_m > n-1$ if $s_1 \in [n-p, n-1]$ and $s_m \in [n-mp, n-(m-1)p-1]$ (because $(m+1)p \leq n-1$).

(2) $\mathcal{N}_1\#\mathcal{N}_m = (\bigcup_{t_1=1}^{p} \Delta_{-t_1})\#(\bigcup_{t_m=(m-1)p+1}^{mp} \Delta_{-t_m}) = (\bigcup_{t_{m+1}=(m-1)p+2}^{(m+1)p} \Delta_{-t_{m+1}}) \cup \{0\}$, since $t_1 + t_m \leq n-1$, if $t_1 \in [1, p]$ and $t_m \in [(m-1)p+1, mp]$. Thus, using the induction assumption,

$$(\mathcal{N}_1\#\mathcal{N}_m)\setminus \bigcup_{u=1}^{m} \mathcal{W}_u = \mathcal{N}_{m+1}.$$

(3) $\mathcal{N}_1\#\mathcal{P}_m = (\bigcup_{t_1=1}^{p} \Delta_{-t_1})\#(\bigcup_{s_m=n-mp}^{n-(m-1)p-1} \Delta_{s_m}) = (\bigcup_{s_{m+1}=n-(m+1)p}^{n-(m-1)p-2} \Delta_{s_{m+1}}) \cup \{0\}$, since if $s_m \in [n-mp, n-(m-1)p-1]$ and $t_1 \in [1, p]$, then we have $s_m + t_1 \leq n$ and $s_m - t_1 \in [n-(m+1)p, n-(m-1)p-2]$. Thus, using the induction assumption,

$$(\mathcal{N}_1\#\mathcal{P}_m)\setminus \bigcup_{u=1}^{m} \mathcal{W}_u = \mathcal{P}_{m+1}.$$

(4) $\mathcal{P}_1\#\mathcal{N}_m = (\bigcup_{s_1=n-p}^{n-1} \Delta_{s_1})\#(\bigcup_{t_m=(m-1)p+1}^{mp} \Delta_{-t_m}) \subseteq (\bigcup_{s_{m+1}=n-(m+1)p}^{n-(m-1)p-2} \Delta_{s_{m+1}}) \cup \{0\}$, and so $(\mathcal{P}_1\#\mathcal{N}_m)\setminus(\bigcup_{u=1}^{m} \mathcal{W}_u) \subseteq \mathcal{P}_{m+1}$.

We have shown that $((\mathcal{P}_1 \cup \mathcal{N}_1)\#(\mathcal{P}_m \cup \mathcal{N}_m))\setminus(\bigcup_{u=1}^{m} \mathcal{W}_u) = \mathcal{P}_{m+1} \cup \mathcal{N}_{m+1}$, and it follows by induction that

$$\bigcup_{u=1}^{v} \mathcal{W}_u = \bigcup_{u=1}^{v} (\mathcal{P}_u \cup \mathcal{N}_u) \cup \{0\}, \quad \text{for } 2 \leq v \leq k.$$

Thus $\bigcup_{u=1}^{v} \mathcal{W}_u = \{E_{i,j} : 1 \leq i, j \leq n\}\setminus\Delta_0$ and so $\mathcal{V}_k \neq M_n(\mathbb{C})$ since $\mathcal{V}_k$ is the linear span of $(\bigcup_{u=1}^{k} \mathcal{W}_u) \cup \{I\}$. Since $\Delta_1\#\Delta_{-1} = \Delta_0 \cup \{0\}$ and since $\Delta_1 \subseteq \mathcal{P}_k \subseteq \bigcup_{u=1}^{k} \mathcal{W}_u$, it follows that $\mathcal{E}_p$ has length $k+1$. This completes the proof. □

Remark 3.9. With notation as in the preceding proof, we have the chain

$$\mathbb{C}I = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{k-1} \subset \mathcal{V}_k \subset \mathcal{V}_{k+1} = M_n(\mathbb{C}),$$

where $\mathcal{V}_m = \operatorname{span}\{\bigcup_{u=1}^{m} (\mathcal{P}_u \cup \mathcal{N}_u)\} + \mathbb{C}I$, for $1 \leq m \leq k$. It is perhaps interesting to note that here the dimension d_m of $\mathcal{V}_m$ is $d_m = mnp + 1$ for $1 \leq m \leq k$, so that the differences in dimension $d_{m+1} - d_m$ all equal np for $0 \leq m \leq k-1$. Also, $d_{k+1} - d_k = n-1$. The msl spanning length of $\mathcal{E}_p$ is also $k+1$.

Finally, notice that the product $|\mathcal{E}_p| \times (\text{Length}(\mathcal{E}_p) - 1) = n(n-1)$, which gives precision to the intuitive idea that length is inversely proportional to cardinality.

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