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On J-self-adjoint operators with stable C-symmetries

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The paper is devoted to the development of the theory of self-adjoint operators in Krein spaces (J-self-adjoint operators) involving some additional properties arising from the existence of C-symmetries. We mainly focus on the recent notion of stable C-symmetry for J-self-adjoint extensions of a symmetric operator S. The general results involve boundary value techniques and reproducing kernel space methods, and they include an explicit functional model for the class of stable C-symmetries. Some of the results are specialized further by studying the case where S has defect numbers $\langle 2, 2\rangle$ in detail.

1. Introduction

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and let J be a non-trivial fundamental symmetry, i.e. $J = J^*$, $J^2 = I$, and $J \neq \pm I$. The space $\mathfrak{H}$ equipped with the indefinite inner product (indefinite metric) $[x, y]_J := (Jx, y)$, $x, y \in \mathfrak{H}$, is called a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$. An operator A acting in $\mathfrak{H}$ is called J-self-adjoint if $A^*J = JA$ i.e. if A is self-adjoint with respect to the indefinite metric $[\cdot,\cdot]_J$.

The development of $\mathcal{PT}$-symmetric quantum mechanics (PTQM) during the last decade (see [6, 32, 35] and the references therein) has given rise to many new mathematical problems in the theory of J-self-adjoint operators. For instance, one of the key aspects of PTQM is the description of a hidden symmetry C for a given pseudo-Hermitian Hamiltonian A in the domain of exact $\mathcal{PT}$-symmetry [6]. By analogy with [6], the definition of C-symmetry in Krein spaces can be formalized as follows.

Definition 1.1. An operator A in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ has the property of C-symmetry if there exists a bounded linear operator C in $\mathfrak{H}$ such that

(i) $C^2 = I$,

(ii) $JC > 0$,

(iii) $AC = CA$.

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Conditions (i) and (ii) are equivalent to the presentation $C = J\mathrm{e}^{Y}$, where Y is a bounded self-adjoint operator that anti-commutes with J. The properties of C are nearly identical to those of the charge conjugation operator in quantum field theory and C determines a new definite inner product $(\cdot,\cdot)_C = [C\cdot,\cdot]_J = (\mathrm{e}^{Y}\cdot,\cdot)$, being equivalent to the initial one $(\cdot,\cdot)$.

If a J-self-adjoint operator A possesses the property of C-symmetry, then A turns out to be self-adjoint with respect to $(\cdot,\cdot)_C$ and the dynamics generated by A is governed by a unitary time evolution. However, the operator C depends on the choice of A and finding it is a non-trivial problem.

There have been many attempts to calculate the operator C [7–10, 22, 30] or the metric operator $\Theta = \mathrm{e}^{Y} = JC$ [26, 27, 31, 32] for various $\mathcal{PT}$-symmetric models of interest. Due to the complexity of the problem, it is unsurprising that the majority of the available formulae are still approximative, usually expressed as leading terms of perturbation series.

We investigate C-symmetries of J-self-adjoint extensions of a densely defined symmetric operator S that commutes with J, i.e. $SJ = JS$. (This means that S is simultaneously symmetric and J-symmetric.)

Definition 1.2. A J-self-adjoint extension A of S has the property of *stable* C-symmetry if A and S have the property of C-symmetry realized by the *same* operator C.

Self-adjoint extensions of S that commute with J are trivial examples of J-self-adjoint extensions with stable C-symmetry. In that case, $C = J$.

The set Σ_J^{st} of all J-self-adjoint extensions of S with stable symmetry can be considered as an analogue of the domain of exact $\mathcal{PT}$-symmetry [6] in the extension theory framework.

If S has defect numbers $\langle 2,2\rangle$, then the non-triviality[1] of Σ_J^{st} is equivalent to the existence of J-self-adjoint extensions of S with empty resolvent set [28]. In that case S commutes with a complex Clifford algebra $\mathcal{CL}_2(J,R)$, where R is an additional fundamental symmetry with $JR = -RJ$ and all stable C-symmetries are expressed in terms of $\mathcal{CL}_2(J,R)$ [2, 18, 28].

We shall propose a general method for the description of all possible stable C-symmetries by applying a boundary value technique and a reproducing kernel Hilbert space model associated with the Weyl function of S (see [5]); this method relies on theorems 3.11, 3.15 and 3.18, below. In some sense this solves the problem of construction of stable C-symmetries for J-self-adjoint extensions of a simple symmetric operator S with arbitrary defect numbers $\langle n,n\rangle$.

J-self-adjoint operators with stable C-symmetries admit detailed spectral analysis [2] and the set Σ_J^{st} may be used for the explanation of exceptional points phenomenon in PTQM, which arises at the boundary of the domain of exact $\mathcal{PT}$-symmetry (see [6,19,32] and the references therein). Our contribution to this promising and self-consistent topic has an introductory character and it consists in the 'phase transition simulation at the boundary' by means of various fundamental decompositions of the Krein space $(\mathfrak{H},[\cdot,\cdot]_J)$ (see § 2.3).

This paper is organized as follows. Section 2, with the exception of § 2.3, contains preliminary results related to Krein space theory and the boundary triplets method

[1]That is, Σ_J^{st} contains not only trivial self-adjoint extensions.

in the extension theory. We have tried to emphasize the usefulness of the Krein space ideology for the description of self-adjoint extensions of a symmetric operator.

In § 3, J-self-adjoint extensions of a symmetric operator S with stable C-symmetry are investigated. Let $\mathfrak{U} = \{C_\alpha\}$ be the collection of operators C (parametrized by a set of indexes $\alpha \in D$) which realize the property of C-symmetry for S (see definition 1.1). The set $\Upsilon_{\mathfrak{U}}$ of J-self-adjoint extensions $A \supset S$ that commutes with any C_α plays a principal role in our considerations. The descriptions for $\Upsilon_{\mathfrak{U}}$ in theorem 3.5 are analogous to the descriptions of self-adjoint extensions $A \supset S$ that commute with a family of unitary operators $\{U_\alpha\}$ satisfying the additional condition $U \in \{U_\alpha\} \iff U^* \in \{U_\alpha\}$ (see [20, 23]).

If $\Upsilon_{\mathfrak{U}}$ is non-empty,[2] then there exists a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S which provides the images $\{\mathcal{C}_\alpha\}$ of $\mathfrak{U} = \{C_\alpha\}$ in $\mathcal{H}$ with the preservation of all principal properties of C_α (lemmas 3.7 and 3.10). This enables us to describe the elements of Σ_J^{st} in terms of stable $\mathcal{J}$-unitary operators of the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$, where $\mathcal{J}$ is the image of J (theorems 3.11 and 3.18). An 'external' description of the family $\{\mathcal{C}_\alpha\}$ (theorem 3.15) is obtained using a reproducing kernel Hilbert space model associated with the Weyl function of S, including a functional analytic model for the family $\mathfrak{U} = \{C_\alpha\}$ itself (see corollary 3.16). This leads to a characterization of the resolvents of operators from Σ_J^{st} (theorem 3.20), which is essential for their spectral analysis.

The description of $\{\mathcal{C}_\alpha\}$ in theorem 3.15 gives a simple method for the construction of stable C-symmetries in the case of a symmetric operator S with arbitrary defect numbers $\langle n, n\rangle$ (see remark 3.17). In the 'exactly solvable' case of defect numbers $\langle 2, 2\rangle$, the set $\mathfrak{U}$ of stable C-symmetries is described explicitly in terms of the Clifford algebra $\mathcal{CL}_2(J, R)$ (see [28]). In § 4 we use this fact for the detailed spectral analysis of operators from Σ_J^{st}. Together with example 4.6 we outline a general scheme of possible applications (§ 4.3) that is particularly useful for the study of differential operators with singular potentials and $\mathcal{PT}$-symmetric boundary conditions [1, 25, 34].

Notation. $\mathcal{D}(A)$ denotes the domain of a linear operator A and $A \upharpoonright \mathcal{D}$ denotes the restriction of A to a set $\mathcal{D}$. The symbols $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$ stand for the commutator and anti-commutator of the operators A and B, respectively. The symbols $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A.

2. Preliminaries

2.1. Elements of the Krein space theory

Let $(\mathfrak{H}, [\cdot,\cdot]_J)$ be a Krein space with a fundamental symmetry J. For the basic theory of Krein spaces and operators acting therein we refer the interested reader to [4].

The operator J determines *the fundamental decomposition* of $\mathfrak{H}$:

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \mathfrak{H}_- = \tfrac{1}{2}(I - J)\mathfrak{H}, \quad \mathfrak{H}_+ = \tfrac{1}{2}(I + J)\mathfrak{H}. \tag{2.1}$$

[2]The case $\Upsilon_{\mathfrak{U}} = \emptyset$ was considered in [29].

A subspace $\mathfrak{L}$ of $\mathfrak{H}$ is called *hypermaximal neutral* if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]} = \{x \in \mathfrak{H} \colon [x, y]_J = 0, \ \forall y \in \mathfrak{L}\}.$$

A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called *uniformly positive* (*respectively, uniformly negative*) if $[x, x]_J \geqslant a^2\|x\|^2$ (respectively, $-[x, x]_J \geqslant a^2\|x\|^2$) $a \in \mathbb{R}, a \neq 0$ for all $x \in \mathfrak{L}$. The subspaces $\mathfrak{H}_\pm$ in (2.1) are examples of uniformly positive and uniformly negative subspaces and, moreover, they are maximal, i.e. $\mathfrak{H}_+$ (respectively, $\mathfrak{H}_-$) is not a proper subspace of a uniformly positive (respectively, negative) subspace.

Let $\mathfrak{L}_+ (\neq \mathfrak{H}_+)$ be a maximal uniformly positive subspace. Then its J-orthogonal complement $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]}$ is a maximal uniformly negative subspace and the direct J-orthogonal sum

$$\mathfrak{H} = \mathfrak{L}_+ [\dot{+}] \mathfrak{L}_- \tag{2.2}$$

gives a fundamental decomposition of $\mathfrak{H}$. The subspaces $\mathfrak{L}_\pm$ in (2.2) can be described as $\mathfrak{L}_+ = (I + X)\mathfrak{H}_+$ and $\mathfrak{L}_- = (I + X^*)\mathfrak{H}_-$, where $X \colon \mathfrak{H}_+ \to \mathfrak{H}_-$ is a strict contraction and $X^* \colon \mathfrak{H}_- \to \mathfrak{H}_+$ is the adjoint of X.

The operator

$$T = XP_+ + X^*P_-$$

acting in $\mathfrak{H}$ is called a *transition operator* from the decomposition (2.1) to the decomposition (2.2). Obviously, T is self-adjoint and a strict contraction, which anti-commutes with J and satisfies $\mathfrak{L}_+ = (I + T)\mathfrak{H}_+, \mathfrak{L}_- = (I + T)\mathfrak{H}_-$.

The projections $P_{\mathfrak{L}_\pm} \colon \mathfrak{H} \to \mathfrak{L}_\pm$ onto $\mathfrak{L}_\pm$ with respect to the decomposition (2.2) are determined by the formulae

$$P_{\mathfrak{L}_-} = (I - T)^{-1}(P_- - TP_+), \qquad P_{\mathfrak{L}_+} = (I - T)^{-1}(P_+ - TP_-).$$

The bounded operator

$$C = P_{\mathfrak{L}_+} - P_{\mathfrak{L}_-} = J(I - T)(I + T)^{-1} \tag{2.3}$$

also describes the subspaces $\mathfrak{L}_\pm$ in (2.2):

$$\mathfrak{L}_+ = \tfrac{1}{2}(I + C)\mathfrak{H}, \qquad \mathfrak{L}_- = \tfrac{1}{2}(I - C)\mathfrak{H} \tag{2.4}$$

and satisfies the conditions $C^2 = I$, $JC > 0$, which are equivalent to the following representation of C (see, for example, [17, lemma 2.8]):

$$C = J\mathrm{e}^Y, \qquad \{J, Y\} = 0, \tag{2.5}$$

where Y is a bounded self-adjoint operator.

Comparing (2.3) and (2.5) we obtain

$$T = (I - \mathrm{e}^Y)(I + \mathrm{e}^Y)^{-1} = \tfrac{1}{2}(\mathrm{e}^{-Y/2} - \mathrm{e}^{Y/2})(\tfrac{1}{2}(\mathrm{e}^{-Y/2} + \mathrm{e}^{Y/2}))^{-1} = -\tanh \tfrac{1}{2}Y.$$

The decomposition $\mathfrak{H} = J\mathfrak{L}_+ [\dot{+}] J\mathfrak{L}_-$ is *dual* to (2.2). Its transition operator coincides with $-T$ and the subspaces $J\mathfrak{L}_\pm$ of the dual decomposition are described by (2.4) with the adjoint operator C^* instead of C.

Let A be an operator in $(\mathfrak{H}, [\cdot, \cdot]_J)$ with the property of C-symmetry (see definition 1.1). In view of (2.3) and (2.4), the operator A can be decomposed with respect to (2.2):

$$A = A_+ \dot{+} A_-, \quad A_+ = A \upharpoonright \mathfrak{L}_+, \quad A_- = A \upharpoonright \mathfrak{L}_-. \tag{2.6}$$

In particular, if A is J-self-adjoint (see the definition in § 1), then its components $A_\pm$ in (2.6) are self-adjoint operators in the Hilbert spaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ with the inner products $[\cdot,\cdot]_J$ and $-[\cdot,\cdot]_J$, respectively. This simple observation leads to the following statement, which is a direct consequence of the Phillips theorem [4, chapter 2, corollary 5.20].

PROPOSITION 2.1. *A J-self-adjoint operator A has the property of C-symmetry if and only if A is similar to a self-adjoint operator in $\mathfrak{H}$. If a J-self-adjoint operator A has C-symmetry, then the adjoint operator A^* has the C^*-symmetry.*

2.2. Boundary triplets technique

Let S be a closed symmetric (densely defined) operator with equal defect numbers in a Hilbert space $\mathfrak{H}$. A triplet $(\mathcal{H},\Gamma_0,\Gamma_1)$, where $\mathcal{H}$ is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings from $\mathcal{D}(S^*)$ into $\mathcal{H}$, is called a *boundary triplet of* S^* if the abstract Green identity

$$(S^*x,y)-(x,S^*y)=(\Gamma_1x,\Gamma_0y)_{\mathcal{H}}-(\Gamma_0x,\Gamma_1y)_{\mathcal{H}},\quad x,y\in\mathcal{D}(S^*) \tag{2.7}$$

holds and the map $(\Gamma_0,\Gamma_1)\colon\mathcal{D}(S^*)\to\mathcal{H}\oplus\mathcal{H}$ is surjective (see [12, 13, 16]).

Fix a boundary triplet $(\mathcal{H},\Gamma_0,\Gamma_1)$ for S^* and consider the linear operators

$$\Omega_+=\frac{1}{\sqrt{2}}(\Gamma_1+\mathrm{i}\Gamma_0),\qquad \Omega_-=\frac{1}{\sqrt{2}}(\Gamma_1-\mathrm{i}\Gamma_0) \tag{2.8}$$

acting from $\mathcal{D}(S^*)$ into $\mathcal{H}$. It follows from (2.7) and (2.8) that

$$(S^*x,y)-(x,S^*y)=\mathrm{i}[(\Omega_+x,\Omega_+y)_{\mathcal{H}}-(\Omega_-x,\Omega_-y)_{\mathcal{H}}]. \tag{2.9}$$

The formula (2.9) can be rewritten as

$$(S^*x,y)-(x,S^*y)=\mathrm{i}[\Psi x,\Psi y]_Z, \tag{2.10}$$

where

$$\Psi=\begin{pmatrix}\Omega_+\\ \Omega_-\end{pmatrix}\colon\mathcal{D}(S^*)\to\mathsf{H}=\begin{pmatrix}\mathcal{H}\\ \mathcal{H}\end{pmatrix}, \tag{2.11}$$

maps $\mathcal{D}(S^*)$ into the Krein space $(\mathsf{H},[\cdot,\cdot]_Z)$ with the indefinite metric

$$[\mathsf{x},\mathsf{y}]_Z=(x_0,y_0)-(x_1,y_1),\quad \mathsf{x}=\begin{pmatrix}x_0\\ x_1\end{pmatrix},\ \mathsf{y}=\begin{pmatrix}y_0\\ y_1\end{pmatrix}\in\mathsf{H}, \tag{2.12}$$

and the fundamental symmetry $Z=\mathrm{diag}(I,-I)$.

An arbitrary closed extension A of S is completely determined by a subspace $\mathsf{L}=\Psi\mathcal{D}(A)$ of H. In particular, due to (2.10), $\Psi\mathcal{D}(A^*)=\mathsf{L}^{[\perp]}$, where $[\perp]$ means the orthogonal complement in the Krein space $(\mathsf{H},[\cdot,\cdot]_Z)$. This leads to the following statement.

LEMMA 2.2. *Self-adjoint extensions of S are in one-to-one correspondence with hypermaximal neutral subspaces of the Krein space $(\mathsf{H},[\cdot,\cdot]_Z)$.*

The Weyl function $M(\cdot)$ and the characteristic function $\Theta(\cdot)$ of S associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$ are defined as follows (see [12, 24]):

$$\left.\begin{aligned} M(\mu)\Gamma_0 f_\mu &= \Gamma_1 f_\mu, \quad \forall f_\mu \in \ker(S^* - \mu I), \quad \forall \mu \in \mathbb{C}_- \cup \mathbb{C}_+, \\ \Theta(\mu)\Omega_+ f_\mu &= \Omega_- f_\mu, \quad \Theta(\bar{\mu})\Omega_- f_{\bar{\mu}} = \Omega_+ f_{\bar{\mu}}, \quad \forall \mu \in \mathbb{C}_+. \end{aligned}\right\} \tag{2.13}$$

Note that $\Theta(\mu)$ and $M(\mu)$ are connected via the Cayley transform

$$\Theta(\mu) = (M(\mu) - \mathrm{i}I)(M(\mu) + \mathrm{i}I)^{-1}, \quad \mu \in \mathbb{C}_+. \tag{2.14}$$

2.3. Fundamental decompositions depending on parameters

In what follows, the decomposition (2.2) is allowed to depend on parameters

$$\mathfrak{H} = \mathfrak{L}_+^\alpha [\dot{+}] \mathfrak{L}_-^\alpha, \quad \alpha \in D \subset \mathbb{R}^m. \tag{2.15}$$

Observe that the subspaces $\mathfrak{L}_\pm^\alpha$ are determined uniquely by the family of transition operators $\{T_\alpha\}_{\alpha \in D}$.

We illustrate what may happen with the decomposition (2.15) when α tends to a certain point α_0 lying on the boundary ∂D of D. More precisely, assume that T_α tends (in the strong sense) to a unitary operator Q in $\mathfrak{H}$ when $\alpha \to \alpha_0$. Then Q is a unitary and self-adjoint operator such that $\{J, Q\} = 0$. In that case, the elements $x_+ + T_\alpha x_+$, $x_+ \in \mathfrak{H}_+$, of $\mathfrak{L}_+^\alpha$ converge to the elements of the subspace $\mathfrak{L} = \{x_+ + Qx_+ : x_+ \in \mathfrak{H}_+\}$.

On the other hand, the elements $x_- + T_\alpha x_-$, $x_- \in \mathfrak{H}_-$, of $\mathfrak{L}_-^\alpha$ converge to the elements $x_- + Qx_-$ of the *same* subspace $\mathfrak{L}$ (since $x_- + Qx_- = x_+ + Qx_+$ for $x_- = Qx_+$). This means that the 'pointwise limit' of $\mathfrak{L}_\pm^\alpha$ (as $\alpha \to \alpha_0$) *coincides with the hypermaximal neutral subspace*

$$\mathfrak{L} = (I + Q)\mathfrak{H}_+ = (I + Q)\mathfrak{H}_- = (I + Q)\mathfrak{H}$$

of the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

Simultaneously, the subspaces $J\mathfrak{L}_\pm^\alpha$ of the dual decomposition

$$\mathfrak{H} = J\mathfrak{L}_+^\alpha [\dot{+}] J\mathfrak{L}_-^\alpha \tag{2.16}$$

'tend' to the dual hypermaximal neutral subspace

$$\mathfrak{L}^\sharp = J\mathfrak{L} = (I - Q)\mathfrak{H}_+ = (I - Q)\mathfrak{H}_- = (I - Q)\mathfrak{H} \quad \text{as } \alpha \to \alpha_0$$

Therefore, the limits $\alpha \to \alpha_0$ of the decompositions (2.15) and (2.16) give rise to a new decomposition

$$\mathfrak{H} = \mathfrak{L} [\dot{+}] \mathfrak{L}^\sharp, \tag{2.17}$$

which turns out to be fundamental in the *new Krein space* $(\mathfrak{H}, [\cdot, \cdot]_Q)$ with the fundamental symmetry Q. The original J and the new Q fundamental symmetries are generators of the Clifford algebra $\mathcal{CL}_2$.

The following examples illustrate the phenomenon described above.

Example 2.3. Let R be a unitary and self-adjoint operator in $\mathfrak{H}$ which anti-commutes with J, i.e. $\{J, R\} = 0$. Then the operator

$$R_\omega = R\mathrm{e}^{\mathrm{i}\omega J} = R[\cos\omega + \mathrm{i}(\sin\omega)J], \quad \omega \in [0, 2\pi), \tag{2.18}$$

is also unitary and self-adjoint in $\mathfrak{H}$ and, furthermore, $\{J, R_\omega\} = 0$.

The operators

$$C_{\chi,\omega} = J\mathrm{e}^{\chi R_\omega} = J[(\cosh\chi)I + (\sinh\chi)R_\omega], \quad \chi \in \mathbb{R}, \tag{2.19}$$

satisfy the conditions $C_{\chi,\omega}^2 = I$ and $JC_{\chi,\omega} > 0$ and they are also defined by (2.3) with

$$T_{\chi,\omega} = -(\tanh \tfrac{1}{2}\chi R_\omega) = -(\tanh \tfrac{1}{2}\chi)R_\omega, \quad \chi \in \mathbb{R}.$$

The operators $C_{\chi,\omega}$ (or $T_{\chi,\omega}$) determine the subspaces

$$\mathfrak{L}_\pm^{\chi,\omega} = \tfrac{1}{2}(I \pm C_{\chi,\omega})\mathfrak{H} = (I + T_{\chi,\omega})\mathfrak{H}_\pm, \quad \alpha = (\chi,\omega) \in D = \mathbb{R} \times [0, 2\pi).$$

If $\chi \to \infty$, then $T_{\chi,\omega}$ tends to a unitary operator $Q = -R_\omega$. The corresponding subspaces $\mathfrak{L}_\pm^{\chi,\omega}$ converge pointwise to the hypermaximal neutral subspace $\mathfrak{L} = (I - R_\omega)\mathfrak{H}$ of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$.

Similarly, when $\chi \to -\infty$, the subspaces of the dual decomposition $J\mathfrak{L}_\pm^{\chi,\omega} = (I - T_{\chi,\omega})\mathfrak{H}_\pm$ tend to the dual hypermaximal neutral subspace $\mathfrak{L}^\sharp = (I + R_\omega)\mathfrak{H}$. The 'limiting' subspaces $\mathfrak{L}$ and $\mathfrak{L}^\sharp$ give rise to the fundamental decomposition of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_Q)$ with $Q = -R_\omega$.

Example 2.4. Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of a symmetric operator S and let $(\mathsf{H}, [\cdot,\cdot]_Z)$ be the corresponding Krein space defined by (2.2). The fundamental decomposition of $(\mathsf{H}, [\cdot,\cdot]_Z)$ has the form

$$\mathsf{H} = \begin{pmatrix} \mathcal{H} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}, \quad \mathsf{H}_+ = \begin{pmatrix} \mathcal{H} \\ 0 \end{pmatrix}, \quad \mathsf{H}_- = \begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix}. \tag{2.20}$$

According to (2.10), the subspaces

$$\mathsf{L}_+^\mu = \Psi \ker(S^* - \mu I), \quad \mathsf{L}_-^\mu = \Psi \ker(S^* - \bar{\mu} I), \quad \mu \in \mathbb{C}_+, \tag{2.21}$$

are, respectively, uniformly positive and uniformly negative in the Krein space $(\mathsf{H}, [\cdot,\cdot]_Z)$ and they form a Z-orthogonal decomposition (see (2.15))

$$\mathsf{H} = \mathsf{L}_+^\mu [+] \mathsf{L}_-^\mu, \quad \mu \in D = \mathbb{C}_+, \tag{2.22}$$

The corresponding family of transition operators $\{T_\mu\}_{\mu \in \mathbb{C}_+}$ from the fundamental decomposition (2.20) to (2.22) has the operator-matrix form (with respect to (2.20))

$$T_\mu = \begin{pmatrix} 0 & \Theta(\bar{\mu}) \\ \Theta(\mu) & 0 \end{pmatrix}, \ \mu \in \mathbb{C}_+, \qquad \Theta(\bar{\mu}) = \Theta^*(\mu), \tag{2.23}$$

where $\Theta(\mu)$ is the characteristic function of S associated with the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$.

Remark 2.5. The decomposition (2.22) depends on the choice of the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ which is determined by the mapping Ψ (see (2.11))

$$\begin{pmatrix} \Gamma_0 x \\ \Gamma_1 x \end{pmatrix} = B\Psi x, \quad x \in \mathcal{D}(S^*), \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i}I & \mathrm{i}I \\ I & I \end{pmatrix}. \tag{2.24}$$

Let U be an arbitrary Z-unitary operator in the Krein space $(\mathsf{H}, [\cdot,\cdot]_Z)$ and let $\Psi_U = U\Psi$. Taking (2.24) into account, we conclude that U determines a new boundary triplet $(\mathcal{H}, \Gamma_0^U, \Gamma_1^U)$ of S^*, where

$$\begin{pmatrix} \Gamma_0^U x \\ \Gamma_1^U x \end{pmatrix} = B\Psi_U x, \quad x \in \mathcal{D}(S^*).$$

In that case, $\mathsf{H} = U\mathsf{L}_+^\mu [+] U\mathsf{L}_-^\mu$, where $\mathsf{L}_\pm^\mu$ are defined by (2.21). The family of transition operators $\{T_\mu^U\}_{\mu\in\mathbb{C}_+}$ is determined in the same manner as (2.23) by the characteristic function $\Theta_U(\mu)$ of S in $(\mathcal{H}, \Gamma_0^U, \Gamma_1^U)$ and $\Theta_U(\mu)$ can be expressed via the Krein–Shmul'yan transformation (see [15])

$$\Theta_U(\mu) = (U_{10} + U_{11}\Theta(\mu))(U_{00} + U_{01}\Theta(\mu))^{-1},$$

where the bounded operators $U_{ij} \in \mathcal{B}[\mathcal{H}]$ originate from the decomposition $U = (U_{ij})_{i,j=0}^1$ with respect to (2.20).

Let r be a real point of regular type of S. Then the operator

$$A_r = S^* \upharpoonright \mathcal{D}(A_r), \qquad \mathcal{D}(A_r) = \mathcal{D}(S) \dotplus \ker(S^* - rI) \tag{2.25}$$

is a self-adjoint extension of S and $\mathsf{L}_r = \Psi\mathcal{D}(A_r)$ is a hypermaximal neutral subspace in $(\mathsf{H}, [\cdot,\cdot]_Z)$. Therefore, $\mathsf{L}_r = (I + X_r)\mathsf{H}_+$, where X_r is a *unitary* mapping of H_+ onto H_-. It follows from the results of [24, 33] that

$$X_r = \Theta(r) = s - \lim_{\mu\to r} \Theta(\mu),$$
$$X_r^{-1} = X_r^* = \Theta^{-1}(r) = s - \lim_{\bar\mu\to r} \Theta(\bar\mu),$$

where $\mu \in \mathbb{C}_+$. This means that the transition operators $\{T_\mu\}_{\mu\in\mathbb{C}_+}$ determined by (2.23) converge (in the strong sense) to

$$Q_r = \begin{pmatrix} 0 & \Theta^{-1}(r) \\ \Theta(r) & 0 \end{pmatrix}.$$

The operator Q_r is self-adjoint and unitary in the Hilbert space H with the inner product $(\cdot,\cdot) = [Z\cdot,\cdot]_Z$ and, moreover, $\{Z, Q_r\} = 0$.

Thus, if $\mu \in \mathbb{C}_+$ tends to a real point of regular type r of S, then the subspaces L_+^μ and L_-^μ in (2.22) converge to the hypermaximal neutral subspace

$$\mathsf{L}_r = (I + \Theta(r))\mathsf{H}_+ = (I + \Theta^{-1}(r))\mathsf{H}_- = (I + Q_r)\mathsf{H}.$$

Furthermore, the dual subspaces $Z\mathsf{L}_\pm^\mu$ tend to the dual hypermaximal neutral subspace

$$\mathsf{L}_r^\sharp = Z\mathsf{L}_r = (I - \Theta(r))\mathsf{H}_+ = (I - \Theta^{-1}(r))\mathsf{H}_- = (I - Q_r)\mathsf{H}$$

of the Krein space $(\mathsf{H}, [\cdot,\cdot]_Z)$. The subspaces L_r and $\mathsf{L}_r^\sharp$ give rise to a decomposition $\mathsf{H} = \mathsf{L}_r [\dotplus] \mathsf{L}_r^\sharp$, which is fundamental in the new Krein space $(\mathsf{H}, [\cdot,\cdot]_{Q_r})$.

3. *J*-self-adjoint operators with stable *C*-symmetry

3.1. Definition of stable *C*-symmetry

Let S be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ with equal defect numbers. In what follows, we suppose that S commutes with J, i.e.

$$[S, J] = 0. \tag{3.1}$$

Denote by $\mathfrak{U} = \{C_\alpha\}_{\alpha \in D}$ the collection of all possible operators C (parametrized by a set of indices $\alpha \in D$) which realize the property of C-symmetry for S (in the sense of definition 1.1).

LEMMA 3.1 (Kuzhel and Trunk [28]). *The set* $\mathfrak{U}$ *is non-empty* (*since* $J \in \mathfrak{U}$) *and* $C \in \mathfrak{U}$ *if and only if* $C^* \in \mathfrak{U}$.

It follows from lemma 3.1 that

$$[S, C] = [S, C^*] = [S^*, C] = [S^*, C^*] = 0, \quad \forall C \in \mathfrak{U}. \tag{3.2}$$

Denote by Σ_J *the set of all J-self-adjoint extensions of* S, i.e.

$$A \in \Sigma_J \iff A \supset S \quad \text{and} \quad A^* J = JA.$$

It follows from (3.1) that $A \in \Sigma_J \iff A^* \in \Sigma_J$.

DEFINITION 3.2. We will say that an operator $A \in \Sigma_J$ belongs to the set Σ_J^{st} of stable C-symmetry if A has the property of C-symmetry realized by an operator $C \in \mathfrak{U}$.

Obviously, this definition is equivalent to the definition 1.2 in § 1, i.e. the condition $A \in \Sigma_J^{\mathrm{st}}$ means that the C-symmetry property of the operators A and S is realized by the same operator C.

LEMMA 3.3. *The following relation holds:* $A \in \Sigma_J^{\mathrm{st}} \iff A^* \in \Sigma_J^{\mathrm{st}}$.

Proof. Let $A \in \Sigma_J^{\mathrm{st}}$. Then there exists an operator $C \in \mathfrak{U}$ commuting with S and A. By (3.2), $[S, C^*] = 0$. On the other hand, $[A^*, C^*] = 0$ due to proposition 2.1. Therefore, S and A^* have the same C^*-symmetry. Lemma 3.3 is proved. □

REMARK 3.4. An arbitrary operator C_α from $\mathfrak{U} = \{C_\alpha\}_{\alpha \in D}$ determines a new definite inner product

$$(\cdot, \cdot)_\alpha = [C_\alpha \cdot, \cdot]_J = (JC_\alpha \cdot, \cdot) \tag{3.3}$$

on the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$, which is equivalent to the initial inner product $(\cdot, \cdot) = [J\cdot, \cdot]_J$. The operator S remains symmetric for any choice of $\alpha \in D$ and an arbitrary $A \in \Sigma_J^{\mathrm{st}}$ is, in fact, a self-adjoint extension of S with respect to a certain choice of $\alpha \in D$.

3.2. The set $\varUpsilon_{\mathfrak{U}}$

Denote by $\varUpsilon_{\mathfrak{U}}$ the set of all J-self-adjoint extensions $A \in \Sigma_J$ which *commute with every operator* $C \in \mathfrak{U}$, i.e.

$$A \in \varUpsilon_{\mathfrak{U}} \iff A \in \Sigma_J \quad \text{and} \quad [A, C] = 0, \ \forall C \in \mathfrak{U}.$$

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Obviously, $\Upsilon_{\mathfrak{U}} \subset \Sigma_J^{\rm st} \subset \Sigma_J$. It follows from remark 3.4 that an operator $A \in \Upsilon_{\mathfrak{U}}$ is a self-adjoint extension of S for any choice of inner product $(\cdot,\cdot)_\alpha$. In particular, A is a self-adjoint extension of S (since $(\cdot,\cdot)_\alpha = (\cdot,\cdot)$ for $C_\alpha = J$).

THEOREM 3.5. *The following statements are true:*

(i) *if r is a real point of regular type of S, then the operator A_r defined by (2.25) belongs to $\Upsilon_{\mathfrak{U}}$;*

(ii) *if $A \in \Upsilon_{\mathfrak{U}}$, then among all self-adjoint extensions $A' \supset S$ transversal to A, there exist at least one belonging to $\Upsilon_{\mathfrak{U}}$;*

(iii) *if S is non-negative, then its Friedrichs and Krein–von Neumann extensions belong to $\Upsilon_{\mathfrak{U}}$.*

Proof. (i) By virtue of (3.2), $[S^*, C] = 0$ for any $C \in \mathfrak{U}$. This implies that

$$C(\ker(S^* - \mu I)) = \ker(S^* - \mu I), \quad \forall \mu \in \mathbb{C} \tag{3.4}$$

(since $C^2 = I$). Using the definition (2.25) of A_r and (3.4) for $\mu = r$, we get $[A_r, C] = 0$. Hence, $A_r \in \Upsilon_{\mathfrak{U}}$.

(ii) According to the classical von Neumann formulae, an arbitrary self-adjoint extension A of S is uniquely determined by a unitary mapping $V\colon \ker(S^* + \mathrm{i}I) \to \ker(S^* - \mathrm{i}I)$ such that

$$\mathcal{D}(A) = \mathcal{D}(S) \dotplus \{x_{-i} + Vx_{-i} \colon \forall x_{-i} \in \ker(S^* + \mathrm{i}I)\}. \tag{3.5}$$

Let $A \in \Upsilon_{\mathfrak{U}}$. Then A is self-adjoint and the corresponding unitary mapping V commutes with any $C \in \mathfrak{U}$ (due to (3.4)). Considering the self-adjoint extension $A' \supset S$ determined by the unitary mapping $V' = -V$ in (3.5), we obtain that $A' \in \Upsilon_{\mathfrak{U}}$ and $\mathcal{D}(A) \cap \mathcal{D}(A') = \mathcal{D}(S)$, $\mathcal{D}(A) \cup \mathcal{D}(A') = \mathcal{D}(S^*)$. Therefore, A and A' are transversal extensions of S.

(iii) The Friedrichs extension $A_{\rm F}$ and the Krein–von Neumann extension $A_{\rm N}$ of S can be characterized as follows (see [3] for the densely defined case and [21] for the general case).

If $\{f, f'\} \in S^*$, then $\{f, f'\} \in A_{\rm F}$ if and only if

$$\inf\{\|f - h\|^2 + (f' - h', f - h) \colon \{h, h'\} \in S\} = 0. \tag{3.6}$$

If $\{f, f'\} \in S^*$, then $\{f, f'\} \in A_{\rm N}$ if and only if

$$\inf\{\|f' - h'\|^2 + (f' - h', f - h) \colon \{h, h'\} \in S\} = 0. \tag{3.7}$$

Since $[S^*, C] = [S, C] = 0$ for any $C \in \mathfrak{U}$, formulae (3.6) and (3.7) imply that $A_{\rm F}$ and $A_{\rm N}$ are decomposed with respect to the decomposition (2.2) (with the subspaces $\mathfrak{L}_\pm$ determined by C):

$$A_{\rm F} = A_{\rm F+} \dotplus A_{\rm F-}, \qquad A_{\rm N} = A_{N+} \dotplus A_{N-},$$

where $A_{\rm F\pm}$ and $A_{\rm N\pm}$ are the Friedrichs extension and the Krein–von Neumann extension of the symmetric operators $S \restriction \mathfrak{L}_\pm$ in the Hilbert spaces $\mathfrak{L}_\pm$, respectively. These decompositions immediately yield the relations $[A_{\rm F}, C] = [A_{\rm N}, C] = 0$. Theorem 3.5 is proved. □

Remark 3.6. It may happen that the set $\Upsilon_{\mathfrak{U}}$ is empty. For instance, let S_1 and S_2 be symmetric operators with defect numbers $\langle 0,1\rangle$ and $\langle 1,0\rangle$ acting in Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then $S = S_1 \oplus S_2$ is a symmetric operator in the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with defect numbers $\langle 1,1\rangle$. The operator S commutes with the fundamental symmetry $J = I \oplus -I$ in $\mathfrak{H}$.

Assume that $A \in \Upsilon_{\mathfrak{U}}$. Then A is a self-adjoint extension of S and $[A,J] = 0$. Therefore, $A \upharpoonright \mathfrak{H}_1$ is a self-adjoint extension of S_1. However, this is impossible. Thus, $\Upsilon_{\mathfrak{U}} = \emptyset$.

3.3. Description of Σ_J^{st}

The description of Σ_J^{st} requires an appropriate boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, in which the images of $\mathfrak{U} = \{C_\alpha\}_{\alpha\in D}$ exist in the parameter space $\mathcal{H}$.

Lemma 3.7. *For each $A \in \Upsilon_{\mathfrak{U}}$, there exists a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^* such that $\mathcal{D}(A) = \ker \Gamma_0$ and the formulae*

$$\mathcal{C}_\alpha \Gamma_0 f = \Gamma_0 C_\alpha f, \quad \mathcal{C}_\alpha \Gamma_1 f = \Gamma_1 C_\alpha f, \quad \forall \alpha \in D,\ \forall f \in \mathcal{D}(S^*) \tag{3.8}$$

correctly define the operator family $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ in $\mathcal{H}$. If S is a simple symmetric operator, then the correspondence $C_\alpha \to \mathcal{C}_\alpha$ established by (3.8) is injective.

Proof. Let $A \in \Upsilon_{\mathfrak{U}}$. Then the corresponding unitary mapping $V\colon \ker(S^* + \mathrm{i}I) \to \ker(S^* - \mathrm{i}I)$ in (3.5) commutes with the family $\mathfrak{U} = \{C_\alpha\}$. Now introduce the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H} = \ker(S^* - \mathrm{i}I)$ and

$$\Gamma_0 x = x_i - V x_{-i}, \quad \Gamma_1 x = \mathrm{i}x_i + \mathrm{i}V x_{-i}, \quad x = u + x_{-i} + x_i \in \mathcal{D}(S^*),$$

with $x_{\pm i} \in \ker(S^* \mp \mathrm{i}I)$.

Since $C_\alpha V = V C_\alpha$, the restriction $\mathcal{C}_\alpha = C_\alpha \upharpoonright \ker(S^* - \mathrm{i}I)$ is an operator in $\mathcal{H}$ and it satisfies the relations in (3.8). By construction, $\mathcal{D}(A) = \ker \Gamma_0$ (see (3.5)).

Let us assume that (3.8) gives the same image $\mathcal{C}$ for two different C-symmetries C_{α_1} and C_{α_2} of S. Then $(C_{\alpha_1} - C_{\alpha_2})\mathcal{D}(S^*) \subset \mathcal{D}(S)$. Combining this relation with (3.4), we conclude that $C_{\alpha_1} f_\mu = C_{\alpha_2} f_\mu$ for every $f_\mu \in \ker(S^* - \mu I)$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$. Since the symmetric operator S is simple, this implies that $C_{\alpha_1} = C_{\alpha_2}$. Lemma 3.7 is proved. □

Remark 3.8. If a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* satisfies (3.8), then the associated transversal extensions $A_0 = S^* \upharpoonright \ker \Gamma_0$ and $A_1 = S^* \upharpoonright \ker \Gamma_1$ belong to $\Upsilon_{\mathfrak{U}}$. This means that *boundary triplets $(\mathcal{H}, \Gamma_0, \Gamma_1)$ with the properties (3.8) exist if and only if the set $\Upsilon_{\mathfrak{U}}$ is non-empty.*

The existence of a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^*, which satisfies the properties in (3.8), will guarantee a couple of useful properties for the operators $\mathcal{C}_\alpha$ and also some important relations between the extensions of S and the parameters corresponding to them in $\mathcal{H}$.

Lemma 3.9. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet for S^* with the properties (3.8). Then the associated γ-field $\gamma(\cdot)$ and Weyl function $M(\cdot)$ satisfy the relations*

$$\gamma(\lambda)\mathcal{C}_\alpha = C_\alpha \gamma(\lambda) \quad \text{and} \quad [\mathcal{C}_\alpha, M(\lambda)] = 0 \tag{3.9}$$

for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $\alpha \in D$.

Proof. Since $C_\alpha(\ker(S^*-\lambda)) = \ker(S^*-\lambda)$ (see (3.4)), each of the identities in (3.9) follows easily from (3.8) by applying the formula $\gamma(\lambda) = (\Gamma_0 \upharpoonright \ker(S^*-\lambda))^{-1}$ and the definition of $M(\lambda)$ in (2.13). □

The next lemma concerns the class of operators $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ appearing in lemma 3.7. When $C_\alpha = J$, the special notation $\mathcal{J}$ is used for the corresponding intertwining operator $\mathcal{C}_\alpha$ in (3.8), i.e. $\mathcal{J}\Gamma_0 = \Gamma_0 J$ and $\mathcal{J}\Gamma_1 = \Gamma_1 J$.

LEMMA 3.10. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with properties (3.8). Then the intertwining operators $\mathcal{C}_\alpha$ satisfy the relations $\mathcal{C}_\alpha^2 = I$, $\mathcal{J}\mathcal{C}_\alpha > 0$, and the operator $\mathcal{J}$ is a fundamental symmetry in the auxiliary Hilbert space $\mathcal{H}$.*

Proof. The identity $\mathcal{C}_\alpha^2 = I$ immediately follows from (3.8) and the corresponding identity $C_\alpha^2 = I$. Furthermore, $[S^*, J] = 0$ due to (3.1). Taking this relation into account and considering (2.9) with Jx and Jy instead of x and y, it follows that $\mathcal{J}$ is unitary in $\mathcal{H}$. This, together with the identity $\mathcal{J}^2 = I$, leads to the fact that $\mathcal{J}$ is a self-adjoint operator. Hence, $\mathcal{J}$ is a fundamental symmetry in $\mathcal{H}$.

Since $[S^*, C_\alpha] = 0$ for every $\alpha \in D$, one can rewrite (2.9) by substituting $JC_\alpha x$ instead of x as follows:

$$(S^*x, y)_\alpha - (x, S^*y)_\alpha = \mathrm{i}[(\mathcal{J}\mathcal{C}_\alpha\Omega_+x, \Omega_+y)_{\mathcal{H}} - (\mathcal{J}\mathcal{C}_\alpha\Omega_-x, \Omega_-y)_{\mathcal{H}}], \tag{3.10}$$

where $(\cdot,\cdot)_\alpha = (JC_\alpha\cdot,\cdot)$. Now, by putting $x = y = f_\mu \in \ker(S^* - \mu I)$, $\mu \in \mathbb{C}_+$, in (3.10) and recalling the definition (2.13) of the characteristic function $\Theta(\mu)$, we obtain

$$2(\operatorname{Im}\mu)(f_\mu, f_\mu)_\alpha = (\mathcal{J}\mathcal{C}_\alpha h, h)_{\mathcal{H}} - (\mathcal{J}\mathcal{C}_\alpha\Theta(\mu)h, \Theta(\mu)h)_{\mathcal{H}}, \tag{3.11}$$

where $h = \Omega_+f_\mu$ is an arbitrary element of $\mathcal{H}$ (since Ω_+ maps $\ker(S^*-\mu I)$ onto $\mathcal{H}$). Due to (2.13) and (3.8), $[\Theta(\mu), \mathcal{C}_\alpha] = 0$ for all $\mu \in \mathbb{C}_\pm$. Hence, (3.11) implies that

$$(\mathcal{J}\mathcal{C}_\alpha(I - \Theta^*(\mu)\Theta(\mu))h, h)_{\mathcal{H}} = (\mathcal{J}\mathcal{C}_\alpha Fh, Fh)_{\mathcal{H}} > 0,$$

where $F = (I - \Theta^*(\mu)\Theta(\mu))^{1/2}$ is an invertible operator in $\mathcal{H}$ (since $\|\Theta(\mu)\| = \|\Theta^*(\mu)\| < 1$) such that $[F, \mathcal{C}_\alpha] = 0$. Therefore, $\mathcal{J}\mathcal{C}_\alpha > 0$. Lemma 3.10 is proved. □

THEOREM 3.11. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with the properties (3.8). Then an arbitrary $A \in \Sigma_J^{\mathrm{st}}$ admits the presentation*

$$A = S^* \upharpoonright \{f \in \mathcal{D}(S^*) \colon \mathcal{K}\Omega_+f = \Omega_-f\}, \tag{3.12}$$

where $\Omega_\pm$ are defined by (2.8) and $\mathcal{K}$ is a stable $\mathcal{J}$-unitary operator in the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$.

Proof. It follows from (2.5) and lemma 3.10 that $\mathcal{C}_\alpha = \mathcal{J}\mathrm{e}^{\mathcal{Y}_\alpha}$, where $\mathcal{Y}_\alpha$ is a bounded self-adjoint operator in $\mathcal{H}$ and $\{\mathcal{J}, \mathcal{Y}_\alpha\} = 0$.

Since the operator S^* is still adjoint for the symmetric operator S with respect to $(\cdot,\cdot)_\alpha$, the formulae (2.8), (2.9) and (3.10) imply that $(\mathcal{H}, \mathrm{e}^{\mathcal{Y}_\alpha/2}\Gamma_0, \mathrm{e}^{\mathcal{Y}_\alpha/2}\Gamma_1)$ is a boundary triplet for S^* acting in the Hilbert space $\mathfrak{H}$ with the inner product $(\cdot,\cdot)_\alpha$.

Let $A \in \Sigma_J^{\mathrm{st}}$. Then $[A, C_\alpha] = 0$ for a certain choice of $\alpha \in D$ and A is a self-adjoint extension of S with respect to $(\cdot,\cdot)_\alpha$. Hence (see [16]),

$$A = S^* \upharpoonright \{f \in \mathcal{D}(S^*) \colon \mathcal{W}\mathrm{e}^{\mathcal{Y}_\alpha/2}\Omega_+f = \mathrm{e}^{\mathcal{Y}_\alpha/2}\Omega_-f\},$$

where $\mathcal{W}$ is a unitary operator in the Hilbert space $\mathcal{H}$. The obtained description of A leads to (3.12) with

$$\mathcal{K} = \mathrm{e}^{-\mathcal{Y}_\alpha/2}\mathcal{W}\mathrm{e}^{\mathcal{Y}_\alpha/2}. \tag{3.13}$$

Since the adjoint[3] operator A^* is determined by (3.12) with $(\mathcal{K}^*)^{-1}$ (see [16]), the relation $A^*J = JA$ means that $(\mathcal{K}^*)^{-1}\mathcal{J} = \mathcal{J}\mathcal{K}$. Therefore, $\mathcal{K}$ is a $\mathcal{J}$-unitary operator in the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$.

It follows from (3.13) that $\|\mathcal{K}^n\| <$ const., for all $n \in \mathbb{Z}$. Hence, $\mathcal{K}$ is a stable $\mathcal{J}$-unitary operator (see, for example, [4]). Theorem 3.11 is proved. □

Proposition 3.12. *Formula* (3.12) *determines an operator* $A \in \Sigma_J^{\mathrm{st}}$, $A \in \Upsilon_{\mathfrak{U}}$, *if and only if the corresponding* $\mathcal{J}$*-unitary operator* $\mathcal{K}$ *has the property of* C*-symmetry realized by some operator* $\mathcal{C}_\alpha$ *from* $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ (*respectively, by every operator* $\mathcal{C}_\alpha$ *from* $\{\mathcal{C}_\alpha\}_{\alpha\in D}$).

Proof. If $A \in \Sigma_J^{\mathrm{st}}$, then $[A, C_\alpha] = 0$ for a certain choice of $\alpha \in D$. This means that $C_\alpha \colon \mathcal{D}(A) \to \mathcal{D}(A)$. By (2.8) and (3.8), $\mathcal{C}_\alpha \Omega_\pm = \Omega_\pm C_\alpha$. Combining this with (3.12) and taking into account that $\mathcal{C}_\alpha^2 = I$, we obtain $[\mathcal{K}, \mathcal{C}_\alpha] = 0$.

Conversely, if A is determined by (3.12) and $[\mathcal{K}, \mathcal{C}_\alpha] = 0$ for a certain operator $\mathcal{C}_\alpha \in \{\mathcal{C}_\alpha\}_{\alpha\in D}$, then, for its 'preimage' C_α (see (3.8)), the relation $C_\alpha \colon \mathcal{D}(A) \to \mathcal{D}(A)$ holds. This means that $[A, C_\alpha] = 0$ (since $[S^*, C_\alpha] = 0$). Hence, $A \in \Sigma_J^{\mathrm{st}}$. □

Remark 3.13. Formula (3.12) establishes *a bijection between the elements of* Σ_J^{st} *and some subset of the set of stable* $\mathcal{J}$*-unitary operators* $\mathcal{K}$ in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$. This subset is uniquely determined by the additional assumption that $\mathcal{K}$ has the property of C-symmetry realized by an operator from the image $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ of the set $\{C_\alpha\}_{\alpha\in D}$ (see lemma 3.7). However, it is not easy to apply this sort of definition. A more appropriate external description for $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ is established in the next subsection using reproducing kernel Hilbert space models associated with Nevanlinna functions.

3.4. Reproducing kernel Hilbert space models

Let $M(\cdot)$ be a Weyl function of the symmetric operator S associated with the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$. The corresponding *Nevanlinna kernel* $\mathsf{N}_M(\xi, \mu)$ on $(\mathbb{C}_+ \cup \mathbb{C}_-) \times (\mathbb{C}_+ \cup \mathbb{C}_-)$ is defined as

$$\mathsf{N}_M(\xi, \mu) := \frac{M(\mu) - M(\bar{\xi})}{\mu - \bar{\xi}}, \quad \mu, \xi \in \mathbb{C}_+ \cup \mathbb{C}_-,\ \xi \neq \bar{\mu}. \tag{3.14}$$

The kernel $\mathsf{N}_M(\xi, \mu)$ is Hermitian, holomorphic and non-negative. The corresponding reproducing kernel Hilbert space will be denoted by $\mathfrak{H}_M$. The space $\mathfrak{H}_M$ consists of $\mathcal{H}$-valued holomorphic vector functions on $\mathbb{C}_+ \cup \mathbb{C}_-$ obtained as the closed linear span of functions $\mu \to \mathsf{N}_M(\xi, \mu)f$, $\xi \in \mathbb{C}_+ \cup \mathbb{C}_-$, $f \in \mathcal{H}$, which is provided with the scalar product determined by

$$\prec \mathsf{N}_M(\xi, \cdot)f, \mathsf{N}_M(\lambda, \cdot)g \succ := (\mathsf{N}_M(\xi, \lambda)f, g)_{\mathcal{H}}, \quad f, g \in \mathcal{H},\ \xi, \lambda \in \mathbb{C}_\pm.$$

The functions $\phi(\cdot) \in \mathfrak{H}_M$ satisfy the reproducing kernel property

$$\prec \phi(\cdot), \mathsf{N}_M(\lambda, \cdot)g \succ = (\phi(\lambda), g)_{\mathcal{H}}, \quad g \in \mathcal{H},\ \lambda \in \mathbb{C}_\pm. \tag{3.15}$$

[3] A^* denotes the adjoint with respect to the initial scalar product $(\cdot,\cdot)$ of $\mathfrak{H}$.

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The reproducing kernel Hilbert space $\mathfrak{H}_M$ gives rise to a useful model representation of the symmetric operator S and the associated boundary mappings. The next statement contains a lot of relevant results (see [5] for a proof and further details).

PROPOSITION 3.14. *Let $M(\cdot)$ be a Weyl function of a simple symmetric operator S. Then*

(i) *the linear relation $S_M = \{\{\phi, \psi\} \in \mathfrak{H}_M^2 \colon \psi(\lambda) = \lambda\phi(\lambda)\}$ is a symmetric operator in $\mathfrak{H}_M$ which is unitarily equivalent to S,*

(ii) *the linear relation*

$$\mathcal{T} = \{\{\phi, \psi\} \in \mathfrak{H}_M^2 \colon \psi(\lambda) - \lambda\phi(\lambda) = c_1 + M(\lambda)c_2,\ c_1, c_2 \in \mathcal{H}\}$$

determines the adjoint S_M^* of S_M in $\mathfrak{H}_M$,

(iii) *the operators*

$$\Gamma_0^M\{\phi, \psi\} = c_2, \quad \Gamma_1^M\{\phi, \psi\} = -c_1, \quad \{\phi, \psi\} \in \mathcal{T}$$

form a boundary triplet $(\mathcal{H}, \Gamma_0^M, \Gamma_1^M)$ for S_M^*,

(iv) *the Weyl function of S_M associated with $(\mathcal{H}, \Gamma_0^M, \Gamma_1^M)$ coincides with $M(\cdot)$.*

Proposition 3.14 yields an explicit description for the set $\{\mathcal{C}_\alpha\}_{\alpha\in D}$.

THEOREM 3.15. *Let S be a simple symmetric operator, let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet for S^* with the properties (3.8), and let $M(\cdot)$ be the associated Weyl function. Then a bounded operator $\mathcal{C}$ in $\mathcal{H}$ belongs to the set $\{\mathcal{C}_\alpha\}_{\alpha\in D}$ if and only if*

$$\mathcal{C}^2 = I, \tag{3.16 a}$$

$$\mathcal{J}\mathcal{C} > 0, \tag{3.16 b}$$

$$[\mathcal{C}, M(\lambda)] = 0, \quad \forall\lambda \in \mathbb{C}_\pm. \tag{3.16 c}$$

Proof. If $\mathcal{C} \in \{\mathcal{C}_\alpha\}_{\alpha\in D}$, then (3.16) holds by lemma 3.9 and lemma 3.10.

Now the converse will be proved. If $\mathcal{C}$ satisfies (3.16), then its adjoint $\mathcal{C}^*$ also satisfies (3.16). Hence, it follows from (3.14) and (3.16 c) that the operators C and C',

$$\mathsf{C}[\mathsf{N}_M(\lambda, \cdot)f] := \mathsf{N}_M(\lambda, \cdot)\mathcal{C}f,$$
$$\mathsf{C}'[\mathsf{N}_M(\lambda, \cdot)f] := \mathsf{N}_M(\lambda, \cdot)\mathcal{C}^*f,$$

are well-defined on the linear span of functions $\{\mathsf{N}_M(\lambda, \cdot)f\}$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, $f \in \mathcal{H}$. Moreover, (3.15) together with (3.14) and (3.16 c) implies that

$$\begin{aligned}\|\mathsf{N}_M(\lambda, \cdot)\mathcal{C}f\|^2_{\mathfrak{H}_M} &= (\mathsf{N}_M(\lambda, \lambda)\mathcal{C}f, \mathcal{C}f)_{\mathcal{H}} \\ &\leqslant \|\mathcal{C}\|^2_{\mathcal{H}}\|(\mathsf{N}_M(\lambda, \lambda)f, f)_{\mathcal{H}} \\ &= \|\mathcal{C}\|^2_{\mathcal{H}}\|\mathsf{N}_M(\lambda, \cdot)f\|^2_{\mathfrak{H}_M}.\end{aligned}$$

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Thus, C and, similarly, C' is continuous. Hence, C and C' can be extended by continuity onto the whole space $\mathfrak{H}_M$. Using (3.15) twice gives

$$\begin{aligned}\prec\phi(\cdot), \mathsf{N}_M(\lambda,\cdot)\mathcal{C}^*f\succ &= (\phi(\lambda), \mathcal{C}^*f)_{\mathcal{H}} \\ &= (\mathcal{C}\phi(\lambda), f)_{\mathcal{H}}\end{aligned}$$

and

$$\begin{aligned}\prec\phi(\cdot), \mathsf{N}_M(\lambda,\cdot)\mathcal{C}^*f\succ &= \prec\phi(\cdot), \mathsf{C}'[\mathsf{N}_M(\lambda,\cdot)f]\succ \\ &= \prec\mathsf{C}'^*[\phi(\cdot)], \mathsf{N}_M(\lambda,\cdot)f\succ \\ &= ((\mathsf{C}'^*[\phi(\cdot)])(\lambda), f)_{\mathcal{H}}.\end{aligned}$$

Comparing the right-hand sides, we obtain $(\mathsf{C}'^*[\phi(\cdot)])(\lambda) = \mathcal{C}\phi(\lambda)$ for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. This means that $\mathsf{C}'^* = \mathsf{C}$ and that the action of C on an arbitrary $\mathcal{H}$-valued function $\phi(\cdot) \in \mathfrak{H}_M$ is realized via the action of $\mathcal{C}$ on the vectors $\phi(\lambda) \in \mathcal{H}$, i.e.

$$(\mathsf{C}[\phi(\cdot)])(\lambda) \equiv \mathcal{C}\phi(\lambda), \quad \forall\lambda \in \mathbb{C}_{\pm}. \tag{3.17}$$

Therefore, $\mathsf{C}^2 = I$ (since $\mathcal{C}^2 = I$) and it is clear from part (i) of proposition 3.14 that C commutes with S_M.

Repeating the arguments above for the case where $\mathcal{C} = \mathcal{J}$, we obtain a fundamental symmetry J in $\mathfrak{H}_M$ defined by the formula

$$(\mathsf{J}[\phi(\cdot)])(\lambda) \equiv \mathcal{J}\phi(\lambda). \tag{3.18}$$

Furthermore, the condition $\mathsf{JC} > 0$ is satisfied. To see this, observe that (3.14)–(3.16) imply that

$$\begin{aligned}\prec(\mathsf{JC})\mathsf{N}_M(\lambda,\cdot)f, \mathsf{N}_M(\lambda,\cdot)f\succ &= (\mathsf{N}_M(\lambda,\lambda)\mathcal{J}\mathcal{C}f, f)_{\mathcal{H}} \\ &= (\mathcal{J}\mathcal{C}\mathsf{N}_M(\lambda,\lambda)^{1/2}f, \mathsf{N}_M(\lambda,\lambda)^{1/2}f)_{\mathcal{H}} \\ &\geqslant \gamma(\mathsf{N}_M(\lambda,\lambda)^{1/2}f, \mathsf{N}_M(\lambda,\lambda)^{1/2}f)_{\mathcal{H}} \\ &= \gamma\prec\mathsf{N}_M(\lambda,\cdot)f, \mathsf{N}_M(\lambda,\cdot)f\succ\end{aligned}$$

for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ and $f \in \mathcal{H}$, and for some $\gamma > 0$, since $\mathcal{J}\mathcal{C} > 0$ in $\mathcal{H}$. This implies that $\mathsf{JC} \geqslant \gamma I$, since JC is continuous in $\mathfrak{H}_M$.

Thus, starting with an operator $\mathcal{C}$ satisfying (3.16), one can construct the operator C, which realizes the property of C-symmetry for the symmetric operator S_M in the Krein space $(\mathfrak{H}_M, [\cdot,\cdot]_{\mathsf{J}})$.

By proposition 3.14, S_M and S have the same Weyl function $M(\cdot)$ associated with the boundary triplets $(\mathcal{H}, \Gamma_0^M, \Gamma_1^M)$ and $(\mathcal{H}, \Gamma_0, \Gamma_1)$, respectively. Therefore, there exists a unitary mapping $U\colon \mathfrak{H}_M \xrightarrow{\text{onto}} \mathfrak{H}$ such that $S_M = U^{-1}SU$ and $\Gamma_j^M = \Gamma_j U$, $j = 0, 1$ (see [14, theorem 3.9]).

Let us show that U can be chosen in such a way that

$$\mathsf{J} = U^{-1}JU, \tag{3.19}$$

where J is defined by (3.18). Indeed, it follows from (3.1) and (3.8) that $M(\cdot) = M_+(\cdot) \oplus M_-(\cdot)$, where the decomposition is with respect to the fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$. Furthermore, $M_{\pm}(\cdot)$ are the

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Weyl functions of the symmetric operators $S_\pm = S \upharpoonright \mathfrak{H}_\pm$ acting in subspaces $\mathfrak{H}_\pm$ of the fundamental decomposition (2.1) of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$.

Let $\mathfrak{H}_{M_\pm}$ be the reproducing kernel Hilbert spaces constructed by $M_\pm(\cdot)$. In view of (3.18), $\mathfrak{H}_M = \mathfrak{H}_{M_+} \oplus \mathfrak{H}_{M_-}$ is the fundamental decomposition of the Krein space $(\mathfrak{H}_M, [\cdot,\cdot]_J)$ and one has $S_M = S_{M_+} \oplus S_{M_-}$ with respect to this decomposition. The pairs of operators S_{M_+}, S_+ and S_{M_-}, S_- have the Weyl functions $M_+(\cdot)$ and $M_-(\cdot)$, respectively. These functions $M_\pm(\cdot)$ are associated with the boundary triplets $(\mathcal{H}_\pm, \Gamma_0^{M_\pm}, \Gamma_1^{M_\pm})$ and $(\mathcal{H}_\pm, \Gamma_0^\pm, \Gamma_1^\pm)$ of $S^*_{M_\pm}$ and $S^*_\pm$, respectively. Here $\Gamma_j^{M_\pm}$ are defined according to statement (iii) of proposition 3.14 and $\Gamma_j^\pm$ are the restrictions of Γ_j onto $\mathcal{D}(S^*_\pm)$. Without loss of generality, one can choose unitary mappings $U_\pm : \mathfrak{H}_{M_\pm} \xrightarrow{\text{onto}} \mathfrak{H}_\pm$ such that $S_{M_\pm} = U_\pm^{-1} S_\pm U_\pm$ and $\Gamma_j^{M_\pm} = \Gamma_j^\pm U_\pm$. In that case the operator $U = U_+ \oplus U_-$ satisfies (3.19).

It follows from (3.19) that the set $\mathcal{U} = \{U^{-1} C_\alpha U\}_{\alpha \in D}$ contains all possible C-symmetries of S_M in the Krein space $(\mathfrak{H}_M, [\cdot,\cdot]_J)$. Therefore, the operator C defined by (3.20) belongs to $\mathcal{U}$ and $\mathsf{C} = U^{-1} C_\alpha U$ for a certain choice of $\alpha \in D$. In that case, taking (3.8) into account, we obtain

$$\Gamma_j^M \mathsf{C} = \Gamma_j^M U^{-1} C_\alpha U = \Gamma_j C_\alpha U = \mathcal{C}_\alpha \Gamma_j U = \mathcal{C}_\alpha \Gamma_j^M, \quad j = 0, 1.$$

On the other hand, in view of (3.17) and statements (ii) and (iii) of proposition 3.14, we have $\Gamma_j^M \mathsf{C} = \mathcal{C} \Gamma_j^M$. Consequently, $\mathcal{C} = \mathcal{C}_\alpha$. Theorem 3.15 is proved. □

Note that (3.16 c) in theorem 3.15 implies the description (3.17) in $\mathfrak{H}_M$ and the fact that the boundary triplet $(\mathcal{H}, \Gamma_0^M, \Gamma_1^M)$ for S^*_M in proposition 3.14 admits the properties in (3.8). In fact, the proof of theorem 3.15 gives *a functional model for the family* $\mathfrak{U}$ *in* $\mathfrak{H}_M$.

Corollary 3.16. *Let $\mathfrak{H}_M$ be the reproducing kernel space associated with the Weyl function $M(\cdot)$ acting in $\mathcal{H}$, and let S_M be as in proposition 3.14. Then the family $\mathfrak{U} = \{\mathsf{C}_\alpha\}$ of all C-symmetries for S_M in $\mathfrak{H}_M$ consists of operators*

$$(\mathsf{C}_\alpha[\phi(\cdot)])(\lambda) \equiv \mathcal{C}_\alpha \phi(\lambda), \quad \phi(\cdot) \in \mathfrak{H}_M, \ \lambda \in \mathbb{C}_\pm, \tag{3.20}$$

where $\mathcal{C}_\alpha$ is a bounded operator in $\mathcal{H}$ satisfying the properties (3.16).

Remark 3.17. In the case of a simple symmetric operator S, theorem 3.15 gives a method for the description of the family $\mathfrak{U} = \{C_\alpha\}$ of stable C-symmetries. First, we determine the collection of all $\mathcal{C}$ in $\mathcal{H}$ that satisfy (3.17) and then construct the family $\mathfrak{U} = \{C_\alpha\}$ of stable C-symmetries for S using some boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for S^* with the properties (3.8) (for example, in $\mathfrak{H}_M$ such a boundary triplet is given by part (iii) of proposition 3.14). Note that this family $\mathfrak{U} = \{C_\alpha\}$ is unitarily equivalent to the family $\mathfrak{U} = \{\mathsf{C}_\alpha\}$ in corollary 3.16.

3.5. Resolvent formula for J-self-adjoint extensions with stable C-symmetry

Combining theorem 3.15 with proposition 3.12, we immediately obtain the following complete description of Σ_J^{st}.

THEOREM 3.18. *Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of S^* with the properties (3.8) and let $M(\cdot)$ be the Weyl function of S. Then $A \in \Sigma_J^{\mathrm{st}}$ if and only if A is defined by (3.12) and the corresponding $\mathcal{J}$-unitary operator $\mathcal{K}$ has the $\mathcal{C}$-symmetry in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ such that $[\mathcal{C}, M(\cdot)] = 0$.*

Another characterization of the class Σ_J^{st} can be obtained by describing the resolvents of $A \in \Sigma_J^{\mathrm{st}}$. Recall from remark 3.4 that if $A \in \Sigma_J^{\mathrm{st}}$, then A is self-adjoint in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_\alpha)$, where the inner product is defined by (3.3). Therefore, the resolvent set of the J-self-adjoint operator $A \in \Sigma_J^{\mathrm{st}}$ is automatically non-empty, since $\mathbb{C}_\pm \subset \rho(A)$. To establish such a characterization, the following definition is needed.

DEFINITION 3.19. Let $\mathcal{B}$ be a (closed linear) relation in a Hilbert space $\mathcal{H}$ and let $\mathcal{C}$ be a bounded operator in $\mathcal{H}$. Then $\mathcal{C}$ is said to commute with $\mathcal{B}$ if the following formula holds:

$$\mathcal{B} = \{\{\mathcal{C}f, \mathcal{C}f'\} \colon \{f, f'\} \in \mathcal{B}\}. \tag{3.21}$$

In this case we can write $[\mathcal{C}, \mathcal{B}] = 0$.

Observe that if $\mathcal{B}$ is an operator, then $\{f, f'\} \in \mathcal{B}$ means that $f' = \mathcal{B}f$. Thus, in this case, (3.21) can be rewritten as $\mathcal{B}\mathcal{C}f = \mathcal{C}\mathcal{B}f$ for all $f \in \mathcal{D}(\mathcal{B})$, i.e. definition 3.19 reduces to the usual definition of commutativity when $\mathcal{B}$ is an operator.

In the next statement, $\gamma(\cdot)$ stands for the γ-field corresponding to the boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ in theorem 3.18 and $A_0 = S^* \upharpoonright \ker \Gamma_0$.

THEOREM 3.20. *Let the assumptions be as in theorem 3.18. Then $A \in \Sigma_J^{\mathrm{st}}$ if and only if*

$$(A - \mu I)^{-1} = (A_0 - \mu I)^{-1} - \gamma(\mu)(M(\mu) - \mathcal{B})^{-1}\gamma^*(\bar{\mu}), \quad \mu \in \rho(A) \cap \rho(A_0), \tag{3.22}$$

where $\mathcal{B}$ is a $\mathcal{J}$-self-adjoint relation,[4] which has the $\mathcal{C}$-symmetry in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ such that $[\mathcal{C}, M(\cdot)] = 0$.

Furthermore, A is disjoint with A_0 (i.e. $\mathcal{D}(A) \cap \mathcal{D}(A_0) = \mathcal{D}(S)$) if and only if $\mathcal{B}$ is an operator with the indicated $\mathcal{C}$-symmetry, and A is transversal with A_0 (i.e. $\mathcal{D}(A) \dotplus \mathcal{D}(A_0) = \mathcal{D}(S^*)$) if and only if $\mathcal{B}$ is a bounded operator with the indicated $\mathcal{C}$-symmetry.

Proof. First assume that $A \in \Sigma_J^{\mathrm{st}}$. Then there exists $C_\alpha \in \mathfrak{U} = \{C_\alpha\}_{\alpha \in D}$ such that $[A, C_\alpha] = 0$. Moreover, $[A_0, C_\alpha] = 0$ (since $A_0 \in \Upsilon_{\mathfrak{U}}$, see remark 3.8). This means that A and A_0 are self-adjoint extensions of the symmetric operator S with respect to the scalar product $(\cdot,\cdot)_\alpha$ (remark 3.4), hence, in particular, $\mathbb{C}_\pm \subset \rho(A) \cap \rho(A_0)$.

Now, rewrite (3.12) as follows:

$$A = S^* \upharpoonright \{f \in \mathcal{D}(S^*) \colon \mathrm{i}(I + \mathcal{K})\Gamma_0 f = (I - \mathcal{K})\Gamma_1 f\}. \tag{3.23}$$

Since $\mathcal{K}$ is $\mathcal{J}$-unitary, this means that A corresponds to the $\mathcal{J}$-self-adjoint relation $\mathcal{B} = \mathrm{i}(I + \mathcal{K})(I - \mathcal{K})^{-1}$ in $\mathcal{H}$, i.e. $\{\Gamma_0, \Gamma_1\}\mathcal{D}(A) = \mathcal{B}$. By theorem 3.18, $\mathcal{K}$ has $\mathcal{C}$-symmetry in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ realized by an operator $\mathcal{C} \in \{\mathcal{C}_\alpha\}_{\alpha \in D}$ such that

[4]We refer the interested reader to [11] for the basic definitions of linear relation theory in the Krein space setting.

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$[\mathcal{C}, M(\cdot)] = 0$. Since $\{h, k\} = \{\varGamma_0 f, \varGamma_1 f\} \in \mathcal{B}$ if and only if $\mathrm{i}(I+\mathcal{K})\varGamma_0 f = (I-\mathcal{K})\varGamma_1 f$ (see (3.23)) and $[\mathcal{C}, \mathcal{K}] = 0$, it is clear that (3.21) is satisfied, so that $[\mathcal{C}, \mathcal{B}] = 0$. This means that $\mathcal{B}$ has the $\mathcal{C}$-symmetry in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$.

Finally, since $\mathcal{B}$ corresponds to A ($\{\varGamma_0, \varGamma_1\}\mathcal{D}(A) = \mathcal{B}$) and $\mathbb{C}_\pm \subset \rho(A)$, [12, § 2, proposition 1] shows that $0 \in \rho(M(\mu) - \mathcal{B})$ for all $\mu \in \mathbb{C}_\pm$ and, moreover, the resolvent formula (3.22) is obtained from [12, § 2, proposition 2] (note that the two propositions in [12] are formulated for an arbitrary closed linear relation $\mathcal{B}$ in $\mathcal{H}$).

To prove the converse statement, assume that A is given by (3.22) for some $\mathcal{J}$-self-adjoint relation $\mathcal{B}$ that has the $\mathcal{C}$-symmetry in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ such that $[\mathcal{C}, M(\cdot)] = 0$. Then $[\mathcal{C}, \mathcal{B}] = 0$ and this is equivalent to $[\mathcal{C}, (\mathcal{B} - M(\mu))^{-1}] = 0$, $\mu \in \mathbb{C}_\pm$, since $[\mathcal{C}, M(\cdot)] = 0$. To see this, observe that $\{f, f'\} \in \mathcal{B}$ is equivalent to

$$\{f' - M(\mu)f, f\} \in (\mathcal{B} - M(\mu))^{-1}$$

and that $\{\mathcal{C}f, \mathcal{C}f'\} \in \mathcal{B}$ is equivalent to

$$\{\mathcal{C}f' - M(\mu)\mathcal{C}f, \mathcal{C}f\} = \{\mathcal{C}(f' - M(\mu)f), \mathcal{C}f\} \in (\mathcal{B} - M(\mu))^{-1},$$

since $[\mathcal{C}, M(\cdot)] = 0$. Here $(\mathcal{B} - M(\mu))^{-1}$ is an operator for $\mu \in \mathbb{C}_\pm$. Hence, we conclude that $[\mathcal{C}, \mathcal{B}] = 0$ if and only if $\mathcal{C}$ commutes with $(\mathcal{B} - M(\mu))^{-1}$. Now it follows from lemma 3.9 that the 'preimage' C_α of $\mathcal{C} = \mathcal{C}_\alpha$, $\alpha \in D$ (see (3.8)), commutes with both of the summands on the right-hand side of (3.22). Thus, $[C_\alpha, (A - \mu I)^{-1}] = 0$, and this is equivalent to $[C_\alpha, A] = 0$. Furthermore, since $J \in \{C_\alpha\}_{\alpha \in D}$ it is clear from (3.22) that A is a J-self-adjoint extension of S. Thus, $A \in \varSigma_J^{\mathrm{st}}$.

The last statement is an immediate consequence of [13, proposition 1.4]. This completes the proof. □

Theorem 3.20 can be used for studying the spectral properties of the operators $A \in \varSigma_J^{\mathrm{st}}$. Recall that if S is simple and $\mu \in \rho(A_0)$, then it follows from (3.22) that, for the components of the spectrum of $A = A_{\mathcal{B}}$, we have (see [12, § 2, proposition 1]).

$$\mu \in \sigma_i(A) \Leftrightarrow 0 \in \sigma_i(M(\mu) - \mathcal{B}), \quad i = p, c, r. \tag{3.24}$$

4. The case of defect numbers $\langle 2, 2\rangle$

Let S be a closed densely defined simple symmetric operator in $\mathfrak{H}$ with defect numbers $\langle 2, 2\rangle$. We recall that S commutes with J and $\varSigma_J$ denotes the collection of all J-self-adjoint extensions of S.

4.1. The descriptions of $\varSigma_J^{\mathrm{st}}$

According to [28], two quite different arrangements for the sets $\varUpsilon_{\mathfrak{U}}$ and $\varSigma_J^{\mathrm{st}}$ can occur.

1. *Every member of $\varSigma_J$ has a non-empty resolvent set.* Then the set $\mathfrak{U} = \{C_\alpha\}$ of all C-symmetries of S consists of the operator J only and $\varUpsilon_{\mathfrak{U}} = \varSigma_J^{\mathrm{st}}$ (see [28, theorem 4.1]). Therefore, operators $A \in \varSigma_J^{\mathrm{st}}$ are just self-adjoint extensions of S, which commute with J.

2. *There are members of $\varSigma_J$ with empty resolvent set.* In that case, there exists an additional ('hidden') fundamental symmetry R in $\mathfrak{H}$ such that $[S, R] = 0$

and $\{J, R\} = 0$. This means that S commutes with the elements of the Clifford algebra $\mathcal{CL}_2(J, R)$ (see [28, theorem 4.3]).

PROPOSITION 4.1. *If Σ_J contains operators with empty resolvent set and if the Weyl function $M(\cdot)$ of S is not constant on $\mathbb{C}_+$, then all operators $C_\alpha \in \mathfrak{U}$ are expressed in terms of the Clifford algebra $\mathcal{CL}_2(J, R)$ and*

$$\mathfrak{U} = \{C_\alpha = J\mathrm{e}^{\chi R_\omega}\}, \quad \forall \alpha = (\chi, \omega) \in D = \mathbb{R} \times [0, 2\pi),$$

where $C_{\chi,\omega} := J\mathrm{e}^{\chi R_\omega}$ are defined by (2.18) and (2.19).

Proof. If $M(\cdot) \not\equiv$ const., then the characteristic function $\Theta(\cdot)$ of S is not a constant on $\mathbb{C}_+$ (see (2.14)). By the definition of the Straus characteristic function $\mathit{Sh}(\cdot)$ (see [33] or [28, § 2.2]), we conclude that $\mathit{Sh} \not\equiv 0$. According to [28, theorem 4.6], this means that $\mathfrak{U} = \{C_{\chi,\omega}\}_{\chi\in\mathbb{R},\omega\in[0,2\pi)}$. □

Otherwise, if $M(\cdot)$ is constant on $\mathbb{C}_+$, then the set $\mathfrak{U}$ increases considerably and it cannot be expressed in terms of $\mathcal{CL}_2(J, R)$. This fact leads to the relation $\Upsilon_{\mathfrak{U}} = \emptyset$ (since hypothetical operators $A \in \Upsilon_{\mathfrak{U}}$ have to satisfy the commutation relation $[A, C_\alpha] = 0$ for all $C_\alpha \in \mathfrak{U}$) and, simultaneously, it gives a 'maximal possible' set of stable C-symmetry Σ_J^{st} in the sense of the next statement.

PROPOSITION 4.2. *If the Weyl function $M(\cdot)$ of S is constant on $\mathbb{C}_+$, then the J-self-adjoint extension A of S belongs to Σ_J^{st} if and only if the spectrum of A is real.*

Proof. In a similar way to the previous proof, we conclude that

$$M(\cdot) \equiv \text{const.} \iff \Theta(\cdot) \equiv \text{const.} \iff \mathit{Sh} \equiv 0.$$

This means that Σ_J contains a two-parameter family of operators with empty resolvent set [28, corollary 3.2] and $A \in \Sigma_J^{\mathrm{st}} \iff \sigma(A) \subset \mathbb{R}$ [28, corollary 4.9]. □

In what follows, we will assume that S *commutes with elements of the Clifford algebra* $\mathcal{CL}_2(J, R)$ *and that* $\Upsilon_{\mathfrak{U}}$ *is non-empty* (or, equivalently, that the conditions of proposition 4.1 hold).

Since $\Upsilon_{\mathfrak{U}}$ is non-empty, there exist boundary triplets for S^* with the properties (3.8). Set one of them at $(\mathcal{H}, \Gamma_0, \Gamma_1)$. Then every $A \in \Sigma_J^{\mathrm{st}}$ is determined by (3.12), where the corresponding $\mathcal{J}$-unitary operator $\mathcal{K}$ is stable in the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ (theorem 3.11). Using proposition 4.1, we can supplement the results in § 3 by giving explicit descriptions for the sets Σ_J^{st} and $\Upsilon_{\mathfrak{U}}$.

In what follows the notation $A_{\mathcal{K}}$ is used for J-self-adjoint-operators A determined by (3.12). Since S has defect numbers $\langle 2, 2\rangle$, the dimension of $\mathcal{H}$ is 2. Hence, the operator $\mathcal{K}$ can be presented as a 2×2-matrix $\mathcal{K} = (k_{ij})$. Since $\mathcal{K}$ is a $\mathcal{J}$-unitary operator, the relation $\mathcal{J} = \mathcal{K}^*\mathcal{J}\mathcal{K}$ holds.

Fix an orthonormal basis of $\mathcal{H}$ in which the matrix representations of $\mathcal{J}$ coincides with the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Then the identity $\mathcal{J} = \mathcal{K}^*\mathcal{J}\mathcal{K}$ takes the form $\sigma_3 = (\bar{k}_{ij})^t\sigma_3(k_{ij})$ and a simple analysis yields the following explicit formula for $\mathcal{K}$:

$$\mathcal{K} = \mathcal{K}(\zeta,\phi,\xi,\omega) = \mathrm{e}^{-\mathrm{i}\xi}\begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi}\cosh\zeta & \mathrm{e}^{-\mathrm{i}\omega}\sinh\zeta \\ -\mathrm{e}^{\mathrm{i}\omega}\sinh\zeta & \mathrm{e}^{\mathrm{i}\phi}\cosh\zeta \end{pmatrix}, \tag{4.1}$$

where $\zeta \in \mathbb{R}$, $\phi \in [0,\pi]$ and $\xi,\omega \in [0,2\pi)$.

THEOREM 4.3. *Assume that S commutes with the Clifford algebra $\mathcal{CL}_2(J,R)$ and that $\Upsilon_{\mathfrak{U}}$ is non-empty. Then the following statements are true:*

(i) *if $A_{\mathcal{K}} \in \Sigma_J$, then its adjoint $A^*_{\mathcal{K}} \in \Sigma_J$ is given by*

$$\mathcal{K}'(\zeta,\phi,\xi,\omega) = \mathcal{K}(-\zeta,\phi,\xi,\omega);$$

(ii) *$A_{\mathcal{K}} \in \Sigma_J$ is self-adjoint if and only if $\zeta = 0$, i.e.*

$$\mathcal{K} = \mathcal{K}(0,\phi,\xi,\omega) = \mathrm{e}^{-\mathrm{i}\xi}\begin{pmatrix} -\mathrm{e}^{-\mathrm{i}\phi} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}; \tag{4.2}$$

(iii) *$A_{\mathcal{K}} \in \Sigma_J$ belongs to $\Upsilon_{\mathfrak{U}}$ if and only if $\zeta = 0$ and $\phi = \frac{1}{2}\pi$, i.e.*

$$\mathcal{K} = \mathcal{K}(0,\tfrac{1}{2}\pi,\xi,\omega) = \mathrm{e}^{-\mathrm{i}\xi}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}; \tag{4.3}$$

(iv) *$A_{\mathcal{K}} \in \Sigma_J$ belongs to $\Sigma_J^{\mathrm{st}} \setminus \Upsilon_{\mathfrak{U}}$ if and only if $\mathcal{K} = \mathcal{K}(\zeta,\phi,\xi,\omega)$, where*

$$|\tanh\zeta| < |\cos\phi|. \tag{4.4}$$

In that case the operator $A_{\mathcal{K}(\zeta,\phi,\xi,\omega)}$ has the $C_{\chi,\omega}$-symmetry, where the parameter χ is (uniquely) determined by the equation

$$\cos\phi\tanh\chi = -\tanh\zeta. \tag{4.5}$$

Proof. (i) This follows from (4.1) by means of the identities $A^*_{\mathcal{K}} = A_{(\mathcal{K}^*)^{-1}}$ and $(\mathcal{K}^{-1})^* = \mathcal{J}\mathcal{K}\mathcal{J}$, where $\mathcal{J}$ is identified with σ_3.

(ii) This follows immediately from (i).

(iv) It follows from proposition 4.1 that $A \in \Sigma_J^{\mathrm{st}}$ if and only if $[A, C_{\chi,\omega}] = 0$ for at least one $C_{\chi,\omega} = J\mathrm{e}^{\chi R_\omega}$, $\chi \in \mathbb{R}$, $\omega \in [0,2\pi)$. Using (2.18), we obtain

$$C_{\chi,\omega} = (\cosh\chi)J + (\sinh\chi)(\cos\omega)JR - \mathrm{i}(\sinh\chi)(\sin\omega)R. \tag{4.6}$$

Since the boundary triplet $(\mathcal{H},\Gamma_0,\Gamma_1)$ of S^* has the properties (3.8), lemma 3.7 implies that the operators $C_{\chi,\omega}$ have images $\mathcal{C}_{\chi,\omega}$ in $\mathcal{H}$ determined by the formula $\mathcal{C}_{\chi,\omega}\Gamma_j = \Gamma_j C_{\chi,\omega}$, $j = 0,1$, where $C_{\chi,\omega}$ have the form (4.6). Considering this formula for $\omega = \frac{1}{2}\pi$ and taking the relation $\mathcal{J}\Gamma_j = \Gamma_j J$ into account, we conclude that $\mathcal{R}\Gamma_j = \Gamma_j R$, where $\mathcal{R}$ is a bounded operator in $\mathcal{H}$. Therefore,

$$\mathcal{C}_{\chi,\omega} = (\cosh\chi)\mathcal{J} + (\sinh\chi)(\cos\omega)\mathcal{J}\mathcal{R} - \mathrm{i}(\sinh\chi)(\sin\omega)\mathcal{R} \tag{4.7}$$

and $\mathcal{C}^2_{\chi,\omega} = I$, $\mathcal{J}\mathcal{C}_{\chi,\omega} > 0$ in $\mathcal{H}$ (see lemma 3.10).

Applying lemma 3.10 to $\mathcal{R}$, instead of $\mathcal{J}$, it can be seen that $\mathcal{R}$ is a fundamental symmetry in $\mathcal{H}$. Moreover, $\{\mathcal{J},\mathcal{R}\}=0$ due to $\{J,R\}=0$. Thus, $\mathcal{J}$ and $\mathcal{R}$ are anti-commuting fundamental symmetries in $\mathcal{H}$ and the operators $\mathcal{C}_{\chi,\omega}=\mathcal{J}\mathrm{e}^{\chi\mathcal{R}_\omega}$ are expressed in terms of the Clifford algebra $\mathcal{CL}_2(\mathcal{J},\mathcal{R})$.

Without loss of generality, we may assume that the matrix representation of $\mathcal{R}$ coincides with the Pauli matrix

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(recall that we have supposed that $\mathcal{J}$ coincides with σ_3). Then the matrix representation of $\mathrm{i}\mathcal{R}\mathcal{J}$ coincides with

$$\sigma_2=\mathrm{i}\sigma_1\sigma_3=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$$

and

$$\begin{aligned}\mathcal{C}_{\chi,\omega}&=(\cosh\chi)\sigma_3+\mathrm{i}(\sinh\chi)(\cos\omega)\sigma_2-\mathrm{i}(\sinh\chi)(\sin\omega)\sigma_1\\ &=\begin{pmatrix}\cosh\chi & (\sinh\chi)\mathrm{e}^{-\mathrm{i}\omega}\\ -(\sinh\chi)\mathrm{e}^{\mathrm{i}\omega} & -\cosh\chi\end{pmatrix}.\end{aligned}\tag{4.8}$$

Since $A=A_{\mathcal{K}}$ is defined by (3.12) (see theorem 3.11) and $[A,C_{\chi,\omega}]=0$, the corresponding operator $\mathcal{K}$ satisfies the relation $[\mathcal{K},\mathcal{C}_{\chi,\omega}]=0$ for certain $\chi\in\mathbb{R}$ and $\omega\in[0,2\pi)$. A routine analysis of the last equality using (4.1) and (4.8) leads to the conclusion that $A_{\mathcal{K}}$ has the C-symmetry ($C\in\{C_{\chi,\omega}\}$) if and only if either $\zeta=0$, $\phi=\frac{1}{2}\pi$ (this case corresponds to the set $\Upsilon_{\mathfrak{U}}$ (see (iii)), or $|\tanh\zeta|<|\cos\phi|$. In the latter case the operator C can be chosen as $C_{\chi,\omega}$, where the parameter χ is uniquely determined by equation (4.5). Hence, (iv) is proved.

(iii) By proposition 4.1 and the definition of $\Upsilon_{\mathfrak{U}}$, we have

$$A\in\Upsilon_{\mathfrak{U}}\iff[A,C_{\chi,\omega}]=0,\quad\forall\chi\in\mathbb{R},\ \omega\in[0,2\pi).$$

The last relation is equivalent to $[\mathcal{K},\mathcal{J}]=[\mathcal{K},\mathcal{R}]=0$ due to (4.7). The first condition $[\mathcal{K},\mathcal{J}]=0$ imposed on the $\mathcal{J}$-unitary operator $\mathcal{K}$ means that $\mathcal{K}$ is unitary. Hence, $A_{\mathcal{K}}$ is self-adjoint, and thus $\zeta=0$ and $\mathcal{K}$ is defined by (4.2) (see (ii)). Now, the second condition $[\mathcal{K},\mathcal{R}]=0$ can be rewritten as $[\mathcal{K}(0,\phi,\xi,\omega),\sigma_1]=0$ (since $\mathcal{R}$ is identified with σ_1). Hence, $-\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\mathrm{i}\phi}=\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{\mathrm{i}\phi}$ and this equality holds if and only if $\phi=\frac{1}{2}\pi$, $\phi\in[0,\pi]$. Theorem 4.3 is proved. □

Remark 4.4. The classical von Neumann formulae were used in [2, theorem 3.2, proposition 3.3] for a similar description of $\varSigma_J^{\mathrm{st}}$ in terms of unitary matrices.

4.2. Spectral analysis of $A\in\varSigma_J^{\mathrm{st}}$

Let $(\mathcal{H},\Gamma_0,\Gamma_1)$ be a boundary triplet for S^* with the properties (3.8) and let $M(\cdot)$ be the corresponding Weyl function. Combining the definition (2.13) of $M(\cdot)$ with the definitions $\mathcal{J}\Gamma_j:=\Gamma_jJ$, $\mathcal{R}\Gamma_j:=\Gamma_jR$, $j=0,1$ of the fundamental symmetries $\mathcal{J}$ and $\mathcal{R}$ in $\mathcal{H}$, we conclude that $[\mathcal{J},M(\cdot)]=[\mathcal{R},M(\cdot)]=0$ (see lemma 3.9). Passing to the matrix representation $\mathcal{M}(\cdot)=(m_{ij}(\cdot))$ of $M(\cdot)$ and using the identification of $\mathcal{J}$ and $\mathcal{R}$ with σ_3 and σ_1, respectively, we get $[\sigma_3, \mathcal{M}(\cdot)] = [\sigma_1, \mathcal{M}(\cdot)] = 0$. This leads to

$$M(\cdot) = m(\cdot)I, \tag{4.9}$$

where $m(\cdot)$ is a scalar function defined on $\rho(A_0)$ $(A_0 = S^* \upharpoonright \ker \Gamma_0)$.

Now consider an arbitrary $C \in \{C_{\chi,\omega}\}$ and the corresponding decomposition

$$\mathfrak{H} = \mathfrak{L}_+^{\chi,\omega}[\dot{+}]\mathfrak{L}_-^{\chi,\omega}, \qquad \mathfrak{L}_\pm^{\chi,\omega} = \tfrac{1}{2}(I \pm C_{\chi,\omega})\mathfrak{H}. \tag{4.10}$$

Since S and S^* commute with $C_{\chi,\omega}$ they are decomposed with respect to (4.10):

$$S = S_+(\chi,\omega) \dot{+} S_-(\chi,\omega), \qquad S^* = S_+^*(\chi,\omega) \dot{+} S_-^*(\chi,\omega), \tag{4.11}$$

where $S_\pm(\chi,\omega) = S \upharpoonright \mathfrak{L}_\pm^{\chi,\omega}$ and $S_\pm^*(\chi,\omega)$ are the adjoints of the symmetric operators $S_\pm(\chi,\omega)$ acting in the spaces[5] $\mathfrak{L}_\pm^{\chi,\omega}$.

Let $A \in \Sigma_J^{\mathrm{st}}$. Then $A = A_{\mathcal{K}}$ is determined by (4.1) and $A_{\mathcal{K}}$ has the $C_{\chi,\omega}$-symmetry for a certain choice of χ and ω (proposition 4.1). Therefore, $A_{\mathcal{K}}$ is decomposed with respect to (4.10):

$$A_{\mathcal{K}} = A_{\mathcal{K}}^+ \dot{+} A_{\mathcal{K}}^-, \tag{4.12}$$

where $S_\pm(\chi,\omega) \subset A_{\mathcal{K}}^\pm \subset S_\pm^*(\chi,\omega)$. In this case, the $\mathcal{J}$-unitary operator $\mathcal{K}$ is decomposed, $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$, with respect to the decomposition

$$\mathcal{H} = \mathcal{H}_+^{\chi,\omega}[+]\mathcal{H}_-^{\chi,\omega}, \qquad \mathcal{H}_\pm^{\chi,\omega} = \tfrac{1}{2}(I \pm \mathcal{C}_{\chi,\omega})\mathcal{H} \tag{4.13}$$

of the Krein space $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{J}})$ (see (4.10)). Since $\dim \mathcal{H} = 2$, the subspaces $\mathcal{H}_\pm^{\chi,\omega}$ are one dimensional. Therefore, $\mathcal{K}_\pm = k_\pm I$ and the eigenvalues $k_\pm$ of $\mathcal{K}$ should satisfy the relations $(\mathcal{K} - k_\pm I)(I \pm \mathcal{C}_{\chi,\omega}) = 0$.

A direct solution of the characteristic equation $\det(\mathcal{K} - kI) = 0$ gives

$$k_\pm = \mathrm{e}^{-\mathrm{i}\xi}[\pm\sqrt{1 - \sin^2\phi\cosh^2\zeta} + \mathrm{i}\sin\phi\cosh\zeta]. \tag{4.14}$$

If, in particular, (4.4) is satisfied, a simple calculation using (4.5) leads to

$$k_+ = -\mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\mathrm{i}t}, \quad k_- = \mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{\mathrm{i}t}, \quad \mathrm{e}^{\mathrm{i}t} := \frac{\cos\phi + \mathrm{i}\sin\phi\cosh\chi}{|\cos\phi + \mathrm{i}\sin\phi\cosh\chi|}. \tag{4.15}$$

In this case, the value of χ in (4.15) is uniquely determined by ζ and ϕ (see (4.5)) and, hence, t can be considered as a function of ζ and ϕ, i.e. $t = t(\zeta,\phi)(\in [0,2\pi))$.

On the other hand, if $\zeta = 0$ and $\phi = \frac{1}{2}\pi$, then, in (4.14), $k_\pm = \mathrm{i}\mathrm{e}^{-\mathrm{i}\xi}$ (see (4.3)). In this case, (4.15) holds with $t = \frac{1}{2}\pi$.

THEOREM 4.5. *Let the conditions of proposition 4.1 be satisfied and assume that $A_{\mathcal{K}} \in \Sigma_J^{\mathrm{st}}$. Then the spectrum of $A_{\mathcal{K}} = A_{\mathcal{K}}(\zeta,\phi,\xi,\omega)$ (see (4.1)) is real and, moreover, $r \in \rho(A_0)$ belongs to the discrete spectrum of $A_{\mathcal{K}}(\zeta,\phi,\xi,\omega)$ if and only if*

$$[\tan\tfrac{1}{2}(\xi + t) + m(r)] \cdot [\cot\tfrac{1}{2}(\xi - t) - m(r)] = 0, \tag{4.16}$$

where $m(\cdot)$ is given by (4.9).

[5]The spaces $\mathfrak{L}_\pm^{\chi,\omega}$ are considered here with the original inner product $(\cdot,\cdot)$ on $\mathfrak{H}$.

If, in particular, $A_{\mathcal{K}} \in \Sigma_J^{\mathrm{st}} \setminus \Upsilon_{\mathfrak{U}}$, then $t = t(\zeta, \phi)$ is determined by (4.5) and (4.15). Furthermore, if $A_{\mathcal{K}} = A_{\mathcal{K}(0,\pi/2,\xi,\omega)} \in \Upsilon_{\mathfrak{U}}$ and $\xi \neq \frac{1}{2}\pi$, then $r \in \rho(A_0)$ belongs to the discrete spectrum of $A_{\mathcal{K}}$ if and only if

$$\tan \tfrac{1}{2}(\xi + \tfrac{1}{2}\pi) + m(r) = 0. \tag{4.17}$$

In this case $A_{\mathcal{K}}$ coincides with self-adjoint operator A_r defined by (2.25).

Proof. The reality of $\sigma(A_{\mathcal{K}})$ is a general property of all J-self-adjoint operators with a C-symmetry (see, for example, proposition 2.1).

Let $A_{\mathcal{K}} \in \Sigma_J^{\mathrm{st}}$. Then $A_{\mathcal{K}}$ admits the decomposition (4.12) for certain χ and ω. In view of (3.12) and (4.11), the corresponding operators $A_{\mathcal{K}}^{\pm}$ are the restrictions of $S_{\pm}^*(\chi, \omega)$ onto (see (3.23))

$$\mathcal{D}(A_{\mathcal{K}}^{\pm}) = \{f \in \mathcal{D}(S_{\pm}^*(\chi, \omega)) \mid k_{\pm}(\Gamma_1 + \mathrm{i}\Gamma_0)f = (\Gamma_1 - \mathrm{i}\Gamma_0)f\}. \tag{4.18}$$

Rewriting the right-hand side of (4.18) as $\mathrm{i}(1 + k_{\pm})/(1 - k_{\pm})\Gamma_0 f_{\pm} = \Gamma_1 f_{\pm}$ and taking into account that

$$\mathrm{i}\frac{1 + k_+}{1 - k_+} = \mathrm{i}\frac{1 - \mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\mathrm{i}t}}{1 + \mathrm{e}^{-\mathrm{i}\xi}\mathrm{e}^{-\mathrm{i}t}} = -\tan \tfrac{1}{2}(\xi + t), \qquad \mathrm{i}\frac{1 + k_-}{1 - k_-} = \cot \tfrac{1}{2}(\xi - t),$$

where t is given by (4.15), we obtain

$$\left.\begin{aligned} \mathcal{D}(A_{\mathcal{K}}^{+}) &= \{f \in \mathcal{D}(S_{+}^*(\chi, \omega)) \mid -\tan \tfrac{1}{2}(\xi + t)\Gamma_0 f = \Gamma_1 f\}, \\ \mathcal{D}(A_{\mathcal{K}}^{-}) &= \{f \in \mathcal{D}(S_{-}^*(\chi, \omega)) \mid \cot \tfrac{1}{2}(\xi - t)\Gamma_0 f = \Gamma_1 f\}. \end{aligned}\right\} \tag{4.19}$$

Using (4.10), (4.13) and recalling that $\mathcal{C}_{\chi,\omega}\Gamma_j = \Gamma_j C_{\chi,\omega}$, it can easily be seen that the restrictions of the original boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ onto the domains $\mathcal{D}(S_{\pm}^*(\chi, \omega))$ give rise to the boundary triplets $(\mathcal{H}_{\pm}^{\chi,\omega}, \Gamma_0, \Gamma_1)$ of $S_{\pm}^*(\chi, \omega)$ in $\mathfrak{L}_{\pm}^{\chi,\omega}$. Moreover, due to (4.9), $m(\cdot)$ is the Weyl function of $S_{\pm}(\chi, \omega)$ associated with the boundary triplets $(\mathcal{H}_{\pm}^{\chi,\omega}, \Gamma_0, \Gamma_1)$. However, by theorem 3.20, the formulae in (4.19) imply that $r \in \rho(A_0)$ is an eigenvalue of $A_{\mathcal{K}}^{+}$ ($A_{\mathcal{K}}^{-}$) if and only if $\tan \frac{1}{2}(\xi + t) + m(r) = 0$ ($\cot \frac{1}{2}(\xi - t) - m(r) = 0$) (see (3.24)). Now (4.16) follows from the decomposition (4.12).

The statement for $A_{\mathcal{K}} \in \Sigma_J^{\mathrm{st}} \setminus \Upsilon_{\mathfrak{U}}$ is clear. Assume that $A_{\mathcal{K}} \in \Upsilon_{\mathfrak{U}}$. Then, according to (4.3) and (4.14), the eigenvalues $k_{\pm}$ of the operator $\mathcal{K}$ coincide and $k_{\pm} = \mathrm{i}\mathrm{e}^{-\mathrm{i}\xi}$. In particular, $k_{\pm} \neq 1$ precisely when $\xi \neq \frac{1}{2}\pi$. Since $\phi = \frac{1}{2}\pi$ and $t = \frac{1}{2}\pi$ in (4.15), we have $-\tan \frac{1}{2}(\xi + \frac{1}{2}\pi) = \cot \frac{1}{2}(\xi - \frac{1}{2}\pi)$. Therefore, (4.16) reduces to (4.17). In this case, the (algebraic) multiplicity of the eigenvalue r is 2 and, hence, $A_{\mathcal{K}} = A_r$. Theorem 4.5 is proved. □

4.3. Example

We start by describing a general procedure which allows us to construct various examples illustrating the results above. The basic ingredients are a symmetric operator S_+ with defect numbers $\langle 1, 1\rangle$ in a Hilbert space $\mathfrak{H}_+$, a boundary triplet $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ for S_+^*, and the Weyl function $m(\cdot)$ of S_+ associated with $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$.

Let $\mathfrak{H}_-$ be a Hilbert space and let W be a unitary map of $\mathfrak{H}_-$ onto $\mathfrak{H}_+$. In the space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$, consider the operators

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & W \\ W^{-1} & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} S_+ & 0 \\ 0 & W^{-1}S_+W \end{pmatrix}. \tag{4.20}$$

Clearly, J and R are anti-commuting fundamental symmetries in $\mathfrak{H}$ and the symmetric operator S has defect numbers $\langle 2, 2\rangle$ in $\mathfrak{H}$ and it commutes with the Clifford algebra $\mathcal{CL}_2(J, R)$.

Let S_+ have real points of regular type. Then the operator S also has real points of regular type. Hence, the set $\Upsilon_{\mathfrak{U}}$ is non-empty (theorem 3.5) and $\mathfrak{U} = \{C_{\chi,\omega}\}$ (proposition 4.1). It follows from (4.6) and (4.7) that a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of S^* has the properties (3.8) if and only if the relations

$$\mathcal{J}\Gamma_j = \Gamma_j J, \qquad \mathcal{R}\Gamma_j = \Gamma_j R \tag{4.21}$$

determine fundamental symmetries $\mathcal{J}$ and $\mathcal{R}$ in $\mathcal{H}$.

Now introduce the mappings $\Gamma_j^- := \Gamma_j^+ W$, $j = 0, 1$. Then $(\mathbb{C}, \Gamma_0^-, \Gamma_1^-)$ is a boundary triplet for $S_-^* = W^{-1}S_+^*W$ and the mappings $\Gamma_j = \Gamma_j^+ \oplus \Gamma_j^-$, $j = 0, 1$, define a boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ for S^* which satisfies (4.21) with

$$\mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \qquad \mathcal{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1. \tag{4.22}$$

Therefore, the boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ satisfies (3.8). Furthermore, the Weyl function $M(\cdot)$ of S associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$ is determined by (4.9), where $m(\cdot)$ is the Weyl function of S_+ associated with $(\mathcal{H}_+, \Gamma_0^+, \Gamma_1^+)$.

With these preparations, the spectral analysis of J-self-adjoint operators with stable C-symmetries can be carried out by a somewhat routine application of theorem 4.5. The corresponding spectral properties depend on the choice of the initial symmetric operator S_+ (or, equivalently, on the choice of the Weyl function $m(\cdot)$ of S_+).

The above considerations are illustrated by the following example.

Example 4.6. Let $\mathfrak{H}_+ = L_2^{\text{even}}(\mathbb{R})$ be the subspace of even functions of $L_2(\mathbb{R})$ and define

$$S_+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \mathcal{D}(S_+) = [\overset{0}{W_2^2}(\mathbb{R}_-) \oplus \overset{0}{W_2^2}(\mathbb{R}_+)] \cap L_2^{\text{even}}(\mathbb{R}).$$

The adjoint $S_+^* = -\mathrm{d}^2/\mathrm{d}x^2$ has the domain $\mathcal{D}(S_+^*) = W_2^2(\mathbb{R} \setminus \{0\}) \cap L_2^{\text{even}}(\mathbb{R})$ and $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ with

$$\Gamma_0^+ u(\cdot) = u(0), \qquad \Gamma_1^+ u(\cdot) = u'(+0) - u'(-0) = 2u'(+0)$$

defines a boundary triplet for S_+^*. With $\mu \in \mathbb{C} \setminus \mathbb{R}$, the defect subspace $\ker(S_+^* - \mu I)$ coincides with the linear span of

$$f_\mu = \begin{cases} \mathrm{e}^{\mathrm{i}\tau x}, & x > 0, \\ \mathrm{e}^{-\mathrm{i}\tau x}, & x < 0, \end{cases}$$

where $\tau = \sqrt{\mu}$ and $\operatorname{Im}\tau > 0$. Since $m(\mu)\Gamma_0^+ f_\mu = \Gamma_1^+ f_\mu$, the Weyl function of S_+ associated with $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ is given by

$$m(\mu) = 2\mathrm{i}\sqrt{\mu}. \tag{4.23}$$

Let $\mathfrak{H}_- = L_2^{\mathrm{odd}}(\mathbb{R})$ be the subspace of odd functions of $L_2(\mathbb{R})$. According to (4.20), the fundamental symmetry J coincides with the space parity operator $\mathcal{P}u(x) = u(-x)$ in $\mathfrak{H} = L_2(\mathbb{R}) = L_2^{\mathrm{even}}(\mathbb{R}) \oplus L_2^{\mathrm{odd}}(\mathbb{R})$. Choosing the unitary map

$$W\colon L_2^{\mathrm{odd}}(\mathbb{R}) \to L_2^{\mathrm{even}}(\mathbb{R}) \quad \text{as } Wu = \operatorname{sgn}(x)u(x),$$

we conclude that the fundamental symmetry R coincides with the multiplication by $\operatorname{sgn}(x)$ in $L_2(\mathbb{R})$. Now, the operator $S = -\mathrm{d}^2/\mathrm{d}x^2$,

$$\mathcal{D}(S) = \overset{0}{W_2^2}(\mathbb{R}_-) \oplus \overset{0}{W_2^2}(\mathbb{R}_+)$$

is symmetric in $L_2(\mathbb{R})$ and its adjoint $S^* = -\mathrm{d}^2/\mathrm{d}x^2$ has the domain $\mathcal{D}(S^*) = W_2^2(\mathbb{R} \setminus \{0\})$.

The boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ for S^*, which is determined by

$$\Gamma_0 f = \Gamma_0(u+v) = \begin{pmatrix} u(0) \\ v(+0) \end{pmatrix}, \qquad \Gamma_1 f = 2\begin{pmatrix} u'(+0) \\ v'(0) \end{pmatrix},$$

where u and v are, respectively, even and odd parts of f, satisfies (4.21) with (4.22). Moreover, all $\mathcal{P}$-self-adjoint operators $A_{\mathcal{K}} \in \Sigma_{\mathcal{P}}^{\mathrm{st}}$ are characterized by parts (iii) and (iv) of theorem 4.3.

The self-adjoint operator $A_0 = S^* \upharpoonright \ker \Gamma_0$ coincides with the Friedrichs extension of S:

$$A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \mathcal{D}(A) = \{f(\cdot) \in W_2^2(\mathbb{R} \setminus \{0\})\colon f(+0) = 0,\ f(-0) = 0\}.$$

Applying theorem 4.5 and taking the relations $\sigma(A_0) = [0, \infty)$ and (4.23) into account, we conclude that *an arbitrary* $A_{\mathcal{K}}(\zeta, \phi, \xi, \omega) \in \Sigma_{\mathcal{P}}^{\mathrm{st}} \setminus \Upsilon_{\mathfrak{U}}$ *has the essential spectrum on* $[0, \infty)$ *and a negative number* r *belongs to the discrete spectrum of* $A_{\mathcal{K}}$ *if and only if*

$$[\tan \tfrac{1}{2}(\xi + t) - 2\sqrt{|r|}] \cdot [\cot \tfrac{1}{2}(\xi - t) + 2\sqrt{|r|}] = 0,$$

where $t = t(\zeta, \phi)$ is determined by (4.5) and (4.15). The algebraic and the geometric multiplicities of r are equal.

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