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HOMOTOPY MODEL THEORY

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Abstract. Drawing on the analogy between any unary first-order quantifier and a "face operator," this paper establishes several connections between model theory and homotopy theory. The concept of simplicial set is brought into play to describe the formulae of any first-order language L, the definable subsets of any L-structure, as well as the type spaces of any theory expressed in L. An adjunction result is then proved between the category of o-minimal structures and a subcategory of the category of linearly ordered simplicial sets with distinguished vertices.

§1. Simplicial ideas.

1.1. Formulae as chains. Homotopy theory dramatically entered the scene of logic through the connections that have been made between Martin-Löf type theory and model categories, since Hofmann and Streicher's seminal paper.[1] This paper aims at focusing on another connection of homotopy theory with logic, based on the following remark: any structure for a first-order language can be turned into a simplicial set. Hence, whereas "Homotopy Type Theory" connects logic with homotopy theory through type theory, "Homotopy Model Theory" is the proposal to connect logic with homotopy theory through model theory. This is the task taken up here.[2]

An intuitive motivation of such perspective is that the notion of *boundary* can easily be transposed to the context of first-order logic, formulae being conceived of as *chains* (in the sense of a formal sum of faces). The boundary of a given formula $\phi(v_0, \dots, v_n)$ with exactly $v_0, \dots, v_n$ as free variables can be defined as follows:

$$\partial\phi := \bigwedge_{i=0}^{n-1} \neg^i \forall x \phi(v_0, \dots, v_{i-1}, x, v_{i+1}, \dots, v_{n-1}),$$

where '$\neg^i$' indicates i consecutive occurrences of the negation symbol. The boundary operator ∂ is not stable under variable renaming, however. To avoid any problem of that kind, we will consider, throughout this paper, a single fixed first-order language

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[1] [5].

[2] The comparison of spaces of types to derived spaces of a topological space has been put forward as early as in [8] (pp. 514–515). Besides, connections between propositional calculus and homotopy types have been explored in [2]. Still, no systematic use of concepts anchored in modern algebraic topology has been made in logic, at the partial exception of [7] and [4].

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L with equality, containing the existential quantifier as a primitive, whose free variable symbols are exactly 'v_i', $i \geq 0$, and whose bound variable symbols are exactly 'x', 'y', 'z', and so on. Each L-formula will be taken up to bound variable renaming, but not up to free variable renaming.

DEFINITION 1.1. An L-formula ϕ is called a *well-formed L-formula* iff the indices of all the free variables occurring in ϕ, regardless of the order of the occurrences of those variables, make up an initial segment of $\mathbb{N}$. From now on, "L-formula" will always refer to a well-formed L-formula only.

Starting with an L-formula $\phi(v_0, v_1, v_2)$, one finds that $\partial(\partial\phi) \equiv \perp$ (since universal quantification commutes with conjunction). This prompts a comparison of ∂ with a genuine boundary operator, in a sense pointing to homotopy theory. To that end, a few explanatory remarks are in order about the concept of *simplicial set*, given the role that it will play in what follows. CW complexes were introduced in algebraic topology in order to analyze nonpathological topological spaces as reconstructible by glueing elementary "cells" along other cells of smaller dimension: the building pattern of CW complexes can be used to compute the homology groups and homotopy groups of that space. Designed to formalize this kind of decomposition of a topological space into cells of increasing dimension, glued together along a common face, a *simplicial set* X is a sequence $(X_n)_{n\in\mathbb{N}}$ of sets, together with maps $d_i : X_n \to X_{n-1}$ $(0 \leq i \leq n)$ and $s_j : X_n \to X_{n+1}$ $(0 \leq j \leq n)$, for each $n \in \mathbb{N}$, satisfying the following *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j, \\ d_i s_j = s_{j-1} d_i & \text{if } i < j, \\ d_j s_j = d_{j+1} s_j = \mathrm{id}, & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1, \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j. \end{cases}$$

The elements of each set X_n, called *n-simplices*, represent the n-cells involved in the construction of the topological space to be built, each map d_i the projection of an n-cell onto one of its faces, and each d_j the upgrading of an n-cell as a *degenerate* $(n+1)$-cell, while the simplicial identities encode the purely combinatorial properties of the building pattern of the space.

Despite being deprived of any topology, simplicial sets are sufficient to capture most features relevant to homotopy theory, since the homotopy category of the category of simplicial sets is a model of homotopy types. As will be seen presently, simplicial sets also supply an interesting connection between the combinatorial aspect of logical syntax and the use of topological methods in model theory.

DEFINITION 1.2. Let F_n be the set of L-formulae with exactly $v_0, \dots, v_n$ as free variables. (F_{-1} may be defined as the set of all L-sentences.) For each $n \in \mathbb{N}$, two systems of maps, $\exists_i : F_n \to F_{n-1}$ $(0 \leq i \leq n)$ and $E_j : F_n \to F_{n+1}$ $(0 \leq j \leq n)$ are, respectively:

$$\exists_i(\phi(v_0, \dots, v_n)) = \exists x \phi(v_0, \dots, v_{i-1}, x, v_i, \dots, v_{n-1}),$$
$$E_j(\phi(v_0, \dots, v_n)) = ((v_j = v_{j+1}) \to \phi(v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{n+1})).$$

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Logical equivalence between L-formulae is considered here as an equivalence relation always restricted to some F_n, in other words as a *graded* logical equivalence. On this condition only, the operators $\exists_i$, E_j and ∂ can be defined up to logical equivalence.[3] Then, up to graded logical equivalence, the maps $\exists_i$ and E_j satisfy the above simplicial identities.

DEFINITION 1.3. To ensure that these simplicial identities be genuine equalities, every L-formula will henceforth be considered up to graded logical equivalence.

THEOREM 1.4. *$F_* = \langle F_n, (\exists_i)_{0 \leq i \leq n}, (E_j)_{0 \leq j \leq n} \rangle_{n \in \mathbb{N}}$ is a simplicial set.*

In this perspective, the existential quantifier can be likened to a "face operator," while the maps E_j are the corresponding "degeneracy operators." In fact, if any L-formula in F_n is seen as the presentation of some n-dimensional subset, $\exists_i$ is exactly a projection, and E_j the converse operation of cylindrification.

REMARK 1.5. In the definition of $\exists_i$, the existential quantifier can be replaced with any first-order unary quantifier Q satisfying $QyQx\phi(y, x, \vec{u}) \equiv QxQy\phi(x, y, \vec{u})$ and $Qx((x = y) \to \phi[x/y]) \equiv \phi(y)$, for every L-formula ϕ. The associated face operators are written d_i^Q and the ensuing simplicial set is written F_*^Q (so $\exists_i = d_i^\exists$ and $F_* = F_*^\exists$).

Given two simplicial sets X and Y, a *simplicial map* $f : X \to Y$ consists of a system of maps $f_n : X_n \to Y_n$ ($n \in \mathbb{N}$) such that both diagrams below

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle d_i^X}\downarrow & & \downarrow{\scriptstyle d_i^Y} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array} \qquad \begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle s_j^X}\downarrow & & \downarrow{\scriptstyle s_j^Y} \\ X_{n+1} & \xrightarrow[f_{n+1}]{} & Y_{n+1} \end{array}$$

commute for every i and every j. The *category* **sSets** *of simplicial sets* is the category whose objects are all simplicial sets and whose arrows are all simplicial maps. Within that category, certain simplicial sets stand out: the *fibrant* ones. Given two n-simplices $x, y \in X_n$ of a simplicial set X, a *shared face* is an $(n-1)$-simplex of X which is a common face of x and y. An *n-horn in X* is a system of n distinct $(n-1)$-simplices of X with a maximal number of $n(n-1)/2$ shared faces. For instance, a 2-horn in X can be represented as a pair of two edges with a shared vertex, in other words as a triangle with one side removed; a 3-horn in X, as a triple of full triangles making up the surface of a tetrahedron with one face removed. A simplicial set X is said to be *fibrant* if, for every $n \geq 1$, any n-horn in X can be extended into an n-simplex of X. This amounts to saying, in combinatorial terms, that if $x_0, \dots, \widehat{x_k}, \dots, x_n$ is an n-tuple of $(n-1)$-simplices of X ('$\widehat{x_k}$' meaning that x_k is omitted) such that $d_i x_j = d_{j-1} x_i$

[3] For example, the two logically equivalent L-formulae $\chi(v_0) := (P(v_0) \vee \neg P(v_0))$ and $\pi(v_0, v_1) := ((P(v_0) \vee \neg P(v_0)) \wedge (P(v_1) \vee \neg P(v_1)))$ give rise to L-formulae $\partial\chi$ and $\partial\pi$ which are not logically equivalent. But $\chi \in F_0$, whereas $\pi \in F_1$, so that χ and π cannot be said to be logically equivalent in the sense meant here.

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for any $i, j \neq k$ with $i < j$, then there is an n-simplex x of X such that $d_i x = x_i$ for any $i \neq k$.

PROPOSITION 1.6. *F_* is fibrant.*

PROOF. The proof derives as a special case from the proof of Theorem 2.10. ⊣

1.2. Syntactic extensions. Following on from this first comparison, it is quite natural to ask about connectives. It turns out that $\wedge$ is characterized, among all symmetric binary connectives, by the condition that $\forall x(\phi x \wedge \psi x) \equiv (\forall x \phi x \wedge \forall x \psi x)$, and $\vee$ in an analogous way. Because binary connectives are at stake, two-dimensional simplicial sets need to be introduced, which is straightforward as soon as it is noticed that a simplicial set can be equivalently defined as a functor $X : \mathbf{\Delta^{op}} \to \mathbf{Sets}$, where $\mathbf{\Delta}$ is the category whose objects are all the finite ordinals and whose morphisms are all order-preserving maps, and where **Sets** is the category of sets and maps between sets. Following this alternative view, one can define a *bisimplicial set* as a functor from $\mathbf{\Delta^{op}} \times \mathbf{\Delta^{op}}$ to **Sets**.

The usual unary and binary connectives can then be described in simplicial and bisimplicial terms. Indeed, the commutative diagram

$$\begin{CD} F_m \times F_p @>{c}>> F_n \\ @V{\langle d_i^{Q}, d_i^{Q'} \rangle}VV @VV{d_i^{Q''}}V \\ F_{m-1} \times F_{p-1} @>>{c'}> F_{n-1} \end{CD}$$

is a way to express that $Q''x(\phi c \psi) \equiv (Q v_i \phi) c' (Q' v_i \psi)$ for any L-formulae $\phi \in F_m$, $\psi \in F_p$ (here $n = max(m, p)$). One has that $\neg$, $\wedge$ and $\vee$ are characterized, respectively, by the commutativity of the three following diagrams:

$$\begin{CD} F_n @>{\neg}>> F_n \\ @V{d_i^{\forall}}VV @VV{\exists_i}V \\ F_{n-1} @>>{\neg}> F_{n-1} \end{CD} \qquad \begin{CD} F_m \times F_p @>{\wedge}>> F_n \\ @V{\langle d_i^{\forall}, d_i^{\forall} \rangle}VV @VV{d_i^{\forall}}V \\ F_{m-1} \times F_{p-1} @>>{\wedge}> F_{n-1} \end{CD} \qquad \begin{CD} F_m \times F_p @>{\vee}>> F_n \\ @V{\langle \exists_i, \exists_i \rangle}VV @VV{\exists_i}V \\ F_{m-1} \times F_{p-1} @>>{\vee}> F_{n-1} \end{CD} \quad .$$

There might seem to be a small problem with conjunction of L-formulae that share the same variables. For instance, how to regiment the L-formula $(R(v_0, v_1) \wedge S(v_0, v_2))$? The L-formula has in fact to forego some transformation, but within its logical equivalence class (although not in the graded sense):

$$\begin{aligned} &(R(v_0, v_1) \wedge S(v_0, v_2)) \\ &\equiv (R(v_0, v_1) \wedge S(v_2, v_3) \wedge (v_0 = v_2)) \\ &\equiv ((R(v_0, v_1) \wedge (v_0 = v_2)) \wedge ((v_0 = v_2) \wedge (v_1 = v_1) \wedge S(v_2, v_3))). \end{aligned}$$

Besides bisimplicial sets, the definition of a simplicial set as a functor $X : \mathbf{\Delta^{op}} \to \mathbf{Sets}$ lends itself to another kind of generalization: a *simplicial object* in an Abelian category $\mathbf{C}$ is a functor $X : \mathbf{\Delta^{op}} \to \mathbf{C}$. A simplicial object in the category of Abelian groups is called a *simplicial group*. Any simplicial object X in any

Abelian category gives rise, in a canonical way, to an associated *chain complex*:[4]
$\dots X_{n+1} \xrightarrow{d^{n+1}} X_n \xrightarrow{d^n} X_{n-1} \dots$, where $d^n := \sum_{i=0}^n (-1)^i d_i$ and $d^n \circ d^{n+1} = 0$. Each F_* turns out to be a simplicial Boolean algebra, with sum and product corresponding to disjunction and conjunction. The category of Boolean algebras, however, is not an Abelian category. An option is to consider each F_n as naturally equipped with a group structure, namely $\langle F_n, \nleftrightarrow, \bot \rangle$, where $\bot$ is any antilogy in F_n and $(\phi \nleftrightarrow \psi) := ((\phi \wedge \neg\psi) \vee (\psi \wedge \neg\phi))$. However, F_*'s being a simplicial group requires more than all the F_n's being groups: it also requires all the $\exists_i$'s and E_j's being morphisms of Abelian groups.

Another, natural option is thus to introduce the free Abelian group $\mathbb{Z}F_n$ generated by the n-simplices, up to the identification $(\phi + \phi) = \phi$. The operators $\exists_i : \mathbb{Z}F_n \to \mathbb{Z}F_{n-1}$ are given by linear extension. The elements of $\mathbb{Z}F_n$ can be seen as *sequents* between L-formulae with $n+1$ free variables, up to the equivalences provided by the structural rules of the sequent calculus for first-order classical logic. Indeed, any element of $\mathbb{Z}F_n$ can be written as follows:

$$\sum_{k=1}^{n} n_k \phi_k(v_0, \dots, v_n) = \sum_{n_k<0} n_k \phi_k \vdash \sum_{n_l>0} n_l \phi_l$$
$$= \phi_{k_1}, \dots, \phi_{k_p} \vdash \phi_{l_1}, \dots, \phi_{l_q}.$$

Any sequent of the form $\phi(v_0, \dots, v_n) \vdash \phi(v_0, \dots, v_n)$ represents the identity element 0 of $\mathbb{Z}F_n$.

Definition 1.7. Given

$$\widetilde{\exists}_i(\phi_1, \dots, \phi_p \vdash \psi_1, \dots, \psi_q) := \exists_i(\phi_1), \dots, \exists_i(\phi_p) \vdash \exists_i(\psi_1), \dots, \exists_i(\psi_q),$$

the n *-th boundary map* associated to $\exists$ is: $\partial_n^{\exists} = \sum_{i=0}^n (-1)^i \widetilde{\exists}_i : \mathbb{Z}F_n \to \mathbb{Z}F_{n-1}$.

Proposition 1.8. *The sequence* $\langle \mathbb{Z}F_n, \partial_n \rangle_{n\in\mathbb{N}}$ *defines a chain complex*:

$$\partial_{n-1}^{\exists} \circ \partial_n^{\exists} = 0 \text{ for all } n \geq 1.$$

Proof. Let S be:

$$\phi_1(v_0, \dots, v_n), \dots, \phi_p(v_0, \dots, v_n) \vdash \psi_1(v_0, \dots, v_n), \dots, \psi_q(v_0, \dots, v_n).$$

Writing ϕ for $\sum_{1\leq k\leq p} \phi_k$ and ψ for $\sum_{1\leq l\leq q} \psi_l$, one has, for n even:

$$\begin{aligned}
&\partial_n^{\exists}(S)\\
&= (\exists x\phi(x, v_0, \dots, v_{n-1}) \vdash \exists x\psi(x, v_0, \dots, v_{n-1}))\\
&- (\exists x\phi(v_0, x, v_1, \dots, v_{n-1}) \vdash \exists x\psi(v_0, x, v_1, \dots, v_{n-1})) + \cdots\\
&\cdots + (\exists x\phi(v_0, \dots, v_{n-1}, x) \vdash \exists x\psi(v_0, \dots, v_{n-1}, x))\\
&= (\exists x\phi(x, v_0, \dots, v_{n-1}), \exists x\psi(v_0, x, v_1, \dots, v_{n-1}), \exists x\phi(v_0, v_1, x, \dots, v_{n-1}),
\end{aligned}$$

[4]This is the "Dold–Kan correspondence." See [12], pp. 265–266 and pp. 270–271.

$\ldots, \exists x\phi(v_0, \ldots, v_{n-1}, x)) \vdash (\exists x\psi(x, v_0, \ldots, v_{n-1}), \exists x\phi(v_0, x, v_1, \ldots, v_{n-1}),$
$\exists x\psi(v_0, v_1, x, \ldots, v_{n-1}), \ldots, \exists x\psi(v_0, \ldots, v_{n-1}, x)).$

Then, writing '$\phi_{01x2\ldots n}$' as a shorthand for $\exists x\phi(v_0, v_1, x, v_2, \ldots, v_n)$, one can see that each $\widetilde{\exists}_i$, for $0 \leq i \leq n-1$, will produce, if i is even (resp. odd), $\phi_{x0\ldots i-1yi\ldots n-2}$, $\psi_{0x1\ldots i-1yi\ldots n-2}$, …, $\phi_{0\ldots i-1yi\ldots x}$ on the left side (resp. right side) of the sequent and $\psi_{x0\ldots \underbrace{i-1y}i\ldots n-2}$, $\psi_{0x1\ldots i-1yi\ldots n-2}$, …, $\psi_{0\ldots i-1yi\ldots x}$ on the right side (resp. left side). So $\widetilde{\exists}_i$ and $\widetilde{\exists}_{i+1}$, put together, will produce the same sequent. Since n is supposed to be even, the n indices i between 0 and $n-1$ can all be considered pairwise, and so in the end the right and left sides of $\partial^{\exists}_{n-1}(\partial^{\exists}_n(S))$ match up. The proof in case n is odd is analogous. ⊣

Definition 1.9. For each $n \in \mathbb{N}$, $Z_n(\exists) := \mathrm{Ker}\ \partial^{\exists}_n$ is the set of all n *-cycles* and $B_n(\exists) := \mathrm{Im}\ \partial^{\exists}_{n+1}$ is the set of all n *-boundaries* associated to $\exists$. The quotient $H_n(\exists) := Z_n(\exists)/B_n(\exists)$ is the n *-th homology group* associated to $\exists$.

The homology groups $(H_n(\exists))_{n\in\mathbb{N}}$ represent the simplicial homology of first-order logic.

§2. Simplicial sets and structures.

2.1. Elementary extensions.

Definition 2.1. For any L-structure M,

$$F^M_* := \langle D_n(M), (\exists^M_i)_{0\leq i\leq n}, (E^M_j)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}, \text{ where:}$$

- $D_n(M)$ is the set of all definable subsets of $|M|^{n+1}$ with parameters;
- $\exists^M_i : D_n(M) \to D_{n-1}(M))$, $\quad A = \{\vec{a} \in |M|^{n+1} : M \vDash \phi(v_0, \ldots, v_n)[\vec{a}]\} \mapsto \{\vec{a'} \in |M|^n : M \vDash \exists x\phi(v_0, \ldots, v_{i-1}, x, v_i, \ldots, v_{n-1})[\vec{a'}]\}$;
- $E^M_j : D_n(M) \to D_{n+1}(M)$, $A \mapsto \{(\vec{x}, y) : \vec{x} \in A \text{ and } y = x_j\}$.

Proposition 2.2. *For any L-structure M, F^M_* is a simplicial set.*

Proof. The simplicial identities between the operators $\exists^M_i$ and E^M_j are the semantic counterpart of the simplicial identities which turn F_* into a simplicial set. ⊣

Definition 2.3. A *simplicial set with distinguished vertices* is a simplicial set X equipped with a selection $X'_0 \subseteq X_0$ of vertices, called *distinguished vertices*.

A *morphism* $f : (X, X'_0) \to (Y, Y'_0)$ *between simplicial sets with distinguished vertices*, or *simplicial morphism* for short, is a simplicial map $f : X \to Y$ such that $f^{-1}(Y'_0) = X'_0$.

Definition 2.4. Given any L-structure M, F^M_* is canonically equipped with a set M'_0 of distinguished vertices, namely the set of all nonempty definable subsets of $|M|$ without parameters. The corresponding simplicial set with distinguished vertices is written M_*.

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THEOREM 2.5. *Let M be a substructure of an L-structure N. The restriction r_n, for each $n \in \mathbb{N}$, being the map which sends ϕ^N to $\phi^N \cap |M|^{n+1}$, for each L-formula $\phi \in F_n$, M is an elementary substructure of N iff the family $(r_n)_{n\in\mathbb{N}}$ of restriction maps induces a simplicial morphism $r_* : N_* \to M_*$.*

PROOF. Let N be an extension of M. Then $r_n : D_n(N) \to N_n^M$, where N_n^M is the set of all $\phi(N, M) := \{\vec{a} \in |M|^{n+1} : N \vDash \phi(v_0, \dots, v_n)[\vec{a}]\}$ for $\phi \in F_n$. Furthermore, the sequence $(N_n^M)_{n\in\mathbb{N}}$ can be endowed with a simplicial structure N_*^M, with $\exists_i^{N^M} : \{\vec{b} \in |M|^{n+1} : N \vDash \phi(\vec{v})[\vec{b}]\} \mapsto \{\vec{b'} \in |M|^n : N \vDash \exists_i(\phi)[\vec{b'}]\}$ and $E_j^{N^M} : \{\vec{b} \in |M|^n : N \vDash \phi(\vec{v})[\vec{b}]\} \mapsto \{(\vec{b}, b_j) \in |M|^{n+1} : N \vDash \phi(\vec{v})[\vec{b}]\}$, for each $\phi \in F_n$.

Now, suppose that r_* is a simplicial morphism:

$$\begin{array}{ccc} D_n(N) & \xrightarrow{r_n} & N_n^M \\ \Big\downarrow{\scriptstyle \exists_i^N} & & \Big\downarrow{\scriptstyle \exists_i^{N^M}} \\ D_{n-1}(N) & \xrightarrow[r_{n-1}]{} & N_{n-1}^M \end{array}$$

Then it turns out that N_*^M is in fact M_*. This point is proved by induction on ϕ. Since M is a substructure of N, $\phi(N, M) = \phi^M$ for each negatomic L-formula $\phi \in F_n$. It is straightforward, by induction, that $(\phi \wedge \psi)(N, M) = (\phi(N, M) \wedge \psi(N, M)) = (\phi^M \wedge \psi^M) = (\phi \wedge \psi)^M$. Finally:

$$\begin{aligned} (\exists x \phi(v_0, \dots, v_{i-1}, x, v_i, \dots, v_n))(N, M) &= r_n(\exists_i^N(\phi(v_0, \dots, v_{n+1}))) \\ &= \exists_i^{N^M}(r_{n+1}(\phi(v_0, \dots, v_{n+1}))). \end{aligned}$$

By induction hypothesis, $r_{n+1}(\phi) \in D_{n+1}(M)$, so $\exists_i^{N^M}(r_{n+1}(\phi)) = (\exists x \phi)(N, M)$ is in $D_n(M)$. Now, the fact that the restriction maps r_n make up a simplicial morphism from N_* to M_* is the direct expression that the Tarski–Vaught test is met. Indeed, suppose $N \vDash \exists x \phi(x, \overline{a_1}, \dots, \overline{a_n})$. Then $(\phi(v_0, \overline{a_1}, \dots, \overline{a_n}))^N \in N_0'$, so $r_0(\phi(v_0, \overline{a_1}, \dots, \overline{a_n})^N) = \phi(v_0, \overline{a_1}, \dots, \overline{a_n})(N, M) \in M_0'$ is nonempty, which means that there is some $a \in |M|$ such that $N \vDash \phi(v_0, \overline{a_1}, \dots, \overline{a_n})[a]$. As a result, M is an elementary substructure of N.

Conversely, suppose that M is an elementary substructure of N. Then, it is routine verification to check that r_* is a simplicial map, with codomain M_*, and thus a simplicial morphism $r : N_* \to M_*$. The elementwise commutativity of the diagram involving r_n and r_{n-1} follows from the Tarski–Vaught test. Indeed, let us consider $\phi \in F_n$ and let us suppose that $(a_1, \dots, a_n) \in |M|^n$. Then:

$N \vDash \exists x \phi(\overline{a_1}, \dots, \overline{a_i}, x, \overline{a_{i+1}}, \dots, \overline{a_n})$
iff there exists $a \in |M|$ such that $N \vDash \phi(\overline{a_1}, \dots, \overline{a_i}, v_0, \overline{a_{i+1}}, \dots, \overline{a_n})[a]$
iff there exists $a \in |M|$ such that $(a_1, \dots, a_i, a, a_{i+1}, \dots, a_n) \in \phi^N \cap |M|^{n+1}$
iff $(a_1, \dots, a_n) \in \exists_i(\phi^N \cap |M|^{n+1})$.

Thus $\exists_i(\phi^N) \cap |M|^n = \exists_i(\phi^N \cap |M|^{n+1})$ for every L-formula $\phi \in F_n$. ⊣

COROLLARY 2.6. *The mapping $(-)_*$ defines a contravariant functor from the category of L-structures and elementary embeddings, to the category of simplicial sets with distinguished vertices and simplicial morphisms.*

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PROOF. Any elementary embedding $f : M \to N$ gives rise to a simplicial morphism $f_* : N_* \to M_*$, in a functorial way, with $f_n : \phi^N \mapsto \{\vec{a} \in |M|^{n+1} : N \vDash \phi[f(\vec{a})]\} = \phi^M$. ⊣

Note that any two elementary equivalent L-structures M and N have isomorphic hierarchies of definable subsets without parameters, but that the corresponding isomorphism $\phi^M \mapsto \phi^N$ between M_* and N_* is not induced in general by any actual map between M and N.

Let M be an elementary substructure of an L-structure N, and let us suppose that $|M|$ is definable in N, by an L-formula $\underline{M}(v_0)$. For any $\phi \in F_n$, the L-formula $\phi^{\underline{M}}$ is defined as $(\underline{M}(v_0) \wedge \cdots \wedge \underline{M}(v_n) \wedge \phi^{(\underline{M})})$, where $\phi^{(\underline{M})}$ is the quantifier relativization of ϕ to $\underline{M}$. This relativization allows one to define, for each $n \in \mathbb{N}$, an extension map $e_n : \phi^M \in D_n(M) \mapsto (\phi^{\underline{M}})^N \in D_n(N)$, the existence of which, as will be seen presently, has a natural simplicial counterpart, if one moves to M_* and N_*. Given two simplicial sets X and Y (resp. two simplicial sets X and Y with distinguished vertices), a *retraction of Y over X* consists of a pair $\langle f, g \rangle$ of simplicial maps (resp. of simplicial morphisms) $f : X \to Y$ and $g : Y \to X$, such that $g \circ f = \mathrm{id}_X$.

THEOREM 2.7. *Let M be an elementary substructure of an L-structure N. Then the domain $|M|$ of M is definable in N iff the family $(e_n)_{n \in \mathbb{N}}$ of extension maps induces a simplicial morphism $e_* : M_* \to N_*$ and $\langle e_*, r_* \rangle$ is a retraction of N_* over M_*.*

PROOF. Suppose $|M|$ is definable in N by some L-formula $\underline{M}(v_0)$. Since $N \vDash \phi[\vec{b}]$ iff $\vec{b} \in |M|^{n+1}$ and $M \vDash \phi[\vec{b}]$, for any $\phi \in F_n$, the diagram

$$\begin{array}{ccc}
\phi^M \in D_n(M) & \xrightarrow{\quad e_n \quad} & (\phi^{\underline{M}})^N \\
\Big\downarrow \exists_i^M & & \Big\downarrow \exists_i^N \\
\{\vec{a}' \in |M|^n : M \vDash \exists_i(\phi)[\vec{a}']\} & \xrightarrow[e_{n-1}]{} & \{\vec{b}' \in |N|^n : N \vDash (\exists_i(\phi))^{\underline{M}}[\vec{b}']\}
\end{array}$$

commutes. Indeed, using the Tarski–Vaught test, one has:

$$\begin{aligned}
N \vDash \exists_i(\phi^{\underline{M}})[\vec{b}'] &\text{ iff } N \vDash \phi^{\underline{M}}[b'_0, \ldots, b'_{i-1}, c_i, b'_i, \ldots, b'_{n-1}] \text{ for some } c_i \in |N| \\
&\text{ iff } N \vDash \phi^{\underline{M}}[b'_0, \ldots, b'_{i-1}, d_i, b'_i, \ldots, b'_{n-1}] \text{ for some } d_i \in |M| \\
&\text{ iff } N \vDash \exists v_i \ (\underline{M}(v_i) \wedge \phi^{\underline{M}})[\vec{b}'] \\
&\text{ iff } N \vDash (\exists_i(\phi))^{\underline{M}}[\vec{b}'].
\end{aligned}$$

That is to say, the collection of maps $(e_n : D_n(M) \to D_n(N))_{n \in \mathbb{N}}$ defines a simplicial map. Now, one easily checks that $(e_n r_n)(\phi^N) = \phi^M$ and that $(r_n e_n)(\phi^M) = \phi^M$ for every $\phi \in F_n$:

$$\begin{aligned}
(e_n r_n)(\phi^N) &= e_n(\{\vec{a} \in |M|^{n+1} : N \vDash \phi[\vec{a}]\}) = e_n(\phi^M) = (\phi^{\underline{M}})^N \\
&= \{\vec{b} \in |N|^{n+1} : N \vDash \phi^{\underline{M}}[\vec{b}]\} = \{\vec{a} \in |M|^{n+1} : N \vDash \phi^{\underline{M}}[\vec{a}]\} = \phi^M, \\
(r_n e_n)(\phi^M) &= r_n((\phi^{\underline{M}})^N) = \{\vec{a} \in |M|^{n+1} : N \vDash \phi^{\underline{M}}[\vec{a}]\} = \phi^M.
\end{aligned}$$

Consequently, $r_* : N_* \to M_*$ defines a retraction of N_* over M_*. Conversely, the very condition $r_* e_* = \mathrm{id}_{M_*}$ supposes that e_* is well defined, and thus that M is a definable elementary substructure of N. As a matter of fact, there must be some L-formula $\chi(v_0)$ such that $e_0(|M|) = e_0((v_0 = v_0)^M) = \chi(v_0)^N$. ⊣

The existence of a retraction $r_* : N_* \to M_*$ implies in particular that M_* and N_* are homotopy equivalent.[5] The emerging analogy is thus between elementary equivalence and homotopy equivalence: as two L-structures are elementary equivalent iff they have isomorphic ultrapowers, two topological spaces are homotopy equivalent iff they are deformation retracts of a single (up to isomorphism) larger space, namely the "mapping cylinder" of their homotopy.

2.2. Types. Another application of the simplicial framework deserves attention, namely types. Given a complete theory T in L, an *n-type* is a complete theory in $L \cup \{v_0, \dots, v_{n-1}\}$ which is consistent with T and whose sentences are considered up to graded logical equivalence. The set of all n-types is written $S_{n-1}(T)$. If T is the theory of some L-structure M and $A \subseteq |M|$, the set of all *n-types over A* (n-types in the language $L(A)$) is written $S_n(A)$.

Definition 2.8. The natural extensions of the maps $\exists_i : F_n \to F_{n-1}$ and $E_j : F_n \to F_{n+1}$ to types as sets of L-formulae are also written $\exists_i : S_n(T) \to S_{n-1}(T)$ and $E_j : S_n(T) \to S_{n+1}(T)$.

Proposition 2.9. *The structure* $S_*(T) = \langle S_n(T), (\exists_i)_{0 \le i \le n}, (E_j)_{0 \le j \le n} \rangle_{n \in \mathbb{N}}$ *is a simplicial set.*

Proof. Given any type $q \in S_n(T)$, there is a type $p \in S_{n+1}(T)$ such that $p \supseteq \{E_i(\psi) : \psi \in q\}$. Indeed, since every finite subset of q is consistent and realized in some model of T, this is obviously the case of $\{E_i(\psi) : \psi \in q\}$ too. But then $\exists_i p \supseteq \{(\exists_i E_i)(\psi) : \psi \in q\} = q$. Since q is supposed to be complete, it follows that $\exists_i p = q$. (However, all the L-formulae in q are not in an existentially quantified form: they are only so up to graded logical equivalence —see Definition 1.3.) As to the simplicial identities, they are verified by straightforward extension from L-formulae to types. ⊣

Theorem 2.10. $S_*(T)$ *is fibrant.*

Proof. In combinatorial terms, one has to show, given n simplices $p_0, \dots, \widehat{p_k}, \dots, p_n \in S_{n-1}(T)$ such that $\exists_i p_j = \exists_{j-1} p_i$ for all $i, j \neq k$ with $i < j$, the existence of $p \in S_n(T)$ such that $\exists_i p = p_i$ for all $i \neq k$. But the complete $(n+1)$-type p generated by the family $\{(v_j = v_j \wedge \phi^j(v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n)) : \phi^j \in p_j,\ j \neq k\}$ matches that condition. A preliminary remark is in order: the hypothesis $\exists_i p_j = \exists_{j-1} p_i$ (for $i < j$ and $i, j \neq k$) means that, for any L-formula $\phi^j(v_0, \dots, v_{n-1}) \in p_j$, there exists an L-formula $\psi^i_\phi(v_0, \dots, v_{n-1}) \in p_i$ such that $\exists x \phi^j(v_0, \dots, v_{i-1}, x, v_i, \dots, v_{n-2})$ is logically equivalent (in the graded sense) with $\exists x \psi^i_\phi(v_0, \dots, v_{j-2}, x, v_{j-1}, \dots, v_{n-2})$. Renaming '$v_{j-1}$', ..., '$v_{n-2}$' is immaterial, so actually, for any variable symbols 'u_1', ..., 'u_{n-1}':

[5] Two simplicial sets X and Y are said to be *homotopy equivalent* if there are simplicial maps $f : X \to Y$ and $g : Y \to X$ which are homotopy inverse to each other, in the sense defined in Section 3.2.

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$$\exists x\phi^j(u_1,\dots,u_i,x,u_{i+1},\dots,u_{j-1},\dots,u_{n-1})$$
$$= \exists x\psi^i_\phi(u_1,\dots,u_{j-1},x,u_j,\dots,u_{n-1}).$$

In the same way, for any L-formula $\phi^i \in p_i$, there exists an L-formula $\chi^j_\phi \in p_j$ such that:

$$\exists x\phi^i(u_1,\dots,u_{j-1},x,u_j,\dots,u_{n-1})$$
$$= \exists x\chi^j_\phi(u_1,\dots,u_i,x,u_{i+1},\dots,u_{j-1},\dots,u_{n-1}).$$

As a consequence, for any $\phi^j \in p_j$ with $j > 0$ and $j \neq k$:

$$\begin{aligned}
&\exists_0((v_j = v_j \wedge \phi^j(v_0,\dots,v_{j-1},v_{j+1},\dots,v_n)))\\
&= (v_{j-1} = v_{j-1} \wedge \exists x\phi^j(x,v_0,\dots,v_{j-2},v_j,\dots,v_{n-1}))\\
&= (v_{j-1} = v_{j-1} \wedge (\exists x\phi^j)[v_j/v_{j-1},\dots,v_{n-1}/v_{n-2}])\\
&= (v_{j-1} = v_{j-1} \wedge (\exists x\psi^0_\phi)[v_j/v_{j-1},\dots,v_{n-1}/v_{n-2}])\\
&= (v_{j-1} = v_{j-1} \wedge \exists x\psi^0_\phi(v_0,\dots,v_{j-2},x,v_j,\dots,v_{n-1})).
\end{aligned}$$

But the latter L-formula is obviously a member of p_0. As to the case $j = 0$ (supposing $k \neq 0$), $\exists_0((v_0 = v_0 \wedge \phi^0(v_1,\dots,v_n))) = \phi^0(v_0,\dots,v_{n-1}) \in p_0$. Gathering both cases ($j > 0$ and $j = 0$), one has that $\exists_0 p = p_0$ (for $k \neq 0$).

Let us prove now that $\exists_n p = p_n$ (for $k \neq n$). As above, for any $\phi^i \in p_i$ with $i < n$ and $i \neq k$:

$$\begin{aligned}
&\exists_n((v_i = v_i \wedge \phi^i(v_0,\dots,v_{i-1},v_{i+1},\dots,v_n)))\\
&= (v_i = v_i \wedge \exists x\phi^i(v_0,\dots,v_{i-1},v_{i+1},\dots,v_{n-1},x))\\
&= (v_i = v_i \wedge (\exists_{n-1}\phi^i)[v_{i+1}/v_i,\dots,v_{n-1}/v_{n-2}])\\
&= (v_i = v_i \wedge (\exists_i\chi^n_\phi)[v_{i+1}/v_i,\dots,v_{n-1}/v_{n-2}])\\
&= (v_i = v_i \wedge \exists x\chi^n_\phi(v_0,\dots,v_{i-1},x,v_{i+1},\dots,v_{n-1})) \in p_n.
\end{aligned}$$

As to the case $i = n$, again $\exists_n((v_n = v_n \wedge \phi^n(v_0,\dots,v_{n-1}))) = \phi^n(v_0,\dots,v_{n-1}) \in p_n$. So $\exists_n p = p_n$ (for $k \neq n$). The proof of the other identities $\exists_j p = p_j$, for all j such that $0 < j < n$ and $j \neq k$, is analogous. ⊣

Given a simplicial set X and $x, y \in X_0$, x is said to be *homotopic* to y, written $x \sim y$, iff there exists a one-simplex $z \in X_1$ such that $d_1 z = x$ and $d_0 z = y$. The one-simplex is said to be a *path* from x to y.

Corollary 2.11. (See [6], Lemmas 3.4.2, 3.4.3 and 3.6.3) *The homotopy relation between 1-types in $S_0(T)$ is an equivalence relation. The quotient $S_0(T)/\sim$ is the set of path components of $S_*(T)$. For $p \in S_0(T)$, the higher homotopy groups $\pi_n(S_*(T), p)$ can be defined as well.*[6]

[6]See [3] (pp. 25–27) and [6] (Lemma 3.4.5, p. 85) for the definition of simplicial homotopy groups and the proof of their being groups.

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Up to now, a simplicial set has been associated to a whole space of types $\langle S_n(T) : n \in \mathbb{N}\rangle$. A single type, however, also induces a simplicial set, provided it has the following property: given some L-structure M and $A \subseteq |M|$, $p \in S_n(A)$ is *definable* iff, for each L-formula $\phi(v_0, \dots, v_n, v_{n+1}, \dots, v_{n+m})$, there is an L$(A)$-formula $(\delta_p\phi)(v_0, \dots, v_{m-1})$ such that, for any tuple $(a_0, \dots, a_{m-1})$ of elements of $|M|$, $\phi(v_0, \dots, v_n, \overline{a_0}, \dots, \overline{a_{m-1}}) \in p$ iff $M \vDash (\delta_p\phi)(v_0, \dots, v_{m-1})[a_0, \dots, a_{m-1}]$. The operator δ^p is called the *definition of* p.

Proposition 2.12. *Given an L-structure M, any definable type $p \in S_0(M)$ induces a fibrant simplicial set.*

Proof. Let $p \in S_0(M)$ be definable: for each L-formula $\phi(v_0, \dots, v_n)$, there is an L-formula $d_i^{p,n}(\phi)(v_0, \dots, v_{n-1})$, possibly with parameters in M, such that, for any $(a_0, \dots, a_{n-1}) \in |M|^n$, $\phi(\overline{a_0}, \dots, \overline{a_{i-1}}, v_0, \overline{a_i}, \dots, \overline{a_{n-1}}) \in p$ iff $M \vDash d_i^{p,n}(\phi)[a_0, \dots, a_{n-1}]$. One has: $d_i^{p,n}(\neg\phi) = \neg d_i^{p,n}(\phi)$ and $d_i^{p,n}(\phi \wedge \psi) = d_i^{p,n}(\phi) \wedge d_i^{p,n}(\psi)$. Besides, each set F_n can be turned into a group, with $\nleftrightarrow$ as the binary law, so each $d_i^{p,n}$ is actually a group homomorphism. Hence $F_*^p := \langle\langle F_n, \nleftrightarrow, \bot\rangle, (d_i^{p,n})_{0\leq i\leq n}, (s_j^n)_{0\leq j\leq n}\rangle_{n\in\mathbb{N}}$ is a simplicial group. Since, according to Moore's theorem,[7] the underlying simplicial set of a simplicial group is fibrant, one concludes that F_*^p is fibrant. ⊣

Proposition 2.13. *Any definable type $p \in S_0(M)$ induces a chain complex.*

Proof. From $\partial^{p,n}(\phi) := (((d_0^{p,n}(\phi) \nleftrightarrow d_1^{p,n}(\phi)) \nleftrightarrow \cdots) \nleftrightarrow d_n^{p,n}(\phi))$ (for each $n \in \mathbb{N}$ and each $\phi \in F_n$), one gets a differentiation operator $\partial^{p,*}$: $\partial^{p,n+1} \circ \partial^{p,n} \equiv \bot$ for every $n \in \mathbb{N}$. In other words, $\langle F_*^p, \partial^{p,*}\rangle$ is a chain complex. ⊣

2.3. Another functorial correspondence. Let $\mathrm{L} = \{R_k^{(n)} : n, k \in \mathbb{N}\} \cup \{<\}$ be a fixed countable first-order language, whose signature contains a distinguished binary relation '$<$'. As above, "definability" will mean, unless otherwise stated, definability with parameters. Moreover, all types in $S_n(M)$, for each $n \in \mathbb{N}$, will be allowed to consist of L(M)-formulae. The simplicial F_*^M associated to any L-structure M (Definition 2.1) does not encode enough of M. In order to improve the functorial correspondence between L-structures and simplicial sets, the simplicial counterpart of the definable subsets of M should be able to mention which of the latter are nonempty. Hence the next definition.

Definition 2.14. A *simplicial set with distinguished simplices*, or *d-simplicial set* for short, is a simplicial set X with a subset $X_n' \subseteq X_n$ of distinguished n-simplices for each $n \in \mathbb{N}$.

A *d-simplicial map* $f : X \to Y$ is a simplicial map $f : X \to Y$ such that, for each $n \in \mathbb{N}$, f_n preserves distinguished n-simplices, i.e., $f_n(X_n') \subseteq Y_n'$.

Lemma 2.15. *Any L-structure M induces a d-simplicial set M^*.*

[7] See [3], Lemma I. 3. 4, p. 12.

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PROOF. Let us define a *complex* of M as a finite collection C of definable subsets of M (i.e., of definable subsets of Cartesian powers of $|M|$) which is closed under the face operator: for any definable subset $(\phi(v_0, \dots, v_n, \overline{a_1}, \dots, \overline{a_l}))^M \in D_n(M)$ (with parameters $a_1, \dots, a_l$), $\phi^M \in C$ implies both

- $\exists_i^M(\phi^M) = (\exists x \phi(v_0, \dots, v_{i-1}, x, v_i, \dots, v_{n-1}, \overline{a_1}, \dots, \overline{a_l}))^M \in C$ and
- $(\exists y \phi(v_0, \dots, v_n, \overline{a_1}, \dots, \overline{a_{j-1}}, y, \overline{a_{j+1}}, \dots, \overline{a_l}))^M \in C$,

for any i with $0 \leq i \leq n$ and any j with $1 \leq j \leq l$. A complex C of M is *of dimension* n if n is the largest integer k such that $D_k(M) \cap C \neq \emptyset$. Given a collection Γ of definable subsets of M, the *complex generated by* Γ, written Γ^c, is the intersection of all complexes of M containing all members of Γ. A complex generated by a single definable subset is called a *principal complex*. For each $n \in \mathbb{N}$, the set of n-simplices of M^* is defined as the set $K(M)_n$ of all complexes of M of dimension n. The maps $\exists_i^M : D_n(M) \to D_{n-1}(M)$ and $E_j^M : D_n(M) \to D_{n+1}(M)$, as defined earlier, are simply extended to collections of parametrically definable subsets. (Note that, given any definable subset ϕ^M of M, $\exists_i(\{\phi^M\}^c) = \{\exists_i(\phi^M)\}^c$ and $E_j(\{\phi^M\}^c) = \{E_j(\phi^M)\}^c$, so that F_*^M can be identified with a simplicial subset of M^*.) The distinguished n-simplices of M^* are the principal complexes of M generated by a nonempty subset of $|M|^{n+1}$ defined by an atomic formula without parameters. Their set is written $K(M)'_n$. ⊣

DEFINITION 2.16. Let M and N be two L-structures. For any homomorphism $f : M \to N$, ${}_nf : D_n(M) \to D_n(N)$, for each $n \in \mathbb{N}$, is the map sending each

$$(\phi(v_0, \dots, v_n, \overline{a_1}, \dots, \overline{a_k}))^M$$

to

$$(\phi(v_0, \dots, v_n, \overline{f(a_1)}, \dots, \overline{f(a_k)}))^N.$$

The extension $C \in K(M)_n \mapsto \{{}_nf(\phi^M) : \phi^M \in C\} \in K(N)_n$ of ${}_nf$ shall also be written ${}_nf$.

In case $f : M \to N$ is an elementary embedding, it is clear that each ${}_nf : D_n(M) \to D_n(N)$ will be injective. Now a simplicial map $f : X \to Y$ is a *cofibration* iff $f_n : X_n \to Y_n$ is an injection for all $n \in \mathbb{N}$. Hence the next lemma.

LEMMA 2.17. *Let M and N be two L-structures and $f : M \xrightarrow{\prec} N$ an elementary embedding. The family $({}_nf : K(M)_n \to K(N)_n)_{n \in \mathbb{N}}$ defines a d-simplicial map $f_! : M^* \to N^*$ which, as a simplicial map, is a cofibration.*

COROLLARY 2.18. *The mapping $M \mapsto M^*$, $f \mapsto f_!$ defines a functor F_0 from the category* **L-Strs** *of L-structures and elementary embeddings, to the category* **d-sSets** *of d-simplicial sets and d-simplicial maps which are cofibrations.*

DEFINITION 2.19. Let M and N be two L-structures and $f : M \xrightarrow{\prec} N$ an elementary embedding. For each $n \in \mathbb{N}$, the *restriction morphism* ${}^nf : D_n(N) \to D_n(M)$ is defined as follows: given any L-formula

$$\phi(v_0, \dots, v_n, \overline{f(a_1)}, \dots, \overline{f(a_k)}, \overline{b_1}, \dots, \overline{b_l})$$

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in F_n with parameters in N, where $b_1, \dots, b_l$ are parameters of ϕ (if any) not in the range of f, ${}^n f$ sends

$$(\phi(v_0, \dots, v_n, \overline{f(a_1)}, \dots, \overline{f(a_k)}, \overline{b_1}, \dots, \overline{b_l}))^N$$

to

$$(\exists x_1 \dots \exists x_l \phi(v_0, \dots, v_n, \overline{a_1}, \dots, \overline{a_k}, x_1, \dots, x_l))^M.$$

The extension $C \in K(N)_n \mapsto \{{}^n f(\phi^N) : \phi^N \in C\} \in K(M)_n$ of ${}^n f$ shall also be written ${}^n f$.

In case $f : M \to N$ is an elementary embedding, the family $({}^n f)_{n \in \mathbb{N}}$ contravariantly associated to f will be something like the opposite of a cofibration. The right notion here is that of "trivial fibration." A *trivial fibration* is a simplicial map $p : X \to Y$ such that: (i) $p_0 : X_0 \to Y_0$ is surjective; (ii) for every $n \geq 1$, given any $(n+1)$-tuple $x_0, \dots, x_n$ of $(n-1)$-simplices of X such that $d_i x_j = d_{j-1} x_i$ (for all i, j with $0 \leq i < j \leq n$) and any n-simplex y of Y such that $d_i y = p_{n-1}(x_i)$ (for all i with $0 \leq i \leq n$), there exists an n-simplex x of X such that $d_i x = x_i$ (for all i with $0 \leq i \leq n$) and $p_n(x) = y$.

LEMMA 2.20. *Let M and N be two L-structures and $f : M \xrightarrow{\prec} N$ an elementary embedding. The family $({}^n f : K(N)_n \to K(M)_n)_{n \in \mathbb{N}}$ defines a d-simplicial map $f^* : N^* \to M^*$ which, as a simplicial map, is a trivial fibration.*

PROOF. Since f is an elementary embedding, f^* obviously satisfies ${}^0 f(K_0(N)) \subseteq K_0(M)$, so it only remains to prove the second condition (ii) in the definition above. For the sake of simplicity, let us consider only the case $n = 1$ and let us suppose that the given simplices are principal complexes. The treatment of the general case is analogous, only more complicated. So, using 'α', 'β' and 'γ' to indicate arbitrary sequences of parameters, let $x_0 = \{(\phi_0(v_0, \overline{f(a_\alpha)}, \overline{b_\alpha}))^N\}^c$, $x_1 = \{(\phi_1(v_0, \overline{f(a_\beta)}, \overline{b_\beta}))^N\}^c$ and $y = \{(\psi(v_0, v_1, \overline{c_\gamma}))^M\}^c$ be such that:

- $M \models \exists x \phi_0(x, \overline{f(a_\alpha)}, \overline{b_\alpha}) \leftrightarrow \exists x \phi_1(x, \overline{f(a_\beta)}, \overline{b_\beta})$,
- $M \models \exists y \psi(y, v_0, \overline{c_\gamma}) \leftrightarrow \exists x_\alpha \phi_0(v_0, \overline{a_\alpha}, x_\alpha)$,
- $M \models \exists z \psi(v_0, z, \overline{c_\gamma}) \leftrightarrow \exists x_\beta \phi_1(v_0, \overline{a_\beta}, x_\beta)$.

Let x be $\{(\psi(v_0, v_1, \overline{f(c_\gamma)}))^N, (\phi_0(v_1, \overline{f(a_\alpha)}, \overline{b_\alpha}))^N, (\phi_1(v_0, \overline{f(a_\beta)}, \overline{b_\beta}))^N\}^c$. One has:

- $p_n(x) = \{(\psi(v_0, v_1, \overline{c_\gamma}))^M, (\exists x_\alpha \phi_0(v_1, \overline{a_\alpha}, x_\alpha))^M, (\exists x_\beta \phi_1(v_0, \overline{a_\beta}, x_\beta))^M\}^c = \{(\psi(v_0, v_1, \overline{c_\gamma}))^M, (\exists y \psi(y, v_1, \overline{c_\gamma}))^M, (\exists z \psi(v_0, z, \overline{c_\gamma}))^M\}^c = y$
- $\exists_0 x = \{(\exists x \psi(x, v_0, \overline{f(c_\gamma)}))^N, (\phi_0(v_0, \overline{f(a_\alpha)}, \overline{b_\alpha}))^N, (\exists x \phi_1(x, \overline{f(a_\beta)}, \overline{b_\beta}))^N\}^c = \{(\exists x_\alpha \phi_0(v_0, \overline{a_\alpha}, x_\alpha))^N, (\phi_0(v_0, \overline{f(a_\alpha)}, \overline{b_\alpha}))^N, (\exists x \phi_0(x, \overline{f(a_\alpha)}, \overline{b_\alpha}))^N\}^c = x_0$
- $\exists_1 x = x_1$ in the same way as just above. ⊣

COROLLARY 2.21. *The mapping $(-)^*$ defines a contravariant functor F from the category* **L-Strs** *of L-structures and elementary embeddings, to the category* **d-sSets** *of simplicial sets with distinguished simplices and d-simplicial maps which are, as simplicial maps, trivial fibrations.*

§3. O-minimal structures as simplicial sets. Given the existence of the functor F, two natural questions are whether a functor can be defined in the other direction (even if it means restricting the two categories **L-Strs** and **d-sSets** to subcategories thereof) and, if so, whether the resulting pair of functors induces an adjunction. The purpose of the rest of this paper is to give a positive answer to both questions (Theorems 3.15 and 3.19).

3.1. O-minimality. It appears that the right restriction of **L-Strs** is to o-minimal L-structures. An L-structure M is said to be *o-minimal* iff $<^M$ is a dense linear order and every definable subset of $\langle M, <^M\rangle$ is a finite union of singletons and open $<^M$-intervals (with endpoints in $|M| \cup \{-\infty, +\infty\}$). The full subcategory of **L-Strs** composed of all o-minimal L-structures is written **oL-Strs**.

LEMMA 3.1. *Suppose X is a simplicial set. Then any linear order on the set X_0 of its vertices induces on each X_n ($n \in \mathbb{N}$) a linear order.*

PROOF. The linear ordering of every X_n is defined by induction. Let us suppose that a linear order $<_n$ has been defined on X_n. For each $E \in X_{n+1}$, let $s(E)$ be the sequence $(d_i(E))_{0\leq i\leq n+1}$. An order $<_{n+1}$ is then defined on X_{n+1} as follows: $E <_{n+1} E'$ iff $s(E) <^l_n s(E')$, where $<^l_n$ is the lexicographic order induced by $<_n$. Since the lexicographical order on a family of linearly ordered sets is a linear extension of their product order, by induction each $<_n$ is a linear order. ⊣

DEFINITION 3.2. A *dense inductively linearly ordered simplicial set with countably many distinguished simplices*, or *dilo-simplicial set* for short, is a d-simplicial set X such that: (i) its set X_0 of vertices is a dense linear order; (ii) for each $n \geq 1$, X_n is endowed with the linear order induced by that on X_0 as above; and (iii) for each $n \in \mathbb{N}$, the subset $X'_n \subseteq X_n$ of distinguished n-simplices is countable.

A *morphism $f : X \to Y$ between dilo-simplicial sets*, or *dilo-simplicial map* for short, is a d-simplicial map $f : X \to Y$ such that, for each $n \in \mathbb{N}$, $f_n|_{X'_n}$ is an order isomorphism from X'_n to Y'_n.

LEMMA 3.3. *For any o-minimal L-structure M, M^* is a dilo-simplicial set.*

PROOF. Since M^* is a d-simplicial set, it is only necessary to define a dense linear ordering of $D_0(M)$. Each definable subset A of $|M|$ can be written $\bigcup_{i=0}^{n_A} \bar{]}a_i, a_{i+1}\bar{[}$ for some $n_A \in \mathbb{N}$, with $a_i \leq^M a_{i+1}$ and the convention:

$$\bar{]}a_i, a_{i+1}\bar{[} = \begin{cases}]a_i, a_{i+1}[\text{ if } a_i <^M a_{i+1}, \\ \{a_i\} \text{ if } a_i = a_{i+1}. \end{cases}$$

The sequence $(a_i)_{0\leq i\leq n_A}$ is called *the sequence associated to A* and written $\sigma(A)$. Then $D_0(M)$ is ordered as follows: $A <^0 A'$ iff $\sigma(A) <^l \sigma(A')$, where $<^l$ is the lexicographic order induced by $<^M$. Finally $K(M)_0$ is ordered as follows: $C <_0 C'$ iff $C_0 <^* C'_0$, where $C_0 = C \cap D_0(M)$, $C'_0 = C' \cap D_0(M)$ and $<^*$ is the lexicographic order induced by $<^0$ on $\wp_f(D_0(M))$. The resulting ordering $<_0$ is a dense linear order. ⊣

A standard result about trivial fibrations[8] is that any trivial fibration $p : X \to Y$ has the right lifting property w.r.t. all cofibrations, which means that, given any cofibration $i : U \to V$, any commutative solid arrow diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ V & \xrightarrow[g]{} & Y \end{array}$$

admits of a lifting $h : V \to X$ satisfying $h \circ i = f$ and $p \circ h = g$. In particular, the lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & = & Y \end{array}$$

has a solution s. Hence any trivial fibration $p : X \to Y$ has a *section*: put differently, there exists a simplicial map $s : Y \to X$ such that $\langle s, p\rangle$ constitutes a retraction pair.

Definition 3.4. Let **dilo-sSets** be the category —a subcategory of **d-sSets**— whose objects are all dilo-simplicial sets and whose morphisms are all dilo-simplicial maps which, as simplicial maps, are trivial fibrations with a unique section. So each morphism p in **dilo-sSets** actually consists of a specific retraction pair $\langle i_p : Y \to X, p : X \to Y\rangle$.

Lemma 3.5. *Let M and N be two L-structures. For any elementary embedding $f : M \xrightarrow{\prec} N$, $\langle f_!, f^*\rangle$ is a retraction pair and $f_!$ is the unique simplicial map g such that $\langle g, f^*\rangle$ is a retraction pair.*

Proof. For each $n \in \mathbb{N}$, the equality ${}^n f \circ {}_n f = \mathrm{id}_{D_n(M)}$ and thus the equality ${}^n f \circ {}_n f = \mathrm{id}_{K(M)_n}$ follow directly from the definitions of ${}_n f$ and ${}^n f$. The uniqueness of $f_!$ follows from the hypothesis that f is an elementary embedding. ⊣

Corollary 3.6. *The restriction $\tilde{F}$ of F to* **oL-Strs**$^{\mathbf{op}}$ *induces a functor F' from* **oL-Strs**$^{\mathbf{op}}$ *to* **dilo-sSets**.

One is now interested in a functor going in the opposite direction. But, given any dilo-simplicial set X, there is an L-structure $\widehat{X}$ naturally associated to it. Indeed, any simplicial set X gives rise, in a canonical way, to a topological space $||X||$, called its *geometric realization*. Actually, from the beginning X is just a recipe for joining polyhedra together so as to obtain $||X||$, of which X can be conceived of as a triangulation. The idea is to realize each n-simplex of X as the n-cell ("standard

[8]See [3], Lemma II.1.1, pp. 67–68.

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n-simplex") $||\Delta^n|| = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_{i=0}^n t_i = 1\}$, and to keep track of the incidences between n-simplices:

$$||X|| := \varinjlim_{\Delta^n \to X} ||\Delta^n|| = \left(\coprod_{n \in \mathbb{N}} X_n \times ||\Delta^n|| \right) / \sim,$$

where $\sim$ expresses the identification of parts that are glued together.[9] Specifically, define $\delta_i : ||\Delta^{n-1}|| \to ||\Delta^n||$ and $\sigma_j : ||\Delta^{n+1}|| \to ||\Delta^n||$ by:

$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$

$\sigma_j(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, t_{n-1})$.

Then $\sim$ is the equivalence relation generated by $(d_i x, u) \sim (x, \delta_i u)$ and $(s_j x, v) \sim (x, \sigma_j v)$, for all $x \in X_n$, $u \in ||\Delta^n||$ and $v \in ||\Delta^{n+1}||$. Thus, an element of $||X||$ is of the form $\overline{(x, u)}$ with $x \in X_n$ and $u = (t_0, \ldots, t_n)$. Note that, by construction, the topological space $||X||$ is a CW complex. On the other hand, any simplicial map $g : X \to Y$ induces a continuous map $||g|| : ||X|| \to ||Y||$, given by: $||g||(\overline{(x, u)}) = \overline{(g_n(x), u)}$. This makes geometric realization functorial.

Definition 3.7. Let X be a dilo-simplicial set and X_n^k the k-th distinguished n-simplex of X, i.e., the k-th member of X_n'. The *image of* X_n^k *in* $||X||$, written $||X||_n^k$, is the image of the composite map:

$\{X_n^k\} \times ||\Delta^n|| \hookrightarrow \coprod_{n \in \mathbb{N}} X_n \times ||\Delta^n|| \twoheadrightarrow ||X||$.

This definition generalizes to any set Y of n-simplices of X: the image $||Y||$ of Y is the union of the images of all the elements of Y.

Definition 3.8. Any dilo-simplicial set X gives rise to an L-structure $\widehat{X}$ defined as follows:

- $|\widehat{X}|$ is $||X||$, the geometric realization of X;
- $(R_k^{(0)})^{\widehat{X}}$ is the image of the k-th distinguished vertex in X_0;
- $(R_k^{(n)})^{\widehat{X}}$, for $n \geq 1$, is $(||X||_n^k)^n$.

Example 3.9. Let X_{S^2} be the canonical simplicial set whose realization is the two-dimensional sphere S^2. Since S^2 is obtained by glueing the boundary of a copy of $||\Delta^2||$ into a single point, X_{S^2} has a single non-degenerate 0-simplex (vertex $*$), a single non-degenerate 2-simplex, and no other non-degenerate simplices. Hence $|\widehat{X_{S^2}}|$ is S^2 and, for all $k \in \mathbb{N}$, $(R_k^{(0)})^{\widehat{X_{S^2}}}$ is $*$, $(R_k^{(1)})^{\widehat{X_{S^2}}}$ is $\{*\}$ and $(R_k^{(2)})^{\widehat{X_{S^2}}}$ is $S^2 \times S^2$.

Definition 3.10. Let $\langle i_p : Y \to X, p : X \to Y \rangle$ be a morphism in **dilo-sSets**. One then defines $\widehat{p} := ||i_p|| : ||Y|| \to ||X||$.

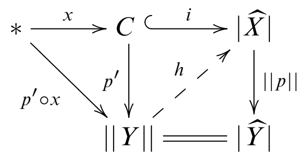
As a result of the definition of a trivial fibration $p : X \to Y$, $||p||$ has the right lifting property w.r.t. all cellular inclusions, i.e., all inclusions $A \hookrightarrow C$ where A

[9] The usual notations for the geometric realization of X are '$|\Delta^n|$' and '$|X|$'. The notations '$||\Delta^n||$' and '$||X||$' chosen in this paper are motivated by the wish not to confuse the domain of an L-structure and the geometric realization of a simplicial set—precisely owing to the connection to be established between both notions (see Definition 3.8 below).

is a subcomplex of some CW complex C.[10] This result will be of use in the proof of the Lemma below.

LEMMA 3.11. *For any morphism $p : X \to Y$ in* **dilo-sSets**, $\widehat{p} : ||Y|| \to ||X||$ *defines an elementary embedding $\widehat{p} : \widehat{Y} \to \widehat{X}$.*

PROOF. Suppose $(y_1, \ldots, y_n) \in (R_k^{(n)})^{\widehat{Y}} = (||Y||_n^k)^n$. Each y_i is of the form $\overline{(\beta_i, u_i)}$ with $\beta_i = Y_n^k$. By definition of a dilo-simplicial map, $p_n \circ (i_p)_n = \mathrm{id}_{Y_n}$ and thus $\widehat{p}(y_i) = \overline{(\alpha_i, u_i)}$ with $\alpha_i = (i_p)_n(\beta_i) = p_n^{-1}(\beta_i) = X_n^k$, so $\widehat{p}(y_i) \in ||X||_n^k$, and so $(\widehat{p}(y_1), \ldots, \widehat{p}(y_n)) \in (R_k^{(n)})^{\widehat{X}}$. Suppose now that $||Y|| \subseteq ||X||$ and that $\widehat{X} \vDash \phi(v_0, \overline{y_1}, \ldots, \overline{y_n})[x]$ with $x \in ||X||$ and $y_1, \ldots, y_n \in ||Y||$. Let C be $(\phi(v_0, \overline{y_1}, \ldots, \overline{y_n}))^{\widehat{X}}$, so that $x \in C$, and let p' be the restriction of $||p||$ to C. Since $p' \circ x$ is a cellular inclusion and p is a trivial fibration, $||p||$ has the right lifting property w.r.t. $p' \circ x$, so there exists a lifting h in the following diagram:



The map h satisfies $h \circ p' \circ x = i \circ x$ and $||p|| \circ h = \mathrm{id}_{||Y||}$. The first condition ensures that there exists $y \in |\widehat{Y}|$ such that $h(y) \in C$ and, by the uniqueness of the section of p, the second ensures that $h(y) = y$. This leads to the existence of $y \in |\widehat{Y}|$ such that $\widehat{X} \vDash \phi(v_0, \overline{y_1}, \ldots, \overline{y_n})[y]$. As a result, the Tarski–Vaught test is verified. $\dashv$

COROLLARY 3.12. *The mapping $\widehat{(-)}$ defines a functor G from* **dilo-sSets**^{op} *to* **L-Strs**.

As an extension of the simplicial setup presented here, a connection can be made with the "independence property" in stability theory. An L-formula $\phi(\vec{x}, \vec{y})$ *does not have the independence property* in some L-structure M if, for any denumerable set $\{\vec{a}_i : i \in \mathbb{N}\}$, there is no family $\{\vec{b}_J : J \subseteq \mathbb{N}\}$ such that, for all $i \in \mathbb{N}$ and all $J \subseteq \mathbb{N}$, $M \vDash \phi(\vec{a}_i, \vec{b}_J)$ iff $i \in J$. A standard result is that an L-formula $\phi(\vec{x}, \vec{y})$ has the independence property in M if and only if, for any indiscernible sequence $(\vec{a}_i)_{i \in I}$ and any tuple $\vec{b}$ in M, there is some cofinal subset $I_0 \subseteq I$ such that either $M \vDash \phi(\vec{a}_i, \vec{b})$ or $M \vDash \neg\phi(\vec{a}_i, \vec{b})$ holds for every $i \in I_0$. (Given a linearly ordered set I, a sequence $(a_i)_{i \in I}$ of tuples in M is *indiscernible* if for every $n \in \mathbb{N}$, whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ are two increasing n-subsequences of I, the two n-tuples $(a_{i_1}, \ldots, a_{i_n})$ and $(a_{j_1}, \ldots, a_{j_n})$ have the same type.)

[10] See [6], pp. 49–58. A cellular inclusion $A \hookrightarrow C$ is a particular case of map in I'-cell (Lemma 2.4.5), and I'-cell $\subseteq I'$-cof (Lemma 2.1.10), so any cellular inclusion is a cofibration in the category **Top** of topological spaces, as defined in Definition 2.4.3. But, for every trivial fibration p in **sSets**, $||p||$ is a trivial Serre fibration in **Top** (Corollary 2.4.20), so it has the right lifting property w.r.t. all maps in J, and thus w.r.t. all cofibrations, since (J-cof)-inj $= J$-inj.

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Proposition 3.13. *Given any dilo-simplicial set X, no L-formula has the independence property in $\widehat{X}$.*

Proof. The combination of Lemma 12.16 and Theorem 12.18 in [10] shows that the proof can be confined to L-formulae $\phi(\vec{x}, \vec{y})$ where $|\vec{x}| = |\vec{y}| = 1$. Now, an indiscernible sequence of elements in $\widehat{X}$ is any sequence of elements of some $||X_n^k|| \setminus \bigcup_{0 \leq i \leq n} ||d_i(X_n^k)||$. So an element $b \in |\widehat{X}|$ can make a difference within an indiscernible sequence $(a_i)_{i \in I}$ only if $b = a_{i_0}$ for some $i_0 \in I$, so I_0 can be taken to be $\{i \in I : i > i_0\}$. ⊣

This result, as no L-formula has the independence property in any o-minimal structure (see [11], ch. 5.), is actually strenghtened by the following Lemma.

Lemma 3.14. *For any dilo-simplicial set X, $\widehat{X}$ is an o-minimal L-structure.*

Proof. The set $\{(n, x, u) : n \in \mathbb{N}, x \in X_n, u \in ||\Delta^n||\}$ can be naturally equipped with a linear order: for $n, n' \in \mathbb{N}$, $x \in X_n$, $x' \in X_{n'}$, $u = (t_0, \dots, t_n) \in ||\Delta^n||$ and $u' = \{(t'_0, \dots, t'_{n'}) \in ||\Delta^{n'}||$, one states:

$$(n, x, u) \prec (n', x', u') \quad \text{iff} \quad \begin{cases} n < n', \\ \text{or } \; n = n' \text{ and } x < x' \text{ in } X_n, \\ \text{or } \; n = n', \; x = x' \text{ and } u < u' \text{ in } ||\Delta^n||, \end{cases}$$

$||\Delta^n||$ being endowed with the lexicographical order.

Then, given $[x, u] := \inf_{\prec}(\{(n_0, x_0, u_0) : x_0 \in X_{n_0} \text{ and } (x_0, u_0) \in \overline{(x, u)}\})$, $|\widehat{X}| = ||X||$ is simply ordered by:

$$\overline{(x, u)} <_X \overline{(x', u')} \quad \text{iff} \quad [x, u] \prec [x', u'].$$

One checks that $<_X$ is a dense linear order which turns $\widehat{X}$ into an o-minimal structure. Indeed, because the interpretation $(R_k^{(1)})^{\widehat{X}}$ of each predicate is a convex set, the structure $\langle \widehat{X}, <_X, (R_k^{(1)})^{\widehat{X}} \rangle$ is weakly o-minimal, by a result given in [1], and actually o-minimal, by construction of $<_X$. Moreover, the n-ary relations for $n \geq 2$ are interpreted by trivial convex subsets (products), so instantiating a variable in an L-formula or quantifying over it simply shifts to another convex subset, and actually to a finite union of singletons and intervals. The same is true about connectives, so any L-formula in F_0 is interpreted in $\widehat{X}$ by a finite union of singletons and intervals. ⊣

Theorem 3.15. *There are two functors* $F' : \mathbf{oL\text{-}Strs}^{\mathbf{op}} \to \mathbf{dilo\text{-}sSets}$ *and* $G' : \mathbf{dilo\text{-}sSets} \to \mathbf{oL\text{-}Strs}^{\mathbf{op}}$ *such that both diagrams below commute*:

$$\underbrace{\mathbf{oL\text{-}Strs}^{\mathbf{op}} \xrightarrow{F'} \mathbf{dilo\text{-}sSets} \hookrightarrow \mathbf{d\text{-}sSets}}_{\tilde{F}},$$

$$\underbrace{\mathbf{dilo\text{-}sSets} \xrightarrow{G'} \mathbf{oL\text{-}Strs}^{\mathbf{op}} \hookrightarrow \mathbf{L\text{-}Strs}^{\mathbf{op}}}_{G^{op}}.$$

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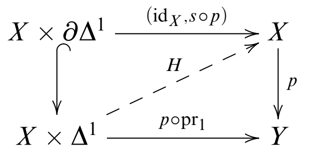
REMARK 3.16. The L-structure $\widehat{X}$ is o-minimal over $||X||$, but any dilo-simplicial set also induces an o-minimal structure $X_{\mathbb{R}}$ over $\mathbb{R}$, defined by $|X_{\mathbb{R}}| = \mathbb{R}$, $(R_k^{(n)})^{X_{\mathbb{R}}} = |X_n^k|$. If $p : X \to Y$ is a trivial fibration, then $p_{\mathbb{R}} := \mathrm{id}_{\mathbb{R}}$ is by construction an elementary embedding of $Y_{\mathbb{R}}$ into $X_{\mathbb{R}}$. The correspondence $(-)_{\mathbb{R}}$ defines a functor, distinct from G', from **dilo-sSets** to **oL-Strs$^{\mathbf{op}}$**.[11]

3.2. Adjunction. The last task ahead is to establish that the functors F' and G' are adjoint functors. To that end, two preliminary results, one model-theoretic, the other homotopy-theoretic, are needed. First, an L-structure P is *prime over* $A \subseteq |P|$ if, for every $N \vDash \mathrm{Th}(P, \overline{\alpha})_{\alpha \in A}$, there is an elementary embedding $f : P \to N$ over A (that is, which is the identity over A). Pillay and Steinhorn[12] proved that, given any o-minimal L-structure M and $A \subseteq |M|$, $\mathrm{Th}(M)$ has a prime model over A, unique up to A-isomorphism.

Secondly, the simplicial set Δ^1 is the simplicial set X with only three nondegenerate simplices: two vertices 0 and 1, and one 1-simplex whose faces are the two vertices 0 and 1. Its realization $||\Delta^1|| = \{(t_0, t_1) \in \mathbb{R}^2 : t_0, t_1 \geq 0, t_0 + t_1 = 1\}$ is a line segment. Given two simplicial maps $f, g : X \to Y$, a *homotopy* $H : f \xrightarrow{\simeq} g$ is a simplicial map $H : X \times \Delta^1 \to Y$ whose restrictions to $X \times \{0\}$ and to $X \times \{1\}$ are f and g, respectively.[13] If there is such an homotopy, the maps f and g are said to be *homotopic*, which is written $f \simeq g$. Two maps $f : X \to Y$ and $f' : Y \to X$ are *homotopy inverse* to each other if $f' \circ f \simeq \mathrm{id}_X$ and $f \circ f' \simeq \mathrm{id}_Y$. Now, as noted above,[14] any trivial fibration has a section. It turns out that any section $s : Y \to X$ of a trivial fibration $p : X \to$ is homotopy inverse to it. Indeed, by hypothesis the lifting problem[15]



has a solution H, which amounts to a homotopy $\mathrm{id}_X \simeq s \circ p$. In particular, for any elementary embedding $f : M \xrightarrow{\prec} N$, $f_!$ is homotopy inverse to f^*.

LEMMA 3.17. *Let X be a dilo-simplicial set. Then there exists a trivial fibration $\varepsilon : (\widehat{X})^* \to X$.*

PROOF. For any vertex x of X, let us consider its realization $\{||x||\} := \{\overline{(x, 0)}\} = \{x\} \times ||\Delta_0|| \subseteq ||X||$, so that the set X_0 of the vertices of X corresponds to a subset

[11]The introduction of that functor would be motivated, in the context of focusing on o-minimal structures over the real line, by the natural comparison of a definable subset of $X_{\mathbb{R}}$ with a simplicial subcomplex of $||X||$ (see [11], ch. 8).

[12]See [9], Theorem 5.1, p. 583.

[13]About $X \times \Delta^1$: the Cartesian product $X \times Y$ of two simplicial sets X and Y is simply defined by: $(X \times Y)_n := X_n \times Y_n$ ($n \in \mathbb{N}$).

[14]See the remark following Lemma 3.3.

[15]The simplicial set $\partial\Delta^1$, called the *boundary* of Δ^1, is the smallest simplicial subset of Δ^1 containing all the 0-simplices (vertices) of the latter, namely 0 and 1.

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of $||X||$. In the same way, each 1-simplex $x \in X_1$ corresponds to an ordered pair of vertices, and thus X_1 to a binary relation on $||X||$. More generally, each $x \in X_n$ corresponds to an $(n+1)$-uple $(x_0, \ldots, x_n)$ of vertices (some of which possibly identical), hence to an $(n+1)$-uple $(||x_0||, \ldots, ||x_n||)$ in $||X||$. The substructure $Y = \langle ||X_0||, (R_k^{(n)})^{\widehat{X}} \cap ||X_0||^n \rangle_{k,n\in\mathbb{N}}$ of $\widehat{X}$ defined by restriction to $||X_0|| = \{||x|| : x \in X_0\}$ is actually an elementary substructure $r : Y \xrightarrow{\prec} \widehat{X}$ of $\widehat{X}$ because, by continuity, every L-formula (with parameters in Y) satisfiable in $\widehat{X}$ is satisfied by a tuple of vertices. Every definable subset $B \in D_n(Y)$ consists of a set of $(n+1)$-tuples of vertices. For $C \in K(Y)_n$, let $\lambda_n(C)$ be $x \in X_n$ if, for each $B \in C \cap D_n(Y)$ and for each $(e_0, e_1, \ldots, e_n) \in B$, the barycenter of the convex hull of $\{e_0, e_1, \ldots, e_n\}$ belongs to $||x||$, and x is the least n-simplex of X with that property; else some fixed n-simplex $x_\star$ of X. Since both the operations of taking the convex hull and the barycenter commute with the face operators, the maps $\lambda_n : K(Y)_n \to X_n$ $(n \in \mathbb{N})$ make up a simplicial map $\lambda : Y^* \to X$, which is easily seen to be a trivial fibration. The map $\varepsilon : (\widehat{X})^* \to X$ is then defined as $\lambda \circ r^*$. Since trivial fibrations are stable under composition, ε is a trivial fibration too. ⊣

Let X be a fibrant simplicial set and $x \in X_0$. The set $\pi_1(X, x)$ is defined as the set of all homotopy equivalence classes of maps $\alpha : \Delta^1 \to X$ satisfying $d_0\alpha = d_1\alpha = x$, where homotopy equivalence between $\alpha : \Delta^1 \to X$ and $\beta : \Delta^1 \to X$ consists in the existence of $\gamma : \Delta^2 \to X$ such that $d_0\gamma = s_0 d_0 \alpha = s_0 d_0 \beta$, $d_1\gamma = \alpha$ and $d_2\gamma = \beta$. The set $\pi_1(X, x)$ can be endowed with a canonical composition law which turns it into a group, called the *first homotopy group* of X at x.

Lemma 3.18. *Let M be an L-structure and $C \in K(M)_0$. Then $\pi_1(M^*, C)$ is trivial only if C is principal.*

Proof. Let $C \in K(M)_0$ be nonprincipal. Without loss of generality one may assume that $C = \{(\phi_1(v_0))^M, (\phi_2(v_0))^M\}^c$ with $(\phi_1(v_0))^M \neq \emptyset$, $(\phi_2(v_0))^M \neq \emptyset$ and $(\phi_1(v_0))^M \neq (\phi_2(v_0))^M$. Let $\alpha := \{E_0((\phi_1(v_0))^M), E_1((\phi_2(v_0))^M)\}^c$ and $\beta := \{(\phi_1(v_0) \wedge \phi_2(v_1))^M, (\phi_1(v_1) \wedge \phi_2(v_0))^M\}^c$. One has: $\exists_0\alpha = \exists_0\beta = \exists_1\alpha = \exists_1\beta = C$. Let us now suppose that there exists $\gamma \in K(M)_2$ such that $\exists_1\gamma = \alpha$ and $\exists_2\gamma = \beta$: then $\{(\phi_1(v_0))^M, (\phi_2(v_0))^M\}^c = \{(\phi_1(v_0) \wedge \exists x \phi_2(x))^M, (\exists x \phi_1(x) \wedge \phi_1(v_0))^M\}^c = \exists_1\beta = \exists_1\exists_2\gamma = \exists_1\exists_1\gamma = \exists_1\alpha = \{(\phi_1(v_0))^M\}^c$, in contradiction with the hypothesis about $(\phi_1(v_0))^M$ and $(\phi_2(v_0))^M$. ⊣

Theorem 3.19. *The functors F' and G' are part of an adjunction*

$$F' : \textbf{oL-Strs}^{\textbf{op}} \rightleftarrows \textbf{dilo-sSets} : G'$$

up to homotopy.

Proof. The more specific statement of the theorem is that, for any objects M of **oL-Strs** and X of **dilo-sSets**, there is a map $\Phi_{M,X} : \mathrm{Hom}_{\textbf{oL-Strs}}(\widehat{X}, M) \to \mathrm{Hom}_{\textbf{dilo-sSets}}(M^*, X)$ with a homotopy inverse $\Psi_{M,X}$ satisfying not only $\Phi \circ \Psi \simeq \mathrm{id}_{\mathrm{Hom}(M^*,X)}$ and $\Psi \circ \Phi \simeq \mathrm{id}_{\mathrm{Hom}(\widehat{X},M)}$, but even $\Psi \circ \Phi = \mathrm{id}_{\mathrm{Hom}(\widehat{X},M)}$. Let $f : \widehat{X} \to M$ be a given elementary embedding in **oL-Strs**. Then $p_f : M^* \to X$ is defined as

$\varepsilon \circ f^*$, where ε is the trivial fibration of Lemma 3.17. Since f^* is a trivial fibration, so is p_f too. And since ε clearly preserves distinguished simplices and respects the ordering of n-simplices for each $n \in \mathbb{N}$, as well as f^*, so does p_f too. Also, p_f is a retraction with a unique section $f_! \circ r_! \circ s$, where $r : Y \xrightarrow{\prec} \widehat{X}$ is the retraction of the previous lemma and $s_n : X_n \to K(Y)_n$ sends each $x \in X_n$ to $\{(||x_0||, \dots, ||x_n||)\}^c$, $x_0, \dots, x_n$ being the vertices corresponding to x in Lemma 3.17. Conversely, any trivial fibration $p : M^* \to X$ in **dilo-sSets** gives rise, for each $x \in |\widehat{X}|$, to a lifting diagram:

$$\begin{array}{ccc} * & \xrightarrow{C_x} & M^* \\ {\scriptstyle \{x\}^c}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ (\widehat{X})^* & \xrightarrow[\varepsilon]{} & X \end{array}$$

where $*$ is (the simplicial set whose realization is) the point and $\{x\}^c$ represents $\{(v_0 = \overline{x})^{\widehat{X}}\}^c \in K(\widehat{X})_0$. Indeed, since p_0 is surjective, there exists $C_x \in K(M)_0$ such that $p_0(C_x) = \varepsilon_0(\{x\}^c)$. One has: $h_0(\{x\}^c) = C_x = i_p \circ \varepsilon \circ \{x\}^c$. Because p and ε are trivial fibrations, and thus weak equivalences, h is a weak equivalence too, i.e., a map between fibrant simplicial sets which induces isomorphisms on all homotopy groups.[16] In particular, $h_1 : \pi_1((\widehat{X})^*, \{x\}^c) \simeq \pi_1(M^*, C_x)$. As a consequence, by Lemma 3.18, $C_x = \{(\phi_x(v_0))^M\}^c$ for some L-formula ϕ_x. For $a \in (\phi_x(v_0))^M$, one has that $(M, a) \vDash \mathrm{Th}(\widehat{X}, \overline{x})$, a theory which (owing to Pillay and Steinhorn's result) has a unique prime model (P, α). Let $e_M : (P, \alpha) \to (M, a)$ and $e_{\widehat{X}} : (P, \alpha) \to (\widehat{X}, x)$ two corresponding elementary embeddings. Even though e_M and $e_{\widehat{X}}$ are not unique themselves, $e_M(\alpha) = a$ and $e_{\widehat{X}}(\alpha) = x$ are ensured to hold. So a (and thus C_x) is uniquely determined. Stating $f_p(x) := a$, one defines $f_p : \widehat{X} \to M$. This is an elementary embedding: indeed, $f_p(|\widehat{X}|) \subseteq |M|$ and types of elements are preserved, since the definition of each is preserved by the corresponding h, which can taken to be $i_p \circ \varepsilon$ for each $x \in |\widehat{X}|$.

The functoriality of $p \mapsto f_p$ and $f \mapsto p_f$ is clear, so it only remains to check that $p_{f_p} \simeq p$ and that $f_{p_f} = f$. Let $p : M^* \to X$ be a given trivial fibration. Then $f_p : \widehat{X} \to M$ is defined by $\{f_p(x)\}^c = (i_p \circ \varepsilon)_0(\{x\}^c)$, and p_{f_p} as the composite $\varepsilon \circ (f_p)^*$. By construction of f_p, $h = (f_p)_!$, so $p_{f_p} = \varepsilon \circ (f_p)^* = p \circ (f_p)_! \circ (f_p)^*$. And since $(f_p)_! \circ (f_p)^* \simeq \mathrm{id}_{M^*}$, one gets: $p_{f_p} \simeq p$. Actually, as $(p_{f_p})_0(\{(v_0 = \overline{f_p(x)})^M\}^c) = (\varepsilon_0 \circ {}^0(f_p))(\{(v_0 = \overline{f_p(x)})^M\}^c) = \varepsilon_0(\{(v_0 = \overline{x})^{\widehat{X}}\}^c) = (p_0 \circ h_0)(\{(v_0 = \overline{x})^{\widehat{X}}\}^c) = p_0(\{(v_0 = \overline{f_p(x)})^M\}^c)$, p and p_{f_p} coincide on $h_0(K(\widehat{X})_0)$ ($\widehat{X}$ and M being o-minimal). The resulting stronger result is written: $p_{f_p} \simeq p$ (rel $h_0(K(\widehat{X})_0)$). On the other hand, let $f : \widehat{X} \to M$ be a given elementary embedding, and $x \in ||X||$. By definition, $f_{p_f}(x)$ is the element b of $|M|$ such that $h_0(\{x\}^c) = \{b\}^c$, where h is the lift of ε along p_f. Besides, $a := f(x)$ is such that ${}^0f(\{a\}^c) = \{x\}^c$, so $(p_f)_0(\{a\}^c) = \varepsilon_0(\{x\}^c) = (p_f)_0(h_0(\{x\}^c)) = (p_f)_0(\{b\}^c)$.

[16]See [3] (p. 32 and p. 39, respectively) for the definition of a weak equivalence and the proof of the "two-out-of-three" property of weak equivalences. See [3] again (pp. 42–45) for the proof that a trivial fibration, as the notion has been defined in this paper, is a weak equivalence.

As a consequence, ${}^0f(\{a\}^c) = {}^0f(\{f(x)\}^c) = {}^0f(\{b\}^c)$. Hence $b = a$, and this holds for each $x \in ||X||$, so $f_{p_f} = f$. ⊣

Theorem 3.19 provides a systematic tool to transpose homotopy theoretic notions to model theory.[17] In particular, a natural avenue to pursue consists in harnessing the internal homotopy theory carried by the category of simplicial sets. Indeed, since any model of a given theory T is associated with a simplicial set in a functorial way, the category of models of T can be conceived of as a subcategory of the category of simplicial sets. Since the latter is a fundamental example of model category, a natural question is: are there theories whose categories of models are model categories? Another, related question is: is it possible to endow **oL-Strs** with a model category structure, so that the adjunction up to homotopy between F' and G' becomes a Quillen adjunction?[18] Regardless of those open questions, the simplicial viewpoint embraced in this paper leads naturally, to conclude, to looking at first-order theories as simplicial presheaves.

PROPOSITION 3.20. *To any L-theory T corresponds a simplicial presheaf* $\mathcal{F}_T$: $\mathbf{L\text{-}Strs}^{\mathbf{op}} \rightarrow \mathbf{sSets}$ *on* **L-Strs**.

PROOF. For each $\phi \in F_n$ and each L-structure M, let $[\phi^M]_T = \{\psi^M : T \vdash \phi \leftrightarrow \psi\}$. It is straightforward to check that $[\phi^M]_T \mapsto [(\exists_i \phi)^M]_T$ and $[\phi^M]_T \mapsto [E_j(\phi^M)]_T$ are well defined and satisfy the simplicial identities, so that $(\mathcal{F}_T(M))_n := \{[\phi^M]_T : \phi \in F_n\}$ $(n \in \mathbb{N})$ defines a simplicial set $\mathcal{F}_T(M)$. In particular, $\mathcal{F}_T(M) = M_*$ iff $M \vDash T$. For any elementary embedding $f : M \rightarrow N$, the definition of $\mathcal{F}_T(f)$ is directly adapted from that of f_*. ⊣

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[17] As an example, the category **oL-Strs** can be shown to admit of a simplicial object up to homotopy. Indeed, the proof of [6], Proposition 3.1.5 (see also Remark 3.1.7) can be transposed to the adjunction up to homotopy of Theorem 3.19.

[18] See [3] and [6] for the definitions of a *model category* and of a *Quillen adjunction*.

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