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Fibonacci Fraction Circles

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We define a *Fibonacci fraction circle* to be a circle passing through an infinite number of points whose coordinates are of the form $\left(\frac{F_k}{F_m}, \frac{F_n}{F_m}\right)$, where the F's are Fibonacci numbers (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …). For instance, the circle

$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4} \tag{1}$$

passes through the points $\left(\frac{1}{1}, \frac{0}{1}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{5}, \frac{3}{5}\right)$, $\left(\frac{1}{13}, \frac{8}{13}\right)$, $\left(\frac{1}{34}, \frac{21}{34}\right)$, …, which can be easily checked. See Figure 1. The purpose of this paper is to find other Fibonacci fraction circles. Several of these circles have been discovered independently by Kocik. [1]

To find Fibonacci fraction circles, we will make use of *Fibonacci hyperbolas*. Kimberling [2] defines a Fibonacci hyperbola to be a hyperbola passing through an infinite number of points of the form (F_m, F_n), whose coordinates are distinct Fibonacci numbers. For example, the hyperbola $x^2 - xy - y^2 = 1 = 0$ passes through the points (1, 0), (2, 1), (5, 3), (13, 8), (34, 21) … . See Figure 2. We will apply a transformation to the Fibonacci hyperbolas and turn them into Fibonacci fraction circles.

We start with a simple example. There is a close relation between the hyperbola $x^2 - xy - y^2 - 1 = 0$ and the circle (1). Let us rewrite the equation of the hyperbola as $X^2 - XY - Y^2 - 1 = 0$, using X and Y instead of x and y. We can rearrange it to put it in the form $1 + Y^2 + XY = X^2$. Dividing both sides by X^2 gives $\frac{1}{X^2} + \frac{Y^2}{X^2} + \frac{Y}{X} = 1$. We now define $X = \frac{1}{X}$ and $Y = \frac{Y}{X}$. Then our equation becomes $x^2 + y^2 + y = 1$. By completing the square, we get (1) of the circle, as desired.

This technique will be used to find the equations of other Fibonacci fraction circles. Kimberling [2] found all Fibonacci hyperbolas, and we will use them. All Fibonacci hyperbolas are either of the form

$$X^2 + (-1)^{n+1} L_n XY + (-1)^n Y^2 + F_n^2 = 0, \text{ for } n = 1, 2, 3, \ldots \tag{2}$$

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or of the form

$$X^2 + (-1)^{n+1} L_n XY + (-1)^n Y^2 - F_n^2 = 0, \text{ for } n = 1, 2, 3, \ldots \quad (3)$$

(or a reflection of one of those in the y-axis), where the L's are the Lucas numbers (1, 3, 4, 7, 11, 18, 29, …), and where we are using X and Y, as above. According to Kimberling [2], hyperbolas satisfying (2) contain points of the form

$$(F_k, F_{k+n}), (F_{k+2}, F_{k+n+2}), (F_{k+4}, F_{k+n+4}), \ldots, \quad (4)$$

where k is an integer such that $k + n$ is odd.

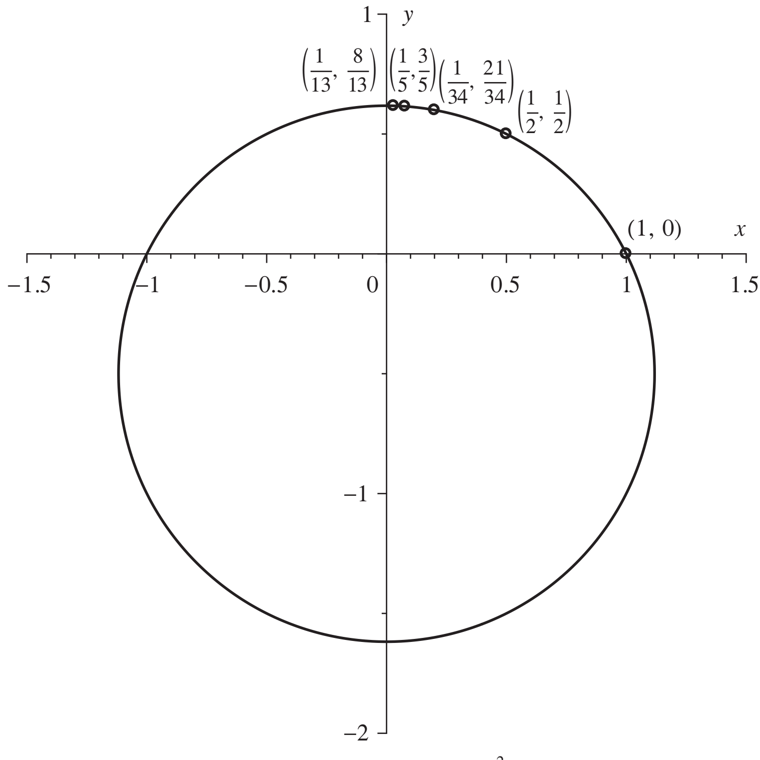

FIGURE 1: The Fibonacci fraction circle $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4}$ showing some points it passes through

Hyperbolas satisfying (3) contain points of the form

$$(F_k, F_{k+n}), (F_{k+2}, F_{k+n+2}), (F_{k+4}, F_{k+n+4}), \ldots, \quad (5)$$

where k is an integer such that $k + n$ is even.

Let us take (2) and divide both sides by Y^2. The result is

$$\frac{X^2}{Y^2} + (-1)^{n+1} L_n \frac{X}{Y} + (-1)^n + \frac{F_n^2}{Y^2} = 0.$$

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We now define

$$x = \frac{F_n}{Y} \qquad \text{and} \qquad y = \frac{X}{Y}. \tag{6}$$

Substituting, we get

$$y^2 + (-1)^{n+1} L_n y + (-1)^n + x^2 = 0. \tag{7}$$

Let us now take the case for which n is an odd number. Then we have

$$x^2 + y^2 + L_n y = 1.$$

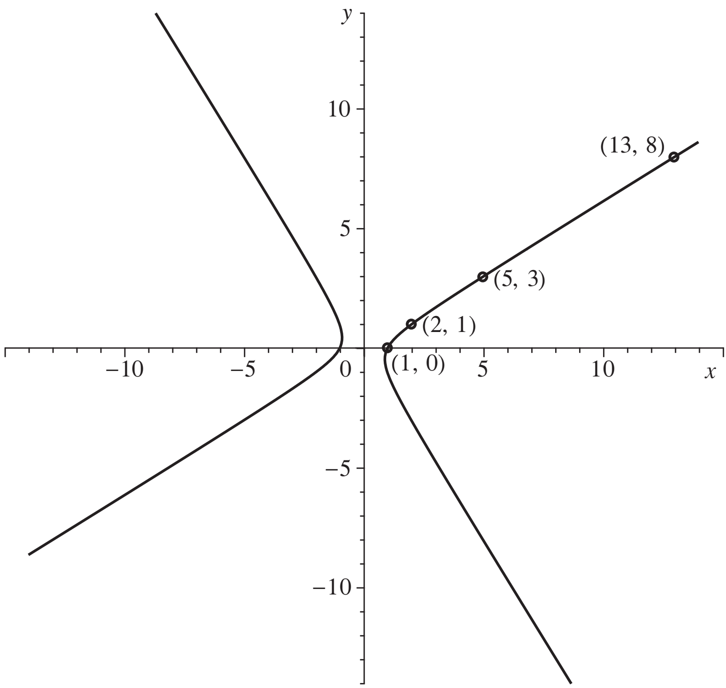

FIGURE 2: The Fibonacci hyperbola $x^2 - xy - y^2 - 1 = 0$ showing some points it passes through

To complete the square, we add $\frac{L_n^2}{4}$ to both sides, and the result is

$$x^2 + \left(y + \frac{L_n}{2}\right)^2 = 1 + \frac{L_n^2}{4}, \qquad n = 1, 3, 5, \ldots . \tag{8}$$

For each n value, we get a Fibonacci fraction circle. Using (4) and (6), we can see that each circle passes through points of the form

$$\left(\frac{F_n}{F_{n+2}}, \frac{F_2}{F_{n+2}}\right), \left(\frac{F_n}{F_{n+4}}, \frac{F_4}{F_{n+4}}\right), \left(\frac{F_n}{F_{n+6}}, \frac{F_6}{F_{n+6}}\right), \ldots . \tag{9}$$

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Table 1 lists several examples, and some points on each circle. The circles are shown in Figure 3.

n	Fibonacci Fraction Circle	Points
1	$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4}$	$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{5}, \frac{3}{5}\right), \left(\frac{1}{13}, \frac{8}{13}\right), \ldots$
3	$x^2 + (y + 2)^2 = 5$	$\left(\frac{2}{5}, \frac{1}{5}\right), \left(\frac{2}{13}, \frac{3}{13}\right), \left(\frac{2}{34}, \frac{8}{34}\right), \ldots$
5	$x^2 + \left(y + \frac{11}{2}\right)^2 = \frac{125}{4}$	$\left(\frac{5}{13}, \frac{1}{13}\right), \left(\frac{5}{34}, \frac{3}{34}\right), \left(\frac{5}{89}, \frac{8}{89}\right), \ldots$

TABLE 1: Fibonacci fraction circles of the form (8), and some representative points

Let us prove that ordered pairs given in (9) satisfy equation (8). Let n be odd and m be even. Then the ordered pairs in (9) are of the form $\left(\frac{F_n}{F_{n+m}}, \frac{F_m}{F_{n+m}}\right)$. Substituting into (8), we wish to show that

$$\left(\frac{F_n}{F_{n+m}}\right)^2 + \left(\frac{F_m}{F_{n+m}} + \frac{L_n}{2}\right)^2 = 1 + \frac{L_n^2}{4}. \tag{10}$$

The left-hand side of (10) can be written

$$\frac{F_n^2}{F_{n+m}^2} + \frac{F_m^2}{F_{n+m}^2} + \frac{F_mL_n}{F_{n+m}} + \frac{L_n^2}{4}$$

$$= \frac{F_n^2 + F_m^2 + F_mF_{n+m}L_n}{F_{n+m}^2} + \frac{L_n^2}{4}. \tag{11}$$

We use Catalan's identity [3, p. 83], $F_{n+m}F_{n-m} - F_n^2 = (-1)^{n+m+1}F_m^2$. Since n is odd and m is even, this becomes $F_n^2 + F_m^2 = F_{n+m}F_{n-m}$. Then (11) becomes

$$\frac{F_{n+m}F_{n-m} + F_mF_{n+m}L_n}{F_{n+m}^2} + \frac{L_n^2}{4}$$

$$= \frac{F_{n-m} + F_mL_n}{F_{n+m}} + \frac{L_n^2}{4}. \tag{12}$$

We now use the identity $F_{n+m} - F_{n+m} = F_mL_n$ (if m is even) [3, p. 97]. This can be written $F_{n-m} + F_mL_n = F_{n+m}$, and then (12) becomes

$$= 1 + \frac{L_n^2}{4}.$$

And so we have verified (10).

We had taken the case of (2) and (7) for which n is odd. Now we will consider the case for which n is even. In that case, (7) becomes

$$x^2 + y^2 - L_ny = -1.$$

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Completing the square gives equations of circles,

$$x^2 + \left(y - \frac{L_n}{2}\right)^2 = \frac{L_n^2}{4} - 1, \qquad n = 2, 4, 6, \ldots . \tag{13}$$

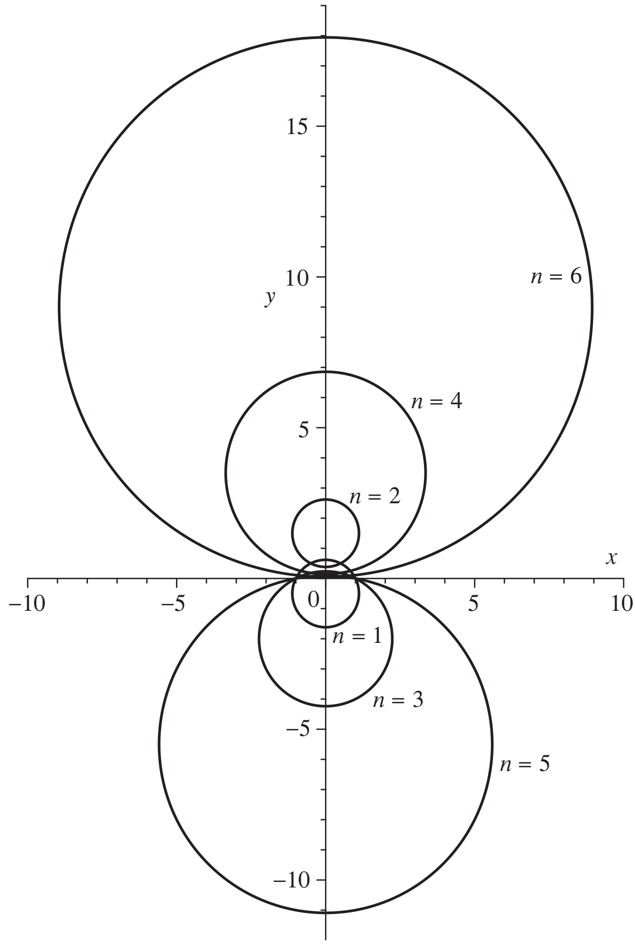

FIGURE 3: Fibonacci fraction circles from Tables 1 and 2

Using (4) and (6), we can see that each circle passes through points of the form

$$\left(\frac{F_n}{F_{n+1}}, \frac{F_1}{F_{n+1}}\right), \left(\frac{F_n}{F_{n+3}}, \frac{F_3}{F_{n+3}}\right), \left(\frac{F_n}{F_{n+5}}, \frac{F_5}{F_{n+5}}\right), \ldots . \tag{14}$$

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Table 2 lists several examples, and some points on each circle. The circles are shown in Figure 3.

n	Fibonacci Fraction Circle	Points
2	$x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{5}{4}$	$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{1}{13}, \frac{5}{13}\right), \ldots$
4	$x^2 + \left(y - \frac{7}{2}\right)^2 = \frac{45}{4}$	$\left(\frac{3}{5}, \frac{1}{5}\right), \left(\frac{3}{13}, \frac{2}{13}\right), \left(\frac{3}{34}, \frac{5}{34}\right), \ldots$
6	$x^2 + (y - 9)^2 = 80$	$\left(\frac{8}{13}, \frac{1}{13}\right), \left(\frac{8}{34}, \frac{2}{34}\right), \left(\frac{8}{89}, \frac{5}{89}\right), \ldots$

TABLE 2: Fibonacci fraction circles of the form (13), and some representative points

The proof that the ordered pairs in (14) satisfy (13) is analogous to the proof of (10), and is left as an exercise for the reader.

We used (2) to produce Fibonacci fraction circles. Let us now use (3) instead. We divide both sides by X^2 and define

$$x = \frac{F_n}{X} \qquad \text{and} \qquad y = \frac{Y}{X}. \tag{15}$$

The result is

$$1 + (-1)^{n+1} L_n y + (-1)^n y^2 - x^2 = 0. \tag{16}$$

We take the case for which n is odd. Eqution (16) becomes

$$x^2 + y^2 - L_n y = 1.$$

By completing the square we get the equations of circles

$$x^2 + \left(y - \frac{L_n}{2}\right) = 1 + \frac{L_n^2}{4}, \qquad n = 1, 3, 5, \ldots . \tag{17}$$

Using (5) and (15), we can see that each circle passes through points of the form

$$\left(\frac{F_n}{F_1}, \frac{F_{n+1}}{F_1}\right), \left(\frac{F_n}{F_3}, \frac{F_{n+3}}{F_3}\right), \left(\frac{F_n}{F_5}, \frac{F_{n+5}}{F_5}\right), \ldots . \tag{18}$$

Table 3 lists several examples, and some points on each circle. The circles are shown in Figure 4.

n	Fibonacci Fraction Circle	Points
1	$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}$	$\left(\frac{1}{1}, \frac{1}{1}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{1}{5}, \frac{8}{5}\right), \ldots$
3	$x^2 + (y - 2)^2 = 5$	$\left(\frac{2}{1}, \frac{3}{1}\right), \left(\frac{2}{2}, \frac{8}{2}\right), \left(\frac{2}{5}, \frac{21}{5}\right), \ldots$
5	$x^2 + \left(y - \frac{11}{2}\right)^2 = \frac{125}{4}$	$\left(\frac{5}{1}, \frac{8}{1}\right), \left(\frac{5}{2}, \frac{21}{2}\right), \left(\frac{5}{5}, \frac{55}{5}\right), \ldots$

TABLE 3: Fibonacci fraction circles of the form (17), and some representative points

The proof that the ordered pairs in (18) satisfy (17) is analogous to the proof of (10), and is left as an exercise for the reader.

We had taken the case of (3) and (16) for which n is odd. Now we will consider the case for which n is even. In that case, (16) becomes

$$x^2 - y^2 + L_n y = 1.$$

This is another hyperbola, not a circle. So it appears that it is not possible to generate Fibonacci fraction circles in this case.

Three of the Fibonacci fraction circles were discovered independently by Kocik [1], namely $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{5}{4}$ from Table 1, $x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{5}{4}$ from Table 2, and $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}$ from Table 3.

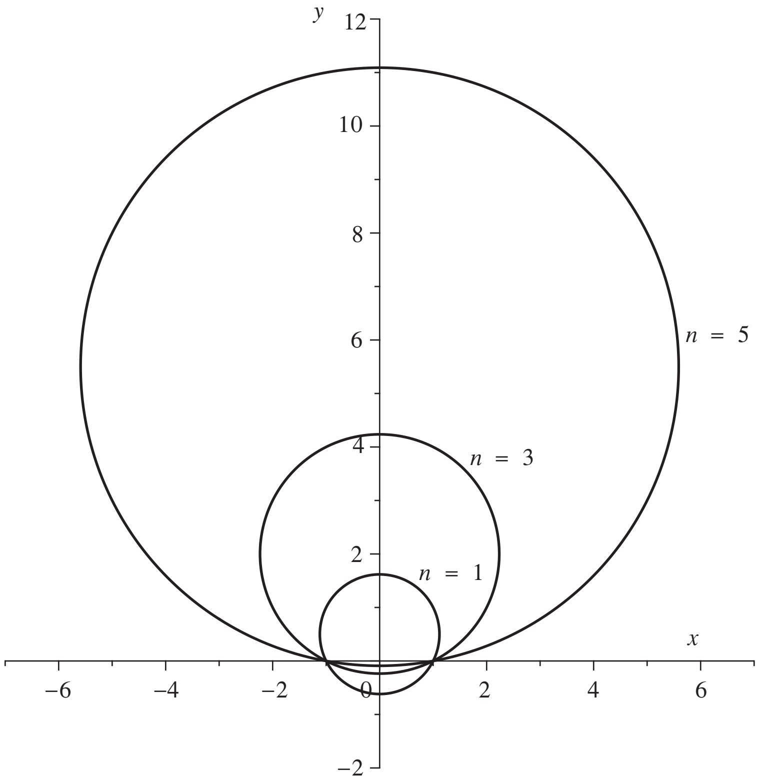

FIGURE 4: Fibonacci fraction circles from Table 3

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MA Presidential Blog

From 14 April 2022, when Dr Colin Foster becomes President of the MA, he will be writing a blog and posting something every other Thursday throughout his year as President. He is intending to address issues of interest and relevance across the entire range from early years through to university – not an easy task, but he is going to do his best! These will follow on from Dr Chris Pritchard's Presidential Essays, but the format and frequency will be different.

Colin hopes that the blog will stimulate conversations within the Association about important issues within mathematics teaching and learning, and there will be a facility to reply/comment underneath the blogposts, so it can support some discussion. Please sign up at

https://blog.foster77.co.uk/

now and take a look at the first post on 14 April.

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