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The Clairvoyant Demon Has a Hard Task

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Consider the integer lattice $L = \mathbb{Z}^2$. For some $m \geqslant 4$, let us colour each column of this lattice independently and uniformly with one of m colours. We do the same for the rows, independently of the columns. A point of L will be called *blocked* if its row and column have the same colour. We say that this random configuration *percolates* if there is a path in L starting at the origin, consisting of rightward and upward unit steps, avoiding the blocked points. As a problem arising in distributed computing, it has been conjectured that for $m \geqslant 4$ the configuration percolates with positive probability. This question remains open, but we prove that the probability that there is percolation to distance n but not to infinity is not exponentially small in n. This narrows the range of methods available for proving the conjecture.

1. Statement of the result

1.1. Introduction

Let $x = (x(0), x(1), \ldots)$ be an infinite sequence and $u = (u(0), u(1), \ldots)$ be a binary sequence with elements in $\{0, 1\}$. Let $s_n = \sum_{i=0}^{n-1} u(i)$. We define the *delayed version* $x^{(u)}$ of x, by

$$x^{(u)}(n) = x(s_n).$$

Thus, if $i = s_n$ then $x^{(u)}(n) = x(i)$, and $x^{(u)}(n+1) = x(i)$ or $x(i+1)$ depending on whether $u(n) = 0$ or 1. If $u(n) = 0$ then we can say that $x^{(u)}$ is *delayed* at time $n+1$. For two infinite sequences x, y we say that they *do not collide* if there is a delay sequence u such that for each n we have

$$x^{(u)}(n) \neq y^{(1-u)}(n).$$

Here, $1 - u$ is the delay sequence complementary to u: thus, $y^{(1-u)}$ is delayed at time n if and only if $x^{(u)}$ is not.

For a given $m > 1$, suppose that $X = (X(0), X(1), \ldots)$ is an infinite sequence of independent random variables, and $Y = (Y(0), Y(1), \ldots)$ is another such sequence, also independent of X, where all variables are uniformly distributed over $\{0, \ldots, m-1\}$.

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Conjecture 1.1 (Winkler, see [2]). *If m is sufficiently large then, with positive probability, X does not collide with Y.*

Actually, the conjecture is open even for $m = 4$. The sequences X, Y can be viewed as two independent random walks on the complete graph K_m, and then the problem is whether a 'clairvoyant demon', that is, a being who knows in advance both infinite sequences X and Y, can introduce delays into these walks in such a way that they never collide.

1.2. Graph reformulation

We define a graph $G = (V, E)$ as follows. $V = \mathbb{Z}^2_{>0}$ is the set of points (i, j) where i, j are positive integers. Let us define the *distance* of two points (i, j), (k, l) as

$$|k - i| + |l - j|$$

(L_1 distance). When representing the set V of points (i, j) graphically, the right direction is the one of increasing i, and the upward direction is the one of increasing j. The set E of edges consists of all pairs of the form $((i, j), (i + 1, j))$ and $((i, j), (i, j + 1))$.

Given X, Y as in the theorem, let us say that a point (i, j) has colour k if $X(i) = Y(j) = k$. Otherwise, it has colour -1, which we will call *white*. It is easy to see that X and Y do not collide if and only if there is an infinite directed path in G starting from $(0, 0)$ and remaining on white points. Indeed, each path corresponds to a delay sequence u such that $u(n) = 1$ if and only if the edge is horizontal. Thus, the two sequences do not collide if and only if the graph of white points 'percolates'. We will say that *there is percolation* if the probability that there is an infinite path is positive.

It has been shown independently in [1] and [3] that if the graph is undirected then there is percolation even for $m = 4$. In traditional percolation theory, when there is percolation then typically (unless the probability of blocking is at a 'critical point') the probability that there is percolation to a distance n but no percolation to infinity is exponentially small in n. This is also the case in the papers cited above, but it is not true for the directed percolation problem we are facing.

Theorem 1.2. *If there is percolation from the origin to infinity with positive probability, then the probability of percolating from the origin to distance n but not to infinity is at least $Cn^{-\alpha}$ for some constants $C, \alpha > 0$ depending only on m.*

2. The proof

Let $b_m = (0, 1, 2, \ldots, m - 1)$ be called the *basic colour sequence* of length m: it is simply the list of all different colours. Let b'_m be the reverse of b_m, that is, $b'_m(i) = b_m(m - i - 1)$. Let $\mathscr{E}_{n,k}$ be the event that, for all $i \in [0, k - 1]$, $j \in [0, m - 1]$, we have

$$Y(n + im + j - 1) = b'_m(j),$$

that is, starting with the index $n - 1$, the sequence Y has k consecutive repetitions of b'_m. We say that i is an index of the occurrence of b_m in the sequence X if $X(i + j) = b_m(j)$

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for $j \in [0, m-1]$. For $i > 0$, let τ_i be the ith index of occurrence of b_m in X. Let $\mathscr{F}_{n,k}$ be the event that, for all $i \in [1, n]$, we have

$$\tau_{i+1} - m - \tau_i \leqslant k - 1, \tag{2.1}$$

and also $\tau_1 \leqslant k - 1$.

Lemma 2.1. *If for some integer $n > 0$ both $\mathscr{E}_{n,k}$ and $\mathscr{F}_{n,k}$ hold, then there is no white infinite directed path.*

Proof. Let us assume that there is a white infinite path. Since there are n consecutive copies of b_m in X, the ith one starting at index τ_i, for each $p \in [1, n]$ there must be a vertical step in the path with an x projection in $[\tau_p, \tau_p + m - 1]$. Therefore the path ascends to the segment

$$[0, \tau_n + m - 1] \times \{n - 1\}$$

before its x projection reaches t_n.

For each $1 \leqslant p \leqslant n$, $0 \leqslant q < k$ there is a diagonally descending sequence of m coloured points

$$\{\, (\tau_p + j,\ n + (q+1)m - j - 2) : j \in [0, m-1] \,\}.$$

For a fixed p there are k such diagonal barriers stacked above each other, forming an impenetrable column of height km. The path would have to ascend between two of these columns, say between column i and $i + 1$. The distance of two consecutive columns from each other is

$$\tau_{i+1} - m - \tau_i \leqslant k - 1.$$

Since there are k consecutive copies of b'_m in Y starting at index $n-1$, for each $q \in [0, k-1]$ there must be a horizontal step in the path with a height in $n + qm - 1 + [0, m-1]$. Since the distance of the two columns is at most $k - 1$, it is not possible for the path to pass between the two columns. (The same holds for the space before the first column.) □

Lemma 2.2. *There is a constant α such that, for all $s > 0$, there is a k with*

$$\mathrm{Prob}(\mathscr{E}_{n,k}) \geqslant m^{-m} n^{-\alpha(s+1)},$$
$$\mathrm{Prob}(\mathscr{F}_{n,k}) \geqslant 1 - n^{-s}.$$

Proof. With

$$p_1 = m^{-m},$$

we have

$$\mathrm{Prob}(\mathscr{E}_{n,k}) = p_1^k.$$

Let us estimate the probability of $\mathscr{F}_{n,k}$. The probability that $\tau_1 > k - 1$ is upper-bounded by the probability that a copy of b_m does not begin at i in X for any i in $\{0, m, \ldots, \lfloor (k-1)/m \rfloor m\}$, which is

$$(1 - p_1)^{\lfloor (k-1)/m \rfloor + 1} \leqslant (1 - p_1)^{k/m} < e^{-p_1 k/m}.$$

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The same estimate holds for the probability of (2.1) assuming that the similar conditions for smaller i have already been satisfied. Hence

$$\mathrm{Prob}(\mathscr{F}_{n,k}) > 1 - ne^{-p_1 k/m}.$$

Let us choose

$$k = \left\lceil \frac{(s+1)m \log n}{p_1} \right\rceil \tag{2.2}$$

for some $s > 0$; then $\mathrm{Prob}(\mathscr{F}_{n,k}) > 1 - n^{-s}$, while

$$\mathrm{Prob}(\mathscr{E}_{n,k}) \geqslant p_1 n^{\frac{m \log p_1}{p_1}(s+1)}. \qquad \square$$

Proof of Theorem 1.2. Assume that there is an infinite path with some positive probability P_0. Choose s such that $n^{-s} < 0.5P_0$ and choose k as a function of s as in (2.2). Then the probability that $\mathscr{F}_{n,k}$ holds and there is an infinite path is at least $0.5P_0$. Let $\mathscr{G}_n$ be the event that there is a path leaving the rectangle $[0, \tau_n] \times [0, n-1]$. Then $\mathrm{Prob}(\mathscr{F}_{n,k} \wedge \mathscr{G}_n) \geqslant 0.5P_0$. Since $\mathscr{F}_{n,k} \wedge \mathscr{G}_n$ is independent of $\mathscr{E}_{n,k}$, we have

$$\mathrm{Prob}(\mathscr{E}_{n,k} \wedge \mathscr{F}_{n,k} \wedge \mathscr{G}_n) \geqslant 0.5P_0 m^{-m} n^{-\alpha(s+1)}. \qquad \square$$

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