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ON THE DEPTH OF SYMBOLIC POWERS OF EDGE IDEALS OF GRAPHS

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Abstract. Assume that G is a graph with edge ideal $I(G)$ and star packing number $\alpha_2(G)$. We denote the sth symbolic power of $I(G)$ by $I(G)^{(s)}$. It is shown that the inequality $\operatorname{depth} S/(I(G)^{(s)}) \geq \alpha_2(G) - s + 1$ is true for every chordal graph G and every integer $s \geq 1$. Moreover, it is proved that for any graph G, we have $\operatorname{depth} S/(I(G)^{(2)}) \geq \alpha_2(G) - 1$.

§1. Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Computing and finding bounds for the depth (or equivalently, projective dimension) of homogenous ideals of S and their powers have been studied by several authors (see, e.g., [3], [4], [5], [8], [9], [11], [12], [13]).

In [7], Fouli *et al.* introduced the notion of *initially regular sequence*. Using this notion, they provided a method for estimating the depth of a homogenous ideal. To be more precise, let $I \subset S$ be a homogenous ideal and let $\{x_{i,j} \mid 1 \leq i \leq q, 0 \leq j \leq t_i\}$ be a subset of distinct variables of S. Suppose $\mathrm{in}_<(I)$ is the initial ideal of I with respect to a fixed monomial order $<$ and assume that $\{u_1, \ldots, u_m\}$ is the set of minimal monomial generators of $\mathrm{in}_<(I)$. It is shown in [7, Theorem 3.11] that $\operatorname{depth} S/I \geq q$, provided that the following conditions hold.

(i) The monomials $u_1, u_2, \ldots, u_m$ are not divisible by $x_{i,j}^2$ for $1 \leq i \leq q$ and $1 \leq j \leq t_i$.
(ii) For $i = 1, 2, \ldots, q$, if a monomial in $\{u_1, \ldots, u_m\}$ is divisible by $x_{i,0}$, then it is also divisible by $x_{i,j}$, for some integer $1 \leq j \leq t_i$.

In Section 2, we provide an alternative proof for this result (see Proposition 2.1). Our proof is based on a short exact sequence argument, while in [7], the authors construct an initially regular sequence to prove their result.

Fouli *et al.* [6] observed that the above result provides a combinatorial lower bound for the depth of edge ideals of graphs. Indeed, for every graph G with edge ideal $I(G)$, we have

$$\operatorname{depth} S/I(G) \geq \alpha_2(G),$$

where $\alpha_2(G)$ denotes the so-called star packing number of G (see Section 2 for the definition of star packing number and see Corollary 2.2 for more details about the above inequality). It is proven in [6, Theorem 3.7] that the above inequality can be extended to powers of $I(G)$ when G is a forest. More precisely, for every forest G and for every integer $s \geq 1$, the inequality

$$\operatorname{depth} S/I(G)^s \geq \alpha_2(G) - s + 1 \qquad (\dagger)$$

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holds. On the other hand, we know from [16, Theorem 5.9] that for every forest G, the sth ordinary and symbolic powers of $I(G)$ coincide. Hence, inequality (†) essentially says that for every forest G and any positive integer s,

$$\operatorname{depth} S/I(G)^{(s)} \geq \alpha_2(G) - s + 1. \tag{‡}$$

In Theorem 3.4, we generalize [6, Theorem 3.7] by proving inequality (‡) for any chordal graph. Moreover, we show that inequality (‡) with $s = 2$ is true for any graph G (see Theorem 4.2).

§2. Preliminaries and known results

In this section, we provide the definitions and the known results which will be used in the next sections.

Let G be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. So, we identify the vertices of G with the variables of S. Also, by abusing the notation, every edge of G will be written by the product of the vertices. For a vertex x_i, the *neighbor set* of x_i is $N_G(x_i) = \{x_j \mid x_i x_j \in E(G)\}$. We set $N_G[x_i] = N_G(x_i) \cup \{x_i\}$ and call it the *closed neighborhood* of x_i. The cardinality of $N_G(x_i)$ is the *degree* of x_i and will be denoted by $\deg_G(x_i)$. For every subset $U \subset V(G)$, the graph $G \setminus U$ has vertex set $V(G \setminus U) = V(G) \setminus U$ and edge set $E(G \setminus U) = \{e \in E(G) \mid e \cap U = \emptyset\}$. A subgraph H of G is called *induced* provided that two vertices of H are adjacent if and only if they are adjacent in G. A graph G is called *chordal* if it has no induced cycle of length at least four. A subset W of $V(G)$ is a *clique* of G if every two distinct vertices of W are adjacent in G. A vertex x of G is a *simplicial vertex* if $N_G(x)$ is a clique. It is well-known that every chordal graph has a simplicial vertex. A subset C of $V(G)$ is a *vertex cover* of G if every edge of G is incident to at least one vertex of C. A vertex cover C is a *minimal vertex cover* if no proper subset of C is a vertex cover of G. The set of minimal vertex covers of G will be denoted by $\mathcal{C}(G)$. A subset A of $V(G)$ is called an *independent subset* of G if there are no edges among the vertices of A. Obviously, A is independent if and only if $V(G) \setminus A$ is a vertex cover of G.

The *edge ideal* of a graph G is defined as

$$I(G) = \big(x_i x_j \,|\, x_i x_j \in E(G)\big) \subset S.$$

For a subset C of $\{x_1, \ldots, x_n\}$, we denote by $\mathfrak{p}_C$, the monomial prime ideal which is generated by the variables belonging to C. It is well-known that for every graph G,

$$I(G) = \bigcap_{C \in \mathcal{C}(G)} \mathfrak{p}_C.$$

Let I be an ideal of S and let $\operatorname{Min}(I)$ denote the set of minimal primes of I. For every integer $s \geq 1$, the sth *symbolic power* of I, denoted by $I^{(s)}$, is defined to be

$$I^{(s)} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(I)} \operatorname{Ker}(S \to (S/I^s)_{\mathfrak{p}}).$$

Let I be a squarefree monomial ideal in S and suppose that I has the irredundant primary decomposition

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

where every $\mathfrak{p}_i$ is an ideal generated by a subset of the variables of S. It follows from [10, Proposition 1.4.4] that for every integer $s \geq 1$,

$$I^{(s)} = \mathfrak{p}_1^s \cap \cdots \cap \mathfrak{p}_r^s.$$

We set $I^{(s)} = S$, for any integer $s \leq 0$.

It is clear that for any graph G and every integer $s \geq 1$,

$$I(G)^{(s)} = \bigcap_{C \in \mathcal{C}(G)} \mathfrak{p}_C^s.$$

As it was mentioned in introduction, Fouli *et al.* [7] detected a method to bound the depth of a homogenous ideal. We provide an alternative proof for their result. Recall that for every monomial u and for every variable x_i, the degree of u with respect to x_i is denoted by $\deg_{x_i}(u)$. Also, for every monomial ideal I, the set of its minimal monomial generators is denoted by $G(I)$.

PROPOSITION 2.1 ([7], Theorem 3.11). *Let I be a proper homogenous ideal of S and let $<$ be a monomial order. Assume that $A = \{x_{i,j} \mid 1 \leq i \leq q, 0 \leq j \leq t_i\}$ is a subset of distinct variables of S, such that the following conditions are satisfied.*

(i) *For every pair of integers $1 \leq i \leq q$, $1 \leq j \leq t_i$ and for every $u \in G(\mathrm{in}_<(I))$, we have $\deg_{x_{i,j}}(u) \leq 1$.*
(ii) *For $i = 1, 2, \ldots, q$, if a monomial $u \in G(\mathrm{in}_<(I))$ is divisible by $x_{i,0}$, then it is also divisible by $x_{i,j}$, for some integer $1 \leq j \leq t_i$.*

Then $\operatorname{depth} S/I \geq q$.

Proof. It is known that $\operatorname{depth} S/I \geq \operatorname{depth} S/\mathrm{in}_<(I)$ (see, e.g., [10, Theorem 3.3.4]). Hence, replacing I by $\mathrm{in}_<(I)$, we may suppose that I is a monomial ideal. We use induction on $|A|$. There is nothing to prove for $|A| = 0$, as in this case $q = 0$. Therefore, assume that $|A| \geq 1$. If $t_i = 0$, for every $i = 1, 2, \ldots, q$, then it follows from condition (ii) that $x_{1,0}, \ldots, x_{q,0}$ do not divide the minimal monomial generators of I. In particular, they form a regular sequences on S/I and the assertion follows. Thus, suppose that $t_i \geq 1$, for some i with $1 \leq i \leq q$. Without loss of generality, suppose $i = 1$. Consider the following short exact sequence:

$$0 \longrightarrow S/(I : x_{1,t_1}) \longrightarrow S/I \longrightarrow S/(I, x_{1,t_1}) \longrightarrow 0.$$

This yields that

$$\operatorname{depth} S/I \geq \min\left\{\operatorname{depth} S/(I : x_{1,t_1}), \operatorname{depth} S/(I, x_{1,t_1})\right\}. \tag{1}$$

By condition (i), the variable x_{1,t_1} does not appear in the minimal monomial generators of $(I : x_{1,t_1})$. In particular, x_{1,t_1} is a regular element on $S/(I : x_{1,t_1})$. Let S' be the polynomial ring obtained from S by deleting the variable x_{1,t_1} (in other words, $S' \cong S/(x_{1,t_1})$). Set $I' := (I : x_{1,t_1}) \cap S'$. It follows that

$$\operatorname{depth} S/(I : x_{1,t_1}) = \operatorname{depth} S/((I : x_{1,t_1}), x_{1,t_1}) + 1 = \operatorname{depth} S'/I' + 1.$$

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Clearly, I' satisfies the assumptions with respect to the set $\{x_{i,j} \mid 2 \le i \le q, 0 \le j \le t_i\}$ of variables. Thus, the induction hypothesis implies that $\operatorname{depth} S'/I' \ge q-1$. Hence, we deduce from the above equalities that

$$\operatorname{depth} S/(I : x_{1,t_1}) \ge q.$$

Using inequality (1), it suffices to prove that $\operatorname{depth} S/(I, x_{1,t_1}) \ge q$. Set $I'' := I \cap S'$. Then $S/(I, x_{1,t_1}) \cong S'/I''$. Put $t_1' := t_1 - 1$ and $t_i' := t_i$, for $i = 2, \ldots, q$. Obviously, I'' satisfies the assumptions with respect to the set $\{x_{i,j} \mid 1 \le i \le q, 0 \le j \le t_i'\}$ of variables. Therefore, we conclude from the induction hypothesis that

$$\operatorname{depth} S/(I, x_{1,t_1}) = \operatorname{depth} S'/I'' \ge q. \qquad \square$$

Let G be a graph and x be a vertex of G. The subgraph $\mathrm{St}(x)$ of G with vertex set $N_G[x]$ and edge set $\{xy \mid y \in N_G(x)\}$ is called a *star with center x*. A *star packing* of G is a family $\mathcal{X}$ of stars in G which are pairwise disjoint, that is, $V(\mathrm{St}(x)) \cap V(\mathrm{St}(x')) = \emptyset$, for $\mathrm{St}(x), \mathrm{St}(x') \in \mathcal{X}$ with $x \ne x'$. The quantity

$$\max\left\{|\mathcal{X}| \mid \mathcal{X} \text{ is a star packing of } G\right\}$$

is called the *star packing number* of G. Following [6], we denote the star packing number of G by $\alpha_2(G)$.

The following corollary is an immediate consequence of Proposition 2.1, and it was indeed observed in [6].

Corollary 2.2 ([6]). *For every graph G, we have*

$$\operatorname{depth} S/I(G) \ge \alpha_2(G).$$

Proof. Let $x_{1,0}, \ldots, x_{q,0}$ be the centers of stars in a largest star packing of G. Moreover, for $1 \le i \le q$, assume that $N_G(x_{i,0}) = \{x_{i,1}, \ldots, x_{i,t_i}\}$. Then the assumptions of Proposition 2.1 are satisfied and it follows that

$$\operatorname{depth} S/I(G) \ge q = \alpha_2(G). \qquad \square$$

§3. Symbolic powers of edge ideals of chordal graphs

In this section, we prove the first main result of this paper, Theorem 3.4 which states that inequality (‡) is true for every chordal graph G and for any integer $s \ge 1$. In order to prove this result, we first need to estimate the star packing number of the graph obtained from G by deleting a certain subset of its vertices. This will be done in the following two lemmas.

Lemma 3.1. *Let G be a graph and let W be a subset of $V(G)$. Then for every $A \subseteq \bigcup_{x \in W} N_G[x]$, we have*

$$\alpha_2(G \setminus A) \ge \alpha_2(G) - |W|.$$

Proof. Let $\mathcal{S}$ be the set of the centers of stars in a largest star packing of G. In particular, $|\mathcal{S}| = \alpha_2(G)$. Since every vertex in A belongs to the closed neighborhood of a vertex in W, it follows from the definition of star packing that $|\mathcal{S} \cap A| \le |W|$. Then the stars in $G \setminus A$

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centered at the vertices in $\mathcal{S}\setminus A$ form a star packing in $G\setminus A$ of size at least $\alpha_2(G)-|W|$. Therefore, $\alpha_2(G\setminus A)\geq\alpha_2(G)-|W|$. ∎

Lemma 3.2. *Assume that G is a graph and $W=\{x_1,\dots,x_d\}$ is a clique of G. Let A be a subset of $V(G)$ such that*

(i) $A\subseteq\bigcup_{i=1}^{d}N_G(x_i)$,
(ii) $N_G(x_1)\setminus\{x_2,\dots,x_d\}\subseteq A$, *and*
(iii) $x_1\notin A$.

Then $\alpha_2(G\setminus A)\geq\alpha_2(G)-d+1$.

Proof. Let $\mathcal{S}$ be the set of the centers of stars in a largest star packing of G. Thanks to (i), similar to the proof of Lemma 3.1, we have $|\mathcal{S}\cap A|\leq d$. If $|\mathcal{S}\cap A|\leq d-1$, then the stars in $G\setminus A$ centered at the vertices in $\mathcal{S}\setminus A$ form a star packing in $G\setminus A$ of size at least $\alpha_2(G)-d+1$. Thus, the assertion follows in this case. Therefore, suppose $|\mathcal{S}\cap A|=d$. In this case, we have

$$x_1,\dots,x_d\in\bigcup_{x\in\mathcal{S}\cap A}N_G(x).$$

It again follows from the definition of star packing that

$$x_1,\dots,x_d\notin\bigcup_{x\in\mathcal{S}\setminus A}N_{G\setminus A}(x).$$

Therefore, we conclude from condition (ii) that

$$N_{G\setminus A}[x_1]\cap\Big(\bigcup_{x\in\mathcal{S}\setminus A}N_{G\setminus A}(x)\Big)\subseteq\{x_1,\dots,x_d\}\cap\Big(\bigcup_{x\in\mathcal{S}\setminus A}N_{G\setminus A}(x)\Big)=\emptyset.$$

As a consequence, the stars in $G\setminus A$ centered at the vertices in $(\mathcal{S}\setminus A)\cup\{x_1\}$ form a star packing in $G\setminus A$ of size $\alpha_2(G)-d+1$. This completes the proof of the lemma. ∎

We are now ready to prove that inequality (‡) holds for any chordal graph. Indeed, we are able to prove the following stronger result.

Proposition 3.3. *Let G be a chordal graph. Suppose H and H' are subgraphs of G with*

$$E(H)\cap E(H')=\emptyset\quad\text{and}\quad E(H)\cup E(H')=E(G).$$

Assume further that H is a chordal graph. Then for every integer $s\geq 1$,

$$\operatorname{depth}S/(I(H)^{(s)}S+I(H')S)\geq\alpha_2(G)-s+1.$$

Proof. As the isolated vertices have no effect on edge ideals, we assume that $V(H)=V(H')=V(G)$ (i.e., we extend the vertex sets of H and H' to $V(G)$). We use induction on $s+|E(H)|$. For $s=1$, we have $I(H)^{(s)}+I(H')=I(G)$ and the assertion follows from Corollary 2.2. Therefore, suppose $s\geq 2$. If $E(H)=\emptyset$, then $I(H')=I(G)$ and again we have the required inequality by Corollary 2.2. Hence, we assume $|E(H)|\geq 1$.

To simplify the notations, we set $I:=I(H)^{(s)}+I(H')$. Since H is a chordal graph, it has a simplicial vertex, say x_1, with nonzero degree. Without loss of generality, suppose $N_H(x_1) = \{x_2, \ldots, x_d\}$, for some integer $d \geq 2$. Consider the following short exact sequence:

$$0 \longrightarrow \frac{S}{(I : x_1 \cdots x_d)} \longrightarrow \frac{S}{I} \longrightarrow \frac{S}{(I, x_1 \cdots x_d)} \longrightarrow 0.$$

Using depth lemma [2, Proposition 1.2.9], we have

$$\operatorname{depth} S/I \geq \min\{\operatorname{depth} S/(I : x_1 \cdots x_d), \operatorname{depth} S/(I, x_1 \cdots x_d)\}. \tag{2}$$

By assumption, for every pair of integers $i \neq j$, with $1 \leq i, j \leq d$ we have $x_i x_j \in E(H)$. Therefore, $x_i x_j$ is not an edge of H'. Set

$$U := \bigcup_{i=1}^{d} N_{H'}[x_i]$$

and

$$U' := \bigcup_{i=1}^{d} N_{H'}(x_i).$$

Then using [14, Lemma 2], we have

$$\begin{aligned}(I : x_1 \cdots x_d) &= \big((I(H)^{(s)} + I(H')) : x_1 \cdots x_d\big)\\ &= I(H)^{(s-d+1)} + I(H' \setminus U) + \big(\text{the ideal generated by } U'\big)\\ &= I(H \setminus U')^{(s-d+1)} + I(H' \setminus U) + \big(\text{the ideal generated by } U'\big).\end{aligned}$$

This yields that

$$\operatorname{depth} S/(I : x_1 \cdots x_d) = \operatorname{depth} S'/(I(H \setminus U')^{(s-d+1)} + I(H' \setminus U)),$$

where $S' = \mathbb{K}[x_i : 1 \leq i \leq n, i \notin U']$. Let G' be the union of $H \setminus U'$ and $H' \setminus U$. In fact, G' is the induced subgraph of G on $V(G) \setminus U'$. Clearly, $N_G(x_1) \setminus \{x_2, \ldots, x_d\}$ is contained in U'. Then the above equality together with Lemma 3.2 and the induction hypothesis implies that

$$\operatorname{depth} S/(I : x_1 \cdots x_d) \geq \alpha_2(G \setminus U') - (s - d + 1) + 1 \geq \alpha_2(G) - s + 1. \tag{3}$$

Using inequalities (2) and (3), it is enough to prove that

$$\operatorname{depth} S/(I, x_1 \cdots x_d) \geq \alpha_2(G) - s + 1.$$

For every integer k with $1 \leq k \leq d-1$, let J_k be the ideal generated by all the squarefree monomials of degree k on variables $x_2, \ldots, x_d$. We continue in the following steps.

Step 1. Let $1 \leq k \leq d-2$ be a fixed integer and assume that $\{u_1, \ldots, u_t\}$ is the set of minimal monomial generators of $x_1 J_k$. In particular, every u_j is divisible by x_1 and $\deg(u_j) = k+1$. For every integer j with $1 \leq j \leq t$, we prove that

$$\begin{aligned}&\operatorname{depth} S/(I + x_1 J_{k+1} + (u_1, \ldots, u_{j-1}))\\ &\geq \min\{\operatorname{depth} S/(I + x_1 J_{k+1} + (u_1, \ldots, u_j)), \alpha_2(G) - s + 1\}.\end{aligned}$$

(Note that for $j = 1$, we have $I + x_1 J_{k+1} + (u_1, \ldots, u_{j-1}) = I + x_1 J_{k+1}$.)

Consider the following short exact sequence.

$$0 \longrightarrow \frac{S}{(I+x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j} \longrightarrow \frac{S}{I+x_1J_{k+1}+(u_1,\dots,u_{j-1})}$$
$$\longrightarrow \frac{S}{I+x_1J_{k+1}+(u_1,\dots,u_j)} \longrightarrow 0.$$

As a consequence,

$$\operatorname{depth} S/(I+x_1J_{k+1}+(u_1,\dots,u_{j-1})) \geq$$
$$\min\big\{\operatorname{depth} S/((I+x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j), \operatorname{depth} S/(I+x_1J_{k+1}+(u_1,\dots,u_j))\big\}.$$

Therefore, to complete this step, it is sufficient to show that

$$\operatorname{depth} S/((I+x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j) \geq \alpha_2(G)-s+1.$$

Set

$$U_j := \{x_i \mid 1 \leq i \leq d \text{ and } x_i \text{ does not divide } u_j\}.$$

For any $x_i \in U_j$, the monomial x_iu_j is a squarefree monomial of degree $k+2$. Hence, x_iu_j belongs to x_1J_{k+1}. This shows that

$$(\text{the ideal generated by } \ U_j) \subseteq \big((x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j\big).$$

We show the reverse inclusion holds too.

Since $x_1J_{k+1}+(u_1,\dots,u_{j-1})$ is a squarefree monomial ideal, it follows that

$$\big((x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j\big)$$

is also a squarefree monomial ideal. On the other hand, every monomial generator of $x_1J_{k+1}+(u_1,\dots,u_{j-1})$ is a monomial over $x_1,\dots,x_d$. This implies that every monomial generator of $\big((x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j\big)$ is also a squarefree monomial over the variables $x_1,\dots,x_d$. Assume that v is a minimal generator of $\big((x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j\big)$. If v is not equal to any of the variables belonging to U_j, then by definition of U_j, every variable dividing v, also divides u_j. As v is a squarefree monomial, we have $v \mid u_j$. Since

$$u_jv \in x_1J_{k+1}+(u_1,\dots,u_{j-1}),$$

we deduce that

$$u_j^2 \in x_1J_{k+1}+(u_1,\dots,u_{j-1}),$$

which implies that

$$u_j \in x_1J_{k+1}+(u_1,\dots,u_{j-1}),$$

because $x_1J_{k+1}+(u_1,\dots,u_{j-1})$ is a squarefree monomial ideal. This is contradiction, as the degree of u_j is strictly less than the degree of any monomial in x_1J_{k+1} and $u_j \notin (u_1,\dots,u_{j-1})$. Hence,

$$\big((x_1J_{k+1}+(u_1,\dots,u_{j-1})):u_j\big) = (\text{the ideal generated by} U_j). \tag{4}$$

Let W_j be the set of variables dividing u_j. In other words, $W_j = \{x_1,\dots,x_d\} \setminus U_j$. We remind that for any pair of integers $1 \leq i,j \leq d$, the vertices x_i and x_j are not adjacent in H'. Set

$$U'_j := \bigcup_{x_i \in W_j} N_{H'}[x_i]$$

and

$$U''_j := \bigcup_{x_i \in W_j} N_{H'}(x_i).$$

Using equality (4), we conclude that

$$\begin{aligned}
&\big((I + x_1 J_{k+1} + (u_1,\ldots,u_{j-1})) : u_j\big)\\
&= \big((I(H)^{(s)} + I(H') + x_1 J_{k+1} + (u_1,\ldots,u_{j-1})) : u_j\big)\\
&= (I(H)^{(s)} : u_j) + I(H' \setminus U'_j) + (\text{the ideal generated by } \ U_j \cup U''_j)\\
&= \big(I\big(H \setminus (U_j \cup U''_j)\big)^{(s)} : u_j\big) + I\big(H' \setminus (U_j \cup U'_j)\big)\\
&+ (\text{the ideal generated by } U_j \cup U''_j).
\end{aligned}$$

Set $H_j := H \setminus (U_j \cup U''_j)$ and $H'_j := H' \setminus (U_j \cup U'_j)$. Then H_j is a chordal graph, and x_1 is a simplicial vertex of H_j. It is also clear that $N_{H_j}[x_1]$ is the set of variables divining u_j. It thus follows from [14, Lemma 2] and the above equalities that

$$\begin{aligned}
&\big((I + x_1 J_{k+1} + (u_1,\ldots,u_{j-1})) : u_j\big) = I(H_j)^{(s-k)} + I(H'_j)\\
&+ (\text{the ideal generated by } \ U_j \cup U''_j).
\end{aligned}$$

This yields that

$$\operatorname{depth} S/((I + x_1 J_{k+1} + (u_1,\ldots,u_{j-1})) : u_j) = \operatorname{depth} S_j/(I(H_j)^{(s-k)} + I(H'_j)),$$

where $S_j = \mathbb{K}[x_i : 1 \le i \le n, i \notin U_j \cup U''_j]$. Let G_j be the union of H_j and H'_j. Then G_j is the induced subgraph of G on $V(G) \setminus (U_j \cup U''_j)$. We conclude from Lemma 3.2 (by considering the clique W_j) that

$$\alpha_2(G_j) \ge \alpha_2(G) - |W_j| + 1 = \alpha_2(G) - k,$$

where the last equality follows from the fact that $\deg(u_j) = k+1$. Hence, the induction hypothesis implies that

$$\begin{aligned}
&\operatorname{depth} S/((I + x_1 J_{k+1} + (u_1,\ldots,u_{j-1})) : u_j) = \operatorname{depth} S_j/(I(H_j)^{(s-k)} + I(H'_j))\\
&\ge \alpha_2(G_j) - (s-k) + 1 \ge \alpha_2(G) - s + 1,
\end{aligned}$$

and this step is complete.

Step 2. Let $1 \le k \le d-2$ be a fixed integer. By a repeated use of Step 1, we have

$$\begin{aligned}
&\operatorname{depth} S/(I + x_1 J_{k+1}) \ge \min\big\{\operatorname{depth} S/(I + x_1 J_{k+1} + (u_1,\ldots,u_t)), \alpha_2(G) - s + 1\big\}\\
&= \min\big\{\operatorname{depth} S/(I + x_1 J_k), \alpha_2(G) - s + 1\big\}.
\end{aligned}$$

Step 3. It follows from Step 2 that

$$\begin{aligned}
&\operatorname{depth} S/(I, x_1 \cdots x_d) = \operatorname{depth} S/(I + x_1 J_{d-1})\\
&\ge \min\big\{\operatorname{depth} S/(I + x_1 J_{d-2}), \alpha_2(G) - s + 1\big\}\\
&\ge \min\big\{\operatorname{depth} S/(I + x_1 J_{d-3}), \alpha_2(G) - s + 1\big\}\\
&\ge \cdots \ge \min\big\{\operatorname{depth} S/(I + x_1 J_1), \alpha_2(G) - s + 1\big\}.
\end{aligned}$$

In particular,

$$\operatorname{depth} S/(I, x_1 \cdots x_d) \geq \min\left\{\operatorname{depth} S/\big(I + (x_1x_2, x_1x_3, \ldots, x_1x_d)\big), \alpha_2(G) - s + 1\right\}. \tag{5}$$

Step 4. Let L be the graph obtained from H, by deleting the edges $x_1x_2, \ldots, x_1x_d$. Then L is the disjoint union of $H \setminus x_1$ and the isolated vertex x_1. In particular, L is a chordal graph. Also, let L' be the graph obtained from H', by adding the edges $x_1x_2, \ldots, x_1x_d$. Then

$$E(L) \cap E(L') = \emptyset \quad \textit{and} \quad E(L) \cup E(L') = E(G).$$

It follows from [15, Lemma 3.2] and the induction hypothesis that

$$\begin{aligned}
&\operatorname{depth} S/(I + (x_1x_2, x_1x_3, \ldots, x_1x_d)) \\
&= \operatorname{depth} S/\big(I(H)^{(s)} + I(H') + (x_1x_2, x_1x_3, \ldots, x_1x_d)\big) \\
&= \operatorname{depth} S/((I(L)^{(s)} + I(L'))) \geq \alpha_2(G) - s + 1.
\end{aligned}$$

Finally, inequality (5) implies that

$$\operatorname{depth} S/(I, x_1 \cdots x_d) \geq \alpha_2(G) - s + 1. \tag{6}$$

Now, inequalities (2), (3), and (6) complete the proof of the proposition. ∎

The following theorem is the main result of this section and follows easily from Proposition 3.3.

Theorem 3.4. *Let G be a chordal graph. Then for every integer $s \geq 1$, we have*

$$\operatorname{depth} S/(I(G)^{(s)}) \geq \alpha_2(G) - s + 1.$$

Proof. The assertion follows from Proposition 3.3 by substituting $H = G$ and $H' = \emptyset$. ∎

§4. Second symbolic power of edge ideals

The aim of this section is to show that inequality (‡) is true for $s = 2$ (Theorem 4.2). To prove this result, we need to bound the depth of ideals of the form $\big(I(G)^{(k)} : xy\big)$, where xy is an edge of G. To achieve this goal, we will use the following lemma in the case of $k = 2$.

Lemma 4.1. *Let G be a graph and xy be an edge of G. Then for any integer $k \geq 2$, we have*

$$\big(I(G)^{(k)} : xy\big) = \big(I(G)^{(k-1)} : x\big) \cap \big(I(G)^{(k-1)} : y\big).$$

Proof. Let u be a monomial in $\big(I(G)^{(k)} : xy\big)$. Then $uxy \in I(G)^{(k)}$. Clearly, this implies that $ux \in I(G)^{(k-1)}$. Therefore, $u \in \big(I(G)^{(k-1)} : x\big)$. Similarly, u belongs to $\big(I(G)^{(k-1)} : y\big)$. Hence,

$$\big(I(G)^{(k)} : xy\big) \subseteq \big(I(G)^{(k-1)} : x\big) \cap \big(I(G)^{(k-1)} : y\big).$$

To prove the reverse inclusion, let v be a monomial in

$$\big(I(G)^{(k-1)} : x\big) \cap \big(I(G)^{(k-1)} : y\big).$$

We must show that $vxy \in I(G)^{(k)}$. It is enough to prove that for any minimal vertex cover C of G, we have $vxy \in \mathfrak{p}_C^k$. So, let C be a minimal vertex cover of G. It follows from $xy \in E(G)$ that C contains at least one of the vertices x and y. Without loss of generality, suppose

$x \in C$. Since $v \in \big(I(G)^{(k-1)} : y\big)$, we have $vy \in I(G)^{(k-1)} \subseteq \mathfrak{p}_C^{k-1}$. This together with $x \in \mathfrak{p}_C$ implies that $vxy \in \mathfrak{p}_C^k$. ∎

The following theorem is the second main result of this paper.

THEOREM 4.2. *For any graph G, we have*

$$\operatorname{depth} S/I(G)^{(2)} \geq \alpha_2(G) - 1.$$

Proof. Set $I := I(G)$ and let $G(I) = \{u_1, \ldots, u_m\}$ be the set of minimal monomial generators of I. For every integer i with $1 \leq i \leq m$ consider the short exact sequence

$$0 \longrightarrow \frac{S}{(I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i} \longrightarrow \frac{S}{I^{(2)} + (u_1, \ldots, u_{i-1})} \longrightarrow \frac{S}{I^{(2)} + (u_1, \ldots, u_i)} \longrightarrow 0,$$

where for $i = 1$, the ideal $(u_1, \ldots, u_{i-1})$ is the zero ideal. It follows from depth Lemma [2, Proposition 1.2.9] that

$$\begin{aligned} &\operatorname{depth} S/(I^{(2)} + (u_1, \ldots, u_{i-1})) \\ &\geq \min\big\{\operatorname{depth} S/((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i), \operatorname{depth} S/(I^{(2)} + (u_1, \ldots, u_i))\big\}. \end{aligned}$$

Using the above inequalities inductively, we have

$$\begin{aligned} &\operatorname{depth} S/I^{(2)} \\ &\geq \min\Big\{\operatorname{depth} S/(I^{(2)} + I), \min\big\{\operatorname{depth} S/((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i) \mid 1 \leq i \leq m\big\}\Big\} \\ &= \min\big\{\operatorname{depth} S/I, \operatorname{depth} S/((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i) \mid 1 \leq i \leq m\big\} \\ &\geq \min\big\{\alpha_2(G), \operatorname{depth} S/((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i) \mid 1 \leq i \leq m\big\}, \end{aligned}$$

where the last inequality follows from Corollary 2.2. Hence, it is enough to show that

$$\operatorname{depth} S/((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i) \geq \alpha_2(G) - 1,$$

for every integer i with $1 \leq i \leq m$.

Fix an integer i with $1 \leq i \leq m$ and assume that $u_i = xy$. By [1, Theorem 4.12], for every pair of integers $1 \leq j < i \leq m$, one of the following conditions holds.

(i) $(u_j : u_i) \subseteq (I^2 : u_i) \subseteq (I^{(2)} : u_i)$; or
(ii) there exists an integer $k \leq i - 1$ such that $(u_k : u_i)$ is generated by a variable, and $(u_j : u_i) \subseteq (u_k : u_i)$.

We know from (i) and (ii) above that

$$\big((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i\big) = (I^{(2)} : u_i) + (\text{some variables}). \tag{7}$$

Let A be the set of variables belonging to $\big((I^{(2)} + (u_1, \ldots, u_{i-1})) : u_i\big)$. Assume that $x \in A$. This means that x^2y belongs to the ideal $I^{(2)} + (u_1, \ldots, u_{i-1})$. Since $u_1, \ldots, u_{i-1}$ do not divide x^2y, we deduce that $x^2y \in I(G)^{(2)}$. But this is a contradiction, as $C := V(G) \setminus \{x\}$ is a vertex cover of G with $x^2y \notin \mathfrak{p}_C^2$. Therefore, $x \notin A$. Similarly, $y \notin A$. It follows from $x, y \notin A$ and equality (7) that

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$$\begin{aligned}\big((I^{(2)}+(u_1,\ldots,u_{i-1})):u_i\big)&=\big(I^{(2)}:u_i\big)+(\text{the ideal generated by } A)\\&=\big(I^{(2)}+(\text{the ideal generated by } A)):u_i\big)\\&=\big(I(G\setminus A)^{(2)}+(\text{the ideal generated by } A)):u_i\big)\\&=\big(I(G\setminus A)^{(2)}:u_i\big)+(\text{the ideal generated by } A).\end{aligned}$$

Therefore,

$$\operatorname{depth} S/((I^{(2)}+(u_1,\ldots,u_{i-1})):u_i)=\operatorname{depth} S_A/(I(G\setminus A)^{(2)}:u_i), \tag{8}$$

where $S_A=\mathbb{K}[x_i:1\le i\le n, x_i\notin A]$. It follows from Lemma 4.1 that

$$(I(G\setminus A)^{(2)}:u_i)=(I(G\setminus A):x)\cap(I(G\setminus A):y).$$

Consider the following short exact sequence.

$$\begin{aligned}0\longrightarrow \frac{S_A}{(I(G\setminus A)^{(2)}:u_i)}\longrightarrow \frac{S_A}{(I(G\setminus A):x)}\oplus\frac{S_A}{(I(G\setminus A):y)}\\ \longrightarrow \frac{S_A}{(I(G\setminus A):x)+(I(G\setminus A):y)}\longrightarrow 0.\end{aligned}$$

Applying depth lemma [2, Proposition 1.2.9] on the above exact sequence, it suffices to prove that

(a) $\operatorname{depth} S_A/(I(G\setminus A):x)\ge \alpha_2(G)-1$,
(b) $\operatorname{depth} S_A/(I(G\setminus A):y)\ge \alpha_2(G)-1$, and
(c) $\operatorname{depth} S_A/((I(G\setminus A):x)+(I(G\setminus A):y))\ge \alpha_2(G)-2$.

To prove (a), note that

$$\begin{aligned}(I(G\setminus A):x)&=I(G\setminus(A\cup N_{G\setminus A}[x]))+(\text{the ideal generated by } N_{G\setminus A}(x))\\&=I(G\setminus(A\cup N_G[x]))+(\text{the ideal generated by } N_{G\setminus A}(x)).\end{aligned}$$

Hence,

$$\operatorname{depth} S_A/(I(G\setminus A):x)=\operatorname{depth} S'/I(G\setminus(A\cup N_G[x])), \tag{9}$$

where $S'=\mathbb{K}[x_i:1\le i\le n, x_i\notin A\cup N_G(x)]$. Obviously, x is a regular element of $S'/I(G\setminus (A\cup N_G[x]))$. Therefore, Corollary 2.2 implies that

$$\operatorname{depth} S'/I(G\setminus(A\cup N_G[x]))\ge \alpha_2(G\setminus(A\cup N_G[x]))+1. \tag{10}$$

Assume that $A\subseteq N_G(x)\cup N_G(y)$. It then follows from Lemma 3.1 that

$$\alpha_2(G\setminus(A\cup N_G[x]))\ge\alpha_2(G)-2.$$

Hence, we conclude from equality (9) and inequality (10) that

$$\operatorname{depth} S_A/(I(G\setminus A):x)\ge\alpha_2(G)-1.$$

Thus, to complete the proof of (a), we only need to show that $A\subseteq N_G(x)\cup N_G(y)$.

Let z be an arbitrary variable in A and suppose $z\notin N_G(x)\cup N_G(y)$. Then the only edge dividing $zu_i=zxy$ is u_i. In particular,

$$z\notin\big((u_1,\ldots,u_{i-1}):u_i\big). \tag{11}$$

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Moreover, since $\{z,x\}$ is an independent subset of G, we conclude that $C' = V(G) \setminus \{z,x\}$ is a vertex cover of G with $zu_i = zxy \notin \mathfrak{p}_{C'}^2$. Thus, $zu_i \notin I^{(2)}$. This means that $z \notin (I^{(2)} : u_i)$. This, together with (11) implies that

$$z \notin \big((I^{(2)}, u_1, \ldots, u_{i-1}) : u_i\big),$$

which is a contradiction. Therefore, $z \in N_G(x) \cup N_G(y)$. Hence, $A \subseteq N_G(x) \cup N_G(y)$ and this completes the proof of (a). The proof of (b) is similar to the proof of (a). We now prove (c).

Note that

$$\begin{aligned}
&(I(G \setminus A) : x) + (I(G \setminus A) : y) \\
&= I(G \setminus (A \cup N_{G\setminus A}[x] \cup N_{G\setminus A}[y])) + (\text{the ideal generated by } N_{G\setminus A}[x] \cup N_{G\setminus A}[y]) \\
&= I(G \setminus (A \cup N_G[x] \cup N_G[y])) + (\text{the ideal generated by } \ N_{G\setminus A}[x] \cup N_{G\setminus A}[y]) \\
&= I(G \setminus (N_G[x] \cup N_G[y])) + (\text{the ideal generated by } \ N_{G\setminus A}[x] \cup N_{G\setminus A}[y]),
\end{aligned}$$

where the last equality follows from $A \subseteq N_G(x) \cup N_G(y)$. We conclude that

$$\operatorname{depth} S_A/((I(G \setminus A) : x) + (I(G \setminus A) : y)) = \operatorname{depth} S''/I(G \setminus (N_G[x] \cup N_G[y])), \qquad (12)$$

where $S'' = \mathbb{K}\big[x_i : 1 \leq i \leq n, x_i \notin N_G[x] \cup N_G[y]\big]$. Using Corollary 2.2 and Lemma 3.1, we deuce that

$$\operatorname{depth} S''/I(G \setminus (N_G[x] \cup N_G[y])) \geq \alpha_2(G \setminus (N_G[x] \cup N_G[y])) \geq \alpha_2(G) - 2.$$

Finally, the assertion of (c) follows from equality (12) and the above inequality. This completes the proof of the theorem. ∎

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