# THE CLASSIFICATION OF ALGEBRAS BY DOMINANT DIMENSION

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**1. Introduction.** Nakayama proposed to classify finite-dimensional algebras $R$ over a field according to how long an exact sequence

$$0 \to R \to X_1 \to \ldots \to X_n$$

of projective and injective $R$–$R$-bimodules $X_i$ they allow. He conjectured that if there exists an infinite sequence of this type, then $R$ must be quasi-Frobenius; and he proved this when $R$ is generalized uniserial **(17)**.

Tachikawa considered similar sequences where the $X_i$ are projective-injective $R$-right-modules, and he called the biggest possible $n$ the dominant dimension of $R$ **(19)**. It is well known that the QF-3 algebras are just those that allow a positive $n$, in both cases **(7; 8)**.

Morita has shown that the endomorphism-ring $R$ of a fully faithful module $X$ over an arbitrary algebra $A$ is QF-3 and that every QF-3 algebra is related to one of that type in a certain way **(13)**. Tachikawa proved Nakayama's above conjecture for endomorphism-rings $R$ of fully faithful modules $X$ over generalized uniserial rings $A$ **(19)**.

Our main results are as follows: (1) Nakayama's dimension, Tachikawa's (right) dominant dimension, and the analogously defined left dominant dimension are equal for any algebra $R$. (2) The class dom. dim. $R \geqslant 2$ is just the class of endomorphism-rings $R$ of fully faithful (finitely generated) modules over arbitrary algebras $A$. (3) The dominant dimension can be characterized, in two different ways, by the Ext-functors. (4) Nakayama's above conjecture holds if $R$ is the endomorphism-ring of a fully faithful module which has an ultimately closed projective resolution (in the sense of Jans **(8)**; this generalizes the results of Nakayama and Tachikawa on the subject).

Most methods used here will work in more general situations, e.g. for rings with suitable chain conditions instead of algebras. We may come back to this question later.

**2. Preliminaries.** All rings under consideration are finite-dimensional algebras with unit element over a field $K$; all modules are unitary and finitely generated. The $K$-dual $\mathrm{Hom}_K(M, K)$ of an $R$-module $M$ is denoted by $M^*$; it is an $R$-module on the opposite side.

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*Definition.* For any $R$-module $M$, we say that its dominant dimension is greater than or equal to $n$, dom. dim. $M \geqslant n$ (where $n$ is a non-negative integer), if there exists an exact sequence $0 \to M \to X_1 \to \ldots \to X_n$ with $R$-modules $X_i$ which are projective and injective.

Such a sequence can be considered as the beginning of an injective resolution. In general, any $R$-module $M$ possesses a so-called *minimal injective resolution* (determined up to isomorphisms)

$$0 \to M \xrightarrow{\alpha_0} Q_0 \xrightarrow{\alpha_1} Q_1 \to \ldots$$

which is constructed by taking as $Q_0$ the injective hull of $M$, and as $Q_{n+1}$ the injective hull of the cokernel $\operatorname{Cok}(\alpha_n)$ **(5)**.

LEMMA 1. *Let*

$$0 \to M \xrightarrow{\alpha_0} Q_0 \xrightarrow{\alpha_1} Q_1 \to \ldots$$

*be the minimal injective resolution and*

$$0 \xrightarrow{\beta_{-1} = 0} M \xrightarrow{\beta_0} Y_0 \xrightarrow{\beta_1} Y_1 \to \ldots$$

*be an arbitrary injective resolution of the $R$-module $M$. Then there exist $R$-modules $Z_0, Z_1, \ldots$ such that $Y_0 \cong Q_0 \oplus Z_0$, $Y_n \cong Q_n \oplus Z_n \oplus Z_{n-1}$ for $n \geqslant 1$, and $\operatorname{Cok}(\beta_k) \cong \operatorname{Cok}(\alpha_k) \oplus Z_k$ for $k \geqslant 0$. $Z_k = 0$ holds if and only if $Y_k$ is the injective hull of $\operatorname{Cok}(\beta_{k-1})$.*

*Proof.* The injective hull is a direct summand in every injective extension; hence $Y_0 \cong Q_0 \oplus Z_0$ and $Z_0 = 0$ if and only if $Y_0$ actually is the injective hull. The map $\beta_0$ and its cokernel may be written as

$$\begin{array}{ccccc} M & \xrightarrow{\alpha_0} & Q_0 & \to & \operatorname{Cok}(\alpha_0) \\ & & \oplus & & \oplus \\ & & Z_0 & \xrightarrow{1} & Z_0 \end{array}$$

The injective hull of $\operatorname{Cok}(\beta_0) = \operatorname{Cok}(\alpha_0) \oplus Z_0$ is $Q_1 \oplus Z_0$; hence we get $Y_1 \cong Q_1 \oplus Z_0 \oplus Z_1$, $Z_1 = 0$ if and only if $Y_1$ is the injective hull of $\operatorname{Cok}(\beta_0)$, and the map $\beta_1$ and its cokernel may be written as

$$\begin{array}{ccccc} Q_0 & \xrightarrow{\alpha_1} & Q_1 & \to & \operatorname{Cok}(\alpha_1) \\ \oplus & & \oplus & & \oplus \\ Z_0 & & Z_1 & \xrightarrow{1} & Z_1 \\ & \overset{1}{\searrow} & \oplus & & \\ & & Z_0 & \xrightarrow{0} & \end{array}$$

Again, the injective hull of $\operatorname{Cok}(\beta_1) \cong \operatorname{Cok}(\alpha_1) \oplus Z_1$ is $Q_2 \oplus Z_1$, so we obtain $Y_2 \cong Q_2 \oplus Z_1 \oplus Z_2$ and $Z_2 = 0$ if and only if $Y_2$ is the injective hull of $\mathrm{Cok}(\beta_1)$. The map $\beta_2$ and its cokernel are

$$\begin{array}{ccccc}
Q_1 & \overset{\alpha_2}{\rightarrow} & Q_2 & \rightarrow & \mathrm{Cok}(\alpha_2) \\
\oplus & & \oplus & & \oplus \\
Z_1 & & Z_2 & \overset{1}{\rightarrow} & Z_2 \\
\oplus & \searrow^{1} & \oplus & & \\
Z_0 & \searrow_{0} & Z_1 & \overset{0}{\rightarrow} & \\
 & \searrow & & &
\end{array}$$

From here on, everything keeps repeating and the lemma is proved.

*Remark.* The lemma immediately implies that if dom. dim. $R \geqslant n$, then the first $n$ modules $Q_i$ of the minimal injective resolution of $M$ are projective.

*Definition.* The dominant dimension of the right (left) $R$-module $R$ will be called the right (left) dominant dimension of the algebra $R$, $r$-dom. dim. $R$ (l-dom. dim. $R$). The dominant dimension of the $R$–$R$-bimodule $R$ will be called the Nakayama dimension of the algebra $R$, N-dim. $R/K$.[1]

*Remark.* Obviously

$$\text{N-dim. } R/K \leqslant \min(\text{r-dom. dim. } R, \text{l-dom. dim. } R).$$

It is known that the following four statements are equivalent: $R$ is QF-3; N-dim. $R/K \geqslant 1$; r-dom dim $R \geqslant 1$; l-dom. dim. $R \geqslant 1$ **(18)**.

**3. The class** dom. dim. $R \geqslant 2$. We first recall some known facts about QF-3 algebras.[2] A *dominant right-ideal* of the algebra $R$ is defined to be a right-ideal generated by an indecomposable idempotent, which is injective. Let $e_1, \ldots, e_k$ be orthogonal idempotents such that $e_1 R, \ldots, e_k R$ are non-isomorphic dominant right-ideals which represent all isomorphism types of dominant right-ideals; further set $e = e_1 + \ldots + e_k$. Then $R$ is QF-3 if and only if the projective-injective $R$-right-module $eR$ is faithful.

That being the case, the $eRe$-left-module $eR$ is fully faithful;[3] denote its endomorphism-ring by $R'$. As $eRe$ is the endomorphism-ring of $eR_R$, $R'$ is the second commutator of $eR_R$ and $R$ may be considered as a subring of $R'$ with the same unit, in a natural way. $R'$ is QF-3. We finally notice that $eRe$ is self-basic.

[1]More precisely, we mean the $R^e$-module $R$, $R^e = R \otimes_K R^0$ being the enveloping algebra of $R$. This notion depends not only on $R$ but also on $K$.

[2]See, for example, **(20**; **21**; **13**, § 17**)**. The theory of QF-3 algebras can be simplified by using some facts from the structure theory of injective modules, e.g. that an indecomposable injective is the injective hull of each of its non-zero submodules, especially of its socle, which is simple **(10)**.

[3]A module ${}_AX$ is called *fully faithful* if every indecomposable projective or injective $A$-left-module is isomorphic to a direct summand of $X$.

THEOREM 2. *The following statements are equivalent for any algebra* $R$:[4] (1) r-dom. dim. $R \geqslant 2$; (2) $R$ *is* QF-3 *and* $R = R'$; (3) $R$ *is the endomorphism-ring of a fully faithful left-module* ${}_AX$.

*Proof.* The implication from (1) to (2) has been shown by Tachikawa (**19**, Theorem 1.4). (2) implies (3) trivially, by the preceding remarks.

Assuming (3), let $\ldots \to P_1 \to P_0 \to X \to 0$ be a (finitely generated) projective resolution of the $A$-module $X$. We get the exact sequence

$$0 \to \mathrm{Hom}_A(X, X) = R \to \mathrm{Hom}_A(P_0, X) \to \mathrm{Hom}_A(P_1, X)$$

of right-$R$-modules and $R$-homomorphisms. $X_R$ is projective and injective (**13**, Theorem 17.2); hence so are $\mathrm{Hom}_A(P_0, X)_R$ and $\mathrm{Hom}_A(P_1, X)_R$. Therefore dom. dim. $R \geqslant 2$.

*Remark.* In the situation of Theorem 2, the algebra $A$ and the module ${}_AX$ are essentially determined by $R$. First of all, $A$ may be replaced by its basic algebra $\epsilon A \epsilon$ and $X$ by $\epsilon X$ without changing the endomorphism-ring; hence $A$ may always be chosen self-basic. But then $R$ determines $A$ up to a ring-isomorphism and ${}_AX$ up to a semilinear isomorphism (**15**, Theorem 3.3).

Particularly if $A$ is self-basic, we have a semilinear isomorphism from ${}_AX$ to ${}_{eRe}eR$ with a ring-isomorphism from $A$ to $eRe$.

LEMMA 3. *Let $R$ be the endomorphism-ring of the fully faithful left-module ${}_AX$. Then* r-dom. dim. $R \geqslant n + 2$ *if and only if* $\mathrm{Ext}_A^k(X, X) = 0$ *for all* $1 \leqslant k \leqslant n$ $(n = 0, 1, 2, \ldots)$.

*Proof.* Assume $\mathrm{Ext}_A^k(X, X) = 0$ for all $1 \leqslant k \leqslant n$, and let

$$\ldots \to P_1 \to P_0 \to X \to 0$$

be a projective resolution of ${}_AX$. We obtain the exact sequence

$$0 \to \mathrm{Hom}_A(X, X) = R \to \mathrm{Hom}_A(P_0, X) \to \ldots \to \mathrm{Hom}_A(P_n, X) \to \mathrm{Hom}_A(P_{n+1}, X)$$

of $R$-right-modules and $R$-homomorphisms, and the modules $\mathrm{Hom}_A(P_i, X)_R$ are projective and injective; hence r-dom. dim. $R \geqslant n + 2$.

Conversely, let r-dom. dim. $R \geqslant n + 2$ and $1 \leqslant m \leqslant n$ such that $\mathrm{Ext}_A^k(X, X) = 0$ for all $1 \leqslant k < m$. We shall show that $\mathrm{Ext}_A^m(X, X) = 0$.

If $P_m \to \ldots \to P_0 \to {}_AX \to 0$ is exact and the $P_i$ are projective, then

$$0 \to \mathrm{Hom}_A(X, X)_R = R_R \to \ldots \xrightarrow{\alpha} \mathrm{Hom}_A(P_m, X)_R$$

is exact and the $\mathrm{Hom}_A(P_i, X)_R$ are projective and injective. Denote by $E$ the injective hull of the cokernel of $\alpha$; then

[4]The counterexample given by Tachikawa (**19**, p. 252) is incorrect. The implication from (2) to (1) has been proved by Mochizuki (**12**): our proof of this is much shorter.

$$0 \to R_R \to \mathrm{Hom}_A(P_0, X)_R \to \ldots \to \mathrm{Hom}_A(P_m, X)_R \to E_R$$

is exact and $E$ is injective and projective, for $E$ is the direct sum of a direct summand of $\mathrm{Hom}_A(P_m, X)_R$ and $Q_{m+1}$ where $0 \to R \to Q_0 \to Q_1 \to \ldots$ is the minimal injective resolution of $R_R$ (Lemma 1), and these two modules are both projective.

Now the injectivity of $X_R$ yields the exact sequence

$$\mathrm{Hom}_R(E, X) \to \mathrm{Hom}_R(\mathrm{Hom}_A(P_m, X), X) \to \ldots \to \mathrm{Hom}_R(R, X) \to 0$$

of left-$A$-modules. Since the ${}_AP_i$ are (finitely generated) projective and ${}_AX$ is fully faithful, and therefore a generator, we obtain the left-$A$-module-isomorphisms

$$\mathrm{Hom}_R(\mathrm{Hom}_A(P_i, X), X) \cong \mathrm{Hom}_R(X, X) \otimes_A P_i \cong A \otimes_A P_i \cong P_i$$

and the exact sequence

$$\mathrm{Hom}_R(E, X) \to P_m \to \ldots \to P_0 \to X \to 0.$$

$E_R$, being projective, injective (and finitely generated) is a finite direct sum of indecomposable projective injective modules, i.e. dominant ideals. Since $X_R$ is projective, injective, and faithful, it contains all isomorphism types of dominant ideals as direct summands, so we get

$$E_R \oplus * \cong \oplus X_R$$

and

$${}_A\mathrm{Hom}_R(E, X) \oplus * \cong \oplus\ {}_A\mathrm{Hom}_R(X, X) \cong \oplus_A A$$

and $\mathrm{Hom}_R(E, X)$ is a projective $A$-module. Hence the above sequence is part of a projective resolution for ${}_AX$.

Applying $\mathrm{Hom}_A(\_, X)$ we obtain

$$0 \to R \to \mathrm{Hom}_A(P_0, X) \to \ldots \to \mathrm{Hom}_A(P_m, X)$$
$$\to \mathrm{Hom}_A(\mathrm{Hom}_R(E, X), X) \cong E \otimes_R \mathrm{Hom}_A(X, X) \cong E,$$

a sequence which is exact by construction of $E$. But its homology is

$$\mathrm{Ext}_A^k(X, X), \qquad 1 \leqslant k \leqslant m;$$

hence $\mathrm{Ext}_A^m(X, X) = 0$.

THEOREM 4. r-dom. dim. $R$ = l-dom. dim. $R$ *for any algebra* $R$.

*Proof.* Assume r-dom. dim. $R \geqslant 2$. By Theorem 2, $R$ is the endomorphism-ring of a fully faithful left-$A$-module ${}_AX$, and r-dom. dim. $R$ is characterized by $\mathrm{Ext}_A^k(X, X)$. Obviously $R \cong \mathrm{Hom}_A(X^*, X^*)$ where $X^*$ is the $K$-dual of X, which is a fully faithful $A$-right-module.[5] Hence l-dom. dim. $R \geqslant 2$, and $\mathrm{Ext}_A^k(X, X) \cong \mathrm{Ext}_A^k(X^*, X^*)$ implies that r-dom. dim. $R$ = l-dom. dim. $R$ in this case.

[5]Endomorphism-rings always operate on the opposite side of the module.

A similar argument holds if l-dom. dim. $R \geqslant 2$. If both r-dom. dim. $R$ and l-dom. dim. $R$ are smaller than two, then either $R$ is QF-3 and we have r-dom. dim. $R = 1 =$ l-dom. dim. $R$, or $R$ is not QF-3 and both dimensions are zero.

*Remark.* Several results by Tachikawa **(19)**, e.g. Theorems 1.8 and 2.8, are immediate consequences of Lemma 3.

**4. All dimensions are equal.** We start with several lemmas which may be of interest by themselves.

LEMMA 5. *Dominant dimension and Nakayama dimension are invariant with respect to arbitrary (not necessarily finite-dimensional) ground-field-extensions.*

*Proof.* Let $M_R$ be an $R$-module and $F$ an extension-field of the ground-field $K$. An exact sequence $0 \to M \to X_1 \to \ldots \to X_n$ of $R$-modules yields an exact sequence

$$0 \to M \otimes_K F \to X_1 \otimes_K F \to \ldots \to X_n \otimes_K F$$

of $(R \otimes_K F)$-modules; and if $X_R$ is injective and projective, then so is $X \otimes F_{R \otimes F}$. (If $X$ is $R$-injective, it is an $R$-direct summand in $\mathrm{Hom}(R_K, X_K)$; hence $X \otimes F$ is an $(R \otimes F)$-direct summand in

$$\mathrm{Hom}(R_K, X_K) \otimes F \cong \mathrm{Hom}(R \otimes F_F, X \otimes F_F)$$

which is $(R \otimes F)$-injective.)

Conversely, if we have an exact sequence $0 \to M \otimes F \to Y_1 \to \ldots \to Y_n$ with injective-projective $(R \otimes F)$-modules $Y_i$, then looking at it as a sequence of $R$-modules only, we obtain

$$M \otimes F \cong \oplus M = M \oplus M' \quad \text{and} \quad 0 \to M \oplus M' \to Y_1 \to \ldots \to Y_n,$$

where the $Y_i$ are $R$-projective-injective **(3**, p. 166**)**. Let $0 \to M \to Q_1 \to Q_2 \to \ldots$ and $0 \to M' \to Q_1' \to Q_2' \to \ldots$ be minimal injective resolutions; then

$$\begin{array}{ccccccc} 0 \to M & \to & Q_1 & \to & Q_2 & \to & \ldots \\ \oplus & & \oplus & & \oplus & & \\ M' & \to & Q'_1 & \to & Q'_2 & \to & \ldots \end{array}$$

is a minimal injective resolution for $M \oplus M'$. Hence $Y_i \cong Q_i \oplus Q'_i \oplus *$ (Lemma 1) and $Q_i$ is $R$-projective $(1 \leqslant i \leqslant n)$.

Taking $M_R = R_R$, we obtain dom. dim. $R =$ dom. dim. $R \otimes F$. Replacing $R$ by the enveloping algebra $R^e$ and choosing $M_{R^e} = {}_R R_R$, we use

$$R^e \otimes_K F \cong R \otimes_K R^0 \otimes_K \mathrm{F} \otimes_F F \cong (R \otimes_K F) \otimes_F (R \otimes_K F)^0 = (R \otimes_K F)^e$$

and we get N-dim. $R/K =$ N-dim. $R \otimes F/F$.

LEMMA 6. dom. dim. $R \otimes_K S = \min(\text{dom. dim. } R, \text{dom. dim. } S)$ *for any two algebras* $R$, $S$.

*Proof.* Given an exact sequence $0 \to R \otimes S \to X_1 \to \ldots \to X_n$ of $(R \otimes S)$-projective-injective modules $X_i$, we obtain dom. dim. $R \geqslant n$, dom. dim. $S \geqslant n$ as before.

Conversely, let $0 \to R \to Y_0 \to \ldots$ and $0 \to S \to Z_0 \to \ldots$ be minimal injective resolutions of $R_R$ or of $S_S$. The tensor product of these two complexes (over $K$) yields an exact complex

$$0 \to R \otimes S \to Y_0 \otimes Z_0 \to Y_0 \otimes Z_1 \oplus Y_1 \otimes Z_0 \to \ldots$$

of $R \otimes S$-modules, and the modules $Y_i \otimes Z_k$ are $(R \otimes S)$-projective-injective whenever $i, k \leqslant \min(\text{dom. dim } R, \text{ dom. dim. } S)$. This implies that

$$\text{dom. dim. } R \otimes S \geqslant \min(\text{dom. dim. } R, \text{ dom. dim. } S).$$

LEMMA 7. *Let $R$ be a* QF-3 *algebra and $M$ an $R$-module. Then* dom. dim. $M \geqslant n$ *if and only if* $\operatorname{Ext}_R^k(B, M) = 0$ *for all* $0 \leqslant k < n$ *and all simple right-$R$-modules $B$ which are not isomorphic to right-ideals of $R$.*

*Proof.* Let

$$0 \to M \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} Q_2 \to \ldots$$

be the minimal injective resolution. Since $R$ is QF-3, the injective hull of any simple (right)-ideal is isomorphic to a dominant ideal.

If $B_1 \oplus \ldots \oplus B_k$ is the socle of $M$, all $B_i$ simple, then $Q_1$ is the direct sum of the injective hulls of the $B_i$'s. Hence $Q_1$ is projective if and only if all the $B_i$'s are isomorphic to right-ideals of $R$.

Analogously, $Q_{k+1}$ will be projective if and only if all simple modules in the socle of $Q_k/\operatorname{Im} \alpha_k$ are isomorphic to ideals.

Given a simple module $B$ in the socle of $Q_k/\operatorname{Im} \alpha_k$, we get an exact sequence $0 \to \operatorname{Im} \alpha_k \to X \to B \to 0$, where $X$ is the inverse image of $B$ under $Q_k \to Q_k/\operatorname{Im} \alpha_k$. Since $Q_k$ is an essential extension of $\operatorname{Im} \alpha_k$, this sequence does not split, whence $\operatorname{Ext}_R^1(B, \operatorname{Im} \alpha_k) \neq 0$.

Conversely, let $B$ be any simple $R$-module such that $\operatorname{Ext}_R^1(B, \operatorname{Im} \alpha_k) \neq 0$. Then there exists a non-splitting exact sequence $0 \to \operatorname{Im} \alpha_k \to Y \to B \to 0$. The monomorphism $\operatorname{Im} \alpha_k \to Y$ is essential, for otherwise we had $0 \neq Y_1 \subset Y$, $Y_1 \cap \operatorname{Im} \alpha_k = 0$; hence $\operatorname{Im} \alpha_k \subsetneqq \operatorname{Im} \alpha_k \oplus Y_1 \subset Y$, $\operatorname{Im} \alpha_k \oplus Y_1 = Y$ as $B$ is simple, and the sequence would split. Since $Q_k$ is the maximal essential extension of $\operatorname{Im} \alpha_k$, $Y$ is contained in $Q_k$ (up to an isomorphism over $\operatorname{Im} \alpha_k$) and $B \cong Y/\operatorname{Im} \alpha_k$ is in the socle of $Q_k/\operatorname{Im} \alpha_k$.

Putting things together, we see that there exists an exact sequence $0 \to M \to X_1 \to \ldots \to X_n$ of injective-projective modules $X_i$ if and only if all the simple modules in the socles of $M$ and the $Q_k/\operatorname{Im} \alpha_k$, $k < n$, are isomorphic to right-ideals, which is the case if and only if

$$\operatorname{Ext}_R^0(B, M) = \operatorname{Hom}_R(B, M) = 0,$$
$$\operatorname{Ext}_R^k(B, M) = \operatorname{Ext}_R^1(B, \operatorname{Im} \alpha_k) = 0, \qquad 1 \leqslant k < n,$$

for all simple modules $B$ which are not isomorphic to right-ideals.

THEOREM 8. N-dim. $R/K$ = dom. dim. $R$ = dom. dim. $R^e$ *for any algebra* $R$.

*Proof*. Since every injective-projective $R^e$-module is injective-projective when considered as an $R$-module, we have dom. dim. $R \geqslant$ N-dim. $R/K$.

To prove the converse inequality, we may assume that $R$ is QF-3, for otherwise both dimensions are zero. Lemma 5 allows to assume that $K$ is algebraically closed. Under these circumstances the simple $R^e$-modules are of the form $B_1 \otimes_K B_2$ where $B_1$, $B_2$ are simple $R$-left-, $R$-right-modules (see, e.g., **2**, § 7, no. 7). If $B_1 \otimes B_2$ is not isomorphic to an $R^e$-(right)-ideal, then at least one of $B_1$, $B_2$ (say $B_2$) is not isomorphic to an $R$-ideal; hence $\mathrm{Hom}_R(B_2, R) = 0$. Assuming dom. dim. $R \geqslant n$ we obtain $\mathrm{Ext}_R^k(B_2, R) = 0$ for all $0 \leqslant k < n$, from Lemma 7. Therefore we get the isomorphism (**3**, p. 167)

$$\mathrm{Ext}_{R^e}^k(B_1 \otimes B_2, R) \cong \mathrm{Ext}_R^k(B_1, \mathrm{Hom}_R(B_2, R)) = 0$$

for $0 \leqslant k < n$; and applying Lemma 7 again we have N-dim. $R/K \geqslant n$.

Finally Lemma 6 yields

$$\text{dom. dim. } R^e = \min(\text{dom. dim. } R; \text{dom. dim. } R^0) = \text{dom. dim. } R,$$

as r-dom. dim. $R^0$ = l-dom. dim. $R$.

**5. Nakayama's conjecture.** This is the following yet unproved proposition: If dom. dim. $R = \infty$, then $R$ is quasi-Frobenius.

It is an old conjecture that the finitistic global projective dimension of any (finite-dimensional) algebra (over a field) is finite (**4**; **9**; **11**). If this were proved, Nakayama's conjecture would follow immediately: since all the kernels in an infinite exact sequence $0 \to R \to X_1 \to X_2 \to \ldots$ with projective-injective $X_i$'s have finite projective dimension, the sequence would split and $R$ would be injective, hence quasi-Frobenius.

Nakayama showed that his conjecture is true when $R$ is generalized uniserial. Tachikawa proved it when $R$ is the endomorphism-ring of a fully faithful module ${}_AX$ over a generalized uniserial ring $A$. Tachikawa's result is a generalization of Nakayama's because of Theorem 2 and the (apparently known, and provable) fact that $A$ must be generalized uniserial if $R$ is. (The converse of the last statement is not true, but $R$ is QF-2 if $A$ is generalized uniserial (see **13**, Theorem 17.4).

We shall prove a generalization of Tachikawa's result. The following notion was introduced by Jans (**8**): a projective resolution $\ldots \to P_1 \to P_0 \to M \to 0$ is called *ultimately closed* if there exists a kernel $I_n \subset P_n$ which is a (finite) direct sum $I_n = \oplus_j I_{nj}$ of submodules $I_{nj}$ each of which is isomorphic to a direct summand of some earlier kernel $I_i$, $i < n$. If every $R$-module possesses an ultimately closed projective resolution, $R$ is said to be ultimately closed.[6]

*Remarks*. 1. Nakayama's conjecture holds for ultimately closed algebras $R$. This is a consequence of Lemma 7 and the following theorem by Jans: *If R*

[6]Recall that all modules under consideration are finitely generated.

*is ultimately closed and M an R-module such that* $\mathrm{Ext}_R^k(M, R) = 0$ *for all* $k \geqslant 0$, *then* $M = 0$ **(8)**.

2. An algebra $R$ is ultimately closed if the square of its radical is zero **(8)**.

3. *A generalized uniserial algebra is ultimately closed.*

*Proof.* Recall that any indecomposable $R$-(right-) module $\neq 0$ may be written, in a unique way, as a factor-module of a primitive ideal:[7] $M \cong eR/I_1$. Hence we get the exact sequence $0 \to I_1 \to eR \to M \to 0$, and since the sub-ideal $I_1$ of $eR$ is indecomposable again, we may continue with $I_1$ instead of $M$ to construct a projective resolution $\ldots \to P_1 \to P_0 \to M \to 0$ where the $P_i$'s are primitive ideals and the kernels are subideals of primitive ideals.[8] Since there are only finitely many of those, say $n$, one kernel has to appear twice among the first $n + 1$ kernels, say $I_i \cong I_j$, $i < j \leqslant n + 1$. The uniqueness of the construction then implies that $I_{i+1} \cong I_{j+1}$; hence $I_k \cong I_{n+1}$ for some $k < n + 1$. An arbitrary $R$-module is the direct sum of indecomposable ones, and constructing a projective resolution as the direct sum of the above resolutions for the indecomposable summands, we see that its $(n + 1)$th kernel has the required property for an ultimately closed resolution.

In view of this last remark, the following is a generalization of Tachikawa's theorem:

THEOREM 9. *Nakayama's conjecture holds for any algebra R which is the endomorphism-ring of a fully faithful module* ${}_AX$ *having an ultimately closed projective resolution.*

*Proof.* dom. dim. $R = \infty$ implies that $\mathrm{Ext}_A^k(X, X) = 0$ for all $k \geqslant 1$ (Lemma 3). Since $X$ is fully faithful, we get $\mathrm{Ext}_A^k(X, A) = 0$; hence the projective dimension of $X$ is either zero or infinite. We shall show that it is indeed finite, and hence zero; consequently $X$ is projective and so is $A^*$, since every indecomposable injective $A$-module is direct in $X$. Therefore $A$ is quasi-Frobenius, and $R$, being the endomorphism-ring of the fully faithful projective module $X$ over the quasi-Frobenius-algebra $A$, is quasi-Frobenius **(13)**.

Taking duals with respect to the ground-field, we see that ${X_A}^*$ possesses an *ultimately closed injective resolution*, which means an injective resolution $0 \to X^* \to Q_0 \to Q_1 \to \ldots$ such that there exists a cokernel $Q_n \to L \to 0$ which is a (finite) direct sum $L = \oplus_j L'_j$ and where each $L' = L'_j$ is a direct summand in some earlier cokernel $Q_i \to L' \oplus L'' \to 0$, $0 \leqslant i < n$.

From the exact sequence $0 \to X^* \to Q_0 \to \ldots \to Q_n \to L \to 0$ we get

$$0 = \mathrm{Ext}_A^k(X, X) \cong \mathrm{Ext}_A^k(X^*, X^*) \cong \mathrm{Ext}_A^{k+n+1}(L, X^*) \cong \mathrm{Ext}_A^k(L, L)$$

for all $k \geqslant 1$, where the middle isomorphism comes from the fact that

[7]Nakayama **(16)**.

[8]If some $I_i = 0$, continue with zeros.

$\operatorname{Ext}_A^k(X^*, X^*) = 0$ and $X^*$ fully faithful implies $\operatorname{Ext}_A^k(Q, X^*) = 0$ for all injective $Q$.

Let $0 \to L' \to Q'_{i+1} \to \ldots \to Q'_n \to C'_n \to 0$ and

$$0 \to L'' \to Q''_{i+1} \to \ldots \to Q''_n \to C''_n \to 0$$

be the beginnings of minimal injective resolutions, then so is

$$0 \to L' \oplus L'' \to Q'_{i+1} \oplus Q''_{i+1} \to \ldots \to Q'_n \oplus Q''_n \to C'_n \oplus C''_n \to 0.$$

Lemma 1 then implies that the cokernel $C_n' \oplus C_n''$ is a direct summand in the corresponding cokernel $L$ of the injective resolution

$$0 \to L' \oplus L'' \to Q_{i+1} \to \ldots \to Q_n \to Q_{n+1} \to \ldots .$$

Hence from $\operatorname{Ext}_A^k(L, L) = 0$, we obtain $\operatorname{Ext}_A^k(C'_n, L') = 0$ for all $k \geqslant 1$.

The diagram

$$\begin{array}{ccccccc} 0 \to L' \to Q'_{i+1} \to \ldots & \searrow & \to & \nearrow & Q'_n \to C'_n \to 0 \\ & & C'_{n-1} & & \\ & 0 \nearrow & & \searrow 0 & \end{array}$$

shows that

$$\operatorname{Ext}_A^1(C'_n, C'_{n-1}) \cong \operatorname{Ext}_A^{n-i}(C'_n, L') = 0,$$

so that $0 \to C'_{n-1} \to Q'_n \to C'_n \to 0$ splits and $C'_n$ is injective. Consequently the injective dimension of $L'$ is not larger than $n - i \leqslant n$; and since this holds for any $L' = L'_j$, we get i-dim. $L \leqslant n$. But then i-dim. $X^* \leqslant 2n + 1$ and p-dim. $X \leqslant 2n + 1 < \infty$, and the theorem is proved.

**6. Examples and supplements.** The following algebra $R_n$ was given in (**6**, p. 92) as an example of a ring with global dimension $n$. Let $e_1, \ldots, e_{n+1}$, $m_1, \ldots, m_n$ be a basis of $R_n$ over the ground-field, and let the multiplication be defined by $e_i^2 = e_i$, $e_i m_i e_{i+1} = m_i$, all other products zero. The algebra $R_n$ has the following properties: $R_n$ is generalized uniserial and dom. dim. $R_n = n$; if $n \geqslant 2$, then $R_n$ is QF-1 and isomorphic to the endomorphism-ring of the minimal fully faithful module over $R_{n-1}$ (**14**; **20**).

This shows that algebras $R$ of arbitrary dominant dimension exist, and that such strong requirements as $R$ being generalized uniserial or QF-1 put no restriction on the possible values of the dominant dimension. (Trivially a QF-1 algebra cannot have dominant dimension one; see Theorem 2.)

One may ask if there is any relation between the dominant dimensions of $A$ and $R$, if the latter is the endomorphism-ring of a fully faithful $A$-module $X$. First of all, it is easy to construct, for any non-semisimple $A$, a fully faithful ${}_AX$ such that dom. dim. $R$ has the smallest possible value, namely two (e.g. take $X = A \oplus A^* \oplus Y \oplus Z$ with $\operatorname{Ext}_A^1(Y, Z) \neq 0$). On the other hand the largest possible value of dom. dim. $R$ for given $A$ is obtained for the minimal fully faithful $A$-module $X$ (Lemma 3). But even if we confine ourselves to this case, there seems to be no relation between dom. dim. $A$ and dom. dim. $R$, as is illustrated by the following example.

Let $B$ be the subalgebra of the full $14 \times 14$ matrix-algebra over a field $K$ generated by the 27 elements

$$c_{11} + c_{22} + c_{33}, \quad c_{44} + c_{55}, \quad c_{66} + c_{77}, \quad c_{88} + c_{99} + c_{10,10},$$
$$c_{11,11} + c_{12,12}, \quad c_{13,13} + c_{14,14}; \quad c_{11,1} + c_{12,3}, \quad c_{13,1} + c_{14,2},$$
$$c_{94} + c_{10,5}, \quad c_{86} + c_{10,7}; \quad c_{26}, \quad c_{28}, \quad c_{34}, \quad c_{39}, \quad c_{51}, \quad c_{5,11}, \quad c_{5,13},$$
$$c_{71}, \quad c_{7,11}, \quad c_{7,13}, \quad c_{10,1}, \quad c_{10,11}, \quad c_{10,13}, \quad c_{12,4}, \quad c_{12,9}, \quad c_{14,6}, \quad c_{14,8}.$$

Then[9] dom. dim. $B = 3$; $B$ is the endomorphism-ring of the minimal fully faithful module over an algebra $A$ with dom. dim. $A = 0$; and the endomorphism-ring $R$ of the minimal fully faithful $B$-module has dominant dimension two.

Nakayama **(17)** used another dimension besides N-dim. $R/K$, defined as follows: E-dim. $R/K \geqslant n$ *if every exact sequence* $0 \to R \to X_1 \to \ldots \to X_k$ *of* $R^e$*-homomorphisms and* $R^e$*-projective modules* $X_i$ *with* $0 \leqslant k < n$ *admits an extension to a similar exact sequence* $0 \to R \to X_1 \to \ldots \to X_k \to X_{k+1}$. This dimension is of interest inasmuch as it does not involve injectivity.

THEOREM 10. N-dim. $R/K$ = E-dim. $R/K$ *for any algebra* $R$.

*Proof.*[10] It is known that the following three statements are equivalent **(18)**: $R$ is QF-3; N-dim. $R/K \geqslant 1$; E-dim. $R/K \geqslant 1$. We can therefore restrict our attention to QF-3 algebras $R$; then $R^e$ is QF-3 too and the injective hull of an $R^e$-module $M$ is projective if and only if all simple $R^e$-submodules of $M$ are isomorphic to $R^e$-ideals.

If a sequence $0 \to R \to X_1 \to \ldots \to X_k$ can be extended as described above, then $X_{k+1}$ can be chosen injective as well as projective; for the cokernel $C$ of $X_{k-1} \to X_k$; being monomorphic to a projective $R^e$-module, contains only such simple submodules that are isomorphic to $R^e$-ideals, so we can take for $X_{k+1}$ the (projective) injective hull of $C$. Assuming now that $E$-dim. $R/K \geqslant n$, we can extend successively $0 \to R$ to $0 \to R \to X_1$ to $0 \to R \to X_1 \to X_2$ up to $0 \to R \to X_1 \to \ldots \to X_n$, always choosing injective-projective $R^e$-modules $X_i$; hence we get N-dim. $R/K \geqslant n$.

Conversely N-dim. $R/K \geqslant n$ implies that $\operatorname{Ext}^k_{R^e}(B, R) = 0$ (Lemma 7) as well as $\operatorname{Ext}^k_R(B, R^e) = 0$ (again Lemma 7, using dom. min. $R^e \geqslant n$ from Theorem 8) for all $0 \leqslant k < n$ and all simple $R^e$-modules $B$ non-isomorphic to $R^e$-ideals. The latter yields $\operatorname{Ext}^k_{R^e}(B, X) = 0$ for all $R^e$-projective $X$. Dividing the exact sequence

$$0 \to R \to X_1 \ldots \to X_k \to C \to 0, \qquad k < n,$$

[9]This example was given by Tachikawa **(19**, p. 252**)**.

[10]The theorem remains true if we allow non-finitely generated modules $X_i$ in the definition of E-dim. $R/K$.

of $R^e$-homomorphisms and $R^e$-projective modules $X_i$ up into short exact sequences and using the long exact sequence for Ext we get

$$\mathrm{Hom}_{R^e}(B, C) = \mathrm{Ext}^0_{R^e}(B, C) \cong \mathrm{Ext}^k_{R^e}(B, R) = 0,$$

for any such $B$. Hence every simple submodule of $C$ is isomorphic to an ideal and the sequence can be extended; consequently E-dim. $R/K \geqslant n$.

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