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NILPOTENCY OF DERIVATIONS

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ABSTRACT. It is shown that the nilpotency of a derivation on a 2-torsion free semiprime ring is always an odd number. Examples are provided to show the necessity of the assumptions.

A derivation ∂ on a ring R is called nilpotent if $\partial^n R = 0$ for some positive integer n. For $n = 2$ and R is a prime ring of characteristic $\neq 2$, a basic theorem of Posner [7] implies that ∂ is actually zero. For $n > 2$, the situation is less clear cut. Herstein in [3] shows that for simple rings with some characteristic restrictions, a nilpotent derivation is identical to an inner derivation $x \to [a, x]$ induced by a nilpotent element a. As a consequence, the index of a nilpotent derivation (for simple rings) must be an odd number; it is equal to $2m - 1$ where m is the nilpotent index of the element a above. A. Kovacs thus asks the possibility of generalizing these to prime rings. We show that the index of nilpotent derivation is indeed an odd number even for semiprime rings without 2-torsion. Our result is the following:

THEOREM. *Let R be a 2-torsion free semiprime ring and ∂ be a derivation of R. If $\partial^{2n} R = 0$ then $\partial^{2n-1} R = 0$.*

In a forthcoming paper [1] this result will be used to prove that, for a prime ring R of characteristic zero, a nilpotent derivation is indeed induced by a nilpotent element in a quotient ring of R, thus generalizing Herstein's result to semiprime ring with minimal restriction on characteristic. In that paper, we also sharpen the well-known result of Kharchenko [5] on derivations satisfying a polynomial identity in a prime ring of characteristic zero, again by application of the main theorem of this paper. Examples are given later in this paper to show the necessity of the assumption of semiprimeness and the restriction on characteristics.

Without loss of generality, we may assume all rings R have 1. For otherwise, R can be adjointed by 1 so that the resulting ring is semiprime and 2-torsion free and, moreover, ∂ can be extended by $\partial 1 = 0$.

Suppose the theorem were false. Then there would exist a *least* positive

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integer n such that $\partial^{2n-1}R \neq 0$ but $\partial^{2n}R = 0$ for some semiprime ring R and some derivation ∂ of R. We shall show that it would lead to a contradiction. Throughout this paper we will let R be such a semiprime ring with such a derivation ∂. For a prime p, we denote $R_p = \{x \in R \mid px = 0\}$. $S = \sum R_p$.

We start with some combinatorial and number theoretical lemmas which will be used in the sequel.

LEMMA 1. *Let*

$$f(m, i) = \begin{vmatrix} 0 & 0 & \cdots & 0 & \binom{m}{0} & \binom{m}{1} \\ 0 & 0 & \cdots & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{i-3} & \binom{m}{i-2} & \binom{m}{i-1} \\ \binom{m}{1} & \binom{m}{2} & \cdots & \binom{m}{i-2} & \binom{m}{i-1} & \binom{m}{i} \end{vmatrix}.$$

Then

$$f(m, i) = (-1)^{i(i-1)/2} \binom{m+i-1}{i}$$

Proof. It is known that, for $r \geq 1$,

$$\sum_{k=0}^{r} (-1)^k \binom{m}{k} \binom{m+r-k-1}{r-k} = 0$$

(see (3.23) in [2]). In the determinant $f(m, i)$, multiplying the k-th column by $(-1)^k \binom{m+i-k-1}{i-k}$ for $k = 1, 2, \ldots, i$, then replacing the last column by their sum and using the above combinatorial identity, we obtain

$$f(m, i) = (-1)^i \begin{vmatrix} 0 & 0 & \cdots & 0 & \binom{m}{0} & 0 \\ 0 & 0 & \cdots & \binom{m}{0} & \binom{m}{1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{m}{0} & \binom{m}{1} & & \binom{m}{i-3} & \binom{m}{i-2} & 0 \\ \binom{m}{1} & \binom{m}{2} & \cdots & \binom{m}{i-2} & \binom{m}{i-1} & -\binom{m+i-1}{i} \end{vmatrix} = (-1)^{i(i-1)/2} \binom{m+i-1}{i}.$$

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LEMMA 2. *For any prime p, R_p is a semiprime ideal as well as a semiprime subring of R and $\partial R_p \subseteq R_p$. Moreover, if $\partial^{2n-1}R_p \neq \{0\}$ then $n = \sum_{i=0}^{N} \alpha_i p^i$, where α_0, $\alpha_N \neq 0$ and $0 \leq \alpha_i \leq (p-1)/2$ for $i = 0, 1, 2, \ldots, N$.*

Proof. The first half of the lemma is routine. We will only show the representation of n by p. Suppose $2n < p$. Pick $N = 0$ and $\alpha_0 = n$. Suppose $2n > p$. Represent $2n = \sum_{i=0}^{N} \beta_i p^i$ where $\beta_N \neq 0$, $0 \leq \beta_i \leq p-1$. We claim that each β_i is an even number for $i = N, N-1, \ldots, 0$. First suppose to the contrary that β_N is odd. Then since $(\beta_N + 1)p^N > 2n$, $\delta^{\beta_N+1}R = 0$ where $\delta = \partial^{p^N}$ is a derivation on R_p. We conclude that δ is a nilpotent derivation on the semiprime ring R_p with index $< 2n$. Note that $2n$ is the least possible such even index. So $\delta^{\beta_N}R_p = 0$ and hence $\partial^{2n-1}R_p = 0$, a contradiction. Thus β_N must be even.

Now, suppose $\beta_N, \beta_{N-1,\ldots,}\beta_i$ are all even where $i > 1$. We want to show that β_{i-1} is also even. If not, then $(\beta_{i-1}+1)p^{i-1} + \beta_i p^i + \beta_{i+1}p^{i+1} + \cdots + \beta_N p^N > 2n$ and $\delta = \partial^{p^{i-1}}$ is a nilpotent derivation on R_p with index $\leq$ the even number $\beta_{i-1} + 1 + \beta_i p + \cdots + \beta_N p^{N-i+1}$ which is less than $2n$. It again contradicts the choice of $2n$. Thus we arrive, by induction, at the conclusion that β_N, $\beta_{N-1}, \ldots, \beta_1$ are even. This implies also that $\beta_0 = 2n - \sum_{i=1}^{N} \beta_i p^i$ is even.

LEMMA 3. *Let p be a prime number and $n = \sum_{i=0}^{N} \alpha_i p^i$, where α_0, $\alpha_N \neq 0$ and $0 \leq \alpha_i < (p-1)/2$ for $i = 0, 1, 2, \ldots, N$. Then $\binom{2n}{n} \not\equiv 0 \pmod{p}$.*

Proof. A well-known result due to E. Lucas [6, pp. 417–420] or [4] says that if $a = \sum a_i p^i$ and $b = \sum b_i p^i$ with $0 \leq a_i$, $b_i < p$ for all i then $\binom{a}{b} \equiv \prod \binom{a_i}{b_i} \pmod{p}$. Using this we immediately obtain $\binom{2n}{n} \equiv \prod \binom{2\alpha_i}{\alpha_i} \pmod{p}$ which is not congruent to zero modulo p since each $\binom{2\alpha_i}{\alpha_i} \not\equiv 0 \pmod{p}$.

LEMMA 4. *Let $a, b \in R$. If $\partial^n xa\, \partial^{2n-1}b = 0$ for all $x \in R$ then $a\, \partial^{2n-1}b = 0$.*

Proof. From $\partial^n(x\, \partial^{n-1}y)a\, \partial^{2n-1}b = 0$ and Leibniz rule, we obtain $\partial^n x\, \partial^{n-1}ya\, \partial^{2n-1}b = 0$. Then using this identity and $\partial^n(\partial x\, \partial^{n-2}y)a\, \partial^{2n-1}b = 0$ we obtain $\partial^{n+1}x\, \partial^{n-2}ya\, \partial^{2n-1}b = 0$. Continuing this process we finally get $\partial^{2n-1}xya\, \partial^{2n-1}b = 0$ for all $x, y \in R$. By the semiprimeness of R, $a\, \partial^{2n-1}b = 0$.

LEMMA 5. *Let $n < m \leq 2n$ and p be a prime.*

(i) *If $a \in R_p$ and $\partial^m xa = 0$ for all $x \in R$ then $\partial^{m-1}x\, \partial^{m-1}ya = 0$ for all $x. y \in R$.*

(ii) *If R is $(2n)!$-torsion free, $a \in R$, and $\partial^m xa = 0$ for all $x \in R$ then $\partial^{m-1}x\, \partial^{m-1}ya = 0$ for all $x, y \in R$.*

Proof. (i) For $i = 0, 1, 2, \ldots, m-2$ and $x, y \in R$, $\partial^m(\partial^i x\, \partial^{m-2-i}y)a = 0$ which

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by Leibniz rule becomes

$$\sum_{j=0}^{m}\binom{m}{j}\partial^{m+i-j}x\,\partial^{m-2-i+j}ya=0.$$

These $m-1$ equations can be expressed in matrix form by using $\partial^m ya=0$:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{m}{0} & \binom{m}{1} \\ 0 & 0 & \cdots & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \binom{m}{2n-m-1} & \binom{m}{2n-m} & \binom{m}{2n-m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{m}{0} & \binom{m}{1} & \cdots & \binom{m}{m-4} & \binom{m}{m-3} & \binom{m}{m-2} \\ \binom{m}{1} & \binom{m}{2} & \cdots & \binom{m}{m-3} & \binom{m}{m-2} & \binom{m}{m-1} \end{bmatrix}\begin{bmatrix} \partial^{2m-3}x\,\partial ya \\ \partial^{2m-4}x\,\partial^2 ya \\ \vdots \\ \partial^{2n}x\,\partial^{2m-2n-2}ya \\ \partial^{2n-1}x\,\partial^{2m-2n-1}ya \\ \vdots \\ \partial^{m}x\quad\partial^{m-2}ya \\ \partial^{m-1}x\,\partial^{m-1}ya \end{bmatrix}=0.$$

We should note that $\partial^{2m-3}x\,\partial ya=\partial^{2m-4}x\,\partial^2 ya=\cdots=\partial^{2n}x\,\partial^{2m-2n-2}ya=0$. Thus in order to show $\partial^{m-1}x\,\partial^{m-1}ya=0$, it suffices to show that for some i, $2n-m+1\le i\le m-1$, the determinant, defined in Lemma 1, $f(m,i)\not\equiv 0 \pmod p$. This can be proved by induction on m $(n<m\le 2n)$. For $m=n+1$, set $i=n$, $f(m,n)=(-1)^{n(n-1)/2}\binom{2n}{n}\not\equiv 0 \pmod p$ by Lemmas 2 and 3. Now we assume $n<k<2n$ and $f(k,i)\not\equiv 0 \pmod p$ for some i where $2n-k+1\le i\le k-1$. By noting that $f(k,i)=f(k+1,i)+(-1)^i f(k+1,i-1)$, at least one of $f(k+1,i)$ and $f(k+1,i-1)$ is not congruent to zero modulo p where $2n-k<i-1<i<k$. Therefore, $f(m,i)\not\equiv 0 \pmod p$ for some i as we desired.

(ii) can be seen analogously by using the fact $f(m,2n-m+1)=(-1)^{(m^2-m-2n)/2}\binom{2n}{m-1}$ which does divide $(2n)!$.

Lemma 6. *Let $a, b\in R$ and $n\le m<2n$. Suppose either $\partial^{2n-1}a\in R_p$ where p is a prime number or $\partial^{2n-1}a\in R$ where R is $(2n)!$-torsion free. If $\partial^m xb\,\partial^{2n-1}a=0$ for all $x\in R$ then $b\,\partial^{2n-1}a=0$.*

Proof. We proceed by induction on m. It is true for $m=n$ by Lemma 4. Now we assume it is true for $m=k$, where $n\le k<2n-1$. If $\partial^{k+1}xb\,\partial^{2n-1}a=0$ for all $x\in R$ then, by Lemma 5, $\partial^k x\,\partial^k yb\,\partial^{2n-1}a=0$ for all $x, y\in R$. Using the induction hypothesis twice we obtain $b\,\partial^{2n-1}a=0$.

LEMMA 7. *R is not torsion free, and $\partial^{2n-1}R_p = 0$ for any prime number p.*

Proof. Recall that $S = \sum R_p$ is an ideal of R. If R were torsion free i.e. $S = (0)$, then by Lemma 5 (ii), $(\partial^{2n-1}R)^2 = (0)$ which by Lemma 6 implies $\partial^{2n-1}R = 0$, a contradiction. Hence R is not torsion free or $S \neq (0)$. But by Lemma 5 (i), $(\partial^{2n-1}R_p)^2 S = (0)$. The semiprimeness of R implies $(\partial^{2n-1}R_p)^2 = (0)$ and hence $\partial^{2n-1}R_p = 0$ by Lemma 6.

Now, we are in a position to prove the theorem. It is easy to see that S is a semiprime ideal of R or R/S is a semiprime ring. ∂ induces naturally a derivation $\bar{\partial}$ on R/S and $\bar{\partial}^{2n} = 0$. Because R/S is torsion free, by Lemma 7 $\bar{\partial}^{2n-1} = (0)$ or equivalently $\partial^{2n-1}R \subseteq S$. For any $x \in R$, since $x\,\partial^{2n-1}R \subseteq S$, by Lemma 7 $\partial^{2n-1}(x\,\partial^{2n-1}R) = (0)$, or $\partial^{2n-1}x\,\partial^{2n-1}R = (0)$ for all $x \in R$. Hence $(\partial^{2n-1}R)^2 = (0)$. Suppose $\partial^{2n-1}R \neq 0$. Then there exists $a \in R$ such that $0 \neq \partial^{2n-1}a \in R_p$ for some prime p. But $\partial^{2n-1}x\,\partial^{2n-1}a = 0$ for all $x \in R$. By Lemma 6, $\partial^{2n-1}a = 0$ a contradiction. Therefore $\partial^{2n-1}R = 0$, again a contradiction. This completes the proof.

The following examples show the necessity of our hypotheses.

EXAMPLE 1. The ring of all two by two upper triangular matrices over the real field is not semiprime but the inner derivations determined by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has nilpotency two. In fact, given any positive integer n one can construct analogously a non-semiprime ring with an inner derivation of nilpotency n. Thus the assumption of semiprimeness is necessary.

EXAMPLE 2. In the simple ring of two by two matrices over Z_2 the inner derivation determined by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has nilpotency two, thus the necessity of the restriction on characteristics.

In conclusion we would like to propose the following generalized Posner's problem. Posner's theorem alluded to above [7] is relating two derivations ∂_1 and ∂_2 in a prime ring. What is an appropriate generalization of his results in our case? More precisely, suppose R is a prime ring of characteristic $\neq 2$ and $\partial_1, \partial_2, \ldots, \partial_{2n}$ are derivations of R such that $\partial_1, \partial_2 \ldots \partial_{2n} = 0$. Can we say something about the products of $2n-1$ such derivations?

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