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IDENTIFICATION OF COVARIANCE STRUCTURES

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The issue of identification of covariance structures, which arises in a number of different contexts, has been so far linked to conditions on the true parameters to be estimated. In this paper, this limitation is removed.

As done by Johansen (1995, *Journal of Econometrics* 69, 112–132) in the context of linear models, the present paper provides necessary and sufficient conditions for the identification of a covariance structure that depends only on the constraints and can therefore be checked independently of estimated parameters.

A *structure condition* is developed, which only depends on the structure of the constraints. It is shown that this condition, if coupled with the familiar order condition, provides a sufficient condition for identification. In practice, because the structure condition holds if and only if a certain matrix, constructed from the constraint matrices, is invertible, automatic software checking for identification is feasible even for large-scale systems.

Most of the paper focuses on structural vector autoregressions, but extensions to other statistical models are also briefly discussed.

1. INTRODUCTION

The aim of this paper is to shed some light on a problem that arises in models that impose some sort of structure on covariance matrices. This is the case with several popular models commonly employed in econometrics and multivariate statistics: the most prominent in econometrics is probably the structural vector autoregression (VAR) model, and much of the present paper will focus on it. Possible extensions of the results presented here to other contexts will be briefly discussed in the final section.

The methodology of structural VARs (SVARs) was pioneered by Sims (1980) and then made popular by countless applications, some of which were highly influential (Blanchard and Quah, 1989, comes to mind). The issues involved in

I thank all the participants at the meeting held in Pavia on June 11, 2004, in honor of Carlo Giannini for their comments; it goes without saying that Carlo himself provided not only acute observations on the day but also the main inspiration for this piece of work. Sadly, Carlo passed away on September 11, 2004, and this paper is dedicated to his memory. Pär Österholm spotted several mistakes in an earlier version and helped me clarify some implementation details. Thanks are also due to Gianni Amisano, Bruce Hansen, Giulio Palomba, Paolo Paruolo, and two anonymous referees. The usual disclaimer obviously applies. Address correspondence to Riccardo Lucchetti, Università Politecnica delle Marche—Facoltà di Economia “Giorgio Fuà”—Dipartimento di Economia; e-mail: r.lucchetti@univpm.it.

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the identification and estimation of such models were thoroughly investigated in Giannini (1992) and Amisano and Giannini (1997), and the present paper builds heavily on that work.

In these models, it is assumed that a valid representation for a vector of n observable variables y_t is a finite-order VAR[1]

$$P(L)y_t = \mu_t + \varepsilon_t, \tag{1}$$

where $P(L)$ is a matrix polynomial in the lag operator, ε_t is a vector white noise sequence with covariance matrix Σ_ε, and the function μ_t may depend on deterministic terms and a given set of exogenous variables. In SVARs, ε_t can be thought of as some function of a vector of n unobservable variables u_t (called the *structural shocks*), which can be assumed independent, or at least uncorrelated, and to have unit variance. The researcher's interest is normally centered on recovering the effect of the u_t's on the y_t's.

In most models of this kind the link between the structural shocks and the system innovations can be written as

$$A(\theta)\varepsilon_t = B(\theta)u_t, \tag{2}$$

where A and B are square nonsingular matrices of order n, which are a function of the parameters to be estimated, θ. In some other cases, the relevant covariance matrix is the so-called long-run covariance matrix of y_t, which satisfies $\Sigma_{LR} = C(1)\Sigma_\varepsilon C(1)'$, with $C(L) = P(L)^{-1}$. In the rest of the paper, we will use the generic notation Σ for the covariance matrix of interest.

We may define a *covariance structure* as a parametric model featuring two matrices A and B such that

$$A(\theta)\Sigma A(\theta)' = B(\theta)B(\theta)'. \tag{3}$$

To avoid cumbersome notation, the reference to θ henceforth will be dropped, and A and B will be implicitly assumed to be functions of θ.

In this paper, we will consider cases when estimation of the parameters θ is carried out by optimizing some objective function:

$$f(\sigma, v(\theta)), \tag{4}$$

where $\sigma = \text{vech}(\Sigma)$ and $v(\theta)$ is the vector function

$$v(\theta) = \text{vech}[A^{-1}BB'(A')^{-1}].$$

Estimation of these models amounts to finding the vector θ such that $A^{-1}BB'(A')^{-1}$ is "closest" (in a sense to be specified) to Σ.

The identification question arises because there is no guarantee that $\hat{\theta}$ is unique, because the objective function is defined in terms of $v(\theta)$ and there may be some $\tilde{\theta} \neq \hat{\theta}$ (and a corresponding, different pair of matrices $A(\tilde{\theta})$ and $B(\tilde{\theta})$) that give rise to the same value of $v(\tilde{\theta})$.

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If the objective function $f(\sigma, v(\theta))$ possesses derivatives up to the second order, identification can be investigated through definiteness of the Hessian matrix at the optimum. As a rule, however, the Hessian matrix is itself a function of the parameters to be estimated, so it cannot be computed before estimation is carried out, which renders this type of check nonoperational.[2] As done by Johansen (1995) in the context of linear models, the present paper provides necessary and sufficient conditions for identification that depend only on the constraints and can therefore be checked independently of estimated parameters: this checking procedure can be made automatic, so that it can be applied to systems of any dimension. Moreover, as a side benefit, the analytical framework put forward here makes it possible to give the researcher an intuitive insight as to *why* a particular model fails to meet the requirements for identification; an example will be given in Section 5.3.

The plan of the paper is as follows. Section 2 provides a brief reminder of some mathematical concepts that will be used in the rest of the paper. Section 3 explores a special case in which the A matrix is fully restricted to establish some concepts and exemplify them more clearly, whereas Section 4 deals with the general case; some examples are given in Section 5. Finally, Section 6 summarizes the results and briefly discusses applications other than those presented here.

2. DEFINITIONS AND NOTATION

In this section, we define some concepts and terms that will be used in the rest of the paper.

2.1. Identification—Generalities

We assume that estimation of the parameters θ is carried out by optimizing an objective function $f(\sigma, v(\theta))$ over a parameter space Θ.

It will be assumed that the objective function has derivatives up to the second order and has at least one optimum $\hat{\theta}$ in the interior of Θ. At $\hat{\theta}$, we obtain an estimate of Σ that obeys

$$\hat{A}\hat{\Sigma}\hat{A}' = \hat{B}\hat{B}'. \tag{5}$$

Obvious examples for $f(\sigma, v(\theta))$ include

- concentrated Gaussian likelihood: $f(\sigma, v(\theta)) = \text{const} - \frac{1}{2}\ln|\hat{\Sigma}| - \frac{1}{2}\operatorname{tr}(\hat{\Sigma}^{-1}\Sigma)$;
- minimum distance estimation (or generalized method of moments [GMM]): $f(\sigma, v(\theta)) = \delta'\Lambda\delta$ where $\delta = v(\theta) - \operatorname{vech}\Sigma$ and Λ is some positive definite matrix.

As stated in the introduction, $\hat{A}$ and $\hat{B}$ satisfying equation (5) are not necessarily unique: in fact, (5) is satisfied by any pair of matrices A_1 and B_1 such that

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$$A_1 = Q\hat{A},$$

$$B_1 = Q\hat{B}H,$$

where Q is invertible and H is orthogonal; it is clear that the matrices A_1 and B_1 can be considered an equivalent reparametrization of the original model. The simplest case that can be taken as an example is when H is a permutation matrix. In this case, B_1 is simply $\hat{B}$ with its columns reordered, that is to say, the ordering of the structural shocks u_t in (2) is changed. The case when H or Q is arbitrary is nevertheless uninteresting, because the practically relevant issues arise when considering whether such matrices Q and H can exist in a neighborhood of the optimum. In the terminology of Rothenberg (1971), therefore, we are dealing with *local*, rather than *global* identification.[3]

To achieve identification, some constraints on A and B must be imposed. In this paper, we will follow a long-standing tradition of imposing a system of $r_a + r_b = r$ linear constraints:[4]

$$R_a \operatorname{vec} A = R_a a = d_a, \tag{6}$$

$$R_b \operatorname{vec} B = R_b b = d_b \tag{7}$$

(lowercase symbols will be used throughout to indicate vectorizations of matrices, i.e., $a \equiv \operatorname{vec} A$). Alternatively, the constraints can be written in explicit form as follows:

$$a = S_a \theta + s_a, \tag{8}$$

$$b = S_b \theta + s_b, \tag{9}$$

where θ are the unconstrained parameters. The number of elements in θ is $p = p_a + p_b$, where $p_a = n^2 - r_a$ and $p_b = n^2 - r_b$. The p columns of the matrix S_a and S_b form bases for the null space of the rows of R_a and R_b, respectively, so that $R_a S_a = R_b S_b = 0$ and the matrices $[S_a | R_a']$ and $[S_b | R_b']$ have column rank n^2 (the symbol | will be used throughout the paper to indicate horizontal stacking of matrices).

If a pair of matrices A and B satisfies the system of constraints (6)–(7), we call them *admissible*. The aim of this paper is to establish conditions under which an optimum of the objective function (4) corresponds to the only admissible pair within a neighborhood.

The standard way of analyzing the constrained optimization problem would be to start from the first-order conditions for an optimum in terms of Lagrange multipliers, as in

$$\frac{\partial f}{\partial v(\theta)} + \lambda' \frac{\partial G}{\partial v(\theta)} = 0,$$

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where $G(v(\theta))$ is the system of constraints induced on $v(\theta)$ by the restrictions (6) and (7). For our purposes, however, it is best to consider an equivalent way of expressing these conditions, involving differentials:

$$\mathrm{d}f = \frac{\partial f}{\partial v(\theta)'} \frac{\partial v(\theta)}{\partial \theta'} \mathrm{d}\theta = J_v^f J_\theta^v \mathrm{d}\theta = 0;$$

if the Jacobian matrix $J_v^f J_\theta^v$ has full column rank, then we have identification at the optimum. This, in turn, requires that the column rank of J_θ^v is full, too. Then, the system is identified if, inside an arbitrarily small neighborhood of $\hat{\theta}$, no other vector θ_1 (and thus, no corresponding pair of admissible matrices A_1 and B_1) exists such that

$$A_1 \hat{\Sigma} A_1' = B_1 B_1'.$$

Obviously, identification imposes an order condition: because the number of elements in σ is $n(n+1)/2$, the column rank of the Jacobian matrix J_θ^v cannot be full if θ has more than $n(n+1)/2$ elements. As is well known, however, this is only a necessary condition.[5] In the next section, another necessary condition, called the *structure condition*, will be developed; it covers some cases of interest where the order condition holds yet the model is underidentified. It will be argued that a sufficient condition for identification can be obtained by requiring that both order and structure conditions hold.

2.2. Decomposition of Square Matrices

As is well known (see, e.g., Lütkepohl, 1996), any $n \times n$ matrix X can be written as

$$X = X_+ + X_-,$$

where X_+ is symmetric and X_- is hemisymmetric[6] and they are defined as

$$X_+ = \frac{1}{2}(X + X'),$$

$$X_- = \frac{1}{2}(X - X').$$

Let us now consider the space $\Omega = \mathbb{R}^{n^2}$: any element of this space can be considered the vectorization of an $(n \times n)$ matrix. In a parallel fashion, the space Ω can be subdivided into two orthogonal subspaces Ω^+ and Ω^-: any vector $x \in \Omega$ can be written as $x = x_+ + x_-$, where $x_+ = \mathrm{vec}(X_+) \in \Omega^+$ and $x_- = \mathrm{vec}(X_-) \in \Omega^-$. It can be shown that Ω^+ has dimension $n(n+1)/2$ and Ω^- has dimension $n(n-1)/2$. We call $\tilde{D}_n$ a $(n^2 \times n(n-1)/2)$ matrix whose

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columns form a basis for Ω^-, so that any hemisymmetric matrix H has a vectorized form that satisfies $h = \tilde{D}_n \varphi$ for some vector φ.

One useful operator[7] in this context is the $n^2 \times n^2$ matrix K_{nn}, which is defined by the property $K_{nn}\text{vec}(A) = \text{vec}(A')$. It can be shown (see, e.g., Magnus and Neudecker, 1988, p. 46) that K_{nn} is symmetric and orthogonal, i.e.,

$$K_{nn} = K'_{nn} \qquad K_{nn}K_{nn} = I.$$

2.3. The Infinitesimal Rotation Operator

A matrix operator that will be useful in the rest of the paper is the so-called infinitesimal rotation operator.[8] Consider a vector x_0: we are interested in an infinitesimal displacement of x_0 that preserves its norm, i.e., a vector $x_1 = x_0 + \mathrm{d}x$ such that $\mathrm{d}\|x\| = 0$. Then it is possible to define a matrix H as the matrix such that $x_1 = (I + H)x_0$. Two properties of H will be of interest.

(1) $(I + H)$ must be orthogonal because $x_1'x_1 = x_0'(I + H)'(I + H)x_0 = x_0'x_0$ must hold for any x_0, and therefore $(I + H)'(I + H) = I$;
(2) H is hemisymmetric because $\mathrm{d}(x'x) = x_0'\mathrm{d}x + (\mathrm{d}x')x_0 = 0$ and therefore $H = -H'$.

In this case, we say that the transformation is an infinitesimal rotation and H is the corresponding infinitesimal rotation operator.

Note that conditions 1 and 2 imply $H'H = 0$; this result is a consequence of H being infinitesimal. An example can be easily given when $x_0 \in \mathbb{R}^2$: in this case, we have

$$x_1 = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} x_0;$$

so

$$\Delta x = x_1 - x_0 = \begin{pmatrix} \cos\delta - 1 & \sin\delta \\ -\sin\delta & \cos\delta - 1 \end{pmatrix} x_0;$$

from a Taylor expansion we get

$$\Delta x = \left[\begin{pmatrix} 0 & \delta \\ -\delta & 0 \end{pmatrix} + R(\delta) \right] x_0;$$

as $\delta \to 0$, Δx becomes $\mathrm{d}x$ and the remainder term $R(\delta)$ disappears; therefore, when δ is infinitesimal, $\mathrm{d}x = Hx$, where H is infinitesimal and hemisymmetric.

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3. IDENTIFICATION IN THE *C*-MODEL

We begin by examining the special case in which $A = I$. In the terminology introduced in Giannini (1992), which will be adopted from here on, this corresponds to the so-called C-model. In this section, therefore, we analyze the special case where all the elements of A are constrained with $R_a = I$ and $d_a = \text{vec}\, I$; because R_a is full rank, S_a does not exist and $s_a = d_a$. In practice, we will only be concerned with constraints on the B matrix: therefore, to avoid clutter, in this section the constraint matrices R_b and S_b will be referred to simply as R and S.

The structure of interest here is $\hat{\Sigma} = B_0 B_0'$. Suppose that another admissible matrix $B_1 = B_0 + \mathrm{d}B$ exists, where $\mathrm{d}B$ is infinitesimal. The model is underidentified if B_1 is observationally equivalent to B_0, i.e., if

$$B_1 B_1' = \hat{\Sigma} = B_0 B_0'. \tag{10}$$

If (10) holds, it is possible to define an infinitesimal rotation matrix H as the matrix such that $B_1 = B_0(I + H)$. From its definition, the matrix B_1 can be written in vectorized form as follows:

$$B_1 = B_0(I + H) \Rightarrow b_1 = b_0 + (H' \otimes I)b_0. \tag{11}$$

If B_1 is to be admissible, then we must have $Rb_1 = d$; however, this implies $Rb_0 + R(H' \otimes I)b_0 = d$, and therefore,[9] using (9),

$$R(H' \otimes I)b_0 = R(H' \otimes I)(S\theta + s) = 0. \tag{12}$$

Consider now the ith row of $R(H' \otimes I)$. If we call e_i the ith column of the identity matrix, we have

$$e_i' R(H' \otimes I) = h'(I \otimes R_i'),$$

where $h = \text{vec}\, H$ and R_i is an $n \times n$ matrix whose vectorization is the ith row of R. To be an infinitesimal rotation matrix, H must be hemisymmetric. Therefore, it is possible to write h as $\tilde{D}_n \varphi$ and equation (12) as

$$\varphi' \tilde{D}_n' (I \otimes R_i')(S\theta + s) = 0 \quad \text{for } i = 1 \ldots p, \tag{13}$$

where $\tilde{D}_n$ is any basis for Ω^- (see Section 2.2). In short, if a nonnull φ exists that satisfies equation (13) for each i, the model is underidentified at θ.

Giannini (1992), in his analysis of the C-model, indicates (p. 28) that identification holds if and only if the matrix

$$R(I \otimes B)\tilde{D}_n \tag{14}$$

has full column rank.[10] This condition was obtained by considering maximum likelihood estimation, because it ensures that the information matrix is positive definite. It is easy to see that this condition is exactly equivalent to requiring

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that equation (13) has no nontrivial solutions: if the matrix in (14) has full rank, then

$$R(I \otimes B)\tilde{D}_n \varphi = R(I \otimes B)h = -R(H \otimes I)b \neq 0$$

for any nonnull φ.

Condition (13) can be rephrased as

$$\varphi' T_i \theta + \varphi' t_i = 0 \quad \text{for } i = 1 \dots p, \tag{15}$$

where

$$T_i \equiv \tilde{D}_n'(I \otimes R_i')S,$$
$$t_i \equiv \tilde{D}_n'(I \otimes R_i')s,$$

and p is the number of constraints (the number of rows in R). In other words, the system is unidentified at θ if some $\varphi \neq 0$ exists that satisfies the preceding equations for every i. It is important to note that, for a given θ, the existence of nonzero solutions to the system (15) (and therefore underidentification) depends only on the matrices T_i and the vectors t_i, *which are functions of the constraints alone*. In general, it could be thought that the existence of a solution to the system (15) for some $\varphi \neq 0$ may depend on θ. However, it will be shown in Section 3.2 that the choice of θ is practically immaterial.

For any given θ, the system (15) can be written as

$$\varphi' \mathbf{T}(\theta) = [0], \tag{16}$$

where

$$\mathbf{T}(\theta) = [T_1\theta + t_1 \mid T_2\theta + t_2 \mid \cdots \mid T_p\theta + t_p].$$

The matrix $\mathbf{T}(\theta)$ has $n(n-1)/2$ rows and p columns, so its rank must be less than or equal to $\min(p, n(n-1)/2)$. Solutions to (16) do not exist (and hence the model is identified) if and only if the rank of $\mathbf{T}(\theta)$ equals $n(n-1)/2$.

3.1. The Structure Condition

The existence of solutions to (16), and therefore underidentification, occurs necessarily whenever the order condition fails, because in this case $p < n(n-1)/2$ and the rank of $\mathbf{T}(\theta)$ is, at most, p. However, there might be cases when (13) holds for any value of $\theta \in \Theta$ even though the order condition might be met. We will call such a model *structurally underidentified*: in these cases, there is at least one H that satisfies equation (13) whatever the choice of θ. Therefore, it must also be true that

$$R'(H' \otimes I)[S|s] = 0 \tag{17}$$

for some infinitesimal rotation H.

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By employing the properties of the Kronecker product and of the vectorization operator in a manner similar to the one used in the derivation of equation (15), it is possible to write the ith column of $(H' \otimes I)S$ as $(H' \otimes I)Se_i = (I \otimes S_i)h$, where S_i is an $n \times n$ matrix whose vectorization is the ith column of S. Similarly, we can transform $(H' \otimes I)s$ into $(H' \otimes I)s = (I \otimes \bar{S})h$.

The problem can now be stated as follows: given the matrix

$$T = \begin{bmatrix} R'(I \otimes S_1) \\ R'(I \otimes S_2) \\ \vdots \\ R'(I \otimes S_p) \\ R'(I \otimes \bar{S}) \end{bmatrix}, \tag{18}$$

if there is an infinitesimal rotation H such that $Th = 0$, then the system is structurally unidentified;[11] in turn, this implies the existence of a nonnull vector φ such that $T\tilde{D}_n\varphi = 0$. As a consequence, *a C-model is structurally identified if and only if the matrix $T\tilde{D}_n$ has full column rank $n(n-1)/2$*. This condition will be henceforth called the *structure condition*.

The structure condition can be more easily checked via the matrix

$$\mathcal{M} = \tilde{D}_n' T' T \tilde{D}_n$$

$$= \sum_{i=1}^{n^2-p} [\tilde{D}_n'(I \otimes S_i')RR'(I \otimes S_i)\tilde{D}_n] + \tilde{D}_n'(I \otimes \bar{S}')RR'(I \otimes \bar{S})\tilde{D}_n.$$

The system is structurally identified if and only if $\mathcal{M}$ is invertible. If not, the matrix H has a vectorization that lies in the right null space of $T\tilde{D}_n$. Note that this condition is wholly independent of θ, so structural identification is a property of the model as such. Therefore, this condition can be checked prior to estimation, like the order condition.

3.2. Sufficiency

Both the order and structure conditions are, by themselves, necessary but not sufficient. However, the order and the structure condition form a quasi-sufficient condition taken together: *if equation (16) is satisfied for some $\varphi \neq 0$ beyond the trivial case of $p < n(n-1)/2$, then the values of θ that satisfy (16) define a set of measure 0, and therefore the model is identified almost everywhere in Θ.*

This assertion can be proven by a line of reasoning similar to that in Johansen (1995, p. 130): consider the matrix $\mathbf{T}(\theta)$ in (16) as a function of θ. For (16) to have nonzero solutions, this matrix must have rank less than $n(n-1)/2$, and therefore the determinant of the following square matrix:

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$$D(\theta) = \mathbf{T}(\theta)\mathbf{T}(\theta)' = \sum_{j=1}^{p} (T_j\theta + t_j)(T_j\theta + t_j)'$$

must be 0. Because the function $|D(\theta)|$ is a polynomial in θ, it is either identically 0 or has a set of solutions that forms a closed set in Θ with zero Lebesgue measure.

Because D has $n(n-1)/2$ rows, its determinant is clearly zero identically if the number of its columns is less than $n(n-1)/2$: this takes us back to the order condition. Nevertheless, $|D| = 0$ may also happen if the order condition is satisfied yet the rank of D is less than $n(n-1)/2$; but in this case, equation (16) is satisfied for any θ, and therefore the model fails to meet the structure condition. As a consequence, when both conditions hold, (16) has no solutions in Θ but for a set of measure 0. Moreover, this set is closed, and the set of points in Θ where the two conditions together are sufficient to ensure identification is open and dense in Θ.

3.3. An Example

Let us analyze a C-model where $n = 2$ and B is lower triangular. As is well known, this case is identified, as B is simply the Cholesky decomposition of Σ, which we assume positive definite. In this case, the number of free parameters is 3 and the parameter space Θ is the subset of $\mathbb{R}^3$ such that B is invertible, namely, $\Theta = \{\theta \in \mathbb{R}^3 : \theta_1\theta_3 \neq 0\}$. The matrix B has the form

$$B = \begin{bmatrix} \theta_1 & 0 \\ \theta_2 & \theta_3 \end{bmatrix},$$

so that the constraint matrices can be written as

$$R = [0 \quad 0 \quad 1 \quad 0],$$

$$d = 0,$$

and the corresponding S matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(the vector s equals 0); as for $\tilde{D}_n$, the only possible choice (up to a scalar) when $n = 2$ is the vector

$$\tilde{D}_n = [0 \quad 1 \quad -1 \quad 0]'.$$

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The order condition is evidently satisfied, as the number of free parameters equals $n(n+1)/2 = 3$. The structure condition can be checked via

$$\mathcal{M} = \tilde{D}_n'\left[\sum_{i=1}^{3}(I \otimes S_i)R'R(I \otimes S_i')\right]\tilde{D}_n = 1,$$

and therefore the model is identified almost everywhere in $\mathbb{R}^3$.

In this case, the dimension of the matrices is small enough to make it feasible to check identification by developing the argument analytically, which is rather instructive. Because B must be lower triangular to be admissible, the identification problem boils down to establishing whether postmultiplying B by an arbitrary infinitesimal rotation may result in another lower triangular matrix. In formulas:

$$\begin{bmatrix}\theta_1 & 0\\ \theta_2 & \theta_3\end{bmatrix}\begin{bmatrix}1 & -\varphi\\ \varphi & 1\end{bmatrix} = \begin{bmatrix}\theta_1 & -\theta_1\varphi\\ \theta_2+\theta_3\varphi & \theta_3-\theta_2\varphi\end{bmatrix}.$$

The resulting matrix is clearly not admissible (i.e., lower triangular) unless $\theta_1\varphi = 0$. If $\varphi \neq 0$, then θ_1 must be 0. This implies that the only region in $\mathbb{R}^3$ where the model is underidentified is the plane $\theta_1 = 0$, which has zero Lebesgue measure and is outside Θ by hypothesis anyway.

The same result can be obtained by considering condition (15) directly; because we have only one constraint, then we have only one T_i matrix and one t_i vector, and these equal

$$\begin{aligned}T_1 &= \tilde{D}_n'(I \otimes R_1')S\\ &= [0 \quad 1 \quad -1 \quad 0]\left[\begin{array}{cc|cc}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &= [1 \quad 0 \quad 0],\end{aligned}$$

whereas $t_1 = 0$. The model is underidentified as long as

$$\varphi'\tilde{D}_n'(I \otimes R_i')(S\theta + s) = \varphi[1 \quad 0 \quad 0]\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \varphi \cdot \theta_1 = 0$$

holds for some $\varphi \neq 0$, but in this case, the only solution is $\theta_1 = 0$. Hence, the model is identified almost everywhere in $\mathbb{R}^3$ and everywhere in Θ.

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4. IDENTIFICATION IN THE *AB*-MODEL

This case is the most general: contrary to the previous section, we analyze the situation where neither A nor B is fully restricted, so the identification question revolves around establishing the existence of two matrices $(A + \mathrm{d}A)$ and $(B + \mathrm{d}B)$ that are still admissible.

The constraints to be put on $A + \mathrm{d}A$ and $B + \mathrm{d}B$ to remain admissible can be specified in a way akin to equation (11) by writing

$$A + \mathrm{d}A = (I + Q)A,$$

$$B + \mathrm{d}B = (I + Q)B(I + H),$$

where $(I + Q)$ is nonsingular and $(I + H)$ is orthogonal, so that

$$\mathrm{d}a = (I \otimes Q)(S_a\theta + s_a), \tag{19}$$

$$\mathrm{d}b = [(I \otimes Q) + (H' \otimes I)](S_b\theta + s_b) \tag{20}$$

(in equation (20), there would be a term $(H' \otimes Q)$, which, however, disappears because both matrices are infinitesimal).

As in the previous section, H has to be an infinitesimal rotation for $(I + H)$ to be orthogonal. The only additional complication with respect to Section 3 is that it is now necessary to take the matrix Q into consideration also. Moreover, Q is not necessarily an infinitesimal rotation operator, because $I + Q$ need not be orthogonal, although it is required that it is invertible. However, if we only consider what happens in a neighborhood of the optimum, it suffices to say that $|I + Q| \neq 0$ for any Q, as long as Q is infinitesimal, by the continuity of the determinant function. Therefore, if we define $q = \mathrm{vec}\, Q$, it is sufficient to consider the condition $q \neq 0$ without any further qualifications.

The equation parallel to (12) is the following system:

$$R_a\mathrm{d}a = R_a(I \otimes Q)(S_a\theta + s_a) = 0, \tag{21}$$

$$R_b\mathrm{d}b = R_b[(I \otimes Q) + (H' \otimes I)](S_b\theta + s_b) = 0; \tag{22}$$

identification holds at θ unless it is possible to find two infinitesimal nonzero matrices Q and H (with $H = -H'$) satisfying the preceding system.

With a little algebra, similar to that used in Section 3, we may reexpress some of the matrix products as

$$e_i' R_a(I \otimes Q) = q' K_{nn}(R_{a,i} \otimes I),$$

$$e_i' R_b(I \otimes Q) = q' K_{nn}(R_{b,i} \otimes I),$$

$$e_i' R(H' \otimes I) = \varphi' \tilde{D}_n(I \otimes R_i'),$$

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because $K_{nn}q = \text{vec}(Q')$ (see Section 2.2) and $h = \tilde{D}_n \varphi$. The symbol $R_{a,i}$ indicates the $(n \times n)$ matrix such that $\text{vec}(R_{a,i})'$ is the ith row of R_a. These rearrangements lead us to examine the following system of equations:

$$q'U_i^a \theta + q'u_i^a = 0 \quad \text{for } i = 1 \dots r_a,$$

$$q'U_j^b \theta + \varphi' T_j^b \theta + q'u_j^b + \varphi' t_j^b = 0 \quad \text{for } j = 1 \dots r_b,$$

where

$$U_i^a \equiv K_{nn}(R'_{a,i} \otimes I)S_a \qquad u_i^a \equiv K_{nn}(R'_{a,i} \otimes I)s_a,$$

$$U_i^b \equiv K_{nn}(R'_{b,i} \otimes I)S_b \qquad u_i^b \equiv K_{nn}(R'_{b,i} \otimes I)s_b,$$

$$T_i^b \equiv \tilde{D}'_n(I \otimes R'_{b,i})S_b \qquad t_i^b \equiv \tilde{D}'_n(I \otimes R'_{b,i})s_b,$$

which admits nonzero solutions if and only if the model is underidentified.

4.1. The Order and Structure Conditions

The order and structure conditions can be stated in a way very similar to Section 3. Again, the order condition requires that the number of free parameters $p_a + p_b$ does not exceed $n(n+1)/2$ or, equivalently, that the number of restrictions put on A and B is at least $n^2 + (n(n-1)/2)$. If this requirement were not met, then there would always be solutions for some nonzero q and/or φ to the equation

$$[q' \mid \varphi'] \mathbf{T}(\theta) = [0], \tag{23}$$

where

$$\mathbf{T}(\theta) = \left[\begin{array}{c|c|c|c|c|c} U_1^a \theta + u_1^a & \cdots & U_{p_a}^a \theta + u_{p_a}^a & U_1^b \theta + u_1^b & \cdots & U_{p_b}^b \theta + u_{p_b}^a \\ \hline 0 & \cdots & 0 & T_1^b \theta + t_1^b & \cdots & T_{p_b}^b \theta + t_{p_b}^a \end{array} \right].$$

The structure condition requires that there are no infinitesimal matrices Q and H (with H hemisymmetric) that satisfy

$$R_a(I \otimes Q)[S_a | s_a] = 0, \tag{24}$$

$$R_b[(I \otimes Q) + (H' \otimes I)][S_b | s_b] = 0. \tag{25}$$

By considering the columns of S_a and S_b one at a time, we get

$$R_a(I \otimes Q)[S_a | s_a]e_i = R_a(S'_{a,i} \otimes I)q = 0 \quad \text{for } i = 1 \dots p$$

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and

$$R_b[(I \otimes Q) + (H' \otimes I)][S_b|s_b]e_i = R_b(S'_{b,i} \otimes I)q + R_b(I \otimes S_{b,i})\tilde{D}_n\varphi = 0$$

for $i = 1 \ldots p$,

so that lack of identification implies (and is implied by) the existence of non-trivial solutions to this system, which evidently parallels the problem of finding solutions to equation (17). One thing that must be noted for the index i is that, in general, it runs from 1 to p, because the number of columns in S_a and S_b is p; however, if the parameter vector θ can be split into $\theta = [\theta'_a|\theta'_b]'$, as is usually the case, then the columns of S_a from the $(p_a + 1)$th onward are all zero and so can be omitted without affecting the existence of solutions to the system (24)–(25); the same applies to the first p_a columns of S_b.

The matrix T equivalent to the one in equation (18) then becomes

$$T = \left[\begin{array}{c|c} R_a(S'_{a,1} \otimes I) & 0 \\ R_a(S'_{a,2} \otimes I) & 0 \\ \vdots & \vdots \\ R_a(S'_{a,p_a} \otimes I) & 0 \\ R_a(\bar{S}'_a \otimes I) & 0 \\ \hline R_b(S'_{b,1} \otimes I) & R_b(I \otimes S_{b,1})\tilde{D}_n \\ R_b(S'_{b,2} \otimes I) & R_b(I \otimes S_{b,2})\tilde{D}_n \\ \vdots & \vdots \\ R_b(S'_{b,p_b} \otimes I) & R_b(I \otimes S_{b,p_b})\tilde{D}_n \\ R_a(\bar{S}'_b \otimes I) & R_b(I \otimes \bar{S}_b)\tilde{D}_n \end{array}\right] = \begin{bmatrix} U_a & 0 \\ U_b & T_b\tilde{D}_n \end{bmatrix}, \tag{26}$$

and the system is structurally identified provided there are no trivial solutions to

$$\begin{bmatrix} U_a & 0 \\ U_b & T_b\tilde{D}_n \end{bmatrix}\begin{bmatrix} q \\ \varphi \end{bmatrix} = 0. \tag{27}$$

As a consequence, an operational procedure for checking the structure condition could simply amount to verifying whether the matrix

$$\mathcal{M} = \begin{bmatrix} U'_a & U'_b \\ 0 & \tilde{D}'_n T'_b \end{bmatrix}\begin{bmatrix} U_a & 0 \\ U_b & T_b\tilde{D}_n \end{bmatrix} = \begin{bmatrix} U'_a U_a + U'_b U_b & U'_b T_b \tilde{D}_n \\ \tilde{D}'_n T'_b U_b & \tilde{D}'_n T'_b T_b \tilde{D}_n \end{bmatrix} \tag{28}$$

is singular.[12]

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The quasi-sufficiency property of the order and structure conditions combined can be assessed by means of an argument similar to that developed in Section 3.2 by noting that the rank of the matrix $\mathbf{T}(\theta)$ in equation (23) is less than $n^2 + (n(n-1)/2)$ either identically or for a set of measure 0 in Θ.

4.2. Special Cases

In certain cases, checking the structure condition involves simpler matrices; the C-model, e.g., clearly emerges as a special case: if we impose the constraint $A = I$, then we have $R_a = I$ and $[S_a | s_a] = \text{vec}\, I$, so the matrix T reduces to

$$T = \left[\begin{array}{c|c} I & 0 \\ \hline U_b & T_b \tilde{D}_n \end{array}\right],$$

which clearly has full column rank if and only if its southeast corner has. This, in turn, is precisely the matrix T in equation (18).

Another notable special case arises when it is assumed that $A\Sigma A' = I$. This case is referred to as the K-model in Giannini (1992), and, because $A\Sigma A' = I$ implies $A'A = \Sigma^{-1}$, it could be conjectured that the requisites for structural identification ought to be similar to those in the C-model. This is indeed the case, as the restrictions put on B parallel those previously put on A, for we have $R_b = I$ and $[S_b | s_b] = \text{vec}\, I$; as a consequence, equation (27) simplifies to

$$\begin{bmatrix} U_a & 0 \\ I & \tilde{D}_n \end{bmatrix} \begin{bmatrix} q \\ \varphi \end{bmatrix} = 0,$$

which implies $q = -\tilde{D}_n \varphi$; therefore, if a nonzero solution exists, the equality

$$U_a \tilde{D}_n \varphi = 0 \qquad \textbf{(29)}$$

must hold. This happens only if the rank of $U_a \tilde{D}_n$ is less than $n(n-1)/2$.

5. A FEW EXAMPLES

5.1. An Unidentified 2 × 2 Case

This is an example of a model reported in Giannini (1992), where it is shown that the order condition is not sufficient, by itself, to ensure identification. In the context of the present paper, it provides a simple yet enlightening example of failure of the structure condition. We have $n = 2$, and A has the following structure:

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$$A = \begin{bmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{bmatrix};$$

in other words, any (2×2) admissible matrix has the same elements on the diagonal, whereas the off-diagonal elements have the opposite sign. In this case, the requirement that A be invertible is accommodated by setting $\Theta = \mathbb{R}^2 \backslash \{0\}$, as the only vector (θ_1, θ_2) that makes A singular is the zero vector.

The fact that $B = I$ by hypothesis allows us to classify this model as a K-model and thus focus on (29) for checking the structure condition, rather than on the more complex equation (28). The constraint matrices can be written as

$$R_a = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$S_a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and both d_a and s_a are suitably shaped zero vectors.

The order condition is obviously met, because the number of free parameters (two) is smaller[13] than $n(n+1)/2 = 3$. As far as the structure condition goes, we can use equation (29); using the same $\tilde{D}_n$ as in Section 3.3, we have

$$\mathcal{M} = \tilde{D}_n' \left[\sum_{i=1}^{2} (S_i \otimes I) R' R (S_i' \otimes I) \right] \tilde{D}_n = 0,$$

which has rank 0, and therefore the model is structurally unidentified.

In this case, the matrices are small enough to analyze in some detail the effect of an infinitesimal rotation on the matrix A symbolically. Because any hemi-symmetric matrix of order 2 can be written as

$$H = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix},$$

then the product $A_1 = A(I + H)$ equals

$$A(I+H) = \begin{bmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} \theta_1 - \lambda\theta_2 & \theta_2 + \lambda\theta_1 \\ -(\theta_2 + \lambda\theta_1) & \theta_1 - \lambda\theta_2 \end{bmatrix},$$

which is clearly admissible. Therefore, there is an infinity of admissible matrices A_1 that satisfy $A_1' A_1 = A'A = \hat{\Sigma}^{-1}$.

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5.2. The "Standard" AB Model

The model analyzed here corresponds to the most common setup in structural VARs: the one where A is lower triangular with ones on the diagonal and B is diagonal.

$$A = \begin{bmatrix} 1 & 0 \\ \theta_1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \theta_2 & 0 \\ 0 & \theta_3 \end{bmatrix}.$$

This example is admittedly rather contrived, as this model could be easily reparametrized into a K-model, but here it just serves the purpose of providing a simple example of constraints put on both A and B. The constraint matrices are

$$R_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [S_a | s_a] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right],$$

$$R_b = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad [S_b | s_b] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The matrix $\mathcal{M}$, computed via equation (28), equals

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Because $\mathcal{M}$ is invertible, the system is structurally identified. Therefore, both order and structure conditions hold, and identification is attained.

5.3. The Blanchard (1989) Model

The model put forward in Blanchard (1989) is possibly one of the most influential applications of structural VARs in the applied macroeconomic literature. Five variables are modeled jointly: gross national product (GNP), unemploy-

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ment, prices, wages, and money supply. The model can be described as an AB model with the following structure:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 & 0 \\ a_{31} & 0 & 1 & a_{34} & 0 \\ 0 & a_{42} & a_{43} & 1 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \sigma_d & c & 0 & 0 & 0 \\ 0 & \sigma_\Theta & 0 & 0 & 0 \\ 0 & c_{32} & \sigma_p & 0 & 0 \\ 0 & c_{42} & 0 & \sigma_w & 0 \\ 0 & 0 & 0 & 0 & \sigma_m \end{bmatrix},$$

where names for the individual parameters are chosen to be consistent with Blanchard's. As can be easily seen, the model is underidentified because the number of free parameters is $17 > n(n+1)/2 = 15$ and therefore the order condition is not satisfied. In the original paper, identification is achieved by setting the two parameters a_{34} and c to fixed values, thereby ensuring that the order condition is met. It can be shown, via the approach presented here, that those restrictions are indeed sufficient for the structure condition to hold and, as a consequence, for the model to be identified.

It may be argued, however, that identification is not necessarily guaranteed by adding any pair of constraints, because the order condition would be met but the structure condition might not be. In fact, an alternative restriction strategy shows such a case:[14] keep a_{34} fixed (as in the original paper) at a value $\overline{a_{34}} \neq 0$ but set the parameter c free and fix a_{21} instead to 1.

Because $n = 5$, the matrix $\mathcal{M}$ has 35 rows and columns and is too big to be reproduced here. However, it can be checked that it is nonsingular and thus that the structure condition holds. Moreover, the matrix $\mathcal{M}$ is block-diagonal and its north-west 9×9 block equals

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The matrix $\mathcal{M}$ is nonsingular, and the model is identified. On the contrary, it is interesting to consider what happens if, instead of setting a_{21} to a fixed

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nonzero value, we set it at 0: in this case, the relevant block of the matrix $\mathcal{M}$ becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

As can be seen, the ninth column is a zero vector, so that $\mathcal{M}$ is singular and the structure condition is not met; as a consequence, in this case the two restrictions $a_{34} = \overline{a_{34}}$ and $a_{12} = 0$ do not suffice to ensure identification.

Again, it is worthwhile analyzing the situation symbolically: because the ninth column of $\mathcal{M}$ is zero, it follows that equation (27) is satisfied by a 10-element vector $\varphi = 0$ and a 25-element vector q whose elements are all 0 except for the ninth one. Hence, the corresponding Q matrix has one nonnull element on the fourth row, second column. By considering the product

$$(I+Q)A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 & 0 \\ a_{31} & 0 & 1 & a_{34} & 0 \\ 0 & a_{42} & a_{43} & 1 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 & 0 \\ a_{31} & 0 & 1 & a_{34} & 0 \\ \lambda a_{21} & \lambda + a_{42} & a_{43} & 1 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & 1 \end{bmatrix},$$

it is easy to see that admissibility of the resulting matrix depends on the value λa_{21}: for $a_{21} = 0$, the matrix is admissible for any $\lambda \neq 0$. The same goes for the product $(I+Q)B$:

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$$(I+Q)B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_d & c & 0 & 0 & 0 \\ 0 & \sigma_\Theta & 0 & 0 & 0 \\ 0 & c_{32} & \sigma_p & 0 & 0 \\ 0 & c_{42} & 0 & \sigma_w & 0 \\ 0 & 0 & 0 & 0 & \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_d & c & 0 & 0 & 0 \\ 0 & \sigma_\Theta & 0 & 0 & 0 \\ 0 & c_{32} & \sigma_p & 0 & 0 \\ 0 & c_{42} + \lambda\sigma_\Theta & 0 & \sigma_w & 0 \\ 0 & 0 & 0 & 0 & \sigma_m \end{bmatrix};$$

again, the matrix $(I + Q)B$ is admissible whatever the value of λ. This means that, despite the fact that the number of free parameters equals $n(n + 1)/2 = 15$, the model is underidentified, because there are infinite pairs of admissible matrices A and B such that $A^{-1}B$ is invariant.

It is interesting to note that an economic interpretation can be given. If the level of economic activity does not have an instantaneous impact on unemployment ($a_{21} = 0$), then it becomes impossible to tell what the instantaneous impact of unemployment on wages ($\lambda + a_{42}$) is: this happens because the restriction $a_{41} = 0$ (output does not affect wages instantaneously) becomes uninformative.

6. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

The main object of this paper is to put forth a method for assessing identification of a covariance structure. The introduction of the structure condition makes it possible to state a sufficient condition for identification solely in terms of the set of constraints.

The ideas were presented here in the context of structural VARs estimation but may prove useful for other multivariate models. A closely related class of models is the structural vector correction models (see, e.g., King, Plosser, Stock, and Watson, 1991), where identification and estimation also take into account the possibility of having to deal with a cointegrated system. This makes it possible to identify a covariance structure by differentiating between permanent and transitory innovations.

Another set of models that lend themselves quite naturally to be analyzed in the present framework is that of LISREL (linear structural relationships) models (or structural equation models). These models are widely used in applied sociology[15] and can be briefly described as models where a vector of observable variables y_t is thought to depend on some exogenous variables x_t through a known number of latent variables λ_t. The assumed data generating process can be summarized as

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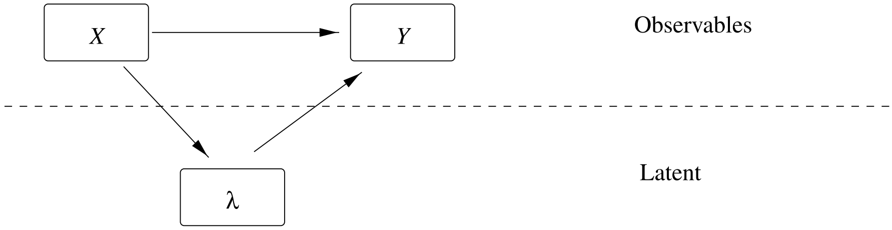
$$y_t = L_y \lambda_t + u_t,$$

$$x_t = L_x x_t + v_t,$$

$$\lambda_t = B\lambda_t + Gx_t + w_t,$$

or graphically:



Estimation of these models is carried out by expressing the covariance matrix of $(y_t|x_t)$ (called the *implied covariance matrix*) in terms of the unknown parameters, so it seems plausible that the concepts in this paper should readily apply.

The identification issue in multivariate generalized autoregressive conditional heteroskedasticity (GARCH) models could also be analyzed along these lines: factor GARCH models have been put forward rather often to overcome problems arising from the massive number of parameters necessary for medium-scale models, and they essentially reproduce equation (2) with conditionally heteroskedastic u_t's; some of the latest examples in this vein are Vrontos, Dellaportas, and Politis (2003), van der Weide (2002), and Lanne and Saikkonen (2005). In these models, however, the very presence of heteroskedasticity makes it impossible to use the same analytical framework presented here, because it is not possible to write the objective function as in (4). It can be said that, in general, heteroskedasticity is actually helpful when it comes to identification: the GO-GARCH (generalized orthogonal GARCH) model in van der Weide (2002), for example, could be considered a C-model with unrestricted B. Another interesting application of this idea was recently proposed by Rigobon (2003). Research by the author is currently under way in this direction.

Some points may deserve further investigation. First, it might be interesting to extend the present approach to make it feasible not only to check whether the model is identified or not but also to compute its overidentification rank. Furthermore, a theoretical issue that may be possible to tackle within the present framework could be to analyze the relationship between the set of points in the parameter space that give rise to singular A and/or B matrices and the set where identification fails, despite the fact that both the order and structure conditions are satisfied. We do know that both are sets with 0 Lebesgue measure; it is natural to ask if anything more stringent can be said in general. Finally, the possibility of generalizing the present approach to nonlinear constraints should be mentioned. Because all the arguments presented applied to an arbitrarily small

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neighborhood of the parameter space, perhaps some kind of linearization of the constraints might be attempted.

NOTES

1. For a complete and detailed survey paper on VARs, see Canova (1995).

2. A procedure is suggested in Giannini (1992) where identification is checked by computing the information matrix at a random point in the parameter space. This procedure hinges on the conjecture that in an identified model the probability of making an incorrect decision is 0. This insight is made more precise in Section 3.2, where it is shown that in such a case underidentification occurs on an area with zero Lebesgue measure in the parameter space.

3. In a globally identified model, the maximum of the objective function is unique. Local identification, on the contrary, stipulates that multiple maxima can exist, but each of these is an isolated point (see also Gourieroux and Monfort, 1995, pp. 88–89). From a statistical viewpoint, the most important requirement is local identification, because it makes it possible to resort to standard proofs for asymptotic properties of extremum estimators. On the other hand, local identification can often be made equivalent to global identification by suitably reshaping the parameter space.

4. Linear constraints are not only easier to manage but are also often the most interesting because they lend themselves very naturally to the representation of some economic theory: a typical example is zero restrictions on some parameters.

5. In fact, this point is sometimes overlooked in the applied literature: e.g., the statement "In the structural VAR approach, B can be any structure as long as it has sufficient restrictions" was taken verbatim from a recent working paper. This sentence seems to imply that the order condition is considered sufficient, not just necessary.

6. A square matrix X for which $X = -X'$ holds is called *hemisymmetric*, or sometimes *skew-symmetric*.

7. An alternate notation for K_{nn}, which is sometimes used, e.g., in Pollock (1979), is Ⓣ.

8. For the properties of the infinitesimal rotation operator, see Weisstein (2004).

9. This reasoning could have been equivalently, and certainly more compactly, put by requiring that, for $B_0 + \mathrm{d}B$ to be admissible, the condition $R\mathrm{d}b = 0$ has to be satisfied, where $\mathrm{d}B = BH$ for some hemisymmetric H; in this case, however, brevity would have possibly come at the expense of clarity.

10. The original notation was slightly altered to match ours.

11. Notice that each row of the matrix T can be written as $\mathrm{vec}(S_i' R_j)'$ for all possible combinations of the columns of R and $[S|s]$. This may help computationally. Another point that may help in a practical software implementation is that, as a rule, a large number of rows in T are zero and clearly can be left out without influencing the rank of T.

12. An OX class for checking the structure condition is available at http://www.econ.univpm.it/lucchetti/SVARident.

13. Or equivalently, the number of constraints (two) is greater than $n(n-1)/2 = 1$.

14. The economic rationale of the choice presented here is obviously irrelevant in the present context.

15. For a survey, see Mueller (1996).

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