Proceedings of the Royal Society of Edinburgh, **151**, 425–450, *2021*
DOI:10.1017/prm.2020.23

Existence results for the Kudryashov–Sinelshchikov–Olver equation

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(MS Received 13 November 2019; Accepted 1 March 2020)

The Kudryashov–Sinelshchikov–Olver equation describes pressure waves in liquids with gas bubbles taking into account heat transfer and viscosity. In this paper, we prove the existence of solutions of the Cauchy problem associated with this equation.

Keywords: Existence; Kudryashov–Sinelshchikov–Olver equation; Cauchy problem

2010 *Mathematics subject classification:* 35G25, 35K55

1. Introduction

In this paper, we investigate the existence of solutions of the following Cauchy problem:

$$\begin{cases} \partial_t u + \kappa u \partial_x u + \alpha \partial_x^3 u + \beta u \partial_x^3 u + \gamma \partial_x u \partial_x^2 u - q^2 \partial_x^2 u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with

$$\kappa, \alpha, \beta, \gamma, q \in \mathbb{R}, \quad \alpha, \beta, \gamma \neq 0. \tag{1.2}$$

The assumption on the initial datum depends on the choice of coefficients κ, β, γ, q. In fact, if we choose

$$(\beta, \gamma) = \left(-\frac{2\kappa}{3}, -\frac{\kappa}{3}\right), \quad \kappa, q \neq 0 \quad \text{or} \tag{1.3}$$

$$\kappa = 0, \quad q \neq 0, \quad \beta = 2\gamma, \tag{1.4}$$

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we assume

$$u_0 \in H^1(\mathbb{R}). \tag{1.5}$$

Instead, if we choose

$$(\kappa, q) = (0, 0), \quad \beta = -2\gamma \quad \text{or} \tag{1.6}$$

$$\kappa = 0, \quad q \neq 0, \quad \beta = -2\gamma, \tag{1.7}$$

we assume

$$u_0 \in H^2(\mathbb{R}). \tag{1.8}$$

From a physical point of view, (1.1) is known as the Kudryashov–Sinelshchikov equation, and describes pressure waves in liquids with gas bubbles taking into account heat transfer and viscosity [27].

Equation (1.1) is a generalization both of the Korteweg-de Vries equation [26]

$$\begin{cases} \partial_t u + \kappa u \partial_x u + \alpha \partial_x^3 u = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

and the Korteweg-de Vries-Burgers one (see [4, 40]):

$$\begin{cases} \partial_t u + \kappa u \partial_x u + \alpha \partial_x^3 u - q^2 \partial_x^2 u = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.10}$$

(1.1) was also derived for water waves by Olver [32] (see also [24]), using Hamiltonian perturbation theory, with further generalization given by Craig and Groves [16].

Mathematical properties of (1.1) were studied recently in many detail, including the existence of the travelling wave solutions [5, 17, 29, 31, 34, 35, 37, 43], the solitary and periodic wave solutions [21, 23], the periodic loop solutions [22], the soliton solutions [46], the quasi-exact solutions [25]. Methods to find exact solutions are in [1–3, 18–20, 30, 36, 38, 42, 44, 45, 47, 48]. Moreover, following [6, 7, 28, 39], under the assumption (1.3), in [8], the authors used the convergence of the solution of (1.1) to the unique entropy solution of the following scalar conservation law:

$$\begin{cases} \partial_t u + \kappa u \partial_x u = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.11}$$

A similar result is proven in [10, 39] and [11] for (1.9) and (1.10), respectively.

The main result of this paper is the following theorem.

Theorem 1.1. *Fix $T > 0$. Assume (1.3) or (1.4) and (1.5). There exists a solution u of (1.1), such that*

$$\cap L^4(0, T; W^{1,4}(\mathbb{R})). \tag{1.12}$$

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If we assume (1.6) *and* (1.8)*, there exists a solution* u *of* (1.1)*, such that*

$$u \in L^{\infty}(0,T;H^2(\mathbb{R})). \tag{1.13}$$

Instead, assuming (1.7) *and* (1.8)*, there exists a solution* u *of* (1.1)*, such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^{\infty}(0,T;H^2(\mathbb{R})) \cap L^2(0,T;H^3(\mathbb{R})), \quad 0 \leqslant t \leqslant T. \tag{1.14}$$

Observe that, if $\kappa = \gamma^2$ with $\gamma \neq 0$, then (1.3) is equivalent to

$$\beta = -\frac{2\gamma^2}{3}, \quad \gamma^2 + 3\gamma = 0. \tag{1.15}$$

Therefore, theorem 1.1 holds also in the case $(\kappa,\beta,\gamma) = (9,-6,-3)$. In general, thanks to [**8**, lemma 2.2], theorem 1.1 holds also in the following cases:

$$\begin{aligned} &\kappa = -3^{1/(2n-1)}, \quad n \neq \frac{1}{2}, \\ &\kappa = (\gamma+\theta)^{2n}, \quad \theta \leqslant 3^{1/(2n-1)}\left(\frac{1}{2n}\right)^{2n/(2n-1)} + \left(\frac{3}{2n}\right)^{1/(2n-1)}. \end{aligned} \tag{1.16}$$

Note that (1.15) and (1.16) do not imply (1.3).

Observe again that theorem 1.1 is based on the Aubin–Lions Lemma (see [**9**, **14**, **15**, **41**]) and the Sobolev Immersion Theorem. It does not give the uniqueness of the solution of (1.1), which is proven in [**33**, theorem 2.2] under the assumption (1.8) and in the appendices A and B assuming (1.10).

The paper is organized as follows. Sections 2, 3, 4, 5 are devoted to the proof of theorem 1.1 under the assumptions (1.3)–(1.5), (1.4) and (1.5), (1.6)–(1.8), (1.7) and (1.8), respectively. In the appendices A and B, we prove the well-posedness of (1.10) under the assumptions (1.5) and (1.8), respectively.

2. Proof of theorem 1.1, under the assumptions (1.3) and (1.5)

In this section, we prove theorem 1.1, under the assumptions (1.3) and (1.5).

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [**12**, **33**]:

$$\begin{cases} \partial_t u_\varepsilon + \kappa u_\varepsilon \partial_x u_\varepsilon + \alpha \partial_x^3 u_\varepsilon \\ \quad +\beta u_\varepsilon \partial_x^3 u_\varepsilon + \gamma \partial_x u_\varepsilon \partial_x^2 u_\varepsilon - q^2 \partial_x^2 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & t>0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leqslant \|u_0\|_{H^1(\mathbb{R})}. \tag{2.2}$$

Let us prove some a priori estimates on u_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

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Following [8, lemma 2.1], we prove the following result.

LEMMA 2.1. *Assume* (1.3). *For each* $t \geqslant 0$,

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2 \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\, \mathrm{d}s \\ + 2q^2 \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\varepsilon \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \\ + 2\varepsilon \int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0.\end{aligned} \tag{2.3}$$

In particular, we have that

$$\|u_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})} \leqslant C_0. \tag{2.4}$$

Proof. Multiplying (2.1) by $2(u_\varepsilon - \partial_x^2 u_\varepsilon)$, we have that

$$\begin{aligned}2(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_t u_\varepsilon + 2\kappa(u_\varepsilon - \partial_x^2 u_\varepsilon)u_\varepsilon\partial_x u_\varepsilon + 2\alpha(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^3 u_\varepsilon \\ + 2\beta(u_\varepsilon - \partial_x^2 u_\varepsilon)u_\varepsilon\partial_x^3 u_\varepsilon + 2\gamma(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon \\ - 2q^2(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^2 u_\varepsilon = -2\varepsilon(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^4 u_\varepsilon.\end{aligned} \tag{2.5}$$

Observe that

$$\begin{aligned}2\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_t u_\varepsilon\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right),\\ 2\kappa\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)u_\varepsilon\partial_x u_\varepsilon\,\mathrm{d}x &= -2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x,\\ 2\alpha\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^3 u_\varepsilon\,\mathrm{d}x &= -2\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon = 0,\\ 2\beta\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x &= -4\beta\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x + \beta\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x,\\ 2\gamma\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x &= 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x - 2\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x,\\ -2q^2\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^2 u_\varepsilon\,\mathrm{d}x &= 2q^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ -2\varepsilon\int_{\mathbb{R}}(u_\varepsilon - \partial_x^2 u_\varepsilon)\partial_x^4 u_\varepsilon &= -2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{2.6}$$

Therefore, by an integration on $\mathbb{R}$ of (2.5) and (2.6), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) + 2q^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ + 2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ = 2(\kappa + 2\beta - \gamma)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x + (2\gamma - \beta)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x.\end{aligned} \tag{2.7}$$

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Thanks to (1.3), we have that

$$\begin{cases} \kappa + 2\beta - \gamma = 0, \\ 2\gamma - \beta = 0. \end{cases} \tag{2.8}$$

Consequently, (2.3) follows from (2.2), (2.8) and an integration on $(0,t)$ of (2.7).

Finally, we prove (2.4). Thanks to (2.3) and the Hölder inequality,

$$\begin{aligned} u_\varepsilon^2(t,x) &= 2\int_{-\infty}^{x} u_\varepsilon \partial_x u_\varepsilon \, \mathrm{d}x \leqslant 2\int_{\mathbb{R}} |u_\varepsilon||\partial_x u_\varepsilon| \, \mathrm{d}x \\ &\leqslant 2\,\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leqslant C_0. \end{aligned}$$

Therefore,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leqslant C_0,$$

which gives (2.4). □

LEMMA 2.2. *Assume* (1.3)*. For each* $t \geqslant 0$,

$$\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})} \, \mathrm{d}s \leqslant C_0. \tag{2.9}$$

The proof of the previous lemma is based on the regularity of the functions u_ε and the following result.

LEMMA 2.3. *For each* $t \geqslant 0$, *we have that*

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \leqslant \frac{3}{2}\left(\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right). \tag{2.10}$$

In particular, we obtain that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \leqslant 6\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \tag{2.11}$$

Proof. We begin by proving (2.10). Since

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} = \int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 \, \mathrm{d}x = \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x u_\varepsilon)^3 \, \mathrm{d}x = -3\int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2 \partial_x^2 u_\varepsilon \, \mathrm{d}x,$$

thanks to the Hölder inequality,

$$\begin{aligned} \|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} &\leqslant 3\int_{\mathbb{R}} |u_\varepsilon(\partial_x u_\varepsilon)^2||\partial_x^2 u_\varepsilon| \, \mathrm{d}x \\ &\leqslant 3\left(\int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^4 \, \mathrm{d}x\right)^{1/2} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\ &\leqslant 3\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\,\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}. \end{aligned}$$

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Due to the Young inequality,

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \leqslant \frac{3}{2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} + \frac{3}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

that is (2.10).

Finally, we prove (2.11). Thanks to the regularity of u_ε and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 &= 2\int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x \leqslant 2\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| \,\mathrm{d}x \\ &\leqslant 2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore,

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} \leqslant 4\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{2.12}$$

It follows from (2.10) that

$$\begin{aligned}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} &\leqslant 6\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\quad + \frac{3}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\leqslant 6\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

which gives (2.11). □

Proof of lemma 2.2. Thanks to (2.3) and (2.11),

$$\begin{aligned}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} &\leqslant 6\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\leqslant C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, integrating on $(0,t)$, by (2.3), we have (2.9). □

The proof of theorem 1.1 is based on the following lemma.

Lemma 2.4. *Fix $T>0$. Then,*

$$\{u_\varepsilon\}_{\varepsilon>0} \textit{ is compact in } L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}). \tag{2.13}$$

Consequently, there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u\in L^2_{\mathrm{loc}}$ $((0,\infty)\times\mathbb{R})$ such that, for each compact subset K of $(0,\infty)\times\mathbb{R}$,

$$u_{\varepsilon_k} \to u \textit{ in } L^2(K) \textit{ and a.e.} \tag{2.14}$$

Moreover, u is a solution of (1.1) satisfying (1.12).

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Proof. To prove (2.13), we rely on the Aubin–Lions Lemma (see [**9**, **14**, **15**, **41**]). We recall that

$$H^1_{\text{loc}}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{\text{loc}}(\mathbb{R}) \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}),$$

where the first inclusion is compact and the second one is continuous. Owing to the Aubin–Lions Lemma [**41**], to prove (2.13), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{\text{loc}}(\mathbb{R})), \tag{2.15}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{\text{loc}}(\mathbb{R})). \tag{2.16}$$

We prove (2.15). Thanks to lemma 2.1,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leqslant C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^1(\mathbb{R})),$$

which gives (2.15).

We prove (2.16). We begin by observing that

$$\beta\partial_x\left(u\partial_x^2 u_\varepsilon\right) = \beta\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \beta u_\varepsilon \partial_x^3 u_\varepsilon = \frac{\beta}{2}\partial_x\left((\partial_x u_\varepsilon)^2\right) + \beta u_\varepsilon \partial_x^3 u_\varepsilon.$$

Therefore,

$$\beta u_\varepsilon \partial_x^3 u_\varepsilon = \beta\partial_x\left(u\partial_x^2 u_\varepsilon\right) - \frac{\beta}{2}\partial_x\left((\partial_x u_\varepsilon)^2\right). \tag{2.17}$$

It follows from (1.1) and (2.17)

$$\partial_t u_\varepsilon = \partial_x\left(-\frac{\kappa}{2}u_\varepsilon^2 - \alpha\partial_x^2 u_\varepsilon - \beta u_\varepsilon\partial_x^2 u_\varepsilon + \frac{\beta-\gamma}{2}(\partial_x u_\varepsilon)^2 + q^2\partial_x u_\varepsilon - \varepsilon\partial_x^3 u_\varepsilon\right). \tag{2.18}$$

We have that

$$\kappa^2\left\|u_\varepsilon^2\right\|^2_{L^2((0,T)\times\mathbb{R})} \leqslant C(T). \tag{2.19}$$

Thanks to lemma 2.1,

$$\begin{aligned}\kappa^2\int_0^T\int_{\mathbb{R}} u_\varepsilon^4 \,\mathrm{d}t\,\mathrm{d}x &\leqslant \kappa^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}} u_\varepsilon^2\,\mathrm{d}t\,\mathrm{d}x \\ &\leqslant C(T)\int_0^T\int_{\mathbb{R}} u_\varepsilon^2\,\mathrm{d}t\,\mathrm{d}x \leqslant C(T).\end{aligned}$$

We claim that

$$\beta^2\left\|u_\varepsilon\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \leqslant C(T). \tag{2.20}$$

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Again by lemma 2.1,

$$\beta^2 \int_0^T \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant \beta^2 \left\| u_\varepsilon \right\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_0^T \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}t\,\mathrm{d}x \\ \leqslant C(T) \int_0^T \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}t\,\mathrm{d}x \leqslant C(T).$$

Moreover, since $0 < \varepsilon < 1$, by lemmas 2.1 and 2.2,

$$\begin{aligned} &\alpha^2 \left\| \partial_x^2 u_\varepsilon \right\|^2_{L^2((0,T)\times\mathbb{R})}, \quad \frac{(\beta-\gamma)^2}{4} \left\| \partial_x u_\varepsilon \right\|^4_{L^4((0,T)\times\mathbb{R})} \leqslant C(T), \\ &q^4 \left\| \partial_x u_\varepsilon \right\|^2_{L^2(\mathbb{R})}, \quad \varepsilon^2 \left\| \partial_x^3 u_\varepsilon \right\|^2_{L^2((0,T)\times\mathbb{R})} \leqslant C(T). \end{aligned} \tag{2.21}$$

Therefore, by (2.19), (2.20) and (2.21)

$$\left\{ -\frac{\kappa}{2} u_\varepsilon^2 - \alpha \partial_x^2 u_\varepsilon - \beta u_\varepsilon \partial_x^2 u_\varepsilon + \frac{\beta - \gamma}{2} (\partial_x u_\varepsilon)^2 + q^2 \partial_x u_\varepsilon - \varepsilon \partial_x^3 u_\varepsilon \right\}_{\varepsilon > 0}$$

is bounded in $H^1((0,T)\times\mathbb{R})$.

Thanks to the Aubin–Lions Lemma, (2.13) and (2.14) hold.

Consequently, u is solution of (1.1) and (1.12) holds. □

3. Proof of theorem 1.1, under the assumptions (1.4) and (1.5)

In this section, we prove theorem 1.1, under the assumptions (1.4) and (1.5). Thanks to (1.4), (1.1) reads

$$\begin{cases} \partial_t u + \alpha \partial_x^3 u + 2\gamma u \partial_x^3 u + \gamma \partial_x u \partial_x^2 u - q^2 \partial_x^2 u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (3.1). Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [**12**, **33**]:

$$\begin{cases} \partial_t u_\varepsilon + \alpha \partial_x^3 u_\varepsilon + 2\gamma u_\varepsilon \partial_x^3 u_\varepsilon + \gamma \partial_x u_\varepsilon \partial_x^2 u_\varepsilon - q^2 \partial_x^2 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{3.2}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that (2.2) holds.

Let us prove some a priori estimates on u_ε.

Lemma 3.1. *For each* $t \geqslant 0$,

$$\left\| \partial_x u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + 2q^2 \int_0^t \left\| \partial_x^2 u_\varepsilon(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\varepsilon \int_0^t \left\| \partial_x^3 u_\varepsilon(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0. \tag{3.3}$$

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Proof. Multiplying (3.2) by $-2\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x \\ &= 2\alpha\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x + 4\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x \\ &\quad + 2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x - 2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\quad + 2\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x \\ &= -2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x + 2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\ &\quad - 2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &= -2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 0.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 0.$$

An integration on $(0,t)$ and (2.2) gives (3.3). □

LEMMA 3.2. *Fix $T > 0$. There exists a constant $C_0 > 0$, independent on ε, such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant C_0. \tag{3.4}$$

In particular, we have that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + 2\varepsilon\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leqslant C(T), \tag{3.5}$$

for every $0 \leqslant t \leqslant T$. Moreover, (2.9) holds for every $0 \leqslant t \leqslant T$.

Proof. Let $0 \leqslant t \leqslant T$. Multiplying (3.2) by $2u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x \\ &= -2\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x - 4\gamma\int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon \,\mathrm{d}x \\ &\quad - 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x + 2q^2\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x - 2\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x\end{aligned}$$

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$$
\begin{aligned}
&= 2\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \, \mathrm{d}x + 6\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \, \mathrm{d}x \\
&\quad - 2q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \, \mathrm{d}x \\
&= 6\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \, \mathrm{d}x - 2q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad - 2\varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 .
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left\| u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + 2q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad = 6\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \, \mathrm{d}x.
\end{aligned} \tag{3.6}
$$

Due to the Young inequality,

$$
\begin{aligned}
&6|\gamma| \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| \, \mathrm{d}x \\
&\quad = 2 \int_{\mathbb{R}} |q \partial_x u_\varepsilon| \left| \frac{3\gamma u_\varepsilon \partial_x^2 u_\varepsilon}{q} \right| \mathrm{d}x \\
&\quad \leqslant q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{9\gamma^2}{q^2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x^2 u_\varepsilon)^2 \, \mathrm{d}x \\
&\quad \leqslant q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{9\gamma^2}{q^2} \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 .
\end{aligned}
$$

If follows from (3.6) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left\| u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + q^2 \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\quad \leqslant \frac{9\gamma^2}{q^2} \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 .
\end{aligned}
$$

Consequently, (2.2), (3.3) and an integration on $(0,t)$, we have that

$$
\begin{aligned}
&\left\| u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}^2 + q^2 \int_0^t \left\| \partial_x u_\varepsilon(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s + 2\varepsilon \int_0^t \left\| \partial_x^2 u_\varepsilon(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s \\
&\quad \leqslant C_0 + \frac{9\gamma^2}{q^2} \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \int_0^t \left\| \partial_x^2 u_\varepsilon(s,\cdot) \right\|_{L^2(\mathbb{R})}^2 \mathrm{d}s \\
&\quad \leqslant C_0 \left(1 + \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 \right).
\end{aligned} \tag{3.7}
$$

We prove (3.4). Thanks to (3.3) and (3.7), and arguing as in lemma 2.1, we obtain the following inequality for $\left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2$:

$$
\left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^4 - C_0 \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}^2 - C_0 \leqslant 0,
$$

which gives (3.4).

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Finally, (3.5) follows from (3.4) and (3.7), while, thanks to (3.3) and (3.5), and arguing as in lemma 2.2, we have (2.9). □

Then, arguing as in § 2, we have theorem 1.1.

4. Proof of theorem 1.1, under the assumptions (1.6) and (1.8)

In this section, we prove theorem 1.1, under the assumptions (1.6) and (1.8). Thanks to (1.6), (1.1) reads

$$\begin{cases}\partial_t u+\alpha\partial_x^3 u-2\gamma u\partial_x^3 u+\gamma\partial_x u\partial_x^2 u=0, & t>0,\ x\in\mathbb{R},\\ u(0,x)=u_0(x), & x\in\mathbb{R}.\end{cases} \tag{4.1}$$

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (4.1).

Fix a small number $0<\varepsilon<1$ and let $u_\varepsilon=u_\varepsilon(t,x)$ be the unique classical solution of the following problem [**12**, **33**]:

$$\begin{cases}\partial_t u_\varepsilon+\alpha\partial_x^3 u_\varepsilon-2\gamma u_\varepsilon\partial_x^3 u_\varepsilon+\gamma\partial_x u_\varepsilon\partial_x^2 u_\varepsilon=-\varepsilon\partial_x^4 u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ u_\varepsilon(0,x)=u_{\varepsilon,0}(x), & x\in\mathbb{R},\end{cases} \tag{4.2}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\left\|u_{\varepsilon,0}\right\|_{H^2(\mathbb{R})}\leqslant\left\|u_0\right\|_{H^2(\mathbb{R})}. \tag{4.3}$$

Let us prove some a priori estimates on u_ε.

Lemma 4.1. *For each $t\geqslant 0$,*

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\leqslant C_0. \tag{4.4}$$

Proof. Multiplying (4.2) by $2\partial_x^4 u_\varepsilon$, we have that

$$2\partial_x^4 u_\varepsilon\partial_t u_\varepsilon=-2\alpha\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon+4\gamma u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon-2\gamma\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon-2\varepsilon(\partial_x^4 u_\varepsilon)^2. \tag{4.5}$$

Since

$$\begin{aligned}2\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t u_\varepsilon\,\mathrm{d}x&=\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ -2\alpha\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\,\mathrm{d}x&=0,\\ 4\gamma\int_{\mathbb{R}}u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\,\mathrm{d}x&=-2\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\,\mathrm{d}x,\\ -2\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\,\mathrm{d}x&=2\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\,\mathrm{d}x,\end{aligned}$$

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integrating (4.5) on $\mathbb{R}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=0.$$

(4.3) and an integration on $(0,t)$ give (4.4). □

LEMMA 4.2. *Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon \mathrm{e}^{(10\gamma^2+5)t/2}\int_0^t \mathrm{e}^{-(10\gamma^2+5)s/2}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\leqslant C(T), \tag{4.6}$$

for every $0\leqslant t\leqslant T$. In particular, we have that

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\ \|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\leqslant C(T), \tag{4.7}$$

for every $0\leqslant t\leqslant T$.

The proof of the previous lemma is based on the regularity of the functions u_ε and the following result.

LEMMA 4.3. *For each $t\geqslant 0$, we have that*

$$\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\leqslant 2\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^7_{L^2(\mathbb{R})}}. \tag{4.8}$$

In particular, we get

$$\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\leqslant \frac{1}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})}+\frac{1}{2}. \tag{4.9}$$

Proof. We begin by observing that

$$\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\leqslant \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (2.12), we obtain that

$$\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\leqslant 2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^3_{L^2(\mathbb{R})}. \tag{4.10}$$

[**13**, lemma 2.3] says that

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}.$$

Therefore, by (4.10),

$$\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\leqslant 2\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^7_{L^2(\mathbb{R})}},$$

that is (4.8).

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Finally, we prove (4.9). Thanks to (4.8) and the Young inequality,

$$\begin{aligned}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x &\leqslant 2\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^7_{L^2(\mathbb{R})}}\\ &\leqslant \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})}\\ &\leqslant \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})} + \frac{1}{2},\end{aligned}$$

that is (4.9). □

Proof of lemma 4.2. Let $0 \leqslant t \leqslant T$. Multiplying (4.2) by $2u_\varepsilon$, we have

$$2u_\varepsilon\partial_t u_\varepsilon = -2\alpha u_\varepsilon\partial_x^3 u_\varepsilon + 4\gamma u_\varepsilon^2\partial_x^3 u_\varepsilon - 2\gamma u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon - 2\varepsilon u_\varepsilon\partial_x^4 u_\varepsilon. \tag{4.11}$$

Observe that

$$\begin{aligned} 2\int_{\mathbb{R}} u_\varepsilon\partial_t u_\varepsilon\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ -2\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x &= 2\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x = 0,\\ 4\gamma\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon\,\mathrm{d}x &= -8\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x,\\ -2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^4 u_\varepsilon\,\mathrm{d}x &= 2\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x = -2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Then an integration on $\mathbb{R}$ of (4.11) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -10\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x. \tag{4.12}$$

Due to (4.4), (4.9) and the Young inequality,

$$\begin{aligned} &10|\gamma|\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon\partial_x^2 u_\varepsilon|\,\mathrm{d}x\\ &\quad\leqslant 5\gamma^2\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 5\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x\\ &\quad\leqslant 5\gamma^2\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{5}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 5\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{5}{2}\\ &\quad\leqslant \frac{10\gamma^2+5}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T).\end{aligned}$$

Consequently, by (4.12),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leqslant \frac{10\gamma^2+5}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T).$$

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The Gronwall Lemma and (4.3) gives

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\, \mathrm{e}^{(10\gamma^2+5)t/2}\int_0^t \mathrm{e}^{-(10\gamma^2+5)s/2}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s$$
$$\leqslant C_0\,\mathrm{e}^{(10\gamma^2+5)t/2} + C(T)\,\mathrm{e}^{(10\gamma^2+5)t/2}\int_0^t \mathrm{e}^{-(10\gamma^2+5)s/2}\,\mathrm{d}s \leqslant C(T),$$

which gives (4.6).

Finally, (4.7) follows from (4.4), (4.6) and [**13**, lemma 2.3]. □

The proof of theorem 1.1 is based on the following lemma.

Lemma 4.4. *Fix $T>0$. Then, (2.13) holds. Consequently, there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u\in L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R})$ such that, for each compact subset K of $(0,\infty)\times\mathbb{R})$, (2.14) holds. Moreover, u is a solution of (4.1) satisfying (1.13).*

Proof. Following lemma 2.4, we prove (2.15) and (2.16).

We prove (2.15). Thanks to lemmas 4.1 and 4.2,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^2(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leqslant C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^2(\mathbb{R})),$$

which gives (2.15).

We prove (2.16). Arguing as in lemma 2.4 thanks to (2.17) and (4.1), we have that

$$\partial_t u_\varepsilon = \partial_x\left(-\alpha\partial_x^2 u_\varepsilon + 2\gamma u_\varepsilon \partial_x^2 u_\varepsilon - \frac{3\gamma}{2}(\partial_x u_\varepsilon)^2 - \varepsilon\partial_x^3 u_\varepsilon\right). \tag{4.13}$$

Observe that, by lemmas 4.1 and 4.2, we have (2.20) with $\beta=-2\gamma$, and $0<\varepsilon<1$,

$$\alpha^2\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})},\ \varepsilon^2\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2(\mathbb{R})} \leqslant C(T). \tag{4.14}$$

We prove that

$$\frac{9\gamma^2}{4}\|\partial_x u_\varepsilon\|^4_{L^4((0,T)\times\mathbb{R})} \leqslant C(T). \tag{4.15}$$

Thanks to lemma 4.2,

$$\frac{9\gamma^2}{4}\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4\,\mathrm{d}t\,\mathrm{d}x \leqslant \frac{9\gamma^2}{4}\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\,\mathrm{d}t\,\mathrm{d}x$$
$$\leqslant C(T)\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\,\mathrm{d}t\,\mathrm{d}x \leqslant C(T).$$

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Therefore, by (2.20), (4.14) and (4.15),

$$\left\{-\alpha\partial_x^2 u_\varepsilon + 2\gamma u_\varepsilon \partial_x^2 u_\varepsilon - \frac{3\gamma}{2}(\partial_x u_\varepsilon)^2 - \varepsilon\partial_x^3 u_\varepsilon\right\}_{\varepsilon>0} \text{ is bounded in } H^1((0,T)\times\mathbb{R}).$$

Thanks to the Aubin–Lions Lemma, (2.13) and (2.14) hold.

Consequently, u is solution of (4.1) and (1.13) holds. □

5. Proof of theorem 1.1, under the assumptions (1.7) and (1.8)

In this section, we prove theorem 1.1, under the assumptions (1.7) and (1.8). Thanks to (1.7), (1.1) reads

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u - 2\gamma u \partial_x^3 u + \gamma\partial_x u \partial_x^2 u - q\partial_x^2 u = 0, & t>0,\ x\in\mathbb{R},\\ u(0,x) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{5.1}$$

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (5.1).

Fix a small number $0<\varepsilon<1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [**12**, **33**]:

$$\begin{cases} \partial_t u_\varepsilon + \alpha\partial_x^3 u_\varepsilon - 2\gamma u_\varepsilon \partial_x^3 u_\varepsilon + \gamma\partial_x u_\varepsilon \partial_x^2 u_\varepsilon - q^2\partial_x^2 u_\varepsilon = -\varepsilon\partial_x^4 u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x\in\mathbb{R}, \end{cases} \tag{5.2}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \leqslant \|u_0\|_{H^2(\mathbb{R})}, \quad \varepsilon\|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leqslant C_0, \tag{5.3}$$

where C_0 is a positive constant, independent on ε.

Let us prove some a priori estimates on u_ε.

We prove the following result

LEMMA 5.1. *For each* $t\geqslant 0$,

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2q^2\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}x + 2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0. \tag{5.4}$$

Proof. We begin by observing that

$$-2q^2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \,\mathrm{d}x \partial_x^4 u_\varepsilon \,\mathrm{d}x = 2q^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Thus, arguing as in lemma 4.1, we have (5.4). □

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Lemma 5.2. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2\,\mathrm{e}^{(10\gamma^2+5)t/2}\int_0^t \mathrm{e}^{-(10\gamma^2+5)s/2}\,\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + 2\varepsilon\,\mathrm{e}^{(10\gamma^2+5)t/2}\int_0^t \mathrm{e}^{-(10\gamma^2+5)s/2}\,\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leqslant C(T), \tag{5.5}$$

for every $0 \leqslant t \leqslant T$. In particular, (4.7) holds.

Proof. Let $0 \leqslant t \leqslant T$. Observe that

$$-2q^2\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon\,\mathrm{d}x = 2q^2\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Arguing as in lemma 4.2, we have (4.7) and (5.5). □

Lemma 5.3. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$q^2\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon\,\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\int_0^t \|\partial_t u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \leqslant C(T), \tag{5.6}$$

for every $0 \leqslant t \leqslant T$.

Proof. Let $0 \leqslant t \leqslant T$. Multiplying (5.2) by $2\partial_t u_\varepsilon$, we have that

$$\begin{aligned} &2(\partial_t u_\varepsilon)^2 - 2q^2\partial_t u_\varepsilon \partial_x^2 u_\varepsilon + 2\varepsilon\partial_t u_\varepsilon\partial_x^4 u_\varepsilon \\ &\quad = -2\alpha\partial_t u_\varepsilon\partial_x^3 u_\varepsilon + 4\gamma\partial_t u_\varepsilon u_\varepsilon\partial_x^3 u_\varepsilon - 2\gamma\partial_t u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon. \end{aligned} \tag{5.7}$$

Since

$$\begin{aligned} -2q^2\int_{\mathbb{R}} \partial_t u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x &= q^2\frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\ 2\varepsilon\int_{\mathbb{R}} \partial_t u_\varepsilon\partial_x^4 u_\varepsilon &= \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\,\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \end{aligned}$$

an integration of (5.7) on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(q^2\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon\,\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2\,\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad = -2\alpha\int_{\mathbb{R}} \partial_t u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x + 4\gamma\int_{\mathbb{R}} \partial_t u_\varepsilon u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x - 2\gamma\int_{\mathbb{R}} \partial_t u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\,\mathrm{d}x. \end{aligned} \tag{5.8}$$

Due to (4.7), (5.4) and the Young inequality,

$$
\begin{aligned}
2|\alpha| \int_{\mathbb{R}} |\partial_t u_\varepsilon||\partial_x^3 u_\varepsilon| \,\mathrm{d}x &= 2\int_{\mathbb{R}} |\partial_t u_\varepsilon||\alpha \partial_x^3 u_\varepsilon| \,\mathrm{d}x \\
&\leqslant \|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \alpha^2 \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|\gamma| \int_{\mathbb{R}} |\partial_t u_\varepsilon||u_\varepsilon \partial_x^3 u_\varepsilon| \,\mathrm{d}x &= \int_{\mathbb{R}} |\partial_t u_\varepsilon||4\gamma u_\varepsilon \partial_x^3 u_\varepsilon| \,\mathrm{d}x \\
&\leqslant \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 8\gamma^2 \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x^3 u_\varepsilon)^2 \,\mathrm{d}x \\
&\leqslant \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + 8\gamma^2 \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
2|\gamma| \int_{\mathbb{R}} |\partial_t u_\varepsilon||\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| \,\mathrm{d}x &= 2\int_{\mathbb{R}} \left|\frac{\partial_t u_\varepsilon}{\sqrt{3}}\right| \left|\sqrt{3}\gamma \partial_x u_\varepsilon \partial_x^2 u_\varepsilon\right| \mathrm{d}x \\
&\leqslant \frac{1}{3}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\gamma^2 \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\
&\leqslant \frac{1}{3}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3\gamma^2 \|\partial_x u_\varepsilon\|^2_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \frac{1}{3}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned}
$$

It follows from (5.8) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(q^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + \frac{1}{6}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad \leqslant C(T) \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned}
$$

(5.3), (5.4) and an integration on $(0,t)$ give

$$
\begin{aligned}
&q^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6}\int_0^t \|\partial_t u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}s \\
&\quad \leqslant C_0 + C(T)\int_0^t \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)t \leqslant C(T),
\end{aligned}
$$

which gives (5.6). □

Using the Sobolev Immersion Theorem, the proof of theorem 1.1 is based on the following lemma.

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Lemma 5.4. *Fix $T > 0$. There exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function u which satisfies (1.14) such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{\text{loc}}((0,T)\times\mathbb{R}),\ 1 \leqslant p < \infty. \tag{5.9}$$

Moreover, u is solution of (5.1).

Proof. Thanks to lemmas 5.1, 5.2 and (5.3)

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}). \tag{5.10}$$

Consequently, (5.10) gives (5.9).

Thanks to lemmas 5.1 and 5.2 we get

$$u \in L^\infty(0,T;H^2(\mathbb{R})),$$

while, lemma 5.4 says that

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\| \mathrm{d}s \leqslant C(T), \quad 0 \leqslant t \leqslant T.$$

Therefore, (1.14) holds and u is solution of (5.1). □

Appendix A. The Korteweg–de Vries–Burgers equation: $u_0 \in H^1(\mathbb{R})$

In this appendix, we prove the well-posedness of the Cauchy problem of (1.10), under the assumption (1.5). Indeed, the main result of this appendix is the following theorem.

Theorem A.1. *Assume (1.5). Fixed $T > 0$, there exists a solution u of (1.10) such that (1.12) holds.*

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.10).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [**12**, **33**]:

$$\begin{cases} \partial_t u_\varepsilon + \kappa u_\varepsilon \partial_x u_\varepsilon + \alpha \partial_x^3 u_\varepsilon - q^2 \partial_x^2 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & t > 0,\ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{A.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that (2.2) holds.

Let us prove some a priori estimates on u_ε.

Arguing as in lemma 2.1, we have the following result.

Lemma A.1. *For each $t \geqslant 0$,*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2 \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\varepsilon \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0. \tag{A.2}$$

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LEMMA A.2. *Fix $T > 0$. There exists a constant $C_0 > 0$, independent on ε, such that (3.4) holds. In particular, we have*

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2 \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\varepsilon \int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0, \tag{A.3}$$

for every $0 \leqslant t \leqslant T$.

Proof. Let $0 \leqslant t \leqslant T$. Multiplying (A.1) by $-2\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x \\
&= 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x - 2\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x \\
&\quad - 2q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x \\
&= 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x + 2\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x \\
&\quad - 2q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x - 2q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad - 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad = 2\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \,\mathrm{d}x.
\end{aligned} \tag{A.4}$$

Due to the Young inequality,

$$\begin{aligned}
2|\kappa| \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon| |\partial_x^2 u_\varepsilon| \,\mathrm{d}x &= 2\int_{\mathbb{R}} \left|\frac{\kappa u_\varepsilon \partial_x u_\varepsilon}{q}\right| \left|q \partial_x^2 u_\varepsilon\right| \\
&\leqslant \frac{\kappa^2}{q^2} \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^2 \,\mathrm{d}x + q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \frac{\kappa^2}{q^2} \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad + q^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

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It follows from (A.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + q^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leqslant \frac{\kappa^2}{q^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

(2.2), (A.2) and an integration on $(0,t)$ gives

$$\begin{aligned}&\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + q^2\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s + 2\varepsilon\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\ &\leqslant C_0 + \frac{\kappa^2}{q^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\ &\leqslant C_0\left(1+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{A.5}$$

Arguing as in lemma 2.1, we have that

$$\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})} - C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} - C_0,$$

which gives (3.4).

Finally, (A.3) follows from (3.4) and (A.5). □

Now, we prove theorem A.1.

Proof of theorem A.1. Arguing as in lemma 2.4, there exists a solution u of (1.10) such that (1.12) holds. □

Appendix B. The Korteweg–de Vries–Burgers equation: $u_0 \in H^2(\mathbb{R})$

In this appendix, we prove the well-posedness of the Cauchy problem of (1.10), under the assumption (1.8). In particular, we prove the following result.

THEOREM B.1. *Assume* (1.8)*. Fixed* $T > 0$*, there exists an unique solution* u *of* (1.10) *such that* (1.14) *holds. Moreover, if* u_1 *and* u_2 *are two solutions of* (1.10)*, we have that*

$$\left\|u_1(t,\cdot) - u_2(t,\cdot)\right\|_{L^2(\mathbb{R})} \leqslant \mathrm{e}^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(\mathbb{R})}, \tag{B.1}$$

for some suitable $C(T) > 0$*, and every* $0 \leqslant t \leqslant T$*.*

To prove theorem B.1, we consider the approximation (A.1), where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that (5.3) holds.

Let us prove some a priori estimates on u_ε.

Since $H^1(\mathbb{R}) \subset H^2(\mathbb{R})$, lemmas A.1 and A.2 are still valid.

We prove the following result.

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Lemma B.1. *Fix $T > 0$. There exists a constant $C_0 > 0$, independent on ε, such that*

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{q^2}{2}\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s + 2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leqslant C_0, \tag{B.2}$$

for every $0 \leqslant t \leqslant T$. In particular, we have that

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leqslant C_0. \tag{B.3}$$

Proof. Let $0 \leqslant t \leqslant T$. Multiplying (A.1) by $2\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon \,\mathrm{d}x \\
&= -2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x - 2\alpha\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x \\
&\quad + 2q\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \,\mathrm{d}x - 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= 2\kappa\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \,\mathrm{d}x + 2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x \\
&\quad - 2q^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2q^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \leqslant 2\kappa\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \,\mathrm{d}x + 2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \,\mathrm{d}x.
\end{aligned} \tag{B.4}$$

Due to (3.4) and the Young inequality,

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 |\partial_x^3 u_\varepsilon| \,\mathrm{d}x &= 2\int_{\mathbb{R}} \left|\frac{\kappa(\partial_x u_\varepsilon)^2}{q}\right| \left|q\partial_x^3 u_\varepsilon\right| \mathrm{d}x \\
&\leqslant \frac{\kappa^2}{q^2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + q^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
2|\kappa|\int_{\mathbb{R}} |u_\varepsilon \partial_x^2 u_\varepsilon||\partial_x^3 u_\varepsilon| \,\mathrm{d}x &= \int_{\mathbb{R}} \left|\frac{2\kappa u_\varepsilon \partial_x^2 u_\varepsilon}{q}\right| \left|q\partial_x^3 u_\varepsilon\right| \mathrm{d}x \\
&\leqslant \frac{2\kappa^2}{q^2}\int_{\mathbb{R}} u_\varepsilon^2 (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x + \frac{q^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leqslant \frac{2\kappa^2}{q^2}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + \frac{q^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\leqslant C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{q^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

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Using (B.4) we gain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{q^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leqslant \frac{\kappa^2}{q^2}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{B.5}$$

By (2.11), (A.2) and (A.3),

$$\begin{aligned}\frac{\kappa^2}{q^2}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} &\leqslant \frac{6\kappa^2}{q^2}\left(\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\leqslant C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, by (B.5), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{q^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \leqslant C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (4.3), (A.3) and an integration on $(0,t)$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{q^2}{2}\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s + 2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\ \leqslant C_0 + C_0\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \leqslant C_0,\end{aligned}$$

which gives (B.2).

Finally, (2.12), (A.3) and (B.2) give (B.3). □

Lemma B.2. *Fix $T > 0$. There exists a constant $C_0 > 0$, independent on ε, such that*

$$q^2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\int_0^t\left\|\partial_t u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \leqslant C_0. \tag{B.6}$$

for every $0 \leqslant t \leqslant T$.

Proof. Let $0 \leqslant t \leqslant T$. Arguing as in lemma 5.3, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(q^2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) + 2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ = -2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon\,\mathrm{d}x - 2\alpha\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t u_\varepsilon\,\mathrm{d}x.\end{aligned} \tag{B.7}$$

Due to (3.4) and the Young inequality,

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|u_\varepsilon\partial_x u_\varepsilon||\partial_t u_\varepsilon|\,\mathrm{d}x &= 2\int_{\mathbb{R}}|\kappa u_\varepsilon\partial_x u_\varepsilon||\partial_t u_\varepsilon|\,\mathrm{d}x\\
&\leqslant \kappa^2\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2\,\mathrm{d}x+\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\leqslant \kappa^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\leqslant C_0\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|\alpha|\int_{\mathbb{R}}|\partial_x^3 u_\varepsilon||\partial_t u_\varepsilon|\,\mathrm{d}x &= \int_{\mathbb{R}}|2\alpha\partial_x^3 u_\varepsilon||\partial_t u_\varepsilon|\\
&\leqslant 2\alpha^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (B.7),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(q^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)+\frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\leqslant C_0\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\alpha^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

(5.3), (A.2), (B.2) and an integration on $(0,t)$ give

$$\begin{aligned}
&q^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\int_0^t\|\partial_t u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\\
&\quad\leqslant C_0+C_0\int_0^t\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s+2\alpha^2\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s\leqslant C_0,
\end{aligned}$$

which gives (B.6). □

Now, we prove theorem B.1.

Proof of theorem B.1. Arguing as in lemma 5.4, there exists a solution u of (1.10) such that (1.14) holds.

We prove (B.1). Let u_1, u_2 be, two solutions of (1.10), that is

$$\begin{cases}\partial_t u_1+2\kappa u_1\partial_x u_1+\alpha\partial_x^3 u_1-q^2\partial_x^2 u_1=0, & t>0,\ x\in\mathbb{R},\\ u_1(0,x)=u_{1,0}(x), & x\in\mathbb{R},\end{cases}$$

$$\begin{cases}\partial_t u_2+2\kappa u_2\partial_x u_2+\alpha\partial_x^3 u_2-q^2\partial_x^2 u_2=0, & t>0,\ x\in\mathbb{R},\\ u_2(0,x)=u_{2,0}(x), & x\in\mathbb{R}.\end{cases}$$

Then, the function

$$\omega=u_1-u_2 \tag{B.8}$$

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is the solution of the following Cauchy problem:

$$\begin{cases}\partial_t\omega+2\kappa(u_1\partial_x u_1-u_2\partial_x u_2)+\alpha\partial_x^3\omega-q^2\partial_x^2\omega=0, & t>0,\ x\in\mathbb{R},\\ \omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x\in\mathbb{R}.\end{cases} \tag{B.9}$$

Observe that

$$u_1\partial_x u_1-u_2\partial_x u_2=u_1\partial_x u_1-u_2\partial_x u_1+u_2\partial_x u_1-u_2\partial_x u_2=\partial_x u_1\omega+u_2\partial_x\omega.$$

Therefore, (B.9) is equivalent to the following equation:

$$\partial_t\omega+\kappa\partial_x u_1\omega+\kappa u_2\partial_x\omega+\alpha\partial_x^3\omega-q^2\partial_x^2\omega=0. \tag{B.10}$$

Since

$$\begin{aligned}2\kappa\int_{\mathbb{R}}u_2\omega\partial_x\omega\,\mathrm{d}x&=-\kappa\int_{\mathbb{R}}\partial_x u_2\omega^2\,\mathrm{d}x,\\ 2\alpha\int_{\mathbb{R}}\omega\partial_x^3\omega\,\mathrm{d}x&=0,\\ -2q^2\int_{\mathbb{R}}\omega\partial_x^2\omega\,\mathrm{d}x&=2q^2\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

multiplying (B.10) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2q^2\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad=-\kappa\int_{\mathbb{R}}\partial_x u_1\omega^2\,\mathrm{d}x+\kappa\int_{\mathbb{R}}\partial_x u_2\omega^2\,\mathrm{d}x\\ &\quad\leqslant|\kappa|\left(\left\|\partial_x u_1\right\|_{L^\infty((0,T)\times\mathbb{R})}+\left\|\partial_x u_2\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{B.11}$$

Fix $T>0$. Observe that, since $u_1,\ u_2\in H^2(\mathbb{R})$, for every $0\leqslant t\leqslant T$, we have that

$$\left\|\partial_x u_1\right\|_{L^\infty((0,T)\times\mathbb{R})},\ \left\|\partial_x u_2\right\|_{L^\infty((0,T)\times\mathbb{R})}\leqslant C(T).$$

Therefore, by (B.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2q^2\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leqslant C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from the Gronwall Lemma and (B.8),

$$\begin{aligned}&\left\|u_1(t,\cdot)-u_2(t,\cdot)\right\|_{L^2(\mathbb{R})}+2q^2\,\mathrm{e}^{C(T)t}\int_0^t\mathrm{e}^{-C(T)s}\left\|\partial_x u_1(s,\cdot)-\partial_x u_2(s,\cdot)\right\|_{L^2(\mathbb{R})}\mathrm{d}s\\ &\quad\leqslant\mathrm{e}^{C(T)t}\left\|u_{1,0}-u_{2,0}\right\|_{L^2(\mathbb{R})},\end{aligned}$$

which gives (B.1). □

https://doi.org/10.1017/prm.2020.23 Published online by Cambridge University Press

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