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The Klein bottle group is not strongly verbally closed, though awfully close to being so

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Abstract. According to Mazhuga's theorem, the fundamental group H of anyconnected surface, possibly except for the Klein bottle, is a retract of each finitely generated group containing H as a verbally closed subgroup. We prove that the Klein bottle group is indeed an exception but has a very close property.

1 Introduction

A subgroup H of a group G is called *verbally closed* [10] (see also [12, 13, 7, 4, 5, 8, 2, 3, 9, 14]) if any equation of the form

$$w(x, y, \dots) = h,$$

where $w(x, y, \dots)$ is an element of the free group $F(x, y, \dots)$ and $h \in H$, having a solution in G has a solution in H. If each finite system of equations with coefficient from H :

$$\{w_1(x, y, \dots) = 1, \dots, w_m(x, y, \dots) = 1\},$$

where $w_i \in H * F(x, y, \dots)$ (and $*$ stands for the free product), having a solution in G has a solution in H, then H is called *algebraically closed* in G.

Algebraic closedness is a stronger property than verbal closedness, but these properties turn out to be equivalent in many cases. A group H is called *strongly verbally closed* [8] if it is algebraically closed in any group containing H as a verbally closed subgroup. Thus, the verbal closedness is a subgroup property, while the strong verbal closedness is an abstract group property. The class of strongly verbally closed groups is fairly wide; see the papers mentioned above. For instance, in [8], it was proved that:

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the fundamental group of any connected surface, possibly except for the Klein bottle, is strongly verbally closed.

The intriguing unique possible exception arose as follows:

- almost all surface groups are similar to free groups in a sense; for such groups, an "industrial" method works; this method goes back to the very first paper [10] on this subject and is based on the use of the Lie words [6] or their analogues;
- the remaining several groups are either
  - abelian and, hence, strongly verbally closed (by a very simple reason) [8],
  - or this exceptional Klein bottle group$K = \langle a, b | a^b = a^{-1} \rangle$, which is neither free-like nor abelian.

We obtain a natural addition to Mazhuga's theorem:

the fundamental group of the Klein bottle is not strongly verbally closed.

A similar situation appeared some time ago. A general theorem with a unique possible exception was proved in [4]:

all non-dihedral virtually free groups containing no non-identity finite normal subgroups are strongly verbally closed

(and the condition of absence of finite normal subgroups cannot be removed); later, it turned out (see [5]) that

the infinite dihedral group is strongly verbally closed too,

and this humble result was more difficult than the above-mentioned general theorem obtained in [4] by the "industrial" method. Generally, if the reader takes their favorite nonabelian group, far from free (*e.g.*, a finite group), then it would likely be difficult to decide whether this group is strongly verbally closed. Proving strong verbal closedness is not easy, nor is disproving this property (actually, it is not too easy to give an example of a group not being strongly verbally closed [4]).

The Klein bottle group $K = \langle a, b | a^b = a^{-1} \rangle$ and the infinite dihedral group $D_\infty = \langle a, b | a^b = a^{-1},\ b^2 = 1 \rangle$ are similar, of course. We use this similarity, apply the result of [5], and conclude that the Klein bottle group, though not strongly verbally closed, has a very similar property.

If a group H is *equationally Noetherian* (*i.e.*, any system of equations over H with finitely many unknowns is equivalent to its finite subsystem), then the algebraic closedness is equivalent to the "local retractness" [5]:

an equationally Noetherian subgroup H of a group G is algebraically closed in G if and only if H is a retract (*i.e.*, the image of an endomorphism ρ such that $\rho \circ \rho = \rho$) of each finitely generated over H subgroup of G (*i.e.*, a subgroup of the form $\langle H \cup X \rangle$, where $X \subseteq G$ is a finite set).

All surface groups are linear [11], and all linear groups are equationally Noetherian [1]. Thus, our main (and sole) result can be stated as follows.

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Theorem *The fundamental group K of the Klein bottle (unlike all other surface groups) embeds into a finitely generated group G as a verbally closed subgroup that is not a retract of G. However, any such G has an index-two subgroup containing K as its retract.*

In the next section, we give an example proving the first assertion of the theorem (*i.e.*, that K is not strongly verbally closed). Section 2 contains auxiliary lemmata. In the last section, we prove the second assertion of the theorem.

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Our notation is fairly standard. Note only that, if x and y are elements of a group, then x^y denotes $y^{-1}xy$, The commutator $[x,y]$ is $x^{-1}y^{-1}xy$. If X is a subset of a group, then $\langle X\rangle$, $\langle\langle X\rangle\rangle$, and $C_H(X)$ stand for the subgroup generated by X, the normal closure of X, and the centraliser of X in H (where H is a subgroup). The symbol $\langle x\rangle_k$ denotes the cyclic group of order k generated by x. The free group with a basis $x_1,\dots,x_n$ is denoted as $F(x_1,\dots,x_n)$.

2 An Example

Let $V_4=\{1,d_1,d_2,d_3\}$ be the Klein four-group (*i.e.*, the noncyclic group of order four). Consider the semidirect product

$$G=\big(V_4\times\langle b\rangle_\infty\big)\ltimes\big(\langle a_1\rangle_\infty\times\langle a_2\rangle_\infty\times\langle a_3\rangle_\infty\big),$$

where the action is $a_i^b=a_i^{-1},\ a_i^{d_i}=a_i,\ a_i^{d_j}=a_i^{-1}$ for all $i\neq j$.

Proposition *The subgroup $K=\langle a,b\rangle\subset G$, where $a=a_1a_2a_3$, (isomorphic to the Klein bottle group) is verbally closed in G, but not a retract.*

Proof The subgroup K is not a retract, because a hypothetical retraction $G\to K$ should map finite-order elements d_i to 1 (as K is torsion-free); then, the relation $a_i^{d_j}=a_i^{-1}$ would show that the images of a_i are also of finite order and, hence, are 1 too; therefore, the image of $a=a_1a_2a_3\in K$ is also 1 that contradicts the fixedness of elements of K under the retraction.

It remains to show that K is verbally closed in G, *i.e.*, any equation of the form

$$w(x,y,\dots)=h,$$

where $h\in K$ and $w(x,y,\dots)$ is an element of the free group $F(x,y,\dots)$, solvable in G, is solvable in K. It is known that, by a change of variables, any such equation can be transformed into the form

$$x^m u(x,y,\dots)=h, \tag{2.1}$$

where $h\in K$ and $u(x,y,\dots)$ lies in the commutator subgroup of the free group $F(x,y,\dots)$. Suppose that equation (2.1) has a solution $(\tilde x,\tilde y,\dots)$ in G. Since there is no principal difference between d_i, we can assume that

(2.2) $$\tilde{x} = d_1^{\varepsilon} b^l a_1^{k_1} a_2^{k_2} a_3^{k_3},$$

where $\varepsilon, l, k_i \in \mathbb{Z}$. We have a homomorphism *the first coordinate*:

$$f\ G \longrightarrow D_\infty = \langle b' \rangle_2 \ltimes \langle a' \rangle_\infty ,$$

where $f(a_1) = a'$, $f(a_2) = f(a_3) = f(d_1) = 1$, $f(b) = f(d_2) = f(d_3) = b'$, and the natural homomorphism *degree*:

$$\deg : G \longrightarrow \mathbb{Z},$$

where $\deg(a_i) = \deg(d_i) = 0$, $\deg(b) = 1$, Applying these homomorphisms to the given equality $\tilde{x}^m u(\tilde{x}, \tilde{y}, \dots) = h$, we obtain

(2.3) $$f\big(\tilde{x}^m u(\tilde{x}, \tilde{y}, \dots)\big) = f(\tilde{x})^m u\big(f(\tilde{x}), f(\tilde{y}), \dots\big) = f(h)$$

and

$$\deg\big(\tilde{x}^m u(\tilde{x}, \tilde{y}, \dots)\big) = m \cdot \deg(\tilde{x}) = \deg(h).$$

Now, consider the elements $\hat{x}, \hat{y}, \dots \in K$ obtained from $\tilde{x}, \tilde{y}, \dots \in G$ by the changes

(2.4) $$a_1 \longmapsto a = a_1 a_2 a_3, \quad a_2 \longmapsto 1, \quad a_3 \longmapsto 1,$$
$$d_1 \longmapsto 1, \quad d_2 \longmapsto b, \quad d_3 \longmapsto b \quad (\text{and } b \longmapsto b),$$

which preserve the first coordinate of any element. For instance, the element $\tilde{x}$, given by expression (2.2), turns into

(2.5) $$\hat{x} = b^l a^{k_1} = b^l a_1^{k_1} a_2^{k_1} a_3^{k_1}.$$

We claim that the tuple $(\hat{x}, \hat{y}, \dots)$ is a solution to equation (2.1) in K. Indeed,

- the first coordinates of $\hat{x}^m u(\hat{x}, \hat{y}, \dots)$ and h are the same:

$$f\big(\hat{x}^m u(\hat{x}, \hat{y}, \dots)\big) = f(\hat{x})^m u\big(f(\hat{x}), f(\hat{y}), \dots\big) \overset{(4)}{=} f(\tilde{x})^m u\big(f(\tilde{x}), f(\tilde{y}), \dots\big) \overset{(3)}{=} f(h)$$

  (where (2.4) and (2.3) imply the corresponding equalities);
- and the degrees of $\hat{x}^m u(\hat{x}, \hat{y}, \dots)$ and h are the same:

$$\deg\big(\hat{x}^m u(\hat{x}, \hat{y}, \dots)\big) = m \cdot \deg(\hat{x}) \overset{(5)}{=} ml \overset{(2)}{=} m \cdot \deg(\tilde{x}) \overset{(3)}{=} \deg(h),$$

It remains to note that an element of $K \subset G$ is uniquely determined by its first coordinate and degree. Thus, $\hat{x}^m u(\hat{x}, \hat{y}, \dots) = h$, we have found a solution to equation (2.1) in K, and this completes the proof. ∎

3 Two Lemmata on Quotient Groups

Lawfication Lemma [13, Lemma 1.1] *If $V(G)$ is a verbal subgroup of a group G, and H is a verbally closed subgroup of G, then $H \cap V(G) = V(H)$ (i.e., the verbal subgroup of H corresponding to the same variety), and the image $H/V(H) \subseteq G/V(G)$ of H under the natural homomorphism $G \to G/V(G)$ is verbally closed in $G/V(G)$.*

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Dihedral-quotient Lemma *If the Klein bottle group $K = \langle a, b | a^b = a^{-1} \rangle$ is a verbally closed subgroup of a group G, then*

(i) $\langle\langle b^2 \rangle\rangle \cap K = \langle b^2 \rangle$, *where* $\langle\langle b^2 \rangle\rangle$ *is the normal closure of* b^2 *in G;*

(ii) *the subgroup* $D_\infty = K / \langle b^2 \rangle \subseteq G / \langle\langle b^2 \rangle\rangle$ *is verbally closed in* $G / \langle\langle b^2 \rangle\rangle$.

Proof We can assume that G satisfies the law $[x^2, y^2] = 1$ (because this law holds in K, and, therefore, the lawfication lemma allows us to replace G with its quotient group by the corresponding verbal subgroup).

For groups with such a law, as well as for all metabelian groups, assertion (i) is a general fact:

> *If A is a normal abelian subgroup of a group G, and the quotient group G/A is also abelian, then the intersection $C_A(X)$ of A and the centraliser of any subset $X \subseteq G$ is normal in G.*

Indeed, $(C_A(X))^g = C_A(X^g) \supseteq C_A(XA) = C_A(X)$.

To obtain assertion (i), we put $A = \langle \{g^2 \mid g \in G\} \rangle$ and $X = K$; we even get more than (i):

the subgroups $\langle\langle b^2 \rangle\rangle$ *and K commute in G.*

Let us prove (ii) now. Suppose that an equation

$$(3.1) \qquad w(x, y, \dots) = h \langle\langle b^2 \rangle\rangle, \quad \text{where } h \in K \text{ and } w(x, y, \dots) \in F(x, y, \dots),$$

is solvable in $G/\langle\langle b^2 \rangle\rangle$. We have to show that this equation is solvable in $D_\infty = K/\langle b^2 \rangle \subseteq G/\langle\langle b^2 \rangle\rangle$.

Case 1: $h = 1$. In this case, equation (3.1) has the trivial solution $(1, 1, \dots)$ in D_∞.

Case 2: $h = b$. In this case, the exponent sum of one of unknowns (say, x) in the word w has to be odd, because otherwise the solvability of equation (3.1) in $G/\langle\langle b^2 \rangle\rangle$ would imply a decomposition of b into a product of squares in G, which contradicts the verbal closedness of K (because b is not a product of squares in K, even modulo $\langle a \rangle$). An equation with odd exponent sum of x and the right-hand side b has in D_∞ an obvious solution: $x = b,\ y = 1,\ z = 1, \dots$.

Case 3: $h = ba^k$. This case is the same as the preceding one, because the dihedral group has an automorphism mapping ba^k to b.

Case 4: (the last case modulo $\langle b^2 \rangle$): $h = a^k$, where $k \neq 0$. Suppose that $(\tilde{x} \langle\langle b^2 \rangle\rangle,\ \tilde{y} \langle\langle b^2 \rangle\rangle, \dots)$ is a solution to equation (3.1) in $G/\langle\langle b^2 \rangle\rangle$. Then the equation

$$[t, (w(x, y, \dots))^2] = [b, h^2] = a^{4k} \quad \text{(where } t \text{ is a new unknown)}$$

has a solution $(\tilde{t} = b,\ \tilde{x},\ \tilde{y}, \dots)$ in G, because $\langle\langle b^2 \rangle\rangle$ commutes with K as noted above. The verbal closedness of K in G implies that this equation has a solution in K, *i.e.*,

$$(3.2) \qquad [\hat{t}, (w(\hat{x}, \hat{y}, \dots))^2] = a^{4k} \quad \text{for some } \hat{t}, \hat{x}, \hat{y}, \dots \in K.$$

This means that:

- $\hat{t} \in b\langle a, b^2\rangle$, because all other elements of K commute with squares; we can even assume that $\hat{t} = b$, since a and b^2 commute with all squares and do not affect commutator (3.2);
- $\big(w(\hat{x}, \hat{y}, \dots)\big)^2 \in \langle b^2\rangle \cup \langle a^2, b^4\rangle$, because only these elements are squares in K;
- $\big(w(\hat{x}, \hat{y}, \dots)\big)^2 \in a^{2k}\langle b^4\rangle$, because, only for such elements of $\langle b^2\rangle \cup \langle a^2, b^4\rangle$, the commutator with $\hat{t} = b$ gives a^{4k};
- $w(\hat{x}, \hat{y}, \dots) \in a^k\langle b^2\rangle$, because each element of the coset $a^{2k}\langle b^4\rangle$ has a unique square root in K.

We have found a solution $(\hat{x}\langle b^2\rangle, \hat{y}\langle b^2\rangle, \dots)$ to equation (3.1) in $D_\infty = K/\langle b^2\rangle$; this completes the proof. ∎

4 Proof of the Second Assertion of the Theorem

Suppose that the fundamental group K of the Klein bottle is a verbally closed subgroup of a finitely generated group G. We have to construct a retraction onto K from an index-at-most-two subgroup of G containing K.

The group G has two normal subgroups: $N_1 = G'$, the commutator subgroup, and $N_2 = \langle\langle b^2\rangle\rangle$, the normal closure of b^2. Taking the quotients transforms K into

$$K/K' = \langle a\rangle_2 \times \langle b\rangle_\infty \subseteq G/N_1 \quad \text{and} \quad K/\langle b^2\rangle = D_\infty \subseteq G/N_2$$

(by the Lawfication and Dihedral-quotient Lemmata); these images of K in G/N_i remain verbally closed in G/N_i (by the same lemmata). Therefore, K/K' and $K/\langle b^2\rangle$ are retracts of G/N_i, because all abelian groups [8] and the infinite dihedral group [5] are strongly verbally closed.

Thus, we obtain the epimorphisms

$$\deg : G \longrightarrow G/G' \longrightarrow K/K' = \langle a\rangle_2 \times \langle b\rangle_\infty \longrightarrow \langle b\rangle_\infty \xrightarrow{\simeq} \mathbb{Z}$$

and

$$f : G \longrightarrow G/\langle\langle b^2\rangle\rangle \longrightarrow K/\langle b^2\rangle = D_\infty$$

such that $\deg(b) = 1$, $f(b) = b\langle b^2\rangle$, $f(a) = a\langle b^2\rangle$.

Combining these two "pseudo-retractions", we construct the homomorphism

$$\Phi : G \longrightarrow \mathbb{Z} \times D_\infty, \quad g \longmapsto \big(\deg(g),\ f(g)\big).$$

The restriction φ of Φ to K is injective, and the image of this restriction is so-called *fibered product*:

$$\varphi(K) = \Phi(K) = \left\{\left(i,\ b^j a^k\langle b^2\rangle\right) \mid i \equiv j \pmod 2\right\},$$

an index-two subgroup of $\mathbb{Z} \times D_\infty$. Therefore, the subgroup $\Phi^{-1}(\Phi(K)) \subseteq G$ has index at most two in G and admits a retraction onto K:

$$G \supseteq \Phi^{-1}(\Phi(K)) \xrightarrow{\Phi} \Phi(K) \xrightarrow{\varphi^{-1}} K.$$

References

[1] G. Baumslag, A. Myasnikov, and V. Remeslennikov, *Algebraic geometry over groups I. Algebraic sets and ideal theory*. J. Algebra **219**(1999), 16–79. https://doi.org/10.1006/jabr.1999.7881

[2] O. Bogopolski, *Equations in acylindrically hyperbolic groups and verbal closedness*. Preprint, 2020. arXiv:1805.08071

[3] O. Bogopolski, *On finite systems of equations in acylindrically hyperbolic groups*. Preprint, 2020. arXiv:1903.10906

[4] A. A. Klyachko and A. M. Mazhuga, *Verbally closed virtually free subgroups.* Sbornik: Mathematics **209**(2018), 850–856. http://dx.doi.org/10.1070/SM8942

[5] A. A. Klyachko, A. M. Mazhuga, and V. Yu. Miroshnichenko, *Virtually free finite-normal-subgroup-free groups are strongly verbally closed.* J. Algebra **510**(2018), 319–330. https://doi.org/10.1016/j.jalgebra.2018.05.028

[6] D. Lee, *On certain C-test words for free groups.* J. Algebra **247**(2002), 509–540. https://doi.org/10.1006/jabr.2001.9001

[7] A. M. Mazhuga, *On free decompositions of verbally closed subgroups of free products of finite groups*. J. Group Theory **20**(2016). https://doi.org/10.1515/jgth-2016-0058

[8] A. M. Mazhuga, *Strongly verbally closed groups.* J. Algebra **493**(2018), 171–184. https://doi.org/10.1016/j.jalgebra.2017.09.021

[9] A. M. Mazhuga, *Free products of groups are strongly verbally closed.* Sbornik: Mathematics **210**(2019), 1456–1492. http://dx.doi.org/10.1070/SM9115

[10] A. Myasnikov and V. Roman'kov, *Verbally closed subgroups of free groups.* J. Group Theory **17**(2014), 29–40. https://doi.org/10.1515/jgt-2013-0034

[11] M. Newman, *A note on Fuchsian groups.* Illinois J. Math. **29**(1985), 682–686. https://dx.doi.org/10.1215/ijm/1256045503

[12] V. A. Roman'kov, *Equations over groups.* Groups Complexity Cryptology **4**(2012), 191–239. https://doi.org/10.1515/gcc-2012-0015

[13] V. A. Roman'kov and N. G. Khisamiev, *Verbally and existentially closed subgroups of free nilpotent groups.* Algebra Logic **52**(2013), 336–351. https://doi.org/10.1007/S10469-013-9245-6

[14] V. A. Roman'kov and Timoshenko, *On verbally closed subgroups of free solvable groups.* Vestnik Omskogo Universiteta **24**(2019), 9–16 [in Russian]. https://doi.org/10.25513/1812-3996.2019.24(1).9-16

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