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COMMENTS ON "ORDERING PROPERTIES OF ORDER STATISTICS FROM HETEROGENEOUS POPULATIONS"

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1. THE SEMINAL RESULTS IN LITERATURE

Balakrishnan and Zhao [1] does an excellent job in this issue at reviewing the recent advances on stochastic comparison between order statistics from independent and heterogeneous observations with proportional hazard rates, gamma distribution, geometric distribution, and negative binomial distributions, the relation between various stochastic order and majorization order of concerned heterogeneous parameters is highlighted. Some examples are presented to illustrate main results while pointing out the potential direction for further discussion.

The set of achieved results in this line of research mainly focus on the independent observations; however, due to the increasing importance of dependence in survival analysis, reliability, network security, financial engineering, and actuarial science etc, one is constantly confronted with observations with interdependence in practical situations, and it is of great interest to study the stochastic comparison of order statistics from dependent and heterogeneous random variables. For example, the isolation times of some users in a network, lifetimes of components of a system sharing a common environmental effect etc. In such a context, both the dependence and the heterogeneity among individual observations play an important role in developing the stochastic orders of order statistics. On the other hand, being alienated from the real background, the pertinent results in this review are short of commensurate application, this to some extent delays the forward pace in this direction. Here we would like to have a discussion on heterogeneous observations with certain dependence structure, and the comments will focus on the usual stochastic order. We hope that our crude comments may draw forth more insightful theories from capable researchers as well as some excellent applications in various related areas.

For ease of reference, we restate some notations and results in literatures in the style conforming to our context.

The random variables $V_1, \ldots, V_n$ are said to follow *proportional hazard rate model* if the survival function of V_i is

$$\bar{F}_{V_i}(x) = \bar{F}^{\lambda_i}(x), \quad i = 1, \ldots, n,$$

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where $\bar{F}(x)$ is the survival function of some baseline random variable. Let $V_{i:n}$ denote the ith order statistic arising from random variables $V_1, \ldots, V_n$. Pledger and Proschan [9] firstly established the following path-breaking result: let $(V_1, \ldots, V_n)$ and $(V_1^*, \ldots, V_n^*)$ be two vectors of independent random variables following proportional hazard rate models with parameter vectors $(\nu_1, \ldots, \nu_n)$ and $(\nu_1^*, \ldots, \nu_n^*)$, respectively. Then,

$$(\nu_1, \ldots, \nu_n) \succ_m (\nu_1^*, \ldots, \nu_n^*) \Longrightarrow V_{k:n} \geq_{\rm st} V_{k:n}^*, \quad \text{for } k = 1, \ldots, n.$$

At a later time, this result was further strengthened by Proschan and Sethuraman [10] from componentwise stochastic order to multivariate stochastic order. That is,

$$(\nu_1, \ldots, \nu_n) \succ_m (\nu_1^*, \ldots, \nu_n^*) \Longrightarrow (V_{1:n}, \ldots, V_{n:n}) \geq_{\rm st} (V_{1:n}^*, \ldots, V_{n:n}^*). \tag{1}$$

These two results actually broke the path for all research work in this direction. Here, we consider the stochastic order in (1) under the assumption of the following two interesting structures of dependence, and as will be seen, they are closely related to the well-known Marshall–Olkin structure of dependence.

Throughout this section, we denote $\bar{F}_i$ the survival function of X_i for $i = 1, \ldots, n$, the terms *increasing* and *decreasing* stand for non-decreasing and non-increasing, respectively. And expectations are implicitly assumed to exist whenever they appear.

2. EXTENSION TO A DEPENDENCE STRUCTURE

Suppose random variables $V_1, \ldots, V_{n+1}$ follow the proportional hazard rate model with parameter vector $(\nu_1, \ldots, \nu_n, \nu)$. Then,

$$X_i = \min\{V_i, V_{n+1}\}, \quad i = 1, \ldots, n, \tag{2}$$

has proportional hazard parameter $\lambda_i = \nu_i + \nu$, and $(X_1, \ldots, X_n)$ has the joint survival function

$$P(X_1 > x_1, \ldots, X_n > x_n) = \bar{F}^{\nu}\left(\bigvee_{i=1}^{n} x_i\right) \prod_{i=1}^{n} \bar{F}^{\nu_i}(x_i). \tag{3}$$

It is plain that $X_1, \ldots, X_n$ are independent when $\nu = 0$.

This formulation actually characterizes the server structure in network design. Imagine $V_1, \ldots, V_n$, and V_{n+1} the *times to failure* of n terminal independent users themselves and that of the server independently providing service to these users, then, $(X_1, \ldots, X_n)$ measures the *times to isolation* of n users in the network. For more on isolation time, one may refer to Li and Li [3], Li, Parker, and Xu [4]. Under the above framework, we present the following generalization of (1).

Theorem 1: *Let $(V_1, \ldots, V_{n+1})$ and $(V_1^*, \ldots, V_{n+1}^*)$ be two set of independent random variables following proportional hazard rate models (2) with proportional parameters $(\nu_1, \ldots, \nu_n, \nu)$ and $(\nu_1^*, \ldots, \nu_n^*, \nu)$, respectively. Then,*

$$(\lambda_1, \ldots, \lambda_n) \succ_m (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \geq_{\rm st} (X_{1:n}^*, \ldots, X_{n:n}^*).$$

Proof: Let $(\lambda_{(1)}, \ldots, \lambda_{(n)})$ and $(\nu_{(1)}, \ldots, \nu_{(n)})$ be the ascending rearrangement of $(\lambda_1, \ldots, \lambda_n)$ and $(\nu_1, \ldots, \nu_n)$, respectively. Since $\lambda_{(i)} = \nu_{(i)} + \nu$ for $i = 1, \ldots, n$, we have

$\sum_{i=1}^{n}\lambda_{(i)} = n\nu + \sum_{i=1}^{n}\nu_i$ and

$$\sum_{i=1}^{k}\lambda_{(i)} = \sum_{i=1}^{k}\nu_{(i)} + k\nu, \quad \text{for } k=1,\ldots,n-1.$$

The majorization $(\lambda_1,\ldots,\lambda_n) \succ_m (\lambda_1^*,\ldots,\lambda_n^*)$ implies $\sum_{i=1}^{n}\lambda_{(i)} = \sum_{i=1}^{n}\lambda_{(i)}^*$,

$$\sum_{i=1}^{k}\lambda_{(i)} \le \sum_{i=1}^{k}\lambda_{(i)}^*, \quad \text{for } k=1,\ldots,n-1,$$

and hence $\sum_{i=1}^{n}\nu_i = \sum_{i=1}^{n}\nu_i^*$,

$$\nu_{(1)} + \cdots + \nu_{(n-k)} \le \nu_{(1)}^* + \cdots + \nu_{(n-k)}^*, \quad \text{for } k=1,\ldots,n-1.$$

That is, $(\nu_1,\ldots,\nu_n) \succ_m (\nu_1^*,\ldots,\nu_n^*)$. By (1), we have

$$(V_{1:n},\ldots,V_{n:n}) \ge_{\text{st}} (V_{1:n}^*,\ldots,V_{n:n}^*).$$

For any increasing function $\phi:\mathbb{R}^n \to \mathbb{R}$, let

$$g(x_1,\ldots,x_n) = \phi(x_1\wedge y,\ldots,x_n\wedge y), \quad \text{for any } y.$$

Because $g(\boldsymbol{x})$ is increasing in $\boldsymbol{x}$, we get

$$\begin{aligned}\mathbb{E}[\phi(V_{1:n}\wedge y,\ldots,V_{n:n}\wedge y)] &= \mathbb{E}[g(V_{1:n},\ldots,V_{n:n})]\\ &\ge \mathbb{E}[g(V_{1:n}^*,\ldots,V_{n:n}^*)]\\ &= \mathbb{E}[\phi(V_{1:n}^*\wedge y\},\ldots,V_{n:n}^*\wedge y)].\end{aligned}$$

Note that V_{n+1} and $(V_{1:n}\cdots,V_{n:n})$ are independent, V_{n+1}^* and $(V_{1:n}^*\cdots,V_{n:n}^*)$ are independent, and V_{n+1} and V_{n+1}^* share a common distribution. By the total expectation, it holds that

$$\mathbb{E}[\phi(V_{1:n}\wedge V_{n+1},\ldots,V_{n:n}\wedge V_{n+1})] \ge \mathbb{E}[\phi(V_{1:n}^*\wedge V_{n+1}^*,\ldots,V_{n:n}^*\wedge V_{n+1}^*)].$$

In view of (2), we have

$$\mathbb{E}[\phi(X_{1:n},\ldots,X_{n:n})] \ge \mathbb{E}[\phi(X_{1:n}^*,\ldots,X_{n:n}^*)].$$

This complete the proof. ∎

Next example tells that the stochastic order in Theorem 1 fails to hold when V_{n+1} and V_{n+1}^* do not have a common distribution.

Example 1: Assume (V_1,V_2,V_3) be a vector of independent exponential random variables with hazard rate vector $(2,7,1)$, and (V_1^*,V_2^*,V_3^*) be a vector of independent exponential random variables with hazard rate vector $(2,5,2)$. Then, $(\lambda_1,\lambda_2)=(3,8)\succ_m(4,7)=(\lambda_1^*,\lambda_2^*)$. However, for all $x\ge 0$,

$$P(X_{1:2}>x) = P(V_1>x,V_2>x,V_3>x) = e^{-10x} < e^{-9x} = P(X_{1:2}^*>x).$$

This implies $X_{1:2}\le_{\text{st}} X_{1:2}^*$ and hence invalidates Theorem 1.

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Boland et al. [2] pointed out that the result of Pledger and Proschan [9] can not be strengthened to the hazard rate order, but they established that

$$(\nu_1, \nu_2) \succ_m (\nu_1^*, \nu_2^*) \Longrightarrow V_{2:2} \geq_{\rm hr} V_{2:2}^* \tag{4}$$

holds for V_i and V_i^* with exponential distributions. Let us extend the above result to the case with dependence.

Theorem 2: *Suppose* (V_1, V_2, V_3) *and* (V_1^*, V_2^*, V_3^*) *be two vectors of independent exponential random variables with hazard rate parameters* (ν_1, ν_2, ν) *and* (ν_1^*, ν_2^*, ν)*, respectively. Then,*

$$(\lambda_1, \lambda_2) \succ_m (\lambda_1^*, \lambda_2^*) \Longrightarrow X_{2:2} \geq_{\rm hr} X_{2:2}^*.$$

Proof: $(\lambda_1, \lambda_2) \succ_m (\lambda_1^*, \lambda_2^*)$ implies $(\nu_1, \nu_2) \succ_m (\nu_1^*, \nu_2^*)$. In view of (4), we have $V_{2:2} \geq_{\rm hr} V_{2:2}^*$, which implies that $P(V_{2:2} > x)/P(V_{2:2}^* > x)$ is increasing in x. Consequently,

$$\frac{P(X_{2:2} > x)}{P(X_{2:2}^* > x)} = \frac{P(V_{2:2} > x, V_3 > x)}{P(V_{2:2}^* > x, V_3^* > x)} = \frac{P(V_{2:2} > x)}{P(V_{2:2}^* > x)}$$

is increasing in x. That is, $X_{2:2} \geq_{\rm hr} X_{2:2}^*$. ■

Set $\bar{F}_i(x_i) = [\bar{F}(x_i)]^{\nu_i + \nu} = u_i$, we get $\bar{F}_i^{-1}(u_i) = \bar{F}^{-1}(u_i^{1/(\nu_i+\nu)})$, for $i = 1, \ldots, n$. In view of (3), $(X_1, \ldots, X_n)$ has the survival copula

$$\begin{aligned}\widehat{C}(u_1, \ldots, u_n) &= \bar{F}^{\nu}\left(\bigvee_{i=1}^{n} \bar{F}^{-1}(u_i^{1/(\nu_i+\nu)})\right) \prod_{i=1}^{n} \bar{F}^{\nu_i}(\bar{F}^{-1}(u_i^{1/(\nu_i+\nu)})) \\ &= \bigwedge_{i=1}^{n} u_i^{\nu/(\nu_i+\nu)} \prod_{i=1}^{n} u_i^{\nu_i/(\nu_i+\nu)}.\end{aligned}$$

It is easy to verify that, for $\nu \geq 0$ and all $u_i \in (0,1)$ with $i = 1, \ldots, n$,

$$\widehat{C}(u_1, \ldots, u_n) \geq \prod_{i=1}^{n} u_i^{\nu/(\nu_i+\nu)} \prod_{i=1}^{n} u_i^{\nu_i/(\nu_i+\nu)} = \prod_{i=1}^{n} u_i.$$

That is, $X_1, \ldots, X_n$ are positively upper orthant dependent. Note that the above equation holds when $\nu = 0$, the dependence structure in (2) does include the independence as one of its special case. By the way, using the same argument as Proposition 7 in Lin and Li [6], we can also show that the random vector $(X_1, \ldots, X_n)$ with survival function (2) is *right corner set increasing* (RCSI), that is, $P(X_1 > x_1, \ldots, X_n > x_n \mid X_1 > t_1, \ldots, X_n > t_n)$ is increasing in $(t_1, \ldots, t_n)$ for all $(x_1, \ldots, x_n)$..

3. EXTENSION TO ONE STRUCTURE OF STRICT DEPENDENCE

Suppose $(V_1, \ldots, V_n)$ be a vector of independent random variables with proportional coefficients vector $(\nu_1, \ldots, \nu_n)$. It is easy to check that

$$X_i = \bigwedge_{j \neq i} V_j, \quad i = 1, \ldots, n, \tag{5}$$

follow the proportional hazard rate model with parameters $\lambda_i = \sum_{j \neq i} \nu_j$, and $(X_1, \ldots, X_n)$ has the survival function

$$\bar{G}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \bar{F}^{\nu_i} \left(\bigvee_{j \neq i} x_j \right), \quad \text{for } i = 1, \ldots, n. \tag{6}$$

The formulation in (5) actually characterizes the P2P (peer-to-peer) structure in network design. Let $V_1, \ldots, V_n$ be times to failure of n terminal independent users themselves. To increase the network security, each node must obtain some $1 \leq k < n$ keys from the other $n-1$ neighbors so as to be entitle to look into the privacy. If $k = n-1$, then, $(X_1, \ldots, X_n)$ above measures the *times to be locked* of n users in the network. For more on isolation time, one may refer to Li and Li [3], Li et al. [4]. Now, let us present main result in this context.

THEOREM 3: *Let $(V_1, \ldots, V_n)$ and $(V_1^*, \ldots, V_n^*)$ be two vector of independent random variables with proportional hazard rate vector $(\nu_1, \ldots, \nu_n)$ and $(\nu_1^*, \ldots, \nu_n^*)$, respectively. Then,*

$$(\lambda_1, \ldots, \lambda_n) \succ_m (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \geq_{\text{st}} (X_{1:n}^*, \ldots, X_{n:n}^*),$$

here $\lambda_i^* = \sum_{j \neq i} \nu_j^*$ and $X_i^* = \bigwedge_{j \neq i} V_j^*$.

PROOF: Denote $\nu_{(i)} = 0$ for $i \geq n+1$. Then,

$$\lambda_{(i)} = \nu_{(1)} + \cdots + \nu_{(n-i)} + \nu_{(n-i+2)} + \cdots + \nu_{(n)}, \quad \text{for } i = 1, \ldots, n.$$

It is easy to verify that $\sum_{i=1}^{n} \lambda_{(i)} = (n-1) \sum_{i=1}^{n} \nu_i$ and

$$\sum_{i=1}^{k} \lambda_{(i)} = (k-1) \sum_{i=1}^{n} \nu_i + \nu_{(1)} + \cdots + \nu_{(n-k)}, \quad \text{for } k = 1, \ldots, n-1.$$

Due to $(\lambda_1, \ldots, \lambda_n) \succ_m (\lambda_1^*, \ldots, \lambda_n^*)$, it holds that $\sum_{i=1}^{n} \lambda_{(i)} = \sum_{i=1}^{n} \lambda_{(i)}^*$ and

$$\sum_{i=1}^{k} \lambda_{(i)} \leq \sum_{i=1}^{k} \lambda_{(i)}^*, \quad \text{for } k = 1, \ldots, n-1.$$

This implies $\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \nu_i^*$ and $\nu_{(1)} + \cdots + \nu_{(n-k)} \leq \nu_{(1)}^* + \cdots + \nu_{(n-k)}^*$ for $k = 1, \ldots, n-1$. That is, $(\nu_1, \ldots, \nu_n) \succ_m (\nu_1^*, \ldots, \nu_n^*)$.

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By (1), we have

$$(V_{1:n},\ldots,V_{n:n}) \geq_{\rm st} (V^*_{1:n},\ldots,V^*_{n:n}).$$

For any increasing function $\psi:\mathbb{R}^n\to\mathbb{R}$,

$$g(x_1,\ldots,x_n)=\psi\left(\bigwedge_{j\neq n} x_j,\ldots,\bigwedge_{j\neq 1} x_j\right)$$

is increasing in $(x_1,\ldots,x_n)$ and thus we have

$$\begin{aligned}\mathbb{E}\left[\psi\left(\bigwedge_{j\neq n} V_{j:n},\cdots,\bigwedge_{j\neq 1} V_{j:n}\right)\right] &= \mathbb{E}[g(V_{1:n},\ldots,V_{n:n})]\\ &\geq \mathbb{E}[g(V^*_{1:n},\ldots,V^*_{n:n})]\\ &= \mathbb{E}\left[\psi\left(\bigwedge_{j\neq n} V^*_{j:n},\cdots,\bigwedge_{j\neq 1} V^*_{j:n}\right)\right].\end{aligned}$$

Denote $V_{k:n}=\infty$ for $k>n$. Then, for $k=1,\ldots,n$,

$$X_{i:n}=\bigwedge_{j\neq n-i+1} V_{j:n},\quad X^*_{i:n}=\bigwedge_{j\neq n-i+1} V^*_{j:n}.$$

As a consequence, it holds that

$$\mathbb{E}[\psi(X_{1:n},\ldots,X_{n:n})]\geq \mathbb{E}[\psi(X^*_{1:n},\ldots,X^*_{n:n})].$$

By the arbitrariness of ψ, this completes the proof. ∎

Setting $\bar{F}_i(x_i)=[\bar{F}(x_i)]^{\sum_{j\neq i}\nu_j}=u_i$, we have $\bar{F}_i^{-1}(u_i)=\bar{F}^{-1}(u_i^{1/\sum_{j\neq i}\nu_j})$, for $i=1,\ldots,n$. Due to (6), $(X_1,\ldots,X_n)$ has the survival copula

$$\begin{aligned}\widehat{C}(u_1,\ldots,u_n) &= \bar{G}(\bar{F}_1^{-1}(u_1),\ldots,\bar{F}_n^{-1}(u_n))\\ &= \prod_{k=1}^n \bar{F}^{\nu_k}\left(\bigvee_{i\neq k}\bar{F}_i^{-1}(u_i)\right)\\ &= \prod_{k=1}^n \bar{F}^{\nu_k}\left(\bigvee_{i\neq k}\bar{F}^{-1}(u_i^{1/\sum_{j\neq i}\nu_j})\right)\\ &= \prod_{k=1}^n \bar{F}^{\nu_k}\left(\bar{F}^{-1}\left(\bigwedge_{i\neq k} u_i^{1/\sum_{j\neq i}\nu_j}\right)\right)\\ &= \prod_{k=1}^n \bigwedge_{i\neq k} u_i^{\nu_k/\sum_{j\neq i}\nu_j}.\end{aligned}$$

It is easy to verify that

$$\widehat{C}(u_1,\ldots,u_n)\geq \prod_{i=1}^n u_i,\quad \text{for all } u_i\in(0,1), i=1,\ldots,n.$$

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For $n = 2$, $X_1 = V_2$ and $X_2 = V_1$ are independent. Then, $\widehat{C}(u, v) = uv$. For $n \geq 3$, $X_1 = \bigwedge_{j \neq 1} V_j$ and $X_2 = \bigwedge_{j \neq 2} V_j$ strictly depend on $(V_3, \ldots, V_n)$. This implies

$$\widehat{C}(u_1, \ldots, u_n) > \prod_{i=1}^{n} u_i.$$

That is, the dependence structure in (5) does not include the independence as one of its special case when $n \geq 3$. Likewise, in the same manner as Proposition 7 in Lin and Li [6], the random vector $(X_1, \ldots, X_n)$ with survival function (6) may be proved to be RCSI.

By the end, we have a discussion on connections between the model in (5) and *Marshall–Olkin* dependence structure.

After Marshal and Olkin [7] firstly proposed the bivariate Marshall–Oikin exponential distribution, Muliere and Scarsini [8] built the Marshall–Olkin proportional hazard distribution. Subsequently, Li and Pellerey [5] introduced generalized bivariate Marshall–Olkin distributions, including the above two models as special cases. Recently, Lin and Li [6] further studied the multivariate generalized Marshall–Olkin distributions.

Denote $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$ and $I_k = I \backslash \{i_k\}$ for $k = 1, \ldots, r$. Let I^c be the complement of I. Then, we have

$$\lim_{\substack{x_i \to 0 \\ i \in I^c}} \bigvee_{i \neq i_k} x_i = \bigvee_{i \in I_k} x_i, \quad \text{for } k = 1, \ldots, r,$$

and

$$\lim_{\substack{x_i \to 0 \\ i \in I^c}} \bigvee_{i \neq k} x_i = \bigvee_{i \in I} x_i, \quad \text{for } k \in I^c.$$

In view of (6), we have

$$\begin{aligned} \bar{G}_I(x_{i_1}, \ldots, x_{i_r}) &= \lim_{\substack{x_i \to 0 \\ i \in I^c}} \bar{G}(x_1, \ldots, x_n) \\ &= \lim_{\substack{x_i \to 0 \\ i \in I^c}} \prod_{k=1}^{n} \bar{F}^{\nu_k} \left(\bigvee_{i \neq k} x_i \right) \\ &= \bar{F}^{\sum_{j \in I^c} \nu_j} \left(\bigvee_{i \in I} x_i \right) \prod_{k=1}^{r} \bar{F}^{\nu_{i_k}} \left(\bigvee_{i \in I_k} x_i \right). \end{aligned} \tag{7}$$

Set $I = \{l, r\} \in \{1, \ldots, n\}$, (7) yields

$$\bar{G}_I(x_l, x_r) = [\bar{F}(x_l \vee x_r)]^{\sum_{j \neq l, r} \nu_j} \bar{F}^{\nu_r}(x_l) \bar{F}^{\nu_l}(x_r).$$

By (1.4) in Lin and Li [6], the two-dimensional marginal follows from Marshall–Olkin proportional hazard model.

Set $I = \{l, r, k\} \in \{1, \ldots, n\}$ in (7), we have

$$\bar{G}_I(x_l, x_r, x_k) = [\bar{F}(x_l \vee x_r \vee x_k)]^{\sum_{j \neq l, r, k} \nu_j} [\bar{F}(x_r \vee x_k)]^{\nu_l} [\bar{F}(x_l \vee x_k)]^{\nu_r} [\bar{F}(x_l \vee x_r)]^{\nu_k},$$

which does not follow Marshall–Olkin proportional hazard model. Moreover, the r-dimensional marginal in (7) does not follow Marshall–Olkin proportional hazard model for $r \geq 4$ either.

References

1. Balakrishnan, N. & Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations. *Probability in the Engineering and Informational Sciences* doi:10.1017/S0269964813000156.
2. Boland, P. J., El-Neweihi, E. & Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics. *Journal of Applied Probability* 31: 180–192.
3. Li, X. & Li, L. (2012). Security analysis of compromised-neighbor-tolerant networks: a stochastic model. *IEEE transactions on Reliability* 61: 749–757.
4. Li, X., Parker, P. & Xu, S. (2011). A stochastic model for quantitative security analysis of networked systems. *IEEE Transactions on Dependable and Secure Computing* 8: 28–43.
5. Li, X. & Pellerey, F. (2011). Generalized Marshall–Olkin distributions, and related bivariate aging properties. *Journal of Multivariate Analysis* 102: 1399–1409.
6. Lin, J. & Li, X. (2012). Multivariate generalized Marshall–Olkin distributions and copulas. *Methodology and Computation in Applied Probability*, DOI: 10.1007/s11009-012-9297-4.
7. Marshall, A.W. & Olkin, I. (1967). A multivariate exponential distribution. *Journal of American Statistical Association* 62: 30–41.
8. Muliere, P. & Scarsini, M. (1987). Characterization of a Marshall–Olkin type class of distributions. *Annals of the Institute of Statistical Mathematics* 39: 429–441.
9. Pledger, P. & Proschan, F. (1971). Comparisions of order statstics and of spacings from heterogeneous distributions. In *Optimizing methods in statistics*. J.S. Rustagi (ed.), Academic Press: New York, pp. 89–113.
10. Proschan, F. & Sethuraman, J. (1976). Stochastic comparisions of order statstics from heterogeneous populations, with applications in reliability. *Journal of Multivariate Analysis* 6: 608–616.