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RANDOM ZEROS ON COMPLEX MANIFOLDS: CONDITIONAL EXPECTATIONS

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Abstract We study the conditional distribution $K_k^N(z \mid p)$ of zeros of a Gaussian system of random polynomials (and more generally, holomorphic sections), given that the polynomials or sections vanish at a point p (or a fixed finite set of points). The conditional distribution is analogous to the pair correlation function of zeros but we show that it has quite a different small distance behaviour. In particular, the conditional distribution does not exhibit repulsion of zeros in dimension 1. To prove this, we give universal scaling asymptotics for $K_k^N(z \mid p)$ around p. The key tool is the conditional Szegő kernel and its scaling asymptotics.

Keywords: random holomorphic sections; zeros of random polynomials; holomorphic line bundle; Kähler manifold; Szegő kernel

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1. Introduction

In this paper we study the conditional expected distribution of zeros of a Gaussian random system $\{s_1, \dots, s_k\}$ of $k \leqslant m$ polynomials of degree N in m variables, given that the polynomials s_j vanish at a point $p \in M$, or at a finite set of points $\{p_1, \dots, p_r\}$. More generally, we consider systems of holomorphic sections of a degree N positive line bundle $L^N \to M_m$ over a compact Kähler manifold of dimension m. The conditional expected distribution is the current $K_k^N(z \mid p) \in \mathcal{D}'^{k,k}(M)$ given by

$$(K_k^N(z \mid p), \varphi) := \boldsymbol{E}_N[(Z_{s_1,\dots,s_k}, \varphi) \mid s_1(p) = \cdots = s_k(p) = 0], \quad \text{for } \varphi \in \mathcal{D}^{m-k,m-k}(M). \tag{1.1}$$

Here, $Z_{s_1,\dots,s_k}$ is the (k,k) current of integration over the simultaneous zeros of the sections, i.e. its pairing with a smooth test form $\varphi \in \mathcal{D}^{m-k,m-k}(M)$ is the integral $\int_{Z_{s_1,\dots,s_k}} \varphi$

of the test form over the joint zero set. The expectation $\boldsymbol{E}_N$ is the standard Gaussian conditional expectation on $\prod_1^k H^0(M, L^N)$, which we condition on the linear random variable $(s_1, \dots, s_k) \mapsto (s_1(p), \dots, s_k(p))$ that evaluates the sections at the point p (see Definition 3.11).

We show that $K_k^N(z \mid p)$ is a smooth (k, k) form away from p (Lemma 5.2), and we determine its asymptotics, both unscaled and scaled, as $N \to \infty$. Our main result on the scaled asymptotics (Theorems 1.2 and 5.1) is that the scaling limit of $K_k^N(z \mid p)$ around the point p is the conditional expected distribution $K_{km}^\infty(z \mid 0)$ of joint zeros given a zero at $z = 0$ in the Bargmann–Fock ensemble of entire holomorphic functions on $\mathbb{C}^m$. Thus, the scaling limit $K_{km}^\infty(z \mid 0)$ is universal, and we give an explicit formula for it. Our study of $K_k^N(z \mid p)$ is parallel to our study of the two-point correlation function $K_{2k}^N(z, p)$ for joint zeros in our prior work with Bleher [**1**, **2**]. There we showed that $K_{2k}^N(z, p)$ similarly has a scaling limit given by the pair correlation function $K_{2km}^\infty(z, 0)$ of zeros in the Bargmann–Fock ensemble. Both $K_{km}^N(z \mid p)$ and $K_{2km}^N(z, p)$ measure a probability density of finding simultaneous zeros at z and at p: $K_{km}^N(z \mid p)$ is the result of conditioning in a Gaussian space (see, for example, [**8**, Chapter 9.3]), while $K_{2km}^N(z, p)$ is a natural conditioning from the viewpoint of random point processes (see § 6.1). Of special interest is the case $k = m$ where the joint zeros are (almost surely) points. In this case, the scaling limit (Bargmann–Fock) conditional density $K_{mm}^\infty(z \mid 0)$ and pair correlation density $K_{2mm}^\infty(z, 0)$ turn out to have quite different short distance behaviour, as discussed in § 1.1 below.

To state our results, we need to recall the definition of a Gaussian random system of holomorphic sections of a line bundle. We let $(L, h) \to (M, \omega_h)$ be a positive Hermitian holomorphic line bundle over a compact complex manifold with Kähler form $\omega_h = \frac{1}{2}\mathrm{i}\Theta_h$. We then let $H^0(M, L^N)$ denote the space of holomorphic sections of the Nth tensor power of L. A special case is when $M = \mathbb{CP}^m$, and $L = \mathcal{O}(1)$ (the hyperplane section line bundle), in which case $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ is the space of homogenous polynomials of degree N. As recalled in § 2, the Hermitian metric h on L induces inner products on $H^0(M, L^N)$ and these induce a Gaussian measure γ_h^N on $H^0(M, L^N)$. A Gaussian random system is a choice of k independent Gaussian random sections, i.e. we endow $\prod_{j=1}^k H^0(M, L^N)$ with the product measure. We refer to $(\prod_{j=1}^k H^0(M, L^N), \prod_{j=1}^k \gamma_h^N)$ as the *Hermitian Gaussian ensemble* induced by h. We let $\boldsymbol{E}_N = \boldsymbol{E}_{(\prod \gamma_h^N)}$ denote the expected value with respect to $\prod \gamma_h^N$. Given $s_1, \dots, s_k \in H^0(M, L^N)$ we denote by $Z_{s_1,\dots,s_k}$ the current of integration over the zero set $\{z \in M \colon s_1(z) = \dots = s_k(z) = 0\}$. Further background is given in § 2 and in [**1**, **11**, **13**].

We first state our result on the unscaled asymptotics of the conditional expectation of the zero current of one section conditioned on the vanishing at one or several points. Recalling (1.1), we consider the conditional expected zero current $K_1^N(z \mid p_1, \dots, p_r) \in \mathcal{D}'^{1,1}(M)$ given by

$$(K_1^N(z \mid p_1, \dots, p_r), \varphi) := \boldsymbol{E}_N[(Z_s, \varphi) \mid s(p_1) = 0, \dots, s(p_r) = 0], \qquad \varphi \in \mathcal{D}^{m-1,m-1}(M), \tag{1.2}$$

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for distinct points $p_1, \dots, p_r \in M$. Our result says that conditioning on the section s vanishing at the p_j only modifies the unconditional zero current by a term of order N^{-m} (where $m = \dim M$).

Theorem 1.1. *Let $(L, h) \to (M, \omega_h)$ be a positive Hermitian holomorphic line bundle over a compact complex manifold of dimension m with Kähler form $\omega_h = \frac{1}{2}\mathrm{i}\Theta_h$, and let $(H^0(M, L^N), \gamma_h^N)$ be the Hermitian Gaussian ensemble. Let let $p_1, \dots, p_r$ be distinct points of M. Then for all test forms $\varphi \in \mathcal{D}^{m-1,m-1}(M)$, we have*

$$(K_1^N(z \mid p_1, \dots, p_r), \varphi) = \boldsymbol{E}_N(Z_s, \varphi) - C_m N^{-m} \sum_{j=1}^{r} \frac{\mathrm{i}\partial\bar{\partial}\varphi(p_j)}{\Omega_M(p_j)} + O(N^{-m-1/2+\varepsilon}),$$

where $\Omega_M = (1/m!)\omega_h^m$ is the volume form of M, and $C_m = \frac{1}{2}\pi^{m-1}\zeta(m+1)$.

Here, ζ denotes the Riemann zeta function $\zeta(t) = \sum_{k=1}^{\infty}(1/k^t)$. As mentioned above, the interesting problem is to rescale the zeros around a fixed point z_0. When $k = m$ the joint zeros of the system are almost surely a discrete set of points which are $1/\sqrt{N}$-dense. Hence, we rescale a $C/\sqrt{N}$-ball around z_0 by $\sqrt{N}$ to make scaled zeros a unit apart on average from their nearest neighbours. If $z_0 \neq p_j$ for any j, the scaled limit density is just the unconditioned scaled density, so we only consider the case where $z_0 = p_{j_0}$ for some j_0. Then the other conditioning points p_j, $j \neq j_0$, become irrelevant to the leading-order term, so we only consider the scaled conditional expectation with one conditioning point. Our main result is the following scaling asymptotics.

Theorem 1.2. *Let $(L, h) \to (M, \omega_h)$ and $(H^0(M, L^N), \gamma_h^N)$ be as in Theorem 1.1, and let $p \in M$. Choose normal coordinates $z = (z_1, \dots, z_m)\colon (M_0, p) \to (\mathbb{C}^m, 0)$ on a neighbourhood M_0 of p, and let $\tau_N = \sqrt{N}z\colon M_0 \to \mathbb{C}^m$ denote the scaled coordinate map.*

Let $K_m^N(z \mid p)$ be the conditional expected zero distribution given by (1.1) and Definition 3.11. Then for a smooth test function $\varphi \in \mathcal{D}(\mathbb{C}^m)$, we have

$$\begin{aligned}(K_m^N(z \mid p), \varphi \circ \tau_N(z)) \\ = \varphi(0) + \int_{\mathbb{C}^m \smallsetminus \{0\}} \varphi(u)\left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}[\log(1 - \mathrm{e}^{-|u|^2}) + |u|^2]\right)^m + O(N^{-1/2+\varepsilon}),\end{aligned}$$

where $u = (u_1, \dots, u_m)$ denotes the coordinates in $\mathbb{C}^m$.

In § 5, we give a similar result (Theorem 5.1) for the conditional expected joint zero current $K_k^N(z \mid p)$ of joint zeros of codimension $k < m$.

Theorem 1.2 may be reformulated (without the remainder estimate) as the following weak limit formula for currents.

Corollary 1.3. *Under the hypotheses and notation of Theorem 1.2,*

$$\begin{aligned}\tau_{N*}(K_m^N(z \mid p)) \to K_{mm}^{\infty}(u \mid 0) &\stackrel{\text{def}}{=} \left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}[\log(1 - \mathrm{e}^{-|u|^2}) + |u|^2]\right)^m \\ &= \delta_0(u) + \frac{1 - (1 + |u|^2)\mathrm{e}^{-|u|^2}}{(1 - \mathrm{e}^{-|u|^2})^{m+1}}\left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}|u|^2\right)^m\end{aligned}$$

weakly in $\mathcal{D}'^{m,m}(\mathbb{C}^m)$, as $N \to \infty$.

The term $\delta_0(u)$ comes of course from the certainty of finding a zero at p given the condition. The form

$$\left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}|u|^2\right)^m$$

is the scaling limit of the unconditioned distribution of zeros.

It follows from the proof that $K^{\infty}_{mm}(u \mid 0)$ is the conditional density of common zeros of m independent random functions in the Bargmann–Fock ensemble of holomorphic functions on $\mathbb{C}^m$ of the form

$$f(u) = \sum_{J\in\mathbb{N}^m} \frac{c_J}{\sqrt{J!}} u^J,$$

where the coefficients c_J are independent complex Gaussian random variables with mean 0 and variance 1. The monomials $(\pi^{-m/2}/\sqrt{J!})u^J$ form a complete orthonormal basis of the Bargmann–Fock space of holomorphic functions that are in $L^2(\mathbb{C}^m, \mathrm{e}^{-|z|^2}\,\mathrm{d}z)$, where $\mathrm{d}z$ denotes Lebesgue measure. (We note that $f(u)$ is a.s. not in $L^2(\mathbb{C}^m, \mathrm{e}^{-|z|^2}\,\mathrm{d}z)$; instead, $f(u)$ is of finite order 2 in the sense of Nevanlinna theory. For further discussion of the Bargmann–Fock ensemble, see [**1**] and § 6 of the first version (arXiv:math/0608743v1) of [**13**].)

1.1. Short distance behaviour of the conditional density

As in the case of the pair correlation function, Corollary 1.3 determines the short distance behaviour of the conditional density of zeros around the conditioning point.

Before describing the results for the conditional density, let us recall the results in [**1**, **2**] for the pair correlation function of zeros. The correlation function $K^N_{nk}(z_1, \dots, z_n)$ is the probability density of finding zeros of a system of k sections at the n points $z_1, \dots, z_n$. For purposes of comparison to the conditional density, we are interested in the pair correlation density $K^N_{2m}(z_1, z_2)$ for a full system of $k = m$ sections. It gives the probability density of finding a pair of zeros of the system at (z_1, z_2). The scaling limit

$$\kappa_{mm}(|u|) := \lim_{N\to\infty} K^N_{1k}(p)^{-2} K^N_{2m}\left(p, p + \frac{u}{\sqrt{N}}\right) \tag{1.3}$$

measures the asymptotic probability of finding zeros at $p, p + (u/\sqrt{N})$. As the notation indicates, it depends only on the distance $r = |u|$ between the scaled points in the scaled metric around p. For small values of r, it is proved in [**1**, **2**] that

$$\kappa_{mm}(r) = \tfrac{1}{4}(m+1)r^{4-2m} + O(r^{8-2m}), \quad \text{as } r \to 0. \tag{1.4}$$

This shows that the pair correlation function exhibits a striking dimensional dependence. When $m = 1$, $\kappa_{mm}(r) \to 0$ as $r \to 0$ and one has 'zero repulsion'. When $m = 2$, $\kappa_{mm}(r) \to \frac{3}{4}$ as $r \to 0$ and zeros neither repel nor attract. With $m \geqslant 3$, $\kappa_{mm}(r) \nearrow \infty$ as $r \to 0$ and there joint zeros tend to cluster, i.e. it is more likely to find a zero at a small distance r from another zero than at a small distance r from a given point.

The probability (density) of finding a pair of scaled zeros at $(p, p + (u/\sqrt{N}))$ sounds similar to finding a second zero at $p + (u/\sqrt{N})$ if there is a zero at p, i.e. the conditional

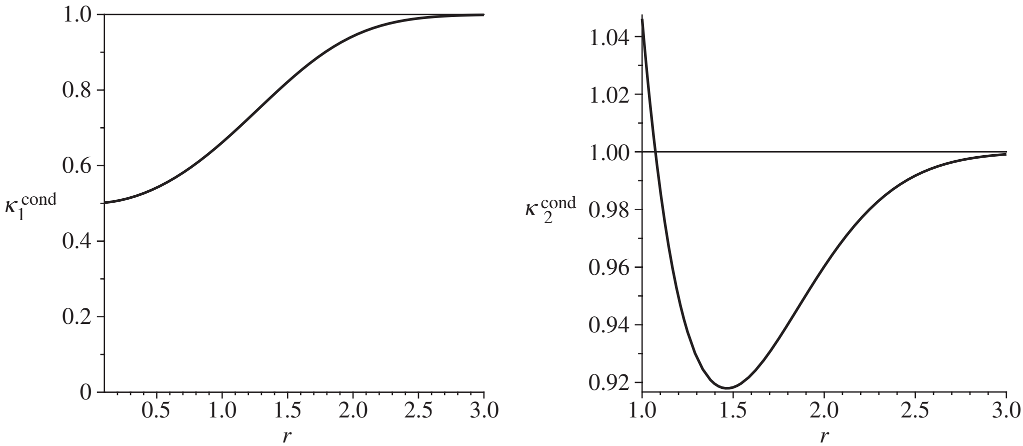

Figure 1. Conditional distribution in dimensions 1 and 2.

probability density. Hence one might expect the scaled conditional probability to resemble the scaled correlation function. But Corollary 1.3 tells a different story. We ignore the term δ_0 (again) since it arises trivially from the conditioning and only consider the behaviour of the coefficient

$$\kappa_m^{\text{cond}}(|u|) := \frac{1-(1+|u|^2)\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^{m+1}} \sim \tfrac{1}{2}|u|^{2-2m} \tag{1.5}$$

of the scaling limit conditional distribution with respect to the Lebesgue density

$$\left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}|u|^2\right)^m$$

near $u = 0$. The shift of the exponent down by 2 in comparison to equation (1.4) has the effect of shifting the dimensional description down by one. In dimension 1, the coefficient is asymptotic to $\frac{1}{2}$ and therefore resembles the neutral situation in our description of the pair correlation function. Thus we do not see 'repulsion' in the one-dimensional conditional density. In dimension two, the conditional density (1.5) is asymptotic to $\frac{1}{2}|u|^{-2}$, and there is a singularly enhanced probability of finding a zero near p similar to that for the pair correlation function in dimension three; and so on in higher dimensions.

Figures 1–3 illustrate the different behaviour of these two conditional zero distributions in low dimensions.

It is well known that conditioning on an event of probability zero depends on the random variable used to define the event. So there is no paradox, but possibly some surprise, in the fact that the two conditional distributions are so different. See § 6 for further discussion of the comparison of the pair correlation and the conditional density.

2. Background

We begin with some notation and basic properties of sections of holomorphic line bundles, Gaussian measures. The notation is the same as in [**1**, **12**, **13**].

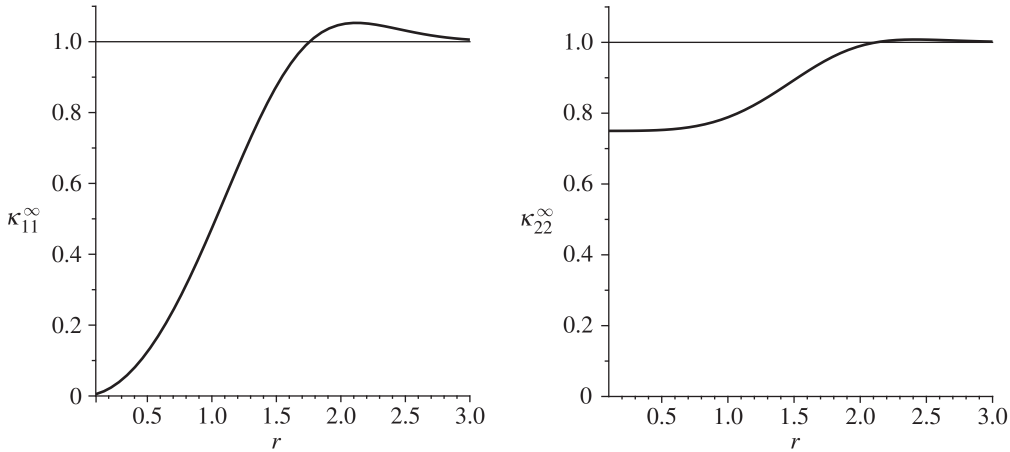

Figure 2. Pair correlation in dimensions 1 and 2.

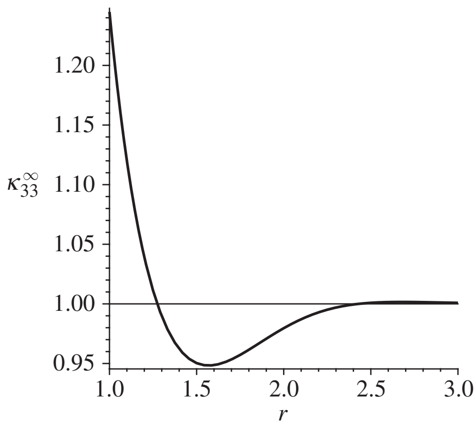

Figure 3. Pair correlation in dimension 3.

2.1. Complex geometry

We denote by $(L, h) \to M$ a Hermitian holomorphic line bundle over a compact Kähler manifold M of dimension m, where h is a smooth Hermitian metric with positive curvature form

$$\Theta_h = -\partial\bar{\partial} \log \|e_L\|_h^2. \tag{2.1}$$

Here, e_L is a local non-vanishing holomorphic section of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ is the h-norm of e_L. As in [**13**], we give M the Hermitian metric corresponding to the Kähler form $\omega_h = \frac{1}{2}\mathrm{i}\Theta_h$ and the induced Riemannian volume form

$$\Omega_M = \frac{1}{m!}\omega_h^m. \tag{2.2}$$

We denote by $H^0(M, L^N)$ the space of holomorphic sections of $L^N = L^{\otimes N}$. The metric h induces Hermitian metrics h^N on L^N given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. We give $H^0(M, L^N)$

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the Hermitian inner product

$$\langle s_1, s_2\rangle = \int_M h^N(s_1,s_2)\Omega_M \quad (s_1,s_2 \in H^0(M,L^N)), \tag{2.3}$$

and we write $\|s\| = \langle s,s\rangle^{1/2}$.

For a holomorphic section $s \in H^0(M,L^N)$, we let $Z_s \in \mathcal{D}'^{1,1}(M)$ denote the current of integration over the zero divisor of s:

$$(Z_s,\varphi) = \int_{Z_s} \varphi, \quad \varphi \in \mathcal{D}^{m-1,m-1}(M),$$

where $\mathcal{D}^{m-1,m-1}(M)$ denotes the set of compactly supported $(m-1,m-1)$ forms on M. (If M has dimension 1, then φ is a compactly supported smooth function.) For $s = ge_L$ on an open set $U \subset M$, the Poincaré–Lelong formula states that

$$Z_s = \frac{\mathrm{i}}{\pi}\partial\bar\partial \log|g| = \frac{\mathrm{i}}{\pi}\partial\bar\partial\log\|s\|_{h^N} + \frac{N}{\pi}\omega_h. \tag{2.4}$$

2.1.1. *The Szegő kernel*

Let $\Pi_N\colon L^2(M,L^N) \to H^0(M,L^N)$ denote the Szegő projector with kernel Π_N given by

$$\Pi_N(z,w) = \sum_{j=1}^{d_N} S_j^N(z) \otimes \overline{S_j^N(w)} \in L_z^N \otimes \bar L_w^N, \tag{2.5}$$

where $\{S_j^N\}_{1\leqslant j\leqslant d_N}$ is an orthonomal basis of $H^0(M,L^N)$.

We shall use the *normalized Szegő kernel*

$$P_N(z,w) := \frac{\|\Pi_N(z,w)\|_{h^N}}{\|\Pi_N(z,z)\|_{h^N}^{1/2}\|\Pi_N(w,w)\|_{h^N}^{1/2}}. \tag{2.6}$$

(Note that $\|\Pi_N(z,w)\|_{h^N} = \sum \|S_j^N(z)\|_{h^N(z)}\|S_j^N(w)\|_{h^N(w)}$, which equals the absolute value of the Szegő kernel lifted to the associated circle bundle, as described in [**12**, **13**].)

We have the $\mathcal{C}^\infty$ diagonal asymptotics for the Szegő kernel [**3**, **17**]:

$$\|\Pi_N(z,z)\|_{h^N} = \frac{N^m}{\pi^m} + O(N^{m-1}). \tag{2.7}$$

Off-diagonal estimates for the normalized Szegő kernel P_N were given in [**13**], using the off-diagonal asymptotics for Π_N from [**1**, **12**]. These estimates are of two types.

(1) 'Far-off-diagonal' asymptotics [**13**, Proposition 2.6]. For $b > \sqrt{j+2k}$, $j,k \geqslant 0$, we have

$$\nabla^j P_N(z,w) = O(N^{-k}) \quad \text{uniformly for } d(z,w) \geqslant b\sqrt{\frac{\log N}{N}}. \tag{2.8}$$

(Here, ∇^j stands for the jth covariant derivative.)

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(2) 'Near-diagonal' asymptotics [**13**, Propositions 2.7, 2.8]. Let $z_0 \in M$. For $\varepsilon, b > 0$, there are constants $C_j = C_j(M, \varepsilon, b)$, $j \geqslant 2$, independent of the point z_0, such that

$$P_N\left(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}\right) = \mathrm{e}^{-|u-v|^2/2}[1 + R_N(u,v)], \tag{2.9}$$

where

$$\left.\begin{aligned} |R_N(u,v)| &\leqslant \tfrac{1}{2} C_2 |u-v|^2 N^{-1/2+\varepsilon}, \\ |\nabla R_N(u)| &\leqslant C_2 |u-v| N^{-1/2+\varepsilon}, \\ |\nabla^j R_N(u,v)| &\leqslant C_j N^{-1/2+\varepsilon}, \quad j \geqslant 2, \end{aligned}\right\} \tag{2.10}$$

for $|u| + |v| < b\sqrt{\log N}$. (Here, u, v are normal coordinates near z_0.)

The limit on the right-hand side of (2.9) is the normalized Szegő kernel for the Bargmann–Fock ensemble (see [**1**]). This is why the scaling limits of the correlation functions and conditional densities coincide with those of the Bargmann–Fock ensemble.

2.2. Probability

If V is a finite-dimensional complex vector space, we shall associate a complex Gaussian probability measure γ to each Hermitian inner product on V as follows. Choose an orthonormal basis $v_1, \dots, v_n$ for the inner product and define γ by

$$\mathrm{d}\gamma(v) = \frac{1}{\pi^n} \mathrm{e}^{-|a|^2}\, \mathrm{d}_{2n} a, \qquad s = \sum_{j=1}^{n} a_j v_j \in V, \tag{2.11}$$

where $\mathrm{d}_{2n}a$ denotes $2n$-dimensional Lebesgue measure. This Gaussian is characterized by the property that the $2n$ real variables $\operatorname{Re} a_j$, $\operatorname{Im} a_j$ $(j = 0, \dots, d_N)$ are independent random variables with mean 0 and variance $\frac{1}{2}$, i.e.

$$\boldsymbol{E}_\gamma a_j = 0, \qquad \boldsymbol{E}_\gamma a_j a_k = 0, \qquad \boldsymbol{E}_\gamma a_j \bar{a}_k = \delta_{jk}.$$

Here and throughout this article, $\boldsymbol{E}_\gamma$ denotes expectation with respect to the probability measure γ: $\boldsymbol{E}_\gamma \varphi = \int \varphi \,\mathrm{d}\gamma$. Clearly, γ does not depend on the choice of orthonormal basis, and each (non-degenerate) complex Gaussian measure on V is associated with a unique (positive definite) Hermitian inner product on V.

In particular, we give $H^0(M, L^N)$ the complex Gaussian probability measure γ_h induced by the inner product (2.3), i.e.

$$\mathrm{d}\gamma_h(s) = \frac{1}{\pi^{d_N+1}} \mathrm{e}^{-|a|^2}\, \mathrm{d}a, \qquad s = \sum_{j=1}^{d_N} a_j S_j^N, \tag{2.12}$$

where $\{S_j^N : 1 \leqslant j \leqslant d_N\}$ is an orthonormal basis for $H^0(M, L^N)$ with respect to (2.3). The probability space $(H^0(M, L^N), \gamma_N)$ is called the *Hermitian Gaussian ensemble*. We regard the currents Z_s (respectively measures $|Z_s|$), as current-valued (respectively measure-valued) random variables on $(H^0(M, L^N), \gamma_N)$, i.e. for each test form (respectively function) φ, (Z_s, φ) (respectively $(|Z_s|, \varphi)$) is a complex-valued random variable.

Since the zero current Z_s is unchanged when s is multiplied by an element of $\mathbb{C}^*$, our results remain the same if we instead regard Z_s as a random variable on the unit sphere $SH^0(M, L^N)$ with Haar probability measure. We prefer to use Gaussian measures in order to facilitate computations.

2.2.1. *Holomorphic Gaussian random fields*

Gaussian random fields are determined by their two-point functions or covariance functions. We are mainly interested in the case where the fields are holomorphic sections of L^N; i.e, our probability space is a subspace $\mathcal{S}$ of the space $H^0(M, L^N)$ of holomorphic sections of L^N and the probability measure on $\mathcal{S}$ is the Gaussian measure induced by the inner product (2.3). If we pick an orthonormal basis $\{S_j\}_{1\leqslant j\leqslant m}$ of $\mathcal{S}$ with respect to (2.3), then we may write $s = \sum_{j=0}^n a_j S_j$, where the coordinates a_j are i.i.d. complex Gaussian random variables. The two-point function

$$\varPi_{\mathcal{S}}(z, w) := \boldsymbol{E}_{\mathcal{S}}(s(z) \otimes \overline{s(w)}) = \sum_{j=1}^n S_j(z) \otimes \overline{S_j(w)} \tag{2.13}$$

is the kernel of the orthogonal projection onto $\mathcal{S}$, and equals the Szegő kernel $\varPi_N(z, w)$ when $\mathcal{S} = H^0(M, L^N)$. The expected zero current $\boldsymbol{E}_{\mathcal{S}}(Z_s)$ for random sections $s \in \mathcal{S}$ is given by the *probabilistic Poincaré–Lelong formula.*

Lemma 2.1. *Let $(L, h) \to M$ be a Hermitian holomorphic line bundle over a compact complex manifold M and let $\mathcal{S} \subset H^0(M, L^N)$ be a Gaussian random field with two-point function $\varPi_{\mathcal{S}}(z, w)$. Then*

$$\boldsymbol{E}_{\mathcal{S}}(Z_s) = \frac{\mathrm{i}}{2\pi}\partial\bar{\partial} \log \|\varPi_{\mathcal{S}}(z, z)\|_{h^N} + \frac{N}{2\pi}\mathrm{i}\varTheta_h.$$

This lemma was given in [**11**, Proposition 3.1] and [**13**, Proposition 2.1] with slightly different hypotheses. For convenience, we include a proof below.

Proof. Let $\{S_j\}_{1\leqslant j\leqslant n}$ be a basis of $\mathcal{S}$ such that $s \in \mathcal{S}$ is of the form $s = \sum_{j=1}^n a_j S_j$, where the a_j are independent standard complex Gaussian random variables, as above. We then have $\|\varPi_{\mathcal{S}}(z, z)\|_{h^N} = \sum_{j=1}^n \|S_j\|_{h^N}^2$. For any $s \in \mathcal{S}$, we write

$$s = \sum_{j=1}^n a_j S_j = \langle a, F\rangle e_L^{\otimes N},$$

where e_L is a local non-vanishing holomorphic section of L, $S_j = f_j e_L^{\otimes N}$, and $F = (f_1, \dots, f_n)$. We then write $F(z) = |F(z)|U(z)$ so that $|U(z)| \equiv 1$ and

$$\log |\langle a, F\rangle| = \log |F| + \log |\langle a, U\rangle|.$$

A key point is that $\boldsymbol{E}(\log |\langle a, U\rangle|)$ is independent of z, and hence $\boldsymbol{E}(\mathrm{d} \log |\langle a, U\rangle|) = 0$. We note that U is well defined a.e. on $M \times \mathcal{S}$; namely, it is defined whenever $s(z) \neq 0$.

Write $\mathrm{d}\gamma = (1/\pi^n)\mathrm{e}^{-|a|^2}\,\mathrm{d}a$. By (2.4), we have

$$(\boldsymbol{E}Z_s, \varphi) = \boldsymbol{E}\biggl(\frac{\mathrm{i}}{\pi}\partial\bar\partial \log|\langle a, F\rangle|, \varphi\biggr) = \frac{\mathrm{i}}{\pi}\int_{\mathbb{C}^n} (\log|\langle a, F\rangle|, \partial\bar\partial\varphi)\,\mathrm{d}\gamma$$
$$= \frac{\mathrm{i}}{\pi}\int_{\mathbb{C}^{d_N}} (\log|F|, \partial\bar\partial\varphi)\,\mathrm{d}\gamma + \frac{\mathrm{i}}{\pi}\int_{\mathbb{C}^n} (\log|\langle a, U\rangle|, \partial\bar\partial\varphi)\,\mathrm{d}\gamma,$$

for all test forms $\varphi \in \mathcal{D}^{m-1,m-1}(M)$. The first term is independent of a, so we may remove the Gaussian integral. The vanishing of the second term follows by noting that

$$\int_{\mathbb{C}^n} (\log|\langle a, U\rangle|, \partial\bar\partial\varphi)\,\mathrm{d}\gamma = \int_{\mathbb{C}^n} \mathrm{d}\gamma \int_M \log|\langle a, U\rangle|\partial\bar\partial\varphi$$
$$= \int_M \int_{\mathbb{C}^n} \log|\langle a, U\rangle|\,\mathrm{d}\gamma\partial\bar\partial\varphi$$
$$= 0,$$

since

$$\int \log|\langle a, U\rangle|\,\mathrm{d}\gamma = \frac{1}{\pi}\int_{\mathbb{C}} \log|a_0|\mathrm{e}^{-|a_0|^2}\,\mathrm{d}a_0$$

is constant, by the $\mathrm{U}(n)$-invariance of $\mathrm{d}\gamma$. Fubini's Theorem can be applied above since

$$\int_{M\times\mathbb{C}^n} |\log|\langle a, U\rangle|\partial\bar\partial\varphi|\,\mathrm{d}\gamma = \frac{1}{\pi}\int_{\mathbb{C}} |\log|a_0||\,\mathrm{e}^{-|a_0|^2}\,\mathrm{d}a_0 \int_M |\partial\bar\partial\varphi| < +\infty.$$

Thus

$$\boldsymbol{E}Z_s = \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log|F|^2$$
$$= \frac{\mathrm{i}}{2\pi}\partial\bar\partial\biggl(\log\sum_{j=1}^n \|S_j\|_h^2 - \log\|e_L\|_h^{2N}\biggr)$$
$$= \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\|\Pi_{\mathcal{S}}(z,z)\|_{h^N} + \frac{\mathrm{i}N}{2\pi}\Theta_h.$$

□

3. Conditioning on the values of a random variable

In this section, we give a precise definition of the conditional expected zero current $\boldsymbol{E}(Z_{s_1,\dots,s_k} \mid s_1(p) = v_1,\ \dots,\ s_k(p) = v_k)$ (Definition 3.11) and give a number of its properties. In particular, we give a formula for the conditional expected zero current $K_1^N(z \mid p_1, \dots, p_r)$ in terms of the conditional Szegő kernel (Lemma 3.9).

3.1. The Leray form

We first give a general formula for the conditional expectation $\boldsymbol{E}(X \mid Y = y)$ of a continuous random variable X with respect to a smooth random variable Y when y is

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a regular value of Y. Our discussion differs from the standard expositions, which do not tend to assume random variables to be smooth.

We begin by recalling the definition of the conditional expectations $\boldsymbol{E}(X \mid \mathcal{F})$ of a random variable X on a probability space $(\Omega, \mathcal{A}, P)$ given a sub-σ-algebra $\mathcal{F} \subset \mathcal{A}$.

Definition 3.1. Let X be a random variable with finite first moment (i.e. $X \in L^1$) on a probability space $(\Omega, \mathcal{A}, P)$, and let $\mathcal{F} \subset \mathcal{A}$ be a σ-algebra. The conditional expectation is a random variable $E(X \mid \mathcal{F}) \in L^1(\Omega, P)$ satisfying:

- $\boldsymbol{E}(X \mid \mathcal{F})$ is measurable with respect to $\mathcal{F}$;
- for all sets $A \in \mathcal{F}$, $\int_A \boldsymbol{E}(X \mid \mathcal{F})\,\mathrm{d}P = \int_A X\,\mathrm{d}P$.

The existence and uniqueness (in L^1) of $E(X \mid \mathcal{F})$ is a standard fact (see, for example, [**9**, Theorem 6.1]).

In this paper, we are interested in the conditional expectation $\boldsymbol{E}(X \mid \sigma(Y))$ of a continuous random variable X on a manifold Ω with respect to a smooth random variable $Y \colon \Omega \to \mathbb{R}^k$. Here, $\sigma(Y)$ denotes the σ-algebra generated by Y, i.e. the pullbacks by Y of the Borel sets in $\mathbb{R}^k$; $\sigma(Y)$ is generated by the sublevel sets $\{Y_j \leqslant t_j,\ j = 1, \dots, k\}$. The condition that $\boldsymbol{E}(X \mid \sigma(Y))$ is measurable with respect to $\sigma(Y)$ implies that it is constant on the level sets of Y. We then write

$$\boldsymbol{E}(X \mid Y = y) := \boldsymbol{E}(X \mid \sigma(Y))(x), \quad x \in Y^{-1}(y).$$

We call $\boldsymbol{E}(X \mid Y = y)$ the *conditional expectation of X given that $Y = y$*. We note that the function $y \mapsto \boldsymbol{E}(X \mid Y = y)$ is in $L^1(\mathbb{R}^k, Y_* P)$, and is not necessarily well defined at each point y. However, in the cases of interest to us, $\boldsymbol{E}(X \mid Y = y)$ will be a continuous function.

To give a geometrical description of $\boldsymbol{E}(X \mid Y)$, we use the language of Gelfand–Leray forms.

Definition 3.2. Let $Y \colon \Omega \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ submersion where Ω is an oriented n-dimensional manifold. Let $\nu \in \mathcal{E}^n(\Omega)$, e.g. a volume form. The Gelfand–Leray form $\mathcal{L}(\nu, Y, y) \in \mathcal{E}^{n-k}(Y^{-1}(y))$ on the level set $\{Y = y\}$ is given by

$$\mathcal{L}(\nu, Y, y) \wedge \mathrm{d}Y_1 \wedge \cdots \wedge \mathrm{d}Y_k = \nu \quad \text{on } Y^{-1}(y), \qquad \text{i.e. } \mathcal{L}(\nu, Y, y) = \frac{\nu}{\mathrm{d}Y_1 \wedge \cdots \wedge \mathrm{d}Y_k}\bigg|_{Y^{-1}(y)}. \tag{3.1}$$

Conditional expectation of a random variable is a form of averaging. The following proposition shows this explicitly: it amounts to averaging X over the level sets of Y.

Proposition 3.3. *Let $\nu \in \mathcal{E}^n(\Omega)$ be a smooth probability measure on a manifold Ω. Let $Y \colon \Omega \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ submersion, and let $X \in L^1(\Omega, \nu)$. Then*

$$\boldsymbol{E}(X \mid Y = y) = \frac{\int_{Y=y} X \mathcal{L}(\nu, Y, y)}{\int_{Y=y} \mathcal{L}(\nu, Y, y)}.$$

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Proof. We first note that

$$\int_{y\in\mathbb{R}^k}\left(\int_{Y^{-1}(y)}|X|\mathcal{L}(\nu,Y,y)\right)\mathrm{d}y_1\cdots\mathrm{d}y_k=\int_X|X|\nu=1,$$

and hence

$$\int_{Y^{-1}(y)}|X|\mathcal{L}(\nu,Y,y)<+\infty$$

for almost all $y\in\mathbb{R}^k$. Furthermore,

$$\int_{Y^{-1}(y)}\mathcal{L}(\nu,Y,y)>0$$

for $Y_*\nu$-almost all $y\in\mathbb{R}^k$, and therefore $\boldsymbol{E}(X\mid Y=y)$ is well defined for $Y_*\nu$-almost all y. Now let

$$\tilde{E}(x)=\frac{\int_{Y=Y(x)}X\mathcal{L}(\nu,Y,Y(x))}{\int_{Y=Y(x)}\mathcal{L}(\nu,Y,Y(x))},\quad\text{for }\nu\text{-almost all }x\in X.$$

The function $\tilde{E}$ is measurable with respect to $\sigma(Y)$ since it is the pullback by Y of a measurable function on $\mathbb{R}^k$.

The only other thing to check is that $\int_A\tilde{E}\nu=\int_A X\nu$ for all $A\in\mathcal{F}$. It suffices to check this for sets A of the form $Y^{-1}(R)$ where R is a rectangle in $\mathbb{R}^k$. But then by the change of variables formula and Fubini's Theorem,

$$\begin{aligned}\int_{Y^{-1}(R)}\tilde{E}\nu&=\int_{y\in R}\left(\int_{Y^{-1}(y)}\tilde{E}\mathcal{L}(\nu,Y,y)\right)\mathrm{d}y_1\cdots\mathrm{d}y_k\\&=\int_{y\in R}\left(\int_{Y^{-1}(y)}X\mathcal{L}(\nu,Y,y)\right)\mathrm{d}y_1\cdots\mathrm{d}y_k\\&=\int_{Y^{-1}(R)}X\,\mathrm{d}\nu.\end{aligned}$$

By uniqueness of the conditional expectation, we then conclude that $\tilde{E}=\boldsymbol{E}(X\mid\sigma(Y))$. □

Example 3.4. Let $\Omega=\mathbb{C}^n$ with Gaussian probability measure $\mathrm{d}\gamma_n=\pi^{-n}\mathrm{e}^{-|a|^2}\,\mathrm{d}a$. Let $\pi_k\colon\mathbb{C}^n\to\mathbb{C}^k$ be the projection $\pi_k(a_1,\dots,a_n)=(a_1,\dots,a_k)$. For $y\in\mathbb{C}^k$ we have

$$\mathcal{L}(\mathrm{d}\gamma_n,\pi_k,y)=\frac{1}{\pi^k}\mathrm{e}^{-(|y_1|^2+\cdots+|y_k|^2)}\,\mathrm{d}\gamma_{n-k}(a_{k+1},\dots,a_n),$$

where

$$\mathrm{d}\gamma_{n-k}(a_{k+1},\dots,a_n)=\mathrm{e}^{-(|a_{k+1}|^2+\cdots+|a_n|^2)}\left(\frac{\mathrm{i}}{2\pi}\right)^{n-k}\mathrm{d}a_{k+1}\wedge\mathrm{d}\bar{a}_{k+1}\wedge\cdots\wedge\mathrm{d}a_n\wedge\mathrm{d}\bar{a}_n$$

is the standard complex Gaussian measure on $\mathbb{C}^{n-k}$. For a bounded random variable X on $\mathbb{C}^n$, let X_y be the random variable on $\mathbb{C}^{n-k}$ given by $X_y(a')=X(y,a')$ for $a'\in\mathbb{C}^{n-k}$. By Proposition 3.3, we then have

$$\boldsymbol{E}_{\gamma_n}(X\mid\pi_k=y)=\boldsymbol{E}_{\gamma_{n-k}}(X_y).\tag{3.2}$$

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This example leads us to the following definition.

Definition 3.5. Let γ be a complex Gaussian measure on a finite-dimensional complex space V, and let W be a subspace of V. We define the *conditional Gaussian measure* γ_W on W to be the Gaussian measure associated with the Hermitian inner product on W induced by the inner product on V associated with γ.

The terminology of Definition 3.5 is justified by the following proposition, which we shall use to define the expected zero current conditioned on the value of a random holomorphic section at a point or points.

Proposition 3.6. *Let $T\colon \mathbb{C}^n \to V$ be a linear map onto a complex vector space V. Let E be a closed subset of $\mathbb{C}^n$ such that $E \cap T^{-1}(y)$ has Lebesgue measure 0 in $T^{-1}(y)$ for all $y \in V$. Let X be a bounded random variable on $\mathbb{C}^n$ such that $X \mid (\mathbb{C}^n \smallsetminus E)$ is continuous. Then $\boldsymbol{E}_{\gamma_n}(X \mid T = y)$ is continuous on V. Furthermore,*

$$\boldsymbol{E}_{\gamma_n}(X \mid T = 0) = \boldsymbol{E}_{\gamma_{\ker T}}(X'),$$

where X' is the restriction of X to $\ker T$ and $\gamma_{\ker T}$ is the conditional Gaussian measure on $\ker T$ as defined above.

Proof. Let $k = \dim V$. We can assume without loss of generality that $\ker T = \{0\} \times \mathbb{C}^{n-k}$. Then the map T has the same fibres as the projection $\pi_k(a_1, \dots, a_n) = (a_1, \dots, a_k)$, and thus $\sigma(T) = \sigma(\pi_k)$. Hence we can assume without loss of generality that $V = \mathbb{C}^k$ and $T = \pi_k$.

Fix $y_0 \in \mathbb{C}^k$ and let $\varepsilon > 0$ be arbitrary. Choose a compact set $K \subset \mathbb{C}^{n-k}$ such that $(\{y_0\} \times K) \cap E = \emptyset$ and $\gamma_{n-k}(\mathbb{C}^{n-k} \smallsetminus K) < \varepsilon / \sup |X|$. Since E is closed, $(\{y\} \times K) \cap E = \emptyset$, for y sufficiently close to y_0. As above, we let $X_y(a') = X(y, a')$ for $a' \in \mathbb{C}^{n-k}$. Since $X_y \to X_{y_0}$ uniformly on K, we have

$$\lim_{y \to y_0} \int_K X_y \,\mathrm{d}\gamma_{n-k} = \int_K X_{y_0} \,\mathrm{d}\gamma_{n-k}. \tag{3.3}$$

It follows from (3.2) that

$$\left| \boldsymbol{E}_{\gamma_n}(X \mid \pi_k = y) - \int_K X_y \,\mathrm{d}\gamma_{n-k} \right| = \left| \int_{\mathbb{C}^{n-k} \smallsetminus K} X_y \,\mathrm{d}\gamma_{n-k} \right| < \varepsilon, \tag{3.4}$$

for all $y \in \mathbb{C}^k$. The first conclusion is an immediate consequence of (3.3), (3.4) and the formula for $\boldsymbol{E}_{\gamma_n}(X \mid T = 0)$ follows from (3.2) with $y = 0$. □

3.2. Conditioning on the values of sections

We now state precisely what is meant by the expected zeros conditioned on sections having specific values at one or several points on the manifold.

Definition 3.7. Let (L, h) be a positive Hermitian holomorphic line bundle over a compact Kähler manifold M with Kähler form ω_h. Let $p_1, \dots, p_r$ be distinct points of

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M. Let $N \gg 0$ and give $H^0(M, L^N)$ the induced Hermitian Gaussian measure γ_N. Let $v_j \in L^N_{p_j}$, for $1 \leqslant j \leqslant r$. We let

$$T\colon H^0(M, L^N) \to L^N_{p_1} \oplus \cdots \oplus L^N_{p_r}, \quad s \mapsto s(p_1) \oplus \cdots \oplus s(p_r).$$

The expected zero current $\boldsymbol{E}(Z_s \mid s(p_1) = v_1, \ldots, s(p_r) = v_r)$ conditioned on the section taking the fixed values v_j at the points p_j is defined by

$$(\boldsymbol{E}_N(Z_s \mid s(p_1) = v_1, \ldots, s(p_r) = v_r), \varphi) = \boldsymbol{E}_{\gamma_N}((Z_s, \varphi) \mid T = v_1 \oplus \cdots \oplus v_r),$$

for smooth test forms $\varphi \in \mathcal{D}^{m-1,m-1}(M)$.

Lemma 3.8. *The mapping*

$$v_1 \oplus \cdots \oplus v_r \mapsto \boldsymbol{E}_N(Z_s \mid s(p_1) = v_1, \ldots, s(p_r) = v_r)$$

is a continuous map from $L^N_{p_1} \oplus \cdots \oplus L^N_{p_r}$ *to* $\mathcal{D}'^{1,1}(M)$.

Proof. Let N be sufficiently large so that T is surjective. Let $\varphi \in \mathcal{D}^{m-1,m-1}(M)$ be a smooth test form, and consider the random variable $X(s) = (Z_s, \varphi)$ on $H^0(M, L^N) \smallsetminus \{0\}$. By [**15**, Theorem 3.8] applied to the projection

$$\{(s, z) \in H^0(M, L^N) \times M \colon s(z) = 0\} \to H^0(M, L^N),$$

the random variable X is continuous on $H^0(M, L^N) \smallsetminus \{0\}$. Furthermore, X is bounded, since we have by (2.4),

$$|X(s)| \leqslant (\sup \|\varphi\|)(Z_s, \omega^{m-1}) = \frac{N}{\pi} (\sup \|\varphi\|) \int_M \omega_h^m.$$

The conclusion follows from Proposition 3.6 with $E = \{0\}$. □

We could just as well condition on the section having specific derivatives, or specific k-jets, at specific points. At the end of this section, we discuss the conditional zero currents of simultaneous sections.

We are particularly interested in the case where the v_j all vanish. In this case, the conditional expected zero current

$$K_1^N(z \mid p_1, \ldots, p_r) = \boldsymbol{E}_N(Z_s \mid s(p_1) = 0, \ldots, s(p_r) = 0)$$

is well defined and we have the following lemma.

Lemma 3.9. *Let* $(L, h) \to (M, \omega_h)$ *and* $(H^0(M, L^N), \gamma_h)$ *be as in Theorem 1.2. Let* $p_1, \ldots, p_r$ *be distinct points of* M *and let* $H_N^{p_1 \cdots p_r} \subset H^0(M, L^N)$ *denote the space of holomorphic sections of* L^N *vanishing at the points* $p_1, \ldots, p_r$. *Then*

$$K_1^N(z \mid p_1, \ldots, p_r) = \boldsymbol{E}_{\gamma_N^{p_1 \cdots p_r}}(Z_s) = \frac{\mathrm{i}}{2\pi} \partial\bar{\partial} \log \|\varPi_N^{p_1 \cdots p_r}(z, z)\|_{h^N} + \frac{N}{\pi} \omega_h,$$

where $\gamma_N^{p_1 \cdots p_r}$ *is the conditional Gaussian measure on* $H_N^{p_1 \cdots p_r}$, *and* $\varPi_N^{p_1 \cdots p_r}$ *is the Szegő kernel for the orthogonal projection onto* $H_N^{p_1 \cdots p_r}$.

Proof. Let $\varphi \in \mathcal{D}^{m-1,m-1}(M)$ be a smooth test form. By Proposition 3.6,

$$(K_1^N(z \mid p_1,\dots,p_r),\varphi) = \boldsymbol{E}_N((Z_s,\varphi) \mid T=0) = \boldsymbol{E}_{\gamma_N^{p_1\cdots p_r}}(Z_s,\varphi),$$

where T is as in Definition 3.7. By Lemma 2.1 with $\mathcal{S} = H_N^{p_1\cdots p_r}$, we then have

$$\boldsymbol{E}_{\gamma_N^{p_1\cdots p_r}}(Z_s,\varphi) = \left(\frac{\mathrm{i}}{2\pi}\partial\bar\partial \log \|\Pi_N^{p_1\cdots p_r}(z,z)\|_{h^N} + \frac{N}{\pi}\omega_h, \varphi\right).$$

□

Recalling the definition of P_N from (2.6), we now prove the following proposition.

Proposition 3.10. *We have*

$$K_1^N(z \mid p) = \boldsymbol{E}_N(Z_s) + \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 - P_N(z,p)^2). \tag{3.5}$$

Proof. As above, we let $H_N^p \subset H^0(M,L^N)$ denote the space of holomorphic sections vanishing at p. Let $\{S_{Nj}^p : j = 1,\dots,d_N - 1\}$ be an orthonormal basis of H_N^p. The Szegő projection Π_N^p is given by

$$\Pi_N^p(z,w) = \sum S_{Nj}^p(z) \otimes \overline{S_{Nj}^p(w)}.$$

By Lemma 3.9 with $r = 1$, we have

$$K_1^N(z \mid p) = \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log \|\Pi_N^p(z,z)\|_{h^N} + \frac{N}{\pi}\omega_h. \tag{3.6}$$

To give a formula for $\Pi_N^p(z,z)$, we consider the *coherent state* at p, $\Phi_N^p(z)$ defined as follows. Let

$$\hat\Phi_N^p(z) := \frac{\Pi_N(z,p)}{\|\Pi_N(p,p)\|_{h^N}^{1/2}} \in H^0(M,L^N) \otimes \bar L_p^N. \tag{3.7}$$

We choose a unit vector $e_p \in L_p$, and we let $\Phi_N^p \in H^0(M,L^N)$ be given by

$$\hat\Phi_N^p(z) = \Phi_N^p(z) \otimes \overline{\mathrm{e}_p^{\otimes N}}. \tag{3.8}$$

The coherent state Φ_N^p is orthogonal to H_N^p, because

$$s \in H_N^p \implies \|\Pi_N(p,p)\|_{h^N}^{1/2}\langle s, \hat\Phi_N^p\rangle = \int_M \Pi_N(p,z)s(z)\Omega_M(z) = s(p) = 0. \tag{3.9}$$

Furthermore, $\|\Phi_N^p\|_{h^N}^2 = 1$, and hence $\{S_{Nj}^p : j = 1,\dots,d_N - 1\} \cup \{\Phi_N^p\}$ forms an orthonormal basis for $H^0(M,L^N)$. Therefore,

$$\Pi_N^p(z,w) = \Pi_N(z,w) - \Phi_N^p(z) \otimes \overline{\Phi_N^p(w)}, \tag{3.10}$$

and in particular

$$\|\Pi_N^p(z,z)\|_{h^N} = \|\Pi_N(z,z)\|_{h^N} - \|\Phi_N^p(z)\|_{h^N}^2. \tag{3.11}$$

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Thus, by (3.11),

$$\log \|\Pi_N^p(z,z)\|_{h^N} = \log\left(\|\Pi_N(z,z)\|_{h^N} - \frac{\|\Pi_N(z,p)\|_{h^N}^2}{\|\Pi_N(p,p)\|_{h^N}}\right)$$
$$= \log\|\Pi_N(z,z)\|_{h^N} + \log(1 - P_N(z,p)^2).$$

By (3.6) and (3.11),

$$K_1^N(z \mid p) = \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log\|\Pi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log(1 - P_N(z,p)^2) + \frac{N}{\pi}\omega_h$$
$$= \boldsymbol{E}_N(Z_s) + \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log(1-P_N(z,p)^2), \tag{3.12}$$

concluding the proof of the Proposition. □

Theorem 1.2 involves the conditional zero current of a system of random sections, which we now define precisely.

Definition 3.11. Let (L,h) be a positive Hermitian holomorphic line bundle over a compact Kähler manifold M with Kähler form ω_h, let $1 \leqslant k \leqslant m = \dim M$, and let $p \in M$. Let $N \gg 0$ and give $H^0(M, L^N)$ the induced Hermitian Gaussian measure γ_N. We let

$$T\colon \bigoplus^k H^0(M, L^N) \to \bigoplus^k L_p^N,$$

where $\bigoplus^k V$ denotes k-tuples in V. The conditional expected zero current $\boldsymbol{E}_N(Z_{s_1,\dots,s_k} \mid s_1(p) = v_1, \dots, s_k(p) = v_k)$ is defined by

$$(\boldsymbol{E}_N(Z_{s_1,\dots,s_k} \mid s_1(p) = v_1,\dots,s_k(p)=v_k), \varphi) = \boldsymbol{E}_{\gamma_N^k}((Z_{s_1,\dots,s_k},\varphi) \mid T = (v_1,\dots,v_k))$$

for smooth test forms $\varphi \in \mathcal{D}^{m-k,m-k}(M)$. The conditional expected zero distribution is the current

$$K_k^N(z \mid p) := \boldsymbol{E}_N(Z_{s_1,\dots,s_k} \mid s_1(p) = 0,\dots,s_k(p) = 0),$$

which is well defined according to the following lemma.

Lemma 3.12. *For $N \gg 0$, the mapping*

$$(v_1,\dots,v_k) \mapsto \boldsymbol{E}_N(Z_{s_1,\dots,s_k} \mid s_1(p) = v_1,\dots,s_k(p) = v_k)$$

is a continuous map from $\bigoplus^k L_p^N$ to $\mathcal{D}'^{m-k,m-k}(M)$.

Proof. Let

$$E = \left\{(s_1,\dots,s_k) \in \bigoplus^k H^0(M,L^N)\colon \dim Z_{s_1,\dots,s_k} = n-k\right\}.$$

Since L is ample, for N sufficiently large, $E \cap T^{-1}(v_1,\dots,v_k)$ is a proper algebraic subvariety of $T^{-1}(v_1,\dots,v_k)$ and hence has Lebesgue measure 0 in $T^{-1}(v_1,\dots,v_k)$, for all $(v_1,\dots,v_k) \in \bigoplus^k L_p^N$. Then Proposition 3.6 applies with $\mathbb{C}^n$ replaced by $\bigoplus^k H^0(M,L^N)$, and continuity follows exactly as in the proof of Lemma 3.8. □

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4. Proof of Theorem 1.1

4.1. Proof for $k = 1$

We first prove Theorem 1.1 when the condition is that $s(p) = 0$ for a single point p.

Proof. Let $\varphi \in \mathcal{D}'^{m-1,m-1}(M)$ be a smooth test form. By Proposition 3.10, we have

$$(K_1^N(z \mid p), \varphi) = (\boldsymbol{E}_N Z_s, \varphi) + \int_M \log(1 - P_N(z,p)^2) \frac{\mathrm{i}}{2\pi} \partial\bar{\partial}\varphi. \tag{4.1}$$

Away from the diagonal, we can write $\log(1 - P_N(z,p)^2) = P_N(z,p)^2 + \frac{1}{2} P_N(z,p)^4 + \cdots$, and we have by (2.8),

$$\log(1 - P_N(z,p)^2) = O(N^{-m-2}) \quad \text{uniformly for } d(z,p) \geqslant b\sqrt{\frac{\log N}{N}}, \tag{4.2}$$

where $b = \sqrt{2m+6}$. Furthermore, by (4.2), we have

$$\int_M \log(1 - P_N(z,p)^2) \frac{\mathrm{i}}{2\pi} \partial\bar{\partial}\varphi = \int_{d(z,p) \leqslant b\sqrt{(\log N)/N}} \log(1 - P_N(z,p)^2) \frac{\mathrm{i}}{2\pi} \partial\bar{\partial}\varphi + O(N^{-m-2}).$$

Using local normal coordinates $(w_1, \dots, w_m)$ centred at p, we write

$$\frac{\mathrm{i}}{2\pi} \partial\bar{\partial}\varphi = \psi(w) \varOmega_0(w), \qquad \varOmega_0(w) = (\tfrac{1}{2}\mathrm{i})^m \mathrm{d}w_1 \wedge \mathrm{d}\bar{w}_1 \wedge \cdots \wedge \mathrm{d}w_m \wedge \mathrm{d}\bar{w}_m.$$

Recalling (2.9), we then have

$$\begin{aligned}\int_M \log(1 - P_N(z,p)^2) \frac{\mathrm{i}}{2\pi} \partial\bar{\partial}\varphi &= \int_{|w| \leqslant b\sqrt{(\log N)/N}} \log[1 - P_N(p + w, p)^2] \psi(w) \varOmega_0(w) + O(N^{-m-2}) \\ &= N^{-m} \int_{|u| \leqslant b\sqrt{\log N}} \log\left[1 - P_N\left(p + \frac{u}{\sqrt{N}}, p\right)^2\right] \psi\left(\frac{u}{\sqrt{N}}\right) \varOmega(u) + O(N^{-m-2}).\end{aligned} \tag{4.3}$$

Let

$$\varLambda_N(z,p) = -\log P_N(z,p), \tag{4.4}$$

so that

$$\log(1 - P_N(z,p)^2) = Y \circ \varLambda_N(z,p), \tag{4.5}$$

where

$$Y(\lambda) := \log(1 - \mathrm{e}^{-2\lambda}) \quad \text{for } \lambda > 0. \tag{4.6}$$

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By (2.9)–(2.10),

$$\Lambda_N\left(p+\frac{u}{\sqrt{N}},p\right)=\tfrac{1}{2}|u|^2+\tilde{R}_N(u), \tag{4.7}$$

where

$$\tilde{R}_N(u)=-\log[1+R_N(u,0)]=O(|u|^2N^{-1/2+\varepsilon})\quad\text{for } |u|<b\sqrt{\log N}. \tag{4.8}$$

We note that

$$0<-Y(\lambda)=-\log(1-\mathrm{e}^{-2\lambda})\leqslant\left(1+\log^+\frac{1}{\lambda}\right), \tag{4.9}$$

$$Y'(\lambda)=\frac{2}{\mathrm{e}^{2\lambda}-1}\leqslant\frac{1}{\lambda}\quad\text{for } \lambda>1. \tag{4.10}$$

Hence, by (4.5)–(4.10),

$$\log\left[1-P_N\left(p+\frac{u}{\sqrt{N}},p\right)^2\right]=\log(1-\mathrm{e}^{-|u|^2})+O(N^{-1/2+\varepsilon})\quad\text{for } |u|<b\sqrt{\log N}. \tag{4.11}$$

Since $\psi(u/\sqrt{N})=\psi(0)+O(u/\sqrt{N})$, we then have

$$\log\left[1-P_N\left(p+\frac{u}{\sqrt{N}},p\right)^2\right]\psi\left(\frac{u}{\sqrt{N}}\right)$$
$$=\psi(0)\log(1-\mathrm{e}^{-|u|^2})+O(N^{-1/2+\varepsilon})+\frac{1}{\sqrt{N}}O(|u|\,|\log(1-\mathrm{e}^{-|u|^2})|)$$
$$\text{for } |u|<b\sqrt{\log N}.$$

Since $O((\log N)^mN^{-1/2+\varepsilon})=O(N^{-1/2+2\varepsilon})$ and $|u|\log(1-\mathrm{e}^{-|u|^2})\in L^1(\mathbb{C}^m)$, we conclude that

$$\int_{|u|\leqslant b\sqrt{\log N}}\log\left[1-P_N\left(p+\frac{u}{\sqrt{N}},p\right)^2\right]\psi\left(\frac{u}{\sqrt{N}}\right)\Omega_0(u)$$
$$=\psi(0)\int_{|u|\leqslant b\sqrt{\log N}}\log[1-\mathrm{e}^{-|u|^2}]\Omega_0(u)+O(N^{-1/2+\varepsilon}).$$

We note that

$$\int_{|u|\geqslant b\sqrt{\log N}}\log[1-\mathrm{e}^{-|u|^2}]\Omega_0(u)=\frac{2\pi^m}{(m-1)!}\int_{b\sqrt{\log N}}^{+\infty}\log(1-\mathrm{e}^{r^2})r^{2m-1}\,\mathrm{d}r=O(N^{-b^2/2}).$$

Since $b>1$, we then have

$$\int_{|u|\leqslant b\sqrt{\log N}}\log\left[1-P_N\left(p+\frac{u}{\sqrt{N}},p\right)^2\right]\psi\left(\frac{u}{\sqrt{N}}\right)\Omega_0(u)$$
$$=\psi(0)\int_{\mathbb{C}^m}\log[1-\mathrm{e}^{-|u|^2}]\Omega_0(u)+O(N^{-1/2+\varepsilon}). \tag{4.12}$$

Combining (4.1), (4.3) and (4.12), we have

$$(\boldsymbol{E}(Z_s\colon s(p)=0),\varphi)=(\boldsymbol{E}Z_s,\varphi)+N^{-m}\psi(0)\int_{\mathbb{C}^m}\log[1-\mathrm{e}^{-|u|^2}]\Omega_0(u)+O(N^{-m-1/2+\varepsilon}). \tag{4.13}$$

We note that

$$\psi(0)=\frac{1}{2\pi}\frac{\mathrm{i}\partial\bar\partial\varphi(p)}{\Omega_M(p)} \tag{4.14}$$

and

$$\begin{aligned}\int_{\mathbb{C}^m}\log[1-\mathrm{e}^{-|u|^2}]\Omega_0(u)&=\frac{2\pi^m}{(m-1)!}\int_0^{+\infty}\log(1-\mathrm{e}^{-r^2})r^{2m-1}\,\mathrm{d}r\\&=\frac{\pi^m}{(m-1)!}\int_0^{+\infty}\log(1-\mathrm{e}^{-t})t^{m-1}\,\mathrm{d}t\\&=-\frac{\pi^m}{(m-1)!}\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{\mathrm{e}^{-nt}}{n}t^{m-1}\,\mathrm{d}t\\&=-\frac{\pi^m}{(m-1)!}\sum_{n=1}^{+\infty}\frac{(m-1)!}{n^{m+1}}\\&=-\pi^m\zeta(m+1).\end{aligned} \tag{4.15}$$

The one-point case ($k=1$) of Theorem 1.1 follows by substituting (4.14), (4.15) into (4.13). □

4.2. The multi-point case

We now condition on vanishing at k points $p_1,\dots,p_k$.

Proof. We let $H^V_N\subset H^0(M,L^N)$ denote the space of holomorphic sections vanishing at the points $p_1,\dots,p_k$. Let $\Phi^{p_j}_N$ be the coherent state at p_j (given by (3.7), (3.8)) for $j=1,\dots,k$. By (3.9), a section $s\in H^0(M,L^N)$ vanishes at p_j if and only if s is orthogonal to $\Phi^{p_j}_N$. Thus $H^0(M,L^N)=H^V_N\oplus\mathrm{Span}\{\Phi^{p_j}_N\}$. Let

$$T\colon H^0(M,L^N)\to L^N_{p_1}\oplus\cdots\oplus L^N_{p_k},\quad s\mapsto s(p_1)\oplus\cdots s(p_k),$$

so that $\ker T=H^V_N$. By Lemma 3.9, the conditional expectation is given by

$$K^N_1(z\mid p_1,\dots,p_k)=\frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\|\Pi^V_N(z,z)\|_{h^N}+\frac{N}{\pi}\omega_h, \tag{4.16}$$

where Π^V_N is the conditional Szegő kernel for the projection onto Π^V_N. We let $\Pi^\perp_N(z,w)$ denote the kernel for the orthogonal projection onto $(H^V_N)^\perp=\mathrm{Span}\{\Phi^{p_j}_N\}$, so that

$$\Pi^V_N(z,w)=\Pi_N(z,w)-\Pi^\perp_N(z,w). \tag{4.17}$$

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Recalling (3.7), (3.8), we have

$$\langle \Phi_N^{p_i}, \Phi_N^{p_j}\rangle \overline{e_{p_i}^{\otimes N}} \otimes e_{p_j}^{\otimes N} = \frac{\langle \sum_\alpha S_\alpha^N(z)\otimes \overline{S_\alpha^N(p_i)}, \sum_\beta S_\beta^N(z)\otimes \overline{S_\beta^N(p_j)}\rangle}{\|\Pi_N(p_i,p_i)\|_{h_N}^{1/2}\|\Pi_N(p_j,p_j)\|_{h_N}^{1/2}}$$

$$= \frac{\overline{\Pi_N(p_i,p_j)}}{\|\Pi_N(p_i,p_i)\|_{h_N}^{1/2}\|\Pi_N(p_j,p_j)\|_{h_N}^{1/2}},$$

and therefore, by (2.8),

$$|\langle \Phi_N^{p_i}, \Phi_N^{p_j}\rangle| = P_N(p_i,p_j) = \delta_i^j + O(N^{-\infty}). \tag{4.18}$$

In particular the Φ_N^j are linearly independent, for $N \gg 0$. Let

$$\langle \Phi_N^{p_i}, \Phi_N^{p_j}\rangle = \delta_i^j + W_{ij}.$$

By (4.18), $W_{ij} = O(N^{-\infty})$. Let us now replace the basis $\{\Phi_N^{p_j}\}$ of $(H_N^V)^\perp$ by an orthonormal basis $\{\Psi_N^j\}$, and write

$$\Psi_N^i = \sum_{j=1}^k A_{ij}\Phi_N^{p_j}.$$

Then

$$\delta_i^j = \langle \Psi_N^i, \Psi_N^j\rangle = \sum_{\alpha,\beta}\langle A_{i\alpha}\Phi^{p_\alpha}, A_{j\beta}\Phi^{p_\beta}\rangle = \sum_{\alpha,\beta} A_{i\alpha}\bar{A}_{j\beta}(\delta_\alpha^\beta + W_{\alpha\beta}),$$

or $I = A(I+W)A^*$.

We have

$$\Pi_N^\perp(z,z) = \sum \Psi_N^j(z)\otimes\overline{\Psi_N^j(z)} = \sum_{j,\alpha,\beta} A_{j\alpha}\bar{A}_{j\beta}\Phi_N^{p_\alpha}\otimes\overline{\Phi_N^{p_\beta}} = \sum_{j\beta} B_{\alpha\beta}\Phi_N^{p_\alpha}\otimes\overline{\Phi_N^{p_\beta}},$$

where

$$B = {}^{\mathrm{t}}A\bar{A} = {}^{\mathrm{t}}(A^*A) = {}^{\mathrm{t}}(I+W)^{-1} = I + O(N^{-\infty}). \tag{4.19}$$

The final equality in (4.19) follows by noting that

$$\|W\|_{\mathrm{HS}} = \eta < 1$$
$$\implies \|(I+W)^{-1} - I\|_{\mathrm{HS}} = \|W - W^2 + W^3 + \cdots\|_{\mathrm{HS}} \leqslant \eta + \eta^2 + \eta^3 + \cdots = \frac{\eta}{1-\eta},$$

where $\|W\|_{\mathrm{HS}} = [\mathrm{Trace}(WW^*)]^{1/2}$ denotes the Hilbert–Schmidt norm. Therefore,

$$\|\Pi_N^\perp(z,z)\| = \sum_{j=1}^k \|\Phi_N^{p_j}(z)\|^2 + O(N^{-\infty}).$$

Repeating the argument of the 1-point case, we then obtain

$$K_1^N(z \mid p_1,\dots,p_k) = (\boldsymbol{E}_N Z_s, \varphi) + \log\Big(1 - \sum P_N(z,p_j)^2\Big) + O(N^{-\infty}). \tag{4.20}$$

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It suffices to verify the theorem in a neighbourhood of an arbitrary point $z_0 \in M$. If $z_0 \notin \{p_1, \dots, p_k\}$, then $\log(1 - \sum P_N(z, p_j)^2) = O(N^{-\infty})$ in a neighbourhood of z_0, and the formula trivially holds. Now suppose $z_0 = p_1$, for example. Then

$$\log\Bigl(1 - \sum P_N(z, p_j)^2\Bigr) = \log(1 - P_N(z, p_1)^2) + O(N^{-\infty})$$

near p_1 and the conclusion holds there by the computation in the 1-point case. □

5. Proof of Theorem 1.2: the scaled conditional expectation

In this section we shall prove Theorem 1.2 together with the following analogous result on the scaling asymptotics of conditional expected zero currents of positive dimension.

Theorem 5.1. *Let $1 \leqslant k \leqslant m-1$. Let $(L, h) \to (M, \omega_h)$ and $(H^0(M, L^N), \gamma_h^N)$ be as in Theorem 1.1. Let $p \in M$, and choose normal coordinates $z = (z_1, \dots, z_m)\colon M_0, p \to \mathbb{C}^m, 0$ on a neighbourhood M_0 of p. Let $\tau_N = \sqrt{N}z\colon M_0 \to \mathbb{C}^m$ be the scaled coordinate map. Then for a smooth test form $\varphi \in \mathcal{D}^{m-k,m-k}(\mathbb{C}^m)$, we have*

$$(K_k^N(z \mid p), \tau_N^* \varphi) = \int_{\mathbb{C}^m \setminus \{0\}} \varphi \wedge \biggl(\frac{\mathrm{i}}{2\pi}\partial\bar\partial[\log(1 - \mathrm{e}^{-|u|^2}) + |u|^2]\biggr)^k + O(N^{-1/2+\varepsilon}),$$

and thus

$$\tau_{N*}(K_k^N(z \mid p)) \to K_{km}^\infty(u \mid 0) := \biggl(\frac{\mathrm{i}}{2\pi}\partial\bar\partial[\log(1 - \mathrm{e}^{-|u|^2}) + |u|^2]\biggr)^k,$$

where $u = (u_1, \dots, u_m)$ denotes the coordinates in $\mathbb{C}^m$.

Just as in Theorem 1.2, $K_{km}^\infty(u \mid 0)$ is the conditional expected zero current of k independent random functions in the Bargmann–Fock ensemble on $\mathbb{C}^m$.

To prove Theorems 1.2 and 5.1, we first note that by (2.7) and Proposition 3.10, we have

$$\begin{aligned} K_1^N(z \mid p) &= \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log \|\varPi_N(z, z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 - P_N(z, p)^2) + \frac{N}{\pi}\omega_h \\ &= \frac{N}{\pi}\omega_h + \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 - P_N(z, p)^2) + O(N^{-1}). \end{aligned} \tag{5.1}$$

In normal coordinates $(z_1, \dots, z_m)$ about p, we have

$$\omega_h = \tfrac{1}{2}\mathrm{i} \sum g_{jl}\, \mathrm{d}z_j \wedge \mathrm{d}\bar z_l, \quad g_{jl}(z) = \delta_j^l + O(|z|). \tag{5.2}$$

Changing variables to $u_j = \sqrt{N}z_j$ gives

$$\frac{N}{\pi}\omega_h = \frac{\mathrm{i}}{2\pi}\sum g_{jl}\biggl(\frac{u}{\sqrt{N}}\biggr)\mathrm{d}u_j \wedge \mathrm{d}\bar u_l = \frac{\mathrm{i}}{2\pi}\partial\bar\partial|u|^2 + \sum O(|u|N^{-1/2})\, \mathrm{d}u_j \wedge \mathrm{d}\bar u_l. \tag{5.3}$$

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We can now easily verify the one-dimensional case of Theorem 1.2. Let $m = 1$. By (4.11), (5.1) and (5.3), we have

$$(K_1^N(z \mid p), \tau_N^*\varphi) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}[\log(1-\mathrm{e}^{-|u|^2}) + |u|^2]\partial\bar{\partial}\varphi + O(N^{-1/2+\varepsilon})$$

for a smooth test function $\varphi \in \mathcal{D}(\mathbb{C})$. We have

$$\begin{aligned}\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}[\log(1-\mathrm{e}^{-|u|^2}) + |u|^2]\partial\bar{\partial}\varphi &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\left[\log\frac{1-\mathrm{e}^{-|u|^2}}{|u|^2} + |u|^2\right]\partial\bar{\partial}\varphi + \varphi(0)\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\varphi\partial\bar{\partial}\left[\log\frac{1-\mathrm{e}^{-|u|^2}}{|u|^2} + |u|^2\right] + \varphi(0)\\ &= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}\smallsetminus\{0\}}\varphi\partial\bar{\partial}[\log(1-\mathrm{e}^{-|u|^2}) + |u|^2] + \varphi(0),\end{aligned}$$

proving Theorem 1.2 for $k = m = 1$.

For the dimension $m > 1$ cases, we first derive some pointwise formulae on $M \smallsetminus \{p\}$. Let

$$\Lambda_N(z) = \Lambda_N(z, p) = -\log P_N(z, p).$$

Recalling (4.5), we have

$$\begin{aligned}\partial\bar{\partial}\log(1 - P_N(z,p)^2) &= \partial\bar{\partial}(Y \circ \Lambda_N)\\ &= Y''(\Lambda_N)\partial\Lambda_N \wedge \bar{\partial}\Lambda_N + Y'(\Lambda_N)\partial\bar{\partial}\Lambda_N\\ &= -\frac{4\mathrm{e}^{-2\Lambda_N}}{(1-\mathrm{e}^{-2\Lambda_N})^2}\partial\Lambda_N \wedge \bar{\partial}\Lambda_N + \frac{2}{\mathrm{e}^{2\Lambda_N}-1}\partial\bar{\partial}\Lambda_N.\end{aligned}$$

By (2.10) and (4.7)–(4.8), we have

$$\begin{aligned}\Lambda_N &= \tfrac{1}{2}|u|^2 + O(|u|^2 N^{-1/2+\varepsilon}),\\ \frac{\partial\Lambda_N}{\partial\bar{u}_j} &= \tfrac{1}{2}u_j + O(|u|N^{-1/2+\varepsilon}),\\ \frac{\partial^2\Lambda_N}{\partial u_j\partial\bar{u}_l} &= \tfrac{1}{2}\delta_j^l + O(N^{-1/2+\varepsilon}).\end{aligned}$$

Thus

$$\bar{\partial}\Lambda_N = \frac{1}{2}\sum[u_j + O(|u|N^{-1/2+\varepsilon})]\,\mathrm{d}\bar{u}_j,$$

and

$$\partial\bar{\partial}\Lambda_N = \Big(\tfrac{1}{2}\partial\bar{\partial}|u|^2 + \sum c_{jl}\,\mathrm{d}u_j \wedge \mathrm{d}\bar{u}_l\Big), \quad c_{jl} = O(N^{-1/2+\varepsilon}).$$

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Since $Y^{(j)}(\lambda) = O(\lambda^{-j})$ for $0 < \lambda < 1$, we then have

$$\begin{aligned}
&\bar\partial \log(1 - P_N(z,p)^2) \\
&\quad = [Y'(\tfrac{1}{2}|u|^2) + O(|u|^{-2}N^{-1/2+\varepsilon})]\Big[\tfrac{1}{2}\bar\partial|u|^2 + \sum O(|u|N^{-1/2+\varepsilon})\,\mathrm{d}\bar u_j\Big] \\
&\quad = \frac{1}{\mathrm{e}^{|u|^2}-1}\bar\partial|u|^2 + \sum O(|u|^{-1}N^{-1/2+\varepsilon})\,\mathrm{d}\bar u_j, \qquad (5.4)
\end{aligned}$$

$$\begin{aligned}
&\partial\bar\partial \log(1 - P_N(z,p)^2) \\
&\quad = -\frac{\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^2}\partial|u|^2 \wedge \bar\partial|u|^2 + \frac{1}{\mathrm{e}^{|u|^2}-1}\partial\bar\partial|u|^2 + \sum O(|u|^{-2}N^{-1/2+\varepsilon})\,\mathrm{d}u_j \wedge \mathrm{d}\bar u_k \\
&\quad = \partial\bar\partial\log(1-\mathrm{e}^{-|u|^2}) + \sum O(|u|^{-2}N^{-1/2+\varepsilon})\,\mathrm{d}u_j \wedge \mathrm{d}\bar u_l, \qquad (5.5)
\end{aligned}$$

for $0 < |u| < b$. Therefore, by (5.1), (5.5) and (5.3),

$$\begin{aligned}
K_1^N(z \mid p) &= \frac{\mathrm{i}}{2\pi}\frac{1}{1-\mathrm{e}^{-|u|^2}}\bigg[-\frac{\mathrm{e}^{-|u|^2}}{1-\mathrm{e}^{-|u|^2}}\partial|u|^2 \wedge \bar\partial|u|^2 + \partial\bar\partial|u|^2\bigg] \\
&\qquad\qquad + \sum O(|u|^{-2}N^{-1/2+\varepsilon})\,\mathrm{d}u_j \wedge \mathrm{d}\bar u_k \\
&= \frac{\mathrm{i}}{2\pi}\partial\bar\partial[\log(1-\mathrm{e}^{-|u|^2}) + |u|^2] + \sum O(|u|^{-2}N^{-1/2+\varepsilon})\,\mathrm{d}u_j \wedge \mathrm{d}\bar u_l, \qquad (5.6)
\end{aligned}$$

for $0 < |u| < b$.

We shall use the following notation. If $R \in \mathcal{D}'^r(M)$ is a current of order 0 (i.e. its coefficients are given locally by measures), we write $R = R_{\text{sing}} + R_{\text{ac}}$, where R_{sing} is supported on a set of (volume) measure 0, and the coefficients of R_{ac} are in L^1_{loc}. We also let $\|R\|$ denote the total variation measure of R:

$$(\|R\|, \psi) := \sup\{|(R,\eta)| \colon \eta \in \mathcal{D}^{2m-r}(M),\ |\eta| \leqslant \psi\} \quad \text{for } \psi \in \mathcal{D}(M).$$

Lemma 5.2. *The conditional expected zero distributions are given by*

$$K_k^N(z \mid p) = \bigg[\frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log(1-P_N(z,p)^2) + \frac{N}{\pi}\omega_h\bigg]^k_{\text{ac}}, \quad \textit{for } 1 \leqslant k \leqslant m-1,$$

$$K_m^N(z \mid p) = \delta_p + \bigg[\frac{\mathrm{i}}{2\pi}\partial\bar\partial\log\|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial\log(1-P_N(z,p)^2) + \frac{N}{\pi}\omega_h\bigg]^m_{\text{ac}}.$$

In particular, the currents $K_k^N(z \mid p)$ are smooth forms on $M \smallsetminus \{p\}$ for $1 \leqslant k \leqslant m$, and only the top-degree current $K_m^N(z \mid p)$ has point mass at p.

Proof. Let

$$T \colon H^0(M, L^N)^k \to (L_p^{\otimes N})^k, \quad (s_1, \dots, s_k) \mapsto (s_1(p), \dots, s_k(p)).$$

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By Proposition 3.6 and Definition 3.11,

$$(K_k^N(z \mid p), \varphi) = \boldsymbol{E}_{(\gamma_N^p)^k}(Z_{s_1,\dots,s_k}, \varphi),$$

for $\varphi \in \mathcal{D}^{m-k,m-k}(M \setminus \{p\})$, where γ_N^p is the conditional Gaussian on H_N^p.

Next, we shall apply Proposition 2.2 in [**13**] to show that

$$K_k^N(z \mid p) = \boldsymbol{E}_{(\gamma_N^p)^k}(Z_{s_1,\dots,s_k}) = [\boldsymbol{E}_{\gamma_N^p} Z_s]^{\wedge k} = [K_1^N(z \mid p)]^{\wedge k} \quad \text{on } M \setminus \{p\}. \tag{5.7}$$

We cannot apply Proposition 2.2 in [**13**] directly, since all sections of H_N^p vanish at p by definition, so H_N^p is not base point free. Instead, we shall apply this result to the blowup $\tilde{M}$ of p. Let $\pi\colon \tilde{M} \to M$ be the blowup map, and let $E = \pi^{-1}(p)$ denote the exceptional divisor. Let $\tilde{L} \to \tilde{M}$ denote the pullback of L, and let $\mathcal{O}(-E)$ denote the line bundle over $\tilde{M}$ whose local sections are holomorphic functions vanishing on E (see [**7**, pp. 136–137]). Thus we have isomorphisms

$$\tau_N\colon H_N^p \xrightarrow{\approx} H^0(\tilde{M}, \tilde{L}^N \otimes \mathcal{O}(-E)), \qquad \tau_N(s) = s \circ \pi. \tag{5.8}$$

(Surjectivity follows from Hartogs's Extension Theorem (see, for example, [**7**, p. 7]).)

Let $\mathcal{I}_p \subset \mathcal{O}_M$ denote the maximal ideal sheaf of $\{p\}$. From the long exact cohomology sequence

$$\cdots \to H^0(M, \mathcal{O}(L^N)) \to H^0(M, \mathcal{O}(L^N) \otimes (\mathcal{O}_M/\mathcal{I}_p^2)) \to H^1(M, \mathcal{O}(L^N) \otimes \mathcal{I}_p^2) \to \cdots$$

and the Kodaira vanishing theorem, it follows that $H^1(M, \mathcal{O}(L^N) \otimes \mathcal{I}_p^2) = 0$ and thus there exist sections of L^N with arbitrary 1-jet at p, for N sufficiently large (see, for example, [**10**, Theorem (5.1)]). Therefore, $\tilde{L}^N \otimes \mathcal{O}(-E)$ is base point free.

We give $H^0(\tilde{M}, \tilde{L}^N \otimes \mathcal{O}(-E))$ the Gaussian measure $\tilde{\gamma}_N := \tau_{N*}\gamma_N^p$. By [**13**, Propositions 2.1, 2.2] applied to the line bundle $\tilde{L}^N \otimes \mathcal{O}(-E) \to \tilde{M}$ and the space $\mathcal{S} = H^0(\tilde{M}, \tilde{L}^N \otimes \mathcal{O}(-E))$, we have $\boldsymbol{E}_{(\tilde{\gamma}_N)^k}(Z_{\tilde{s}_1,\dots,\tilde{s}_k}) = (\boldsymbol{E}_{\tilde{\gamma}_N} Z_{\tilde{s}_1})^{\wedge k}$ (where the $\tilde{s}_j$ are independent random sections in $\mathcal{S}$). Equation (5.7) then follows by identifying $\tilde{M} \setminus E$ with $M \setminus \{p\}$ and $H^0(\tilde{M}, \tilde{L}^N \otimes \mathcal{O}(-E))$ with H_N^p. By (5.1) and (5.7), we then have

$$K_k^N(z \mid p) = \left[\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log \|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log(1 - P_N(z,p)^2) + \frac{N}{\pi}\omega_h\right]^k$$
$$\text{on } M \setminus \{p\} \text{ for } 1 \leqslant k \leqslant m.$$

Since $K_k^N(z \mid p)$ is a current of order 0, to complete the proof of the lemma it suffices to show that

(i) $\|K_k^N(z \mid p)\|(\{p\}) = 0$ for $k < m$,

(ii) $K_m^N(z \mid p)(\{p\}) = 1$.

We first verify (ii). Let $\{\varphi_n\}$ be a decreasing sequence of smooth functions on M such that $0 \leqslant \varphi_n \leqslant 1$ and $\varphi_n \to \chi_{\{p\}}$ as $n \to \infty$. We consider the random variables $X_n^m\colon (H_p^N)^m \to \mathbb{R}$ given by

$$X_n^m(\boldsymbol{s}) = (Z_{\boldsymbol{s}}, \varphi_n), \quad \boldsymbol{s} = (s_1, \dots, s_m).$$

Every m-tuple $\boldsymbol{s} \in (H^p_N)^m$ has a zero at p by definition, and almost all $\boldsymbol{s}$ have only simple zeros; therefore $X^m_n(\boldsymbol{s}) \to Z_{\boldsymbol{s}}(\{p\}) = 1$ a.s. Furthermore, $1 \leqslant X^m_n(\boldsymbol{s}) \leqslant (Z_{\boldsymbol{s}}, 1) = N^m c_1(L)^m$. Therefore, by dominated convergence,

$$\begin{aligned} K^N_m(z \mid p)(\{p\}) &= \lim_{n\to\infty} (K^N_m(z \mid p), \varphi_n) \\ &= \lim_{n\to\infty} \int X^m_n \,\mathrm{d}(\gamma^p_N)^m \\ &= \int \lim_{n\to\infty} X^m_n \,\mathrm{d}(\gamma^p_N)^m \\ &= 1. \end{aligned}$$

To verify (i), we note that $\|K^N_k(z \mid p)\|\Omega_M \leqslant C K^N_k(z \mid p) \wedge \omega_h^{m-k}$ (where the constant C depends only on k and m), and thus it suffices to show that

(i′) $(K^N_k(z \mid p) \wedge \omega_h^{m-k})(\{p\}) = 0$ for $k < m$.

For $k < m$, we let

$$X^k_n(\boldsymbol{s}) = (Z_{\boldsymbol{s}} \wedge \omega_h^{m-k}, \varphi_n) \leqslant \pi^{m-k} N^m c_1(L)^m, \quad \boldsymbol{s} = (s_1, \dots, s_k),$$

where φ_n is as before. But this time, $X^k_n(\boldsymbol{s}) = \int_{Z_{\boldsymbol{s}}} \varphi_n \omega_h^{m-k} \to 0$ a.s. Equation (i′) now follows exactly as before. (Equation (i) is also an immediate consequence of Federer's support theorem for locally flat currents [**6**, 4.1.20].) □

We now complete the proof of Theorem 5.1. By Lemma 5.2 and the asymptotic formula (5.6), we have

$$\begin{aligned} K^N_k(z \mid p) &= K^N_k(z \mid p)_{\mathrm{ac}} \\ &= \left[\frac{\mathrm{i}}{2\pi}\partial\bar\partial \log \|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 - P_N(z,p)^2) + \frac{N}{\pi}\omega_h\right]^k_{\mathrm{ac}} \\ &= \left(\frac{\mathrm{i}}{2\pi}\right)^k \left[-\frac{\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^2}\partial|u|^2 \wedge \bar\partial|u|^2 + \frac{\partial\bar\partial|u|^2}{(1-\mathrm{e}^{-|u|^2})}\right]^k \\ &\qquad + \sum O(|u|^{-2k} N^{-1/2+\varepsilon})\,\mathrm{d}u_{j_1} \wedge \mathrm{d}u_{l_1} \wedge \dots \wedge \mathrm{d}u_{j_k} \wedge \mathrm{d}u_{l_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} &(K^N_k(z \mid p), \tau^*_N\varphi) \\ &\quad = \int_{M_0 \smallsetminus \{p\}} \left[\frac{\mathrm{i}}{2\pi}\partial\bar\partial \log \|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar\partial \log(1 - P_N(z,p)^2) + \frac{N}{\pi}\omega_h\right]^k \wedge \tau^*_N\varphi \\ &\quad = \left(\frac{\mathrm{i}}{2\pi}\right)^k \int_{\mathbb{C}^m \smallsetminus \{0\}} \left[-\frac{\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^2}\partial|u|^2 \wedge \bar\partial|u|^2 + \frac{\partial\bar\partial|u|^2}{(1-\mathrm{e}^{-|u|^2})}\right]^k \wedge \varphi \\ &\qquad + N^{-1/2+\varepsilon}\|\varphi\|_\infty \int_{\mathrm{Supp}(\varphi)} O(|u|^{-2k})(\mathrm{i}\partial\bar\partial|u|^2)^m, \end{aligned} \tag{5.9}$$

which verifies Theorem 5.1. □

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To prove Theorem 1.2, we need to integrate by parts, since if $k = m$, the integral in the last line of (5.9) does not *a priori* converge. To begin the proof, by Lemma 5.2 we have

$$(K_m^N(z \mid p), \varphi \circ \tau_N) = \varphi(0) + \int_{M_0 \smallsetminus \{p\}} \varphi(\sqrt{N}z)\left[\frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log\|\varPi_N(z,z)\|_{h^N} + \frac{\mathrm{i}}{2\pi}\partial\bar{\partial}\log(1-P_N(z,p)^2) + \frac{N}{\pi}\omega_h\right]^m. \tag{5.10}$$

Writing

$$\omega_h = \tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\rho, \qquad \rho(z) = |z|^2 + O(|z|^3), \tag{5.11}$$

we then have

$$(K_m^N(z \mid p), \varphi \circ \tau_N) = \varphi(0) + \int_{\mathbb{C}^m \smallsetminus \{0\}} \varphi \cdot \left(\frac{\mathrm{i}}{2\pi}\partial\bar{\partial} f_N\right)^m, \tag{5.12}$$

where

$$f_N(u) = \log\left\|\varPi_N\left(\frac{u}{\sqrt{N}}, \frac{u}{\sqrt{N}}\right)\right\|_{h^N} - m\log(N/\pi) + \log\left(1 - P_N\left(\frac{u}{\sqrt{N}}, 0\right)^2\right) + N\rho\left(\frac{u}{\sqrt{N}}\right).$$

By (2.7), (4.11) and (5.11),

$$f_N(u) = \log(1 - \mathrm{e}^{-|u|^2}) + |u|^2 + O(N^{-1/2+\varepsilon}). \tag{5.13}$$

Again recalling (5.6), we have

$$\partial\bar{\partial} f_N = -\frac{\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^2}\partial|u|^2 \wedge \bar{\partial}|u|^2 + \frac{\partial\bar{\partial}|u|^2}{(1-\mathrm{e}^{-|u|^2})} + O(|u|^{-2}N^{-1/2+\varepsilon}). \tag{5.14}$$

We now integrate (5.12) by parts. Let

$$\alpha_N = f_N(\partial\bar{\partial} f_N)^{m-1}. \tag{5.15}$$

Then for $\delta > 0$,

$$\int_{|u|>\delta} \varphi\partial\bar{\partial}\alpha_N = \int_{|u|>\delta} \alpha_N \wedge \partial\bar{\partial}\varphi + \tfrac{1}{2}\mathrm{i}\int_{|u|=\delta} (\varphi\,\mathrm{d}^c\alpha_N - \alpha_N \wedge \mathrm{d}^c\varphi), \tag{5.16}$$

where $\mathrm{d}^c = \mathrm{i}(\bar{\partial} - \partial)$. By (5.13)–(5.15),

$$\alpha_N = \alpha_\infty + O(|u|^{-2m+2}\log(|u| + |u|^{-1})N^{-1/2+\varepsilon}), \tag{5.17}$$

where

$$\begin{aligned}\alpha_\infty &= [\log(1-\mathrm{e}^{-|u|^2}) + |u|^2]\{\partial\bar{\partial}[\log(1-\mathrm{e}^{-|u|^2}) + |u|^2]\}^{m-1} \\ &= [\log(1-\mathrm{e}^{-|u|^2}) + |u|^2]\left[-\frac{\mathrm{e}^{-|u|^2}}{(1-\mathrm{e}^{-|u|^2})^2}\partial|u|^2 \wedge \bar{\partial}|u|^2 + \frac{\partial\bar{\partial}|u|^2}{(1-\mathrm{e}^{-|u|^2})}\right]^{m-1}.\end{aligned}$$

In particular,

$$\alpha_N = O(|u|^{-2m+2}\log(|u|+|u|^{-1})), \tag{5.18}$$

and therefore

$$\lim_{\delta\to 0}\int_{|u|=\delta}\alpha_N\wedge \mathrm{d}^c\varphi = 0.$$

Futhermore, by (5.4) and (5.14),

$$\mathrm{d}^c\alpha_N = \frac{\mathrm{d}^c|u|^2\wedge(\partial\bar\partial|u|^2)^{m-1}}{(1-\mathrm{e}^{-|u|^2})^m} + O(|u|^{-2m+1}N^{-1/2+\varepsilon}).$$

Therefore,

$$\begin{aligned}\left(\frac{\mathrm{i}}{2\pi}\right)^m \tfrac{1}{2}\mathrm{i}\int_{|u|=\delta}\varphi\,\mathrm{d}^c\alpha_N &= -\frac{\delta\cdot\delta^{2m-1}}{(1-\mathrm{e}^{-\delta^2})^m}\mathrm{Average}_{|u|=\delta}(\varphi) + O(N^{-1/2+\varepsilon})\sup_{|u|=\delta}|\varphi| \\ &\to -\varphi(0)[1+O(N^{-1/2+\varepsilon})].\end{aligned}$$

Thus,

$$\left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\varphi\partial\bar\partial\alpha_N = \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\alpha_N\partial\bar\partial\varphi - \varphi(0)[1+O(N^{-1/2+\varepsilon})]. \tag{5.19}$$

Combining (5.12), (5.17) and (5.19),

$$\begin{aligned}(K_m^N(z\mid p),\varphi\circ\tau_N) &= \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\alpha_N\partial\bar\partial\varphi + O(N^{-1/2+\varepsilon}) \\ &= \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\alpha_\infty\partial\bar\partial\varphi + O(N^{-1/2+\varepsilon}).\end{aligned}$$

Repeating the integration by parts argument using α_∞, we conclude that

$$\left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\varphi\partial\bar\partial\alpha_\infty = \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\alpha_\infty\partial\bar\partial\varphi - \varphi(0). \tag{5.20}$$

Therefore,

$$\begin{aligned}&(K_m^N(z\mid p),\varphi\circ\tau_N) \\ &\quad= \varphi(0) + \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\varphi\partial\bar\partial\alpha_\infty + O(N^{-1/2+\varepsilon}) \\ &\quad= \varphi(0) + \left(\frac{\mathrm{i}}{2\pi}\right)^m\int_{\mathbb{C}^m\smallsetminus\{0\}}\varphi(u)\{\partial\bar\partial[\log(1-\mathrm{e}^{-|u|^2})+|u|^2]\}^m + O(N^{-1/2+\varepsilon}),\end{aligned}$$

which completes the proof of Theorem 1.2. □

Remark 5.3. In fact, it follows from Demailly's Comparison Theorem for generalized Lelong numbers [**5**, Theorem 7.1], applied to the plurisubharmonic functions $f_N(u)$ and $\log|u|^2$ and closed positive current $T = 1$, that the two measures $\mathrm{i}^m\partial\bar\partial\alpha_N$ and

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$\mathrm{i}^m \partial\bar{\partial} \log |u|^2$ impart the same mass to the point 0, and therefore we can replace (5.19) by the precise identity

$$\left(\frac{\mathrm{i}}{2\pi}\right)^m \int_{\mathbb{C}^m \smallsetminus \{0\}} \varphi \partial\bar{\partial}\alpha_N = \left(\frac{\mathrm{i}}{2\pi}\right)^m \int_{\mathbb{C}^m \smallsetminus \{0\}} \alpha_N \partial\bar{\partial}\varphi - \varphi(0).$$

Equation (5.20) similarly follows from the same argument.

6. Comparison of pair correlation density and conditional density

We conclude with further discussion of the comparison between the pair correlation function and conditional Gaussian density of zeros.

6.1. Comparison in dimension 1

We now explain the sense in which the pair correlation $K^N_{1m}(p)^{-2}K^N_{2m}(z,p)$ of [**1**, **2**] may be viewed as a conditional probability density.

We begin with the case of polynomials, i.e. $M = \mathbb{CP}^1$. The possible zero sets of a random polynomial form the configuration space

$$(\mathbb{CP}^1)^{(N)} = \mathrm{Sym}^N \mathbb{CP}^1 := \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{N} / S_N$$

of N points of $\mathbb{CP}^1$, where S_N is the symmetric group on N letters. We define the joint probability current of zeros as the pushforward

$$\boldsymbol{K}^N_N(\zeta_1, \dots, \zeta_N) := \mathcal{D}_* \gamma^N_h \tag{6.1}$$

of the Gaussian measure on the space $\mathcal{P}_N$ of polynomials of degree N under the 'zero set' map $\mathcal{D} \colon \mathcal{P}_N \to (\mathbb{CP}^1)^{(N)}$ taking s_N to its zero set. An explicit formula for it in local coordinates is

$$\boldsymbol{K}^N_N(\zeta_1, \dots, \zeta_N) = \frac{1}{Z_N(h)} \frac{|\Delta(\zeta_1, \dots, \zeta_N)|^2 \, \mathrm{d}_2\zeta_1 \cdots \mathrm{d}_2\zeta_N}{(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z - \zeta_j)|^2 \mathrm{e}^{-N\varphi(z)} \, \mathrm{d}\nu(z))^{N+1}}, \tag{6.2}$$

where $Z_N(h)$ is a normalizing constant. We refer to [**16**] for further details.

As in [**4**, § 5.4, (5.39)], the pair correlation function is obtained from the joint probability distribution by integrating out all but two variables. If we fix the second variable of $K^N_{21}(z,p)$ at p and divide by the density $K^N_{11}(p)$ of zeros at p, we obtain the same density as if we fixed the first variable $\zeta_1 = p$ of the density of $\boldsymbol{K}^N_N(\zeta_1, \dots, \zeta_N)$, integrated out the last $N-2$ variables and divided by the density at p. But fixing $\zeta_1 = p$ and dividing by $K^N_{11}(p) d_2\zeta_1$ is the conditional probability distribution of zeros defined by the random variable ζ_1. Thus in dimension 1, $K^N_{11}(p)^{-2}K^N_{21}(z,p)$ is the conditional density of zeros at z given a zero at p if we condition using $\zeta_1 = p$ in the configuration space picture. This use of the term 'conditional expectation of zeros given a zero at p' can be found, for example, in [**14**].

6.2. Comparison in higher dimensions

The above configuration space approach is difficult to generalize to higher dimensions and full systems of polynomials. In particular, it is difficult even to describe the configuration of joint zeros of a system as a subset of the symmetric product. Indeed, the number of simultaneous zeros of m sections is almost surely $c_1(L)^m N^m$ so the variety C_N of configurations of simultaneous zeros is a subvariety of the symmetric product $M^{(c_1(L)^m N^m)}$. Since C_N is the image of the zero set map

$$\mathcal{D}\colon G(m, H^0(N, L^N)) \to M^{(c_1(L)^m N^m)}$$

from the Grassmannian of m-dimensional subspaces of $H^0(N, L^N)$, its dimension (given by the Riemann–Roch formula) is quite small compared with the dimension of the symmetric product:

$$\dim C_N = \frac{c_1(L)^m}{(m-1)!} N^m + O(N^{m-1}) \sim \frac{1}{m!} \dim M^{(c_1(L)^m N^m)}.$$

Under the zero set map, the probability measure on systems pushes forward to C_N, but to our knowledge there is no explicit formula for $\boldsymbol{K}_N^N$ as in (6.2).

We now provide an intuitive and informal comparison of the two scaling limits without using our explicit formulae. Let $B_\delta(p) \subset \mathbb{C}^m$ be the ball of radius δ around p, let $\boldsymbol{s} = (s_1, \dots, s_m)$ be an m-tuple of independent random sections in $H^0(M, L^N)$, and let 'Prob' denote the probability measure $(\gamma_h^N)^m$ on the space of m-tuples $\boldsymbol{s}$. We define the events,

$$U_\delta^p = \{\boldsymbol{s}\colon \boldsymbol{s} \text{ has a zero in } B_\delta(p)\}, \qquad U_\varepsilon^q = \{\boldsymbol{s}\colon \boldsymbol{s} \text{ has a zero in } B_\varepsilon(q)\}.$$

Now the probability interpretation of the pair correlation function is based on the fact that, as $\delta, \varepsilon \to 0$,

$$\int_{B_\delta(P)\times B_\varepsilon(q)} \boldsymbol{E}[Z_{\boldsymbol{s}}(z) Z_{\boldsymbol{s}}(w)] = \operatorname{Prob}(U_\delta^p \cap U_\varepsilon^q)[1 + o(1)],$$

since the probability of having two or more zeros in a small ball is small compared with the probability of having one zero.

It follows that

$$\lim_{\varepsilon,\delta\to 0} \frac{1}{\operatorname{Vol}(B_\delta(p)) \times \operatorname{Vol}(B_\varepsilon(q))} \operatorname{Prob}(U_\delta^p \cap U_\varepsilon^q) = K_{2mm}^\infty(p, q).$$

Similarly,

$$\lim_{\delta\to 0} \operatorname{Prob}(U_\delta^p) \simeq \frac{1}{\operatorname{Vol} B_\delta(p)} \int_{B_\delta(p)} \boldsymbol{E} Z_{\boldsymbol{s}}(z) = K_{1mm}^\infty(p).$$

Hence, as $\varepsilon, \delta \to 0$,

$$\operatorname{Prob}(U_\varepsilon^q \mid U_\delta^p) \simeq \frac{(\int_{B_\delta(p)\times B_\varepsilon(q)} \boldsymbol{E} Z_{\boldsymbol{s}}(z) Z_{\boldsymbol{s}}(w))}{(\int_{B_\delta(p)} \boldsymbol{E} Z_{\boldsymbol{s}}(z))} = \frac{(\int_{B_\delta(p)\times B_\varepsilon(q)} K_{2mm}^\infty(z, w))}{(\int_{B_\delta(p)} K_{1mm}^\infty(z))},$$

so that

$$\lim_{\varepsilon,\delta\to 0}\frac{1}{\operatorname{Vol}B_\varepsilon(q)}\operatorname{Prob}(U_\varepsilon^q\mid U_\delta^p)=\frac{K_{2mm}^\infty(p,q)}{K_{1mm}^\infty(p)}.$$

By comparison,

$$K_1^\infty(q\mid p)=\lim_{\varepsilon\to 0}\frac{1}{\operatorname{Vol}B_\varepsilon(q)}\operatorname{Prob}(U_\varepsilon^q\mid \boldsymbol{s}(p)=0)=\lim_{\varepsilon,\delta\to 0}\frac{1}{\operatorname{Vol}B_\varepsilon(q)}\frac{\operatorname{Prob}(U_\varepsilon^q\cap\mathcal{F}_\delta^p)}{\operatorname{Prob}(\mathcal{F}_\delta^p)},$$

where

$$\mathcal{F}_\delta^p=\Big\{(s_1,\dots,s_m)\colon\Big(\sum|s_j(p)|_{h^N}^2\Big)^{1/2}<\delta\Big\}.$$

Thus, the difference between the Gaussian conditional density and the pair correlation density corresponds to the difference between the family of systems $\mathcal{F}_\delta^p$ and the family of systems U_ε^p. This comparison of the pair correlation density and the Gaussian conditional density shows that in a probabilistic sense, the conditions '$\boldsymbol{s}(p)$ is small' and '$\boldsymbol{s}$ has a zero near p' are mutually singular.

6.3. Comparison of the conditional expectation and pair correlation in codimension 1

We take a different approach to comparing $K_1^N(z\mid p)$ and $K_{21}^N(z,p)$. The scaling asymptotics of $K_1^N(z\mid p)$ and $K_{21}^N(z,p)$ are both given by universal expressions in the normalized Szegő kernel or two-point function $P_N(z,w)$ (defined in (2.6)). This is to be expected since the two-point function is the only invariant of a Gaussian random field. Indeed, Proposition 3.10 says that

$$K_1^N(z\mid p)=\boldsymbol{E}_N(Z_s)+\frac{1}{2\pi}(\mathrm{i}\partial\bar\partial)_zY(-\log P_N(z,p)),\tag{6.3}$$

where $Y(\lambda)=\log(1-\mathrm{e}^{-2\lambda})$ (recall (4.6)).

We now review the approach to the pair correlation current $K_{21}^N(z,p)$ given in [**13**]. The *pair correlation current* of zeros Z_s is given by $\boldsymbol{E}_N(Z_s\boxtimes Z_s)$, and the *variance current* is given by

$$\mathbf{Var}_N(Z_s):=\boldsymbol{E}_N(Z_s\boxtimes Z_s)-\boldsymbol{E}_N(Z_s)\boxtimes\boldsymbol{E}_N(Z_s)\in\mathcal{D}'^{2k,2k}(M\times M).\tag{6.4}$$

Here we write

$$S\boxtimes T=\pi_1^*S\wedge\pi_2^*T\in\mathcal{D}'^{p+q}(M\times M),\quad \text{for } S\in\mathcal{D}'^p(M),\ T\in\mathcal{D}'^q(M),$$

where $\pi_1,\pi_2\colon M\times M\to M$ are the projections to the first and second factors, respectively.

In [**13**], the first two authors gave a *pluri-bipotential* for the variance current in codimension 1, i.e. a function $Q_N\in L^1(M\times M)$ such that

$$\mathbf{Var}_N(Z_s)=(\mathrm{i}\partial\bar\partial)_z(\mathrm{i}\partial\bar\partial)_wQ_N(z,w).\tag{6.5}$$

The bipotential $Q_N\colon M\times M\to[0,+\infty)$ is given by

$$Q_N(z,w)=\tilde G(P_N(z,w)),\quad \tilde G(t)=-\frac{1}{4\pi^2}\int_0^{t^2}\frac{\log(1-s)}{s}\,\mathrm{d}s.\tag{6.6}$$

The analogue to (6.3) for the pair correlation current can be written

$$K_{21}^N(z,p) = \boldsymbol{E}_N(Z_s \boxtimes Z_s) = \boldsymbol{E}_N(Z_s) \boxtimes \boldsymbol{E}_N(Z_s) + \partial\bar{\partial}_z\partial\bar{\partial}_p F(-\log P_N(z,p)), \qquad (6.7)$$

where F is the anti-derivative of the function $(1/2\pi^2)Y$:

$$F(\lambda) = \tilde{G}(\mathrm{e}^{-\lambda}) = -\frac{1}{2\pi^2}\int_\lambda^\infty \log(1-\mathrm{e}^{-2s})\,\mathrm{d}s, \quad \lambda \geqslant 0. \qquad (6.8)$$

That is, $(1/2\pi^2)Y(-\log P_N(z,p))$ is the relative potential between the conditioned and unconditioned distribution of zeros, while $F(-\log P_N(z,p))$ is the relative *bi-potential* for the pair correlation current $\boldsymbol{E}_N(Z_s \boxtimes Z_s)$.

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