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# A polynomial-exponential variation of Furstenberg's ×2×3 theorem

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*Abstract*. Furstenberg's $\times 2\times 3$ theorem asserts that the double sequence $(2^m 3^n\alpha)_{m,n\geq 1}$ is dense modulo one for every irrational $\alpha$. The same holds with 2 and 3 replaced by any two multiplicatively independent integers. Here we obtain the same result for the sequences $(\binom{m+n}{d}a^m b^n\alpha)_{m,n\geq 1}$ for any non-negative integer $d$ and irrational $\alpha$, and for the sequence $(P(m)a^m b^n)_{m,n\geq 1}$, where $P$ is any polynomial with at least one irrational coefficient. Similarly to Furstenberg's theorem, both results are obtained by considering appropriate dynamical systems.

Key words: density modulo 1, topological dynamics, Furstenberg's diophantine theorem
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## 1. *Introduction*

By a well-known theorem of Kronecker, the sequence $(n\alpha)_{n=1}^{\infty}$ is dense modulo one for all irrational $\alpha$. As shown by Weyl, this sequence is even uniformly distributed modulo one. Hardy and Littlewood [**9**] generalized the result of Kronecker, showing that the sequence $(n^r\alpha)_{n=1}^{\infty}$ is dense modulo one for every positive integer $r$ and irrational $\alpha$. (In fact, it is also uniformly distributed modulo one.) A significant generalization in this direction, for multiplicative semigroups of integers, was obtained by Furstenberg [**7**]. A semigroup $\Sigma\subseteq\mathbb{Z}$ is *lacunary* if there is an integer $s$ such that every positive $t\in S$ is a power of $s$, and it is *non-lacunary* otherwise. Furstenberg proved that if $\Sigma$ is a non-lacunary semigroup of integers, then the set $\Sigma\alpha$ is dense modulo one for every irrational $\alpha$.

We may reformulate Furstenberg's result basically as follows. Two integers $a$, $b$ with $|a|, |b|\geq 2$ are *multiplicatively dependent* if $a^k = b^l$ for some non-zero integers $k, l$, and they are *multiplicatively independent* otherwise. Furstenberg's result claims that if

$a, b$ are multiplicatively independent, then the set $\{a^m b^n \alpha \mid m, n \in \mathbb{N}\}$ is dense modulo one for every irrational $\alpha$. On the other hand, there exist 'many' irrationals $\alpha$ for which $\{a^m \alpha \mid m \in \mathbb{N}\}$ is not dense modulo one (although it is dense for a set of $\alpha$ of full Lebesgue measure).

Furstenberg's proof used topological dynamics. (We note, in passing, that an elementary proof is also available; see Boshernitzan [**5**].) The key is to view the set of numbers $\Sigma\alpha$ modulo one as the orbit of the point $\alpha$ under the semigroup corresponding to $\Sigma$ of endomorphisms of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The theorem is (almost) equivalent to the statement that if $\Sigma$ is a semigroup of endomorphisms of $\mathbb{T}$ containing two multiplicatively independent endomorphisms, then any closed, $\Sigma$-invariant set is either a finite set of rationals or $\mathbb{T}$ itself.

One may study the analogous questions on more general groups. Let $\Sigma$ be a commutative semigroup of endomorphisms of $\mathbb{T}^d$. Under what conditions on $\Sigma$ is it the case that the only infinite, closed, $\Sigma$-invariant subset of $\mathbb{T}^d$ is $\mathbb{T}^d$? Necessary and sufficient conditions for $\Sigma$ to satisfy this property were obtained in [**1**]. (See Muchnik [**12**] for an analogue in the case where $\Sigma$ is not necessarily commutative.) A generalization to semigroups of endomorphisms of a more general family of compact abelian groups was proved in [**2**].

These topological dynamics results have, in turn, number-theoretical implications. Specifically, the results of [**1**] and [**2**] led to a characterization of multiplicative semigroups in real number fields, every dilation of which (with 'very few' constructible exceptions) is dense modulo one. Similarly, Kra [**10**] and Meiri [**11**] were able to prove the density modulo one of sets of the form $\{\sum_{i=1}^{k} a_i^m b_i^n \alpha \mid m, n \in \mathbb{N}\}$ for pairs of multiplicatively independent integers $(a_i, b_i)$. Urban [**13–15**], dealing with similar sums with algebraic numbers $a_i, b_i$ instead of integers, was able to generalize simultaneously both of the above results in some cases.

Gorodnik and Kadyrov significantly strengthened the result of Urban. Denote the algebraic closure of the $p$-adic field $\mathbb{Q}_p$ by $\overline{\mathbb{Q}_p}$. For convenience, put $\mathbb{Q}_\infty = \mathbb{R}$. A semigroup $\Sigma$ of algebraic numbers is *hyperbolic* if, for every prime $p$ (including $p = \infty$), there is a field embedding $\theta : \mathbb{Q}(\Sigma) \longrightarrow \overline{\mathbb{Q}_p}$ such that $\theta(\Sigma) \nsubseteq \{|z|_p \leq 1\}$. Then $\rho(\Sigma) \nsubseteq \{|z|_p \neq 1\}$ for all field embeddings $\rho : \mathbb{Q}(\Sigma) \longrightarrow \overline{\mathbb{Q}_p}$. In [**8**] they proved the following theorem. Let $(\lambda_i, \mu_i)$, $i = 1, 2, \ldots, k$, be multiplicatively independent real algebraic numbers of modulus greater than one. Assume that, for all $\theta \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $(\theta^k(\lambda_i), \theta^k(\mu_i)) \neq (\lambda_i^k, \mu_i^k)$ for all $k \in \mathbb{N}$ and $i \neq j$ and that the semigroup generated by each of the pairs $(\lambda_i, \mu_i)$ is hyperbolic. Then $\{\sum_{i=1}^{k} \lambda_i^m \mu_i^n \xi_i \mid m, n \in \mathbb{N}\}$ is dense modulo one, where $\xi_i$, $i = 1, 2, \ldots, k$ are real numbers, at least one of which satisfies $\xi_i \notin \mathbb{Q}(\lambda_i, \mu_i)$.

Bergelson and Simmons [**4**] have recently considered sets related to $\{a^m b^n \mid m, n \in \mathbb{N}\}$, where $a, b \geq 2$ are again multiplicatively independent integers, but $m$ and $n$ are restricted to certain 'special' types of sets. We mention here some of their results. A set $A \subseteq \mathbb{N}$ is *dispersing* if $A\alpha = \{n\alpha \mid n \in A\}$ is dense modulo one for every irrational $\alpha$. A set $S = \{s_k \mid k \in \mathbb{N}\} \subseteq \mathbb{N}$ with $s_1 < s_2 < \cdots$ is *syndetic* if there exists a positive integer $d$ such that $s_{k+1} - s_k \leq d$ for every $k$, *thick* if it contains arbitrarily long intervals and *piecewise syndetic* if it is an intersection of a thick set and a syndetic set. With this terminology, they

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were able to show that if $M$ is syndetic, $N$ is piecewise syndetic and $I \subseteq \mathbb{N}$ is any infinite set, then $\{a^m b^n l \mid m \in M, n \in N, l \in I\}$ is dispersing [**4**, Corollary 1.29]. A set $S \subseteq \mathbb{N}$ is a *Bohr* set if there exist $d \in \mathbb{N}$, $v \in \mathbb{T}^d$ and an open set $\emptyset \neq U \subseteq \mathbb{T}^d$ such that $\emptyset \neq \{n \in \mathbb{N} \mid nv \in U\} \subseteq S$. They showed that, for a Bohr set $M$ and a piecewise syndetic set $N$, the set $\{a^m b^n \mid m \in M, n \in N\}$ is dispersing [**4**, Corollary 1.30]. For a set $A \subseteq \mathbb{N}$, the *finite sum set* of $A$ is $\mathrm{FS}(A) = \{\sum_{n \in F} n \mid \emptyset \neq F \subseteq A \text{ finite}\}$. It was also shown that, for a syndetic set $M$ and for $N = \mathrm{FS}(R)$, where $R$ is a set such that $\{(r/k)\log_b a \mid r \in R, (r/k) \in \mathbb{N}\}$ is dense modulo one for all $k$, the set $\{a^m b^n \mid m \in M, n \in N\}$ is dispersing [**4**, Theorem 1.31].

In this paper, we deal with yet another direction of generalization of Furstenberg's theorem. Namely, instead of exponential double sequences $(a^m b^n \alpha)_{m,n=0}^{\infty}$, we consider polynomial-exponential sequences of the form $(\binom{m+n}{d} a^m b^n \alpha)_{m,n=0}^{\infty}$ for any fixed $d$. We are able to show that if $a$, $b$ are multiplicatively independent, then this sequence is dense modulo one for every irrational $\alpha$. We also show that, under the same assumptions on $a$ and $b$, the double sequence $(P(m)a^m b^n)_{m,n=0}^{\infty}$ is dense modulo one for every polynomial $P$ with at least one irrational coefficient. These results will follow from a corresponding result in topological dynamics, which may be of independent interest.

In §2, we formulate our main results. In §3, the proofs are presented.

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## 2. *The main results*

Let $d > 0$ and $k \geq 0$ be integers. Denote by $I_d(k)$ the $d \times d$ matrix $A = (a_{ij})_{i,j=0}^{d-1}$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j = i + k, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $I_d(0)$ is the identity matrix, and $I_d(k) = 0$ if $k \geq d$.) It can be easily verified that $I_d(k) = I_d(1)^k$ and therefore $I_d(k)I_d(l) = I_d(k + l)$. If there is no possible confusion, we write just $I(k)$ instead of $I_d(k)$ and $I$ instead of $I_d$.

Let $\mathbf{s} = (s_0, s_1, \ldots, s_{d-1})$, $\mathbf{t} = (t_0, t_1, \ldots, t_{d-1}) \in \mathbb{Z}^d$ be such that $s_0$ and $t_0$ are multiplicatively independent and that either $s_1$ or $t_1$ is non-zero. The semigroup generated by $\sigma = \sum_{j=0}^{d-1} s_j I(j)$ and $\tau = \sum_{j=0}^{d-1} t_j I(j)$ will be denoted by $\Sigma_{\mathbf{s},\mathbf{t}}$.

THEOREM 2.1. *Let* $\mathbf{s} = (s_0, s_1, \ldots, s_{d-1})$, $\mathbf{t} = (t_0, t_1, \ldots, t_{d-1}) \in \mathbb{Z}^d$ *be such that* $s_0$ *and* $t_0$ *are multiplicatively independent and that either* $s_1$ *or* $t_1$ *is non-zero. Let* $A \subseteq \mathbb{T}^d$ *be an infinite, closed and* $\Sigma_{\mathbf{s},\mathbf{t}}$*-invariant set. Then* $\pi_1(A) = \mathbb{T}$*, where* $\pi_1 : \mathbb{T}^d \longrightarrow \mathbb{T}$ *is the projection on the first coordinate.*

*Remark 2.2.* The condition that at least one of $s_1$ or $t_1$ is not zero cannot be waived. Indeed, if $s_1 = t_1 = 0$, then the set $\{0\} \times \mathbb{T}^{d-1}$ is infinite, closed and $\Sigma$-invariant, but $\pi_1(A) = \{0\}$.

THEOREM 2.3. *Let* $a$, $b$ *be multiplicatively independent integers and let* $d$ *be an arbitrary fixed non-negative integer. Then the set* $\{\binom{m+n}{d} a^m b^n \alpha \mid m, n \in \mathbb{N}\}$ *is dense modulo one for every irrational number* $\alpha$.

THEOREM 2.4. *Let* $a, b$ *be multiplicatively independent integers and let* $P(x)$ *be a polynomial with at least one irrational coefficient. Then the set* $\{P(m)a^m b^n \mid m, n \in \mathbb{N}\}$ *is dense modulo one.*

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3. *Proofs*

Let $d$ be a positive integer. Let $T:\mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation and let $v \in \mathbb{R}^d$. We write $T^n(v) \xrightarrow[n\to\infty]{} \infty$ if $\|T^n(v)\| \xrightarrow[n\to\infty]{} \infty$ (for any norm $\|\cdot\|$ on $\mathbb{R}^d$).

LEMMA 3.1. *Let $T:\mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation whose eigenvalues are all of absolute value greater than one. Then $T^n(v) \xrightarrow[n\to\infty]{} \infty$ for every non-zero $v \in \mathbb{R}^d$.*

*Proof.* Use the Jordan normal form. □

LEMMA 3.2. *Let $T:\mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation. Assume that all eigenvalues of $T$ are of absolute value greater than one. Let $A \subseteq \mathbb{R}^d$ be a closed $T$-invariant set containing zero as an accumulation point. Then there exists a non-zero point $v \in A$ such that $T^{-n}(v) \in A$ for all $n \in \mathbb{N}$.*

*Proof.* Let $(v_k)_{k=1}^\infty$ be a sequence of non-zero elements of $A$ converging to zero. We may assume that $\|v_k\| < 1/\|T\|^k$ for every $k$, where $\|\cdot\|$ denotes both some norm on $\mathbb{R}^d$ and the induced matrix norm. By the previous lemma, for every $k$ there exists a positive integer $n$ such that $T^n(v_k) \notin B(0,1)$, where $B(x,r)$ is the open ball of radius $r$ centered at $x$. Let $n_k$ be the minimal such integer. Clearly, $n_k > k$ for each $k$. Note that the sequence $(T^{n_k}(v_k))_{k=1}^\infty$ is bounded. Indeed, since $\|T^{n_k-1}(v_k)\| < 1$, we have $\|T^{n_k}(v_k)\| = \|T(T^{n_k-1}(v_k))\| \le \|T\|\|T^{n_k-1}(v_k)\| < \|T\|$. As the set $K = \overline{B(0,\|T\|)} \setminus B(0,1)$ is compact, there is a subsequence of $(T^{n_k}(v_k))_{k=1}^\infty$ that converges to some vector $v \in A \cap K$. Replacing $(v_k)_{k=1}^\infty$ by the corresponding subsequence, we may assume that $T^{n_k}(v_k) \xrightarrow[k\to\infty]{} v$. For an arbitrary fixed $n$, we have $n_k \ge n$ for $k \ge n$. Since $T^{n_k-n}(v_k) \xrightarrow[k\to\infty]{} T^{-n}(v)$, we have $T^{-n}(v) \in A$ for each $n$. □

LEMMA 3.3. *Let $\sigma = \sum_{j=0}^{d-1} s_j I(j)$, where $s_0 \ne 0$. Then, for every integer $m$, we have $\sigma^m = s_0^m \sum_{j=0}^{d-1} P_j(m) I(j)$, where each $P_j$ is a polynomial of degree at most $j$ and $P_0 = 1$. Moreover, denoting by $j_0$ the least $j$ (if any) for which $s_j \ne 0$, we have:*

(a) $P_{j_0}(m) = (s_{j_0}/s_0)m$; *and*

(b) *if $j_0 = 1$, then the leading monomial of each $P_j(m)$ is $(1/j!)(s_1/s_0)^j m^j$.*

*Remark 3.4.* The $s_i$ in the lemma are any real numbers, and we consider $\sigma$ as a real matrix and not as an endomorphism of $\mathbb{T}^d$.

*Proof of Lemma 3.3.* First assume that $m > 0$. Denote by $\mathbb{Z}_{\ge 0}$ the set of non-negative integers and by $\|\cdot\|$ the 1-norm. For $\mathbf{m} = (m_0, m_1, \ldots, m_{d-1}) \in \mathbb{Z}_{\ge 0}^d$, with $\|\mathbf{m}\| = m$, let $\binom{m}{\mathbf{m}} = \binom{m}{m_0, m_1, \ldots, m_{d-1}}$ and $\mathbf{s}^{\mathbf{m}} = s_0^{m_0} s_1^{m_1} \cdots s_{d-1}^{m_{d-1}}$. By the multinomial theorem, and since $I(j_1)I(j_2) = I(j_1 + j_2)$ for $j_1, j_2 \ge 0$,

$$\sigma^m = \sum_{\mathbf{m}:\|\mathbf{m}\|=m} \binom{m}{\mathbf{m}} \mathbf{s}^{\mathbf{m}} I\bigg(\sum_{j=0}^{d-1} j m_j\bigg).$$

The entries of $\sigma^m$ along the diagonal located $k$ places to the right of the main diagonal (namely, the diagonal consisting of the entries $(1, k+1), (2, k+2), \ldots, (d-k, d)$) are

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therefore

$$p_j = \sum_{\substack{\mathbf{m}:\|\mathbf{m}\|=m \\ m_1+2m_2+\cdots+(d-1)m_{d-1}=j}} \binom{m}{\mathbf{m}} \mathbf{s}^{\mathbf{m}}. \tag{1}$$

A necessary condition for the contribution of the term indexed by **m** on the right-hand side to be non-zero is that $m_i \leq j/i$ for each $1 \leq i \leq d-1$. Hence the actual number of terms in the sum is bounded. Hence the right-hand side of (1) may be written in the form $s_0^m P_j(m)$ for an appropriate polynomial $P_j$.

Let $\mathbf{m} = (m_0, m_1, \ldots, m_{d-1}) \in \mathbb{Z}^d_{\geq 0}$ be such that $\|\mathbf{m}\| = m$ and

$$m_1 + 2m_2 + \cdots + (d-1)m_{d-1} = j.$$

Then $m_1 + m_2 + \cdots + m_{d-1} \leq j$, and therefore $m_0 \geq m - j$, so that $\deg P_j \leq j$. The contribution of the term indexed by **m** on the right-hand side of (1) to the sum is

$$\binom{m}{m_0, m_1, \ldots, m_{d-1}} s_0^{m_0} s_{j_0}^{m_1} \cdots s_{d-1}^{m_{d-1}}.$$

The only term on the right-hand side of (1), which contributes to the diagonal located $j_0$ places to the right of the main diagonal, comes from $\mathbf{m} = (m-1, 0, \ldots, 0, 1, 0, \ldots, 0)$. Its contribution is $(s_{j_0}/s_0)m$, which proves (a).

If $j_0 = 1$, then $\mathbf{m} = (m-j, j, 0, \ldots, 0)$ contributes the term $\binom{m}{j}(s_1/s_0)^i$ of maximal degree to $P_j$. This proves (b).

Now assume that $m < 0$. Using the Cayley–Hamilton theorem, we get

$$\sigma^{-1} = \frac{1}{s_0} \sum_{i=0}^{d-1} (-1)^{d-i-1} \binom{d}{i} \frac{1}{s_0^{d-i-1}} \sigma^{d-i-1}.$$

By the previous case, the entries of each $\sigma^{d-i-1}$ on the diagonal located $j_0$ places to the right of the main diagonal are $q_{j_0} = ks_0^{k-1} s_{j_0}$. Combining these two facts, we get that the $(1, d + j_0)$ entry of $\sigma^{-1}$ is $(s_{j_0}/s_0^2) \sum_{i=0}^{d-2} (-1)^{d-i-1}(d-i-1)\binom{d}{d-i} = -s_{j_0}/s_0^2$. Now employ what we have proved above to the matrix $\sigma^{-1}$ and the exponent $-m$.

The case $m = 0$ is trivial. □

Let $\Sigma$ be a commutative semigroup of endomorphisms of $\mathbb{T}^d$. For a positive integer $l$, denote $\Sigma^l = \{\sigma^l \mid \sigma \in \Sigma\}$.

LEMMA 3.5. *Let $\Sigma$ be a finitely generated commutative semigroup of endomorphisms of $\mathbb{T}^d$. Let $B \subseteq \mathbb{T}^d$ be a $\Sigma$-invariant set containing a rational vector $v_1$. Then there exist a positive integer $l$ and a rational vector $u \in B$ such that $\sigma(u) = u$ for every $\sigma \in \Sigma^l$.*

*Proof.* Let $S = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a set of generators of $\Sigma$. Since $v_1$ is a rational vector, the orbit of $v_1$ under $\Sigma$ is finite. Hence, there exist integers $i_1 > j_1 \geq 0$ such that $\sigma_1^{i_1}(v_1) = \sigma_1^{j_1}(v_1)$. Denote $v_2 = \sigma_1^{j_1}(v_1)$. Now, there exist integers $i_2 > j_2 \geq 0$ such that $\sigma_2^{i_2}(v_2) = \sigma_2^{j_2}(v_2)$. Denote $v_3 = \sigma_2^{j_2}(v_2)$. Continuing this process for all members of $S$, we finally arrive at a vector $v_{r+1} = \sigma_r^{j_r}(v_r)$. Let $l = \operatorname{lcm}\{i_k - j_k \mid k = 1, 2, \ldots, r\}$. We have $\sigma^l(v_{r+1}) = v_{r+1}$ for all $\sigma \in \Sigma$. □

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LEMMA 3.6. *Let $\Sigma$ be a finitely generated commutative semigroup of endomorphisms of $\mathbb{T}^d$. Suppose that for every positive integer $l$ and every infinite closed and $\Sigma^l$-invariant subset $A$ of $\mathbb{T}^d$ containing zero as an accumulation point, we have $\pi_1(A) = \mathbb{T}$. Then every infinite closed and $\Sigma$-invariant subset $B$ of $\mathbb{T}^d$ containing a rational vector as an accumulation point satisfies $\pi_1(B) = \mathbb{T}$.*

*Proof.* Pick up a rational vector $v \in B'$. By the preceding lemma, there exist a rational vector $u = (a_0/k, a_1/k, \ldots, a_{d-1}/k)^T$ and a positive integer $l$ such that $\sigma^l(u) = u$ for every $\sigma \in \Sigma$. Define $A = B - u$. Obviously, $A$ is an infinite, closed and $\Sigma^l$-invariant set containing zero as an accumulation point, and therefore $\pi_1(A) = \mathbb{T}$. Now $\pi_1(A) = \pi_1(B) - a_0/k$, so that $\pi_1(B) = \mathbb{T}$. □

*Proof of Theorem 2.1.* Assume, say, that $s_1 \neq 0$. Without loss of generality, we may assume that $s_0$ and $t_0$ are positive. Indeed, if, say, $s_0 < 0$, then by replacing $\sigma$ by $\sigma^2$, we revert to the case where the entries on the main diagonal are positive. Moreover, note that then the entries on the diagonal $\{(i, i+1) : 0 \leq i \leq d-2\}$ change from $s_1$ to $2s_0s_1$. In particular, if $s_1 \neq 0$, then $\sigma^2$ enjoys the analogous property. Finally, the validity of the theorem for the subsemigroup $\Sigma^2_{\mathbf{s},\mathbf{t}}$, generated by $\sigma^2$ and $\tau^2$, certainly implies its validity for $\Sigma_{\mathbf{s},\mathbf{t}}$ itself.

Take an arbitrary fixed positive integer $l$. Let $A$ be an infinite, closed and $\Sigma^l_{\mathbf{s},\mathbf{t}}$-invariant set. We will first show that if, in addition, $A$ contains zero as an accumulation point, then $\pi_1(A) = \mathbb{T}$.

Since $\Sigma^l_{\mathbf{s},\mathbf{t}}$ satisfies the same properties as $\Sigma_{\mathbf{s},\mathbf{t}}$, it suffices to prove that $\pi_1(A) = \mathbb{T}$ for $l = 1$. Thus, let $A$ be an infinite, closed and $\Sigma_{\mathbf{s},\mathbf{t}}$-invariant subset of $\mathbb{T}^d$. By Lemma 3.2, there exists a non-zero point $v = (x_0, x_1, \ldots, x_{d-1}) \in \tilde{A}$ such that $\tau^{-n}(v) \in \tilde{A}$ for every $n \geq 0$, where $\tilde{A}$ is the lifting of $A$ to $\mathbb{R}^d$. Without loss of generality, assume that $x_{d-1} \neq 0$. (If $j \leq d-2$ is the largest index for which $x_j \neq 0$, then replace $\mathbb{T}^d$ by its subgroup $\mathbb{T}^{j+1} \times \{0\}^{d-j-1}$ and change $A$ and $\Sigma_{\mathbf{s},\mathbf{t}}$ accordingly.) By Lemma 3.3, for non-negative integer $m$,

$$\sigma^m = s_0^m \begin{pmatrix} 1 & P_1(m) & P_2(m) & \cdots & P_{d-2}(m) & P_{d-1}(m) \\ 0 & 1 & P_1(m) & \cdots & P_{d-3}(m) & P_{d-2}(m) \\ 0 & 0 & 1 & \cdots & P_{d-4}(m) & P_{d-3}(m) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & P_1(m) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where each $P_k$ may be written in the form

$$P_k(m) = \frac{1}{k!}\left(\frac{s_1}{s_0}\right)^k m^k + \tilde{p}_{k-1}(m)$$

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for an appropriate polynomial $\tilde{p}_{k-1}$ with $\deg \tilde{p}_{k-1} < k$. (Here $\tilde{p}_{-1} = 0$.) The same lemma also yields

$$\tau^{-n} = t_0^{-n} \begin{pmatrix} 1 & Q_1(n) & Q_2(n) & \cdots & Q_{d-2}(n) & Q_{d-1}(n) \\ 0 & 1 & Q_1(n) & \cdots & Q_{d-3}(n) & Q_{d-2}(n) \\ 0 & 0 & 1 & \cdots & Q_{d-4}(n) & Q_{d-3}(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & Q_1(n) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where each $Q_k$ may be written in the form

$$Q_k(n) = \frac{1}{k!}\left(-\frac{t_1}{t_0}\right)^k n^k + \tilde{q}_{k-1}(n)$$

for an appropriate polynomial $\tilde{q}_{k-1}$ with $\deg \tilde{q}_{k-1} < k$. (Again, $\tilde{q}_{-1} = 0$.) Hence

$$\sigma^m \tau^{-n} = s_0^m t_0^{-n} \begin{pmatrix} 1 & R_1(m,n) & R_2(m,n) & \cdots & R_{d-2}(m,n) & R_{d-1}(m,n) \\ 0 & 1 & R_1(m,n) & \cdots & R_{d-3}(m,n) & R_{d-2}(m,n) \\ 0 & 0 & 1 & \cdots & R_{d-4}(m,n) & R_{d-3}(m,n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & R_1(m,n) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where each $R_k(m, n)$ may be written in the form

$$\begin{aligned} R_k(m,n) &= \sum_{j=0}^{k} P_j(m) Q_{k-j}(n) \\ &= \sum_{j=0}^{k} \frac{1}{j!(k-j)!}\left(\frac{s_1}{s_0}\right)^j \left(-\frac{t_1}{t_0}\right)^{k-j} m^j n^{k-j} + \tilde{r}_{k-1}(m,n) \\ &= \frac{1}{k!}\left(m\frac{s_1}{s_0} - n\frac{t_1}{t_0}\right)^k + \tilde{r}_{k-1}(m,n) \end{aligned}$$

for an appropriate polynomial $\tilde{r}_{k-1}$ with $\deg \tilde{r}_{k-1} < k$.

Let

$$n(m) = \lfloor m \log_{t_0} s_0 + \log_{t_0} m^{d-1} \rfloor. \tag{2}$$

We have

$$s_0^m t_0^{-n(m)} = \frac{t_0^{\{m \log_{t_0} s_0 + \log_{t_0} m^{d-1}\}}}{m^{d-1}},$$

where $\{x\}$ denotes the fractional part of a real number $x$. Put

$$\alpha_m = \{m \log_{t_0} s_0 + \log_{t_0} m^{d-1}\}.$$

Then

$$\pi_1(\sigma^m \tau^{-n(m)}(v)) = t_0^{\alpha_m} \frac{1}{m^{d-1}} (U_0(m)x_0 + U_1(m)x_1 + \cdots + U_{d-1}(m)x_{d-1}), \tag{3}$$

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where the functions $U_k(m)$ are obtained from restrictions of the $R_k$ above, that is,

$$U_k(m) = R_k\ (m, n(m))\ .$$

We have

$$\begin{aligned}\lim_{m\to\infty} \frac{U_l(m)}{m^k} &\\ = \lim_{m\to\infty} &\frac{(1/l!)(m(s_1/s_0) - \lfloor m \log_{t_0} s_0 + \log_{t_0} m^{d-1} \rfloor \cdot (t_1/t_0))^l + \tilde{r}_{l-1}(m, n(m))}{m^k} \\ = &\begin{cases} 0 & \text{if } 0 \le l \le k-1, \\ \dfrac{1}{k!}\left(\dfrac{s_1}{s_0} - \dfrac{t_1}{t_0} \log_{t_0} s_0\right)^k & \text{if } l = k. \end{cases}\end{aligned}$$

Since $s_0$ and $t_0$ are multiplicatively independent, $(t_1/t_0) \log_{t_0} s_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$, and since $s_1/s_0 \in \mathbb{Q} \setminus \{0\}$, $(1/k!)(s_1/s_0 - t_1/t_0 \log_{t_0} s_0)^k \ne 0$. Denote

$$\gamma_m = \frac{1}{m^{d-1}}(U_0(m)x_0 + U_1(m)x_1 + \cdots + U_{d-1}(m)x_{d-1}).$$

Since $x_{d-1} \ne 0$, we have

$$\gamma_m \xrightarrow[m\to\infty]{} \gamma = \frac{1}{(d-1)!}\left(\frac{s_1}{s_0} - \frac{t_1}{t_0} \log_{t_0} s_0\right)^{d-1} x_{d-1} \ne 0.$$

Without loss of generality, we may assume that $\gamma > 0$. By (3), and since $\lim_{m\to\infty}(t_0^{\alpha_m}\gamma_m - t_0^{\alpha_m}\gamma) = 0$, we may look at the limit points of the sequence $(t_0^{\alpha_m}\gamma)_{m=1}^\infty$ instead of those of the sequence $(t_0^{\alpha_m}\gamma_m)_{m=1}^\infty$.

By [**6**, Theorem 1.10], the sequence $(\alpha_m)_{m=1}^\infty$ is well distributed modulo one and, in particular, is dense modulo one. Hence $(t_0^{\alpha_m}\gamma)_{m=1}^\infty$ is dense in $[\gamma, t_0\gamma)$. Changing (2) by putting $n(m) = \lfloor m \log_{t_0} s_0 + \log_{t_0} m^{d-1} \rfloor - r$ for some sufficiently large non-negative constant integer $r$, we see, similarly to the above, that $(t_0^{\alpha_m}\gamma)_{m=1}^\infty$ is dense modulo one. Hence the sequence $(t_0^{\alpha_m}\gamma_m)_{m=1}^\infty$ is dense modulo one and, consequently, so is the sequence $(\pi_1(\sigma^m\tau^{-n(m)}(v)))_{m=1}^\infty$.

Now let $A$ be a general infinite closed and $\Sigma$-invariant subset of $\mathbb{T}^d$. Since $\Sigma$ is hyperbolic and multi-parameter, by [**3**, Theorem 2.1], there is a rational accumulation point $v \in A$. By the previous lemma, $\pi_1(A) = \mathbb{T}$, as needed. □

*Proof of Theorem 2.3.* Consider the semigroup $\Sigma$ generated by the endomorphisms of $\mathbb{T}^{d+1}$ given by

$$\sigma = a(I + I(1)), \qquad \tau = b(I + I(1)).$$

By Lemma 3.3, we have $\sigma^m = a^m \sum_{j=0}^d \binom{m}{j} I(j)$ and $\tau^n = b^n \sum_{j=0}^d \binom{n}{j} I(j)$. Hence the $(1, d+1)$-entry of $\sigma^m\tau^n$ is $a^m b^n \sum_{j=0}^d \binom{m}{j}\binom{n}{d-j} = a^m b^n \binom{m+n}{d}$. Let $v = (0, \ldots, 0, \alpha)^T \in \mathbb{T}^{d+1}$, where $\alpha$ is any irrational. Put $A = \overline{\Sigma v}$. It is clear that $A$ is infinite, closed and $\Sigma$-invariant. By Theorem 2.1, we have $\pi_1(A) = \mathbb{T}$. As

$$\pi_1(A) = \overline{\left\{\binom{m+n}{d} a^m b^n \alpha \mid m, n \in \mathbb{N}\right\}},$$

this proves the theorem. □

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*Proof of Theorem 2.4.* Let $d = \deg P$. Consider the semigroup $\Sigma$ generated by the endomorphisms

$$\sigma = a(I + I(1)), \quad \tau = bI$$

of $\mathbb{T}^{d+1}$. By Lemma 3.3, we have $\sigma^m = a^m \sum_{j=0}^{d} \binom{m}{j} I(j)$. Hence the first row of $\sigma^m \tau^n$ is

$$a^m b^n \left( \binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{d} \right).$$

Since the polynomials $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{k}$ form a basis of the linear space $\mathbb{R}_{\leq d}[x]$ consisting of all polynomials of degree up to $d$ over $\mathbb{R}$, there exist scalars $\lambda_0, \lambda_1, \ldots, \lambda_d$ such that $P(x) = \sum_{j=0}^{d} \lambda_j \binom{x}{j}$. Putting $v = (\lambda_0, \lambda_1, \ldots, \lambda_d)^T \in \mathbb{T}^{d+1}$ and $A = \overline{\Sigma v}$, we therefore get

$$\pi_1(A) = \overline{\{P(m) a^m b^n \mid m, n \in \mathbb{N}\}}.$$

By Theorem 2.1, we have $\pi_1(A) = \mathbb{T}$, which proves the theorem. □

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