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Embedding Spanning Bipartite Graphs of Small Bandwidth

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Böttcher, Schacht and Taraz (*Math. Ann.*, 2009) gave a condition on the minimum degree of a graph G on n vertices that ensures G contains every r-chromatic graph H on n vertices of bounded degree and of bandwidth $o(n)$, thereby proving a conjecture of Bollobás and Komlós (*Combin. Probab. Comput.*, 1999). We strengthen this result in the case when H is bipartite. Indeed, we give an essentially best-possible condition on the degree sequence of a graph G on n vertices that forces G to contain every bipartite graph H on n vertices of bounded degree and of bandwidth $o(n)$. This also implies an Ore-type result. In fact, we prove a much stronger result where the condition on G is relaxed to a certain robust expansion property. Our result also confirms the bipartite case of a conjecture of Balogh, Kostochka and Treglown concerning the degree sequence of a graph which forces a perfect H-packing.

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1. Introduction

A central problem in graph theory is to establish conditions on a graph G which ensure that G contains another graph H as a spanning subgraph. Perhaps the best-known example of such a problem is when H is a Hamilton cycle. Dirac's theorem [12] states that any graph G on n vertices with minimum degree $\delta(G) \geqslant n/2$ contains a Hamilton cycle. The Pósa–Seymour conjecture (see [13] and [28]) states that any graph G on n vertices with $\delta(G) \geqslant rn/(r+1)$ contains the rth power of a Hamilton cycle. (The rth power of a Hamilton cycle C is obtained from C by adding an edge between every pair of vertices of distance at most r on C.) Komlós, Sárközy and Szemerédi [20] proved this conjecture for sufficiently large graphs.

There has also been significant attention on establishing minimum degree conditions which ensure a graph contains a *perfect H-packing*: given a graph H, a perfect H-packing in a graph G is a collection of vertex-disjoint copies of H which covers all the vertices in G. (Perfect H-packings are also referred to as *H-factors* or *perfect H-tilings*.) A seminal result in the area is the Hajnal–Szemerédi theorem [14], which states that every graph G whose order n

is divisible by r contains a perfect K_r-packing provided that $\delta(G) \geqslant (r-1)n/r$. (Corrádi and Hajnal [10] had earlier proved this result in the case when $r=3$.) Notice that in the case when $r+1$ divides $|G|$, a necessary condition for a graph G to contain the rth power of a Hamilton cycle is that G contains a perfect K_{r+1}-packing. Thus, the Pósa–Seymour conjecture implies the Hajnal–Szemerédi theorem. Kühn and Osthus [23, 24] characterized, up to an additive constant, the minimum degree which ensures a graph G contains a perfect H-packing for an arbitrary graph H. (This improved previous bounds of Alon and Yuster [1] and Komlós, Sárközy and Szemerédi [21].)

It is desirable to find conditions that ensure a graph G contains H as a spanning subgraph where H is *any* graph from a large collection of graphs. That is, rather than finding individual results for specific graphs H, one seeks more general, wide-reaching results. A graph H on n vertices is said to have *bandwidth at most b* if there exists a labelling of the vertices of H by the numbers $1,\dots,n$ such that for every edge $ij \in E(H)$ we have $|i-j| \leqslant b$. Clearly every graph H has bandwidth at most $|H|-1$. Thus, a perfect H-packing has bandwidth at most $|H|-1$. Further, a Hamilton cycle has bandwidth 2, and in general the rth power of a Hamilton cycle has bandwidth at most $2r$. Böttcher, Preussmann, Taraz and Würfl [5] proved that every planar graph H on n vertices with bounded maximum degree has bandwidth at most $O(n/\log n)$.

The following result of Böttcher, Schacht and Taraz [7] gives a condition on the minimum degree of a graph G on n vertices that ensures G contains *every* r-chromatic graph on n vertices of bounded degree and of bandwidth $o(n)$, thereby proving a conjecture of Bollobás and Komlós [18].

Theorem 1.1 (Böttcher, Schacht and Taraz [7]). *Given any $r, \Delta \in \mathbb{N}$ and any $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is an r-chromatic graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. If G is a graph on n vertices with*

$$\delta(G) \geqslant \left(\frac{r-1}{r} + \gamma\right) n,$$

then G contains a copy of H.

Prior to the proof of Theorem 1.1, Csaba [11] and Hàn [15] proved the case when H is bipartite and Böttcher, Schacht and Taraz [6] proved the case when $\chi(H) = 3$. In this paper our focus is on strengthening Theorem 1.1 in the case when H is bipartite.

1.1. Degree sequence conditions

Dirac's theorem and the Hajnal–Szemerédi theorem are best possible in the sense that the minimum degree conditions in both these results cannot be lowered. However, this does not mean that one cannot strengthen these results. Indeed, Chvátal [9] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that the degrees of the graph G are $d_1 \leqslant \cdots \leqslant d_n$. If $n \geqslant 3$ and $d_i \geqslant i+1$ or $d_{n-i} \geqslant n-i$ for all $i < n/2$ then G is Hamiltonian. Notice that Chvátal's theorem is much stronger than Dirac's theorem since it allows for almost half of the vertices of G to have degree less than $n/2$.

Balogh, Kostochka and Treglown [2] proposed the following two conjectures concerning the degree sequence of a graph which forces a perfect H-packing.

Conjecture 1.2 (Balogh, Kostochka and Treglown [2]). *Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices with degree sequence $d_1 \leqslant \cdots \leqslant d_n$ such that:*

- $d_i \geqslant (r-2)n/r + i$ *for all* $i < n/r$,
- $d_{n/r+1} \geqslant (r-1)n/r$.

Then G contains a perfect K_r-packing.

Note that Conjecture 1.2, if true, is much stronger than the Hajnal–Szemerédi theorem since the degree condition allows for n/r vertices to have degree less than $(r-1)n/r$.

Conjecture 1.3 (Balogh, Kostochka and Treglown [2]). *Suppose $\gamma > 0$ and H is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If G is a graph whose order $n \geqslant n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leqslant \cdots \leqslant d_n$ satisfies*

- $d_i \geqslant (r-2)n/r + i + \gamma n$ *for all* $i < n/r$,

then G contains a perfect H-packing.

In this paper we prove the following result, which gives a condition on the degree sequence of a graph G on n vertices that ensures G contains every bipartite graph on n vertices of bounded degree and of bandwidth $o(n)$.

Theorem 1.4. *Given any $\Delta \in \mathbb{N}$ and any $\gamma > 0$, there exists constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a bipartite graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. Let G be a graph on n vertices with degree sequence $d_1 \leqslant \cdots \leqslant d_n$. If*

- $d_i \geqslant i + \gamma n$ *or* $d_{n-i-\gamma n} \geqslant n - i$ *for all* $i < n/2$,

then G contains a copy of H.

The degree sequence condition in Theorem 1.4 is similar to that in Chvátal's theorem, except that now we have two error terms in the condition. Notice that Theorem 1.4 is much stronger than the bipartite case of Theorem 1.1. Furthermore, in the case when $r = 2$, Conjecture 1.3 is implied by Theorem 1.4.

Theorem 1.4 is, up to the error terms, best-possible for many graphs H. Indeed, suppose that H is a bipartite graph on an even number n of vertices that contains a perfect matching. Suppose that $m \in \mathbb{N}$ such that $m < n/2$. Let G be a graph on n vertices with vertex classes V_1, V_2, V_3 of sizes m, $m-1$ and $n-2m+1$ respectively, and whose edge set contains all possible edges except for those in V_1 and between V_1 and V_3. Let $d_1 \leqslant \cdots \leqslant d_n$ denote the degree sequence of G. Then

- $d_i \geqslant i - 1$ and $d_{n-i+2} \geqslant n - i$ for all $i < n/2$,

but since $|V_1| > |V_2|$, G does not contain a perfect matching and therefore H.

1.2. Ore-type degree conditions

Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices of a graph. The name comes from Ore's theorem [27], which states that a graph G of order $n \geqslant 3$ contains a

Hamilton cycle if $d(x) + d(y) \geqslant n$ for all non-adjacent $x \neq y \in V(G)$. Recently, Châu [8] proved an Ore-type analogue of the Pósa–Seymour conjecture in the case of the square of a Hamilton cycle (*i.e.*, when $r = 2$).

The following Ore-type result of Kierstead and Kostochka [17] implies the Hajnal–Szemerédi theorem. Let $n, r \in \mathbb{N}$ such that r divides n. Suppose that G is a graph on n vertices such that for all non-adjacent $x \neq y \in V(G)$, $d(x) + d(y) \geqslant 2(r-1)n/r - 1$. Then G contains a perfect K_r-packing. Kühn, Osthus and Treglown [25] characterized, asymptotically, the Ore-type degree condition which ensures a graph G contains a perfect H-packing for an arbitrary graph H.

It is natural to seek an Ore-type analogue of Theorem 1.1. The following result provides such an analogue in the case when H is bipartite.

Theorem 1.5. *Given any $\Delta \in \mathbb{N}$ and any $\gamma > 0$, there exists constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a bipartite graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. Let G be a graph on n vertices such that, for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geqslant (1+\gamma)n.$$

Then G contains a copy of H.

In Section 2.2 we show that Theorem 1.5 is a direct consequence of Theorem 1.4. Note that Theorem 1.5 is best-possible up to the error term for bipartite graphs H on n vertices which do not contain an isolated vertex. Indeed, let G consist of a copy of K_{n-1} and an isolated vertex. Then G does not contain H but $d(x) + d(y) = n - 2$ for all non-adjacent $x \neq y \in V(G)$.

In light of Theorem 1.5, we propose the following Ore-type analogue of Theorem 1.1.

Conjecture 1.6. *Given any $r, \Delta \in \mathbb{N}$ and any $\gamma > 0$, there exists constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is an r-chromatic graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. Let G be a graph on n vertices such that, for all non-adjacent $x \neq y \in V(G)$,*

$$d(x) + d(y) \geqslant 2\left(\frac{r-1}{r} + \gamma\right)n.$$

Then G contains a copy of H.

If true, Conjecture 1.6 is stronger than Theorem 1.1. Böttcher and Müller [3, 4] have proved the conjecture in the case when $r = 3$.

1.3. Robustly expanding graphs

An important and well-studied notion in graph theory is *graph expansion*. We will consider the following stronger notion of 'robust expansion'. Roughly speaking, a graph G on n vertices is a robust expander if, for every 'reasonably sized' set $S \subseteq V(G)$, G contains at least $|S| + o(n)$ vertices that are adjacent to 'many' vertices in S. More formally, let $0 < \nu \leqslant \tau < 1$. Suppose that G is a graph on n vertices and $S \subseteq V(G)$. Then the *ν-robust neighbourhood $RN_{\nu,G}(S)$ of S* is the set of vertices $v \in V(G)$ such that $|N(v) \cap S| \geqslant \nu n$. We say that G is a *robust (ν,τ)-expander* if every $S \subseteq V(G)$ with $\tau n \leqslant |S| \leqslant (1-\tau)n$ satisfies $|RN_{\nu,G}(S)| \geqslant |S| + \nu n$.

The notion of robustly expanding (di)graphs was first introduced by Kühn, Osthus and Treglown in [26]. The following result is an immediate consequence of Theorem 16 from [26].

Theorem 1.7 (Kühn, Osthus and Treglown [26]). *Given positive constants $\nu \leqslant \tau \ll \eta < 1$, there exists a positive integer n_0 such that the following holds. Let G be a graph on $n \geqslant n_0$ vertices with $\delta(G) \geqslant \eta n$ which is a robust (ν,τ)-expander. Then G contains a Hamilton cycle.*

(Throughout the paper, we write $0 < \alpha \ll \beta \ll \gamma$ to mean that we can choose the constants α, β, γ from right to left. More precisely, there are increasing functions f and g such that, given γ, whenever we choose some $\beta \leqslant f(\gamma)$ and $\alpha \leqslant g(\beta)$, all calculations needed in our proof are valid. Hierarchies of other lengths are defined in the obvious way.)

We will use Theorem 1.7 to prove the following result concerning embedding bipartite graphs of small bandwidth.

Theorem 1.8. *Given $\Delta \in \mathbb{N}$ and positive constants $\nu \leqslant \tau \ll \eta < 1$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a bipartite graph on $n \geqslant n_0$ vertices with $\Delta(H) \leqslant \Delta$ and bandwidth at most βn. Let G be a graph on n vertices with $\delta(G) \geqslant \eta n$ which is a robust (ν,τ)-expander. Then G contains a copy of H.*

In Section 2.2 we show that Theorem 1.8 implies Theorem 1.4 and that Theorem 1.4 implies Theorem 1.5. Thus, we only prove Theorem 1.8 directly.

Note that Theorem 1.8 is very general in the sense that it allows for the graph G to have small minimum degree (although $\delta(G)$ must be linear). Furthermore, there are examples of graphs G that satisfy the hypothesis of Theorem 1.8 and whose *maximum degree* is also small. Indeed, let $0 < \nu \ll \tau \ll \eta < 1$ such that $1/\eta$ is an odd integer. Further, choose $n \in \mathbb{N}$ such that $\eta n \in \mathbb{N}$. Define G to be the blow-up of a cycle on $1/\eta$ vertices, such that each vertex class of G contains ηn vertices. Thus, $|G| = n$ and $\delta(G) = \Delta(G) = 2\eta n$. It is easy to check that G is a robust (ν,τ)-expander. Given constants $0 < \nu \ll \tau \ll p < 1$, with high probability $G(n,p)$ is a robust (ν,τ)-expander with minimum degree at least $pn/2$ and maximum degree at most $2pn$.

Theorem 1.8 therefore implies that, with high probability, $G(n,p)$ contains all bipartite graphs H on n vertices of bounded degree and bandwidth $o(n)$. A result of Huang, Lee and Sudakov [16] actually implies that, with high probability, any spanning subgraph G' of $G(n,p)$ with minimum degree $\delta(G') \geqslant (1/2 + o(1))np$ contains all such H.

2. Notation and preliminaries

2.1. Notation

Throughout this paper we omit floors and ceilings whenever this does not affect the argument. We write $|G|$ for the order of a graph G, $\delta(G)$ and $\Delta(G)$ for its minimum and maximum degrees respectively and $\chi(G)$ for its chromatic number. The degree of a vertex $x \in V(G)$ is denoted by $d(x)$ and its neighbourhood by $N(x)$. Given $S \subseteq V(G)$, we define $N(S) := \bigcup_{v \in S} N(v)$.

Given disjoint $A, B \subseteq V(G)$, the number of edges with one endpoint in A and one endpoint in B is denoted by $e_G(A,B)$. We write $(A,B)_G$ for the bipartite subgraph of G with vertex classes

A and B whose edges are precisely those edges in G with one endpoint in A and the other in B. Often we will write (A,B), for example, if this is unambiguous.

2.2. Degree sequence and Ore-type conditions forcing robust expansion

The following result is an immediate consequence of Lemma 13 from [26].

Lemma 2.1 ([26]). *Given positive constants* $\tau \ll \eta < 1$ *there exists an integer* n_0 *such that, whenever* G *is a graph on* $n \geqslant n_0$ *vertices with*

$$d_i \geqslant i + \eta n \quad \text{or} \quad d_{n-i-\eta n} \geqslant n - i \quad \text{for all } i < n/2,$$

then $\delta(G) \geqslant \eta n$ *and* G *is a robust* (τ^2, τ)*-expander.*

Notice that Lemma 2.1 together with Theorem 1.8 implies Theorem 1.4. We now show that Theorem 1.4 implies Theorem 1.5.

Lemma 2.2. *Let* $\gamma > 0$. *Suppose* G *is a graph on* n *vertices such that, for all non-adjacent* $x \neq y \in V(G)$,

$$d(x) + d(y) \geqslant (1 + 2\gamma)n.$$

Let $d_1 \leqslant \cdots \leqslant d_n$ *denote the degree sequence of* G. *Then*

$$d_i \geqslant i + \gamma n \quad \text{or} \quad d_{n-i-\gamma n} \geqslant n - i \quad \text{for all } i < n/2.$$

Proof. Firstly note that for $(1-\gamma)n/2 \leqslant i < n/2$ we wish to show that either $d_{n-i'-\gamma n} \geqslant n - i'$ or $d_{i'} \geqslant i' + \gamma n$, where $i' := n - i - \gamma n$. Notice that $n/2 - \gamma n < i' \leqslant n/2 - \gamma n/2$. Thus, it suffices to only consider i such that $1 \leqslant i \leqslant (1-\gamma)n/2$.

Suppose there is some $1 \leqslant i \leqslant (1-\gamma)n/2$ such that the statement does not hold. Then there is a set A of i vertices, each of degree less than $i + \gamma n \leqslant n/2 + \gamma n/2$. So for any $x, y \in A$, $d(x) + d(y) < (1 + 2\gamma)n$ and hence $G[A]$ is a clique. Set $B := V(G)\backslash A$. Note that $e_G(A,B) < (\gamma n + 1)i$. Hence, there is a vertex $x \in B$ that receives less than $\min\{\gamma n + 1, i\}$ edges from A. Therefore, there is a vertex $y \in A$ such that $xy \notin E(G)$. Thus, $d(x) + d(y) < (n - i - 1 + \gamma n + 1) + (i + \gamma n) \leqslant (1 + 2\gamma)n$, contradicting our assumption. □

3. Outline of the proof of Theorem 1.8

3.1. Proof overview

The overall strategy is similar to that of the proof of Theorem 1.1 in [7]. Indeed, as in [7] the proof is split into two main lemmas: the *Lemma for* G and the *Lemma for* H. However, many of the methods used in [7] break down in our setting so our argument proceeds somewhat differently.

The role of the Lemma for G (Lemma 7.1) is to obtain some special structure within G so that it will be suitable for embedding H into. By applying Theorem 1.7, we show that G contains a spanning subgraph G' which 'looks' like the blow-up of a cycle $C = V_1 V_2 \dots V_{2k} V_1$. More precisely, there is a partition $V_1, \dots, V_{2k}$ of $V(G)$ such that:

(i) $(V_{2i-1}, V_{2i})_{G'}$ is a 'super-regular' pair of density at least $d > 0$ for each $1 \leqslant i \leqslant k$,
(ii) $(V_{2i}, V_{2i+1})_{G'}$ is an 'ε-regular' pair of density at least d for each $1 \leqslant i \leqslant k$.

Furthermore, there are even integers $1 \leqslant i_1 \neq j_1 \leqslant 2k$ such that:

(iii) $(V_{i_1}, V_{j_1})_{G'}$ is 'ε-regular' with density at least d.

(So $V_{i_1}V_{j_1}$ can be thought of as a chord of C.) Crucially, this partition is 'robust' in the sense that one can modify the sizes of each partition class V_i somewhat without destroying the properties (i)–(iii). (This is made precise by the *Mobility Lemma* given in Section 6.)

Set $c := V_{i_1}V_{j_1}$. The role of the Lemma for H (Lemma 8.1) is to construct a graph homomorphism f from H to $C \cup \{c\}$ in such a way that 'most' of the edges of H are mapped to edges of the form $V_{2i-1}V_{2i}$ for some i. (Recall that these are the edges which correspond to super-regular pairs in G'.) The homomorphism f is such that every $V_i \in C$ receives roughly $|V_i|$ vertices of H. So f can be viewed as a 'guide' as to which vertex class $V_i \subseteq V(G)$ each vertex from H is embedded into. In particular, since the partition $V_1, \dots, V_{2k}$ is 'robust', we can alter the sizes of the classes V_i such that (i)–(iii) still hold and so that now $|f^{-1}(V_i)| = |V_i|$ for all i. Properties (i)–(iii) then allow us to apply the Blow-Up Lemma [19] to embed H into G' and thus G. (Actually we apply a result from [3] which is a consequence of the Blow-Up Lemma.)

3.2. Techniques for the Lemma for G

In order to obtain the partition $V_1, \dots, V_{2k}$ of $V(G)$ we modify a partition $V'_0, V'_1, \dots, V'_{2k}$ obtained by applying Szemerédi's Regularity Lemma [30] to G. Roughly speaking, $V'_1, \dots, V'_{2k}$ will satisfy (i)–(iii). Thus, we need to redistribute the vertices from V'_0 into the other vertex classes whilst retaining these properties. We also require our partition $V_1, \dots, V_{2k}$ to satisfy $|V_{2i-1}| \approx |V_{2i}|$ for each $1 \leqslant i \leqslant k$. So we need to redistribute vertices in a 'balanced' way. In the Lemma for G in [7], the minimum degree condition of Theorem 1.1 is heavily relied on to achieve this. However, our graph G may have very small minimum degree. So instead we introduced the notion of a 'shifted M-walk' to help us redistribute vertices: given a perfect matching M in a graph R, a shifted M-walk is a walk whose edges alternate between edges of M and edges of $R \backslash M$ (see Section 5.1 for the precise definition). Since G is a robust expander, we can find short shifted M-walks in a reduced graph R of G. (Here, M will be the perfect matching in R that corresponds to the super-regular pairs from (i) above.) These walks act as a 'guide' as to how we redistribute vertices amongst the vertex classes.

3.3. Techniques for the Lemma for H

In [7] the techniques used are actually strong enough to prove a more general result than Theorem 1.1 (and so Theorem 1.1 is not proved directly). For example, in the case when $r = 2$, their result concerns not only bipartite H but also a special class of 3-colourable graphs H where the third colour class is very small (see Theorem 2 in [7] for precise details). One example of such a graph H is a Hamilton cycle C' with a chord between two vertices of distance 2 on C'. H is 3-colourable and has bounded bandwidth. However, H cannot be embedded into *every* graph G satisfying the hypothesis of Theorem 1.8. Indeed, consider the graph G defined at the end of Section 1.3.

In particular, this means we have to approach the proof of the Lemma for H differently. Since H has bandwidth $o(n)$ we can chop $V(H)$ into small linear-sized segments $A_1, B_1, \dots, A_m, B_m$ where all the edges of H lie in pairs of the form $(A_i, B_i)_H$ and $(B_i, A_{i+1})_H$ and such that $A := \cup_{i=1}^{m} A_i$ and $B := \cup_{i=1}^{m} B_i$ are the colour classes of H. Ideally we would want to construct f to map the vertices of A_1 into V_1, the vertices of B_1 into V_2 and so on, continuing around C many times until all the vertices have been assigned. However, since $|A|$ and $|B|$ may vary widely, this would map vertices in an unbalanced way. That is, the total number of vertices mapped to 'odd' classes V_{2i-1} would differ widely from the total number of vertices mapped to 'even' classes V_{2i}. We get around this problem by using the chord $c = V_{i_1}V_{j_1}$ to 'flip' halfway in the process. So after this, vertices from the B_i are mapped to 'odd' classes V_{2i-1} and vertices from the A_i are mapped to the 'even' classes V_{2i}. We also 'randomize' part of the mapping procedure to ensure that the number of vertices of H assigned to each V_i is approximately $|V_i|$. (A randomization technique of a similar flavour was used in [22].)

4. The Regularity Lemma

In the proof of the Lemma for G (Lemma 7.1) we will use Szemerédi's Regularity Lemma [30]. In this section we will introduce all the information we require about this result. To do this, we first introduce some more notation. The *density* of a bipartite graph G with vertex classes A and B is defined to be

$$d(A,B) := \frac{e(A,B)}{|A||B|}.$$

Given any $\varepsilon, d > 0$, we say that G is (ε,d)-*regular* if $d(A,B) \geqslant d$ and, for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geqslant \varepsilon|A|$ and $|Y| \geqslant \varepsilon|B|$, we have $|d(A,B) - d(X,Y)| < \varepsilon$. We say that G is (ε,d)-*super-regular* if additionally every vertex $a \in A$ has at least $d|B|$ neighbours in B and every vertex $b \in B$ has at least $d|A|$ neighbours in A. We also say that (A,B) is an (ε,d)-*(super-)regular pair*. We will frequently use the following simple fact.

Fact 4.1. *Let $\varepsilon, d > 0$. Suppose that $G = (A,B)$ is an (ε,d)-regular pair. Let $B' \subseteq B$ be such that $|B'| \geqslant \varepsilon|B|$. Then there are at most $\varepsilon|A|$ vertices in A with fewer than $(d-\varepsilon)|B'|$ neighbours in B'.* □

We will also require the next simple proposition which allows us to modify a (super-)regular pair without destroying its (super-)regularity (see, *e.g.*, [6, Proposition 8]).

Proposition 4.2. *Let (A,B) be an (ε,d)-regular pair, and let A' and B' be vertex sets with $|A'\Delta A| \leqslant \alpha|A|$ and $|B'\Delta B| \leqslant \beta|B|$. Then (A',B') is an (ε',d')-regular pair where*

$$\varepsilon' := \varepsilon + 3(\sqrt{\alpha} + \sqrt{\beta}) \quad \text{and} \quad d' := d - 2(\alpha+\beta).$$

If, moreover, (A,B) is (ε,d)-super-regular and each vertex in A' has at least $d|B'|$ neighbours in B' and each vertex in B' has at least $d|A'|$ neighbours in A', then (A',B') is (ε',d')-super-regular.

We will use the following degree form of Szemerédi's Regularity Lemma [30] which can be easily derived from the classical version.

Lemma 4.3 (Regularity Lemma). *For every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $K_0 = K_0(\varepsilon, k_0)$ such that for every $d \in [0,1]$ and for every graph G on $n \geqslant K_0$ vertices there exists a partition*

$V_0, V_1, \dots, V_k$ of $V(G)$ and a spanning subgraph G' of G, such that the following conditions hold:

(i) $k_0 \leqslant k \leqslant K_0$,
(ii) $d_{G'}(x) \geqslant d_G(x) - (d+\varepsilon)n$ *for every* $x \in V(G)$,
(iii) *the subgraph* $G'[V_i]$ *is empty for all* $1 \leqslant i \leqslant k$,
(iv) $|V_0| \leqslant \varepsilon n$,
(v) $|V_1| = |V_2| = \dots = |V_k|$,
(vi) *for all* $1 \leqslant i < j \leqslant k$ *either* $(V_i, V_j)_{G'}$ *is an* (ε, d)*-regular pair or* $G'[V_i, V_j]$ *is empty.*

We call $V_1, \dots, V_k$ *clusters*, V_0 the *exceptional set* and the vertices in V_0 *exceptional vertices*. We refer to G' as the *pure graph*. The *reduced graph R of G with parameters ε, d and k_0* is the graph whose vertices are $V_1, \dots, V_k$ and in which V_iV_j is an edge precisely when $(V_i, V_j)_{G'}$ is (ε, d)-regular.

The following result implies that the property of a graph G being a robust expander is 'inherited' by the reduced graph R of G. It is an immediate consequence of Lemma 14 from [26].

Lemma 4.4 ([26]). *Let k_0, n_0 be positive integers and let $\varepsilon, d, \eta, \nu, \tau$ be positive constants such that $1/n_0 \ll \varepsilon \ll d \ll \nu, \tau, \eta < 1$ and such that $k_0 \ll n_0$. Let G be a graph on $n \geqslant n_0$ vertices with $\delta(G) \geqslant \eta n$ and such that G is a robust (ν, τ)-expander. Let R be the reduced graph of G with parameters ε, d and k_0. Then $\delta(R) \geqslant \eta |R|/2$ and R is a robust $(\nu/2, 2\tau)$-expander.*

5. Useful results

5.1. Shifted walks and robust expanders

Let G be a graph containing a perfect matching M. A *shifted M-walk* in G with endpoints $a = v_1$ and $b = v_{2\ell}$ is a walk $v_1 v_2 \dots v_{2\ell}$ in G such that $v_{2i}v_{2i+1} \in M$ for every $1 \leqslant i \leqslant \ell - 1$ and $v_{2i-1}v_{2i} \notin M$ for any $1 \leqslant i \leqslant \ell$. A shifted M-walk is *simple* if it contains each edge of M at most twice. Note that a path containing a single edge $v_1v_2 \notin M$ is a (simple) shifted M-walk for any perfect matching M.

Lemma 5.1. *Let G be a graph containing a perfect matching M, and let W be a shifted M-walk in G with endpoints a and b. Then W contains a simple shifted M-walk W' with endpoints a and b.*

Proof. We proceed by induction. Let $W = v_1 \dots v_{2\ell}$. If W is already simple then we set $W' := W$; otherwise, there exists an edge $xy \in M$ which appears at least three times in W. Let the first three appearances of xy be $v_{i_1}v_{i_1+1}$, $v_{i_2}v_{i_2+1}$ and $v_{i_3}v_{i_3+1}$ (in order). Now each of v_{i_1}, v_{i_2} and v_{i_3} is either x or y, and so without loss of generality we can assume that $v_{i_1} = v_{i_2} = x$. Now we have a shorter shifted M-walk $v_1 \dots v_{i_1} v_{i_2+1} \dots v_{2\ell}$. □

Lemma 5.2. *Let M be a perfect matching in a graph G and let $A \subseteq V(M)$ be a set containing at most one vertex from each edge of M. Suppose that W is a shifted M-walk both of whose endpoints lie in A. Then W contains a shifted M-walk W' such that the endpoints of W' both lie in A and no other vertices of W' lie in A.*

Proof. Let $W = v_1 \ldots v_{2\ell}$. Let $1 \leqslant i_1 \leqslant \ell$ be minimal such that $v_{2i_1} \in A$, and let $1 \leqslant i_2 \leqslant i_1$ be maximal such that $v_{2i_2-1} \in A$. Now $v_{2i_2-1}v_{2i_2} \ldots v_{2i_1-1}v_{2i_1}$ is the desired shifted M-walk. □

In the proof of the Lemma for G we will use shifted walks in the reduced graph R of G as a 'guide' as to how to redistribute vertices in G. Since the reduced graph R will be a robust expander, the following result ensures we can find our desired shifted walks.

Let G be a graph containing a perfect matching M, and let $A \subseteq V(G)$. For each $v \in V(G)$, let $v' \in V(G)$ be the unique vertex such that $vv' \in M$. The *shifted M-neighbourhood* of A is the set $SN_M(A) = \{v' \mid v \in N(A)\}$. $SN_M^r(A)$ is defined recursively by $SN_M^1(A) := SN_M(A)$ and $SN_M^r(A) := SN_M(SN_M^{r-1}(A))$ for $r \geqslant 2$.

Lemma 5.3. *Let $0 < \nu \leqslant \tau < \eta \ll 1$ be constants. Suppose G is a graph on n vertices with $\delta(G) \geqslant \eta n$ which is a robust (ν, τ)-expander, and let M be a perfect matching in G. Then, for any $a \in V(G)$, G contains a shifted M-walk of length at most $3/\nu$ which both starts and finishes at a.*

Proof. The minimum degree condition implies that $|SN_M(a)| = |N(a)| \geqslant \eta n \geqslant \tau n$. Since G is a robust (ν, τ)-expander,

$$|SN_M^r(a)| = |N(SN_M^{r-1}(a))| \geqslant \min\{|SN_M^{r-1}(a)| + \nu n, (1 - \tau + \nu)n\},$$

for all $r \geqslant 2$. Hence $|SN_M^{1/2\nu}(a)| \geqslant (\tau + 1/2 - \nu)n$ and so

$$|N(SN_M^{1/2\nu}(a))| \geqslant (1/2 + \tau)n.$$

Thus, there exists some edge $vv' \in M$ such that both v and v' lie in $N(SN_M^{1/2\nu}(a))$. This implies that there exists a shifted M-walk P with endpoints a and v and a shifted M-walk P' with endpoints a and v', each of length at most $1/\nu + 1$. Now $P \cup vv' \cup P'$ forms a shifted M-walk of length at most $2/\nu + 3 \leqslant 3/\nu$ which starts and finishes at a. □

The next lemma allows us to delete a small number of vertices from a robust expander without destroying this property.

Lemma 5.4. *Let $0 < \alpha < \nu \leqslant \tau \ll 1$ be constants. Suppose that G is a graph on n vertices which is a robust (ν, τ)-expander and let $S \subseteq V(G)$ be a set of size αn. Then $G - S$ is a robust $(\nu - \alpha, \tau + \alpha)$-expander.*

Proof. Let $G' := G - S$ and $n' := |G'|$. Consider any $A \subseteq V(G')$ such that $(\tau + \alpha)n' \leqslant |A| \leqslant (1 - \tau - \alpha)n'$. Set $A' := A \cup S$. Then

$$\tau n \leqslant (\tau + \alpha)n' + \alpha n \leqslant |A'| \leqslant (1 - \tau - \alpha)n' + \alpha n \leqslant (1 - \tau)n.$$

So $|RN_{\nu,G}(A')| \geqslant |A'| + \nu n$. Now every vertex of $RN_{\nu,G}(A')$ has at least νn neighbours in A' and since $|S| = \alpha n$, at least $(\nu - \alpha)n \geqslant (\nu - \alpha)n'$ of these must lie in A. Hence every vertex of $RN_{\nu,G}(A') \backslash S$ lies in $RN_{\nu-\alpha,G'}(A)$, and so $|RN_{\nu-\alpha,G'}(A)| \geqslant |A'| + \nu n - |S| \geqslant |A| + (\nu - \alpha)n'$, as desired. □

5.2. Probabilistic bounds

The following two probabilistic bounds are used in the proof of the Lemma for H (Lemma 8.1).

Lemma 5.5 ([22], Lemma 2.1). *Suppose that* $1/k \ll p, (1-p), \varepsilon$, *that* $n \geqslant k^3/6$, *and that* $X \sim \mathrm{Bin}(n,p)$. *Then, for any* $0 \leqslant r \leqslant k-1$,

$$\frac{1-\varepsilon}{k} \leqslant \mathbb{P}(X \equiv r \mod k) \leqslant \frac{1+\varepsilon}{k}.$$

Lemma 5.6 ([29], Proposition 1.1). *Let* $X_1, \dots, X_n$ *be random variables taking values in* $[0,1]$, *such that, for each* $1 \leqslant k \leqslant n$,

$$\mathbb{E}[X_k \mid X_{k-1}, \dots, X_1] \leqslant a_k.$$

Let $\mu := \sum_{k=1}^n a_k$. *Then, for any* $0 < \delta \leqslant 1$,

$$\mathbb{P}\left[\sum_{k=1}^n X_k > (1+\delta)\mu\right] \leqslant e^{-\frac{\delta^2\mu}{3}}.$$

We also require the following expectation bound.

Lemma 5.7. *Suppose that* X *and* Y *are integer-valued random variables and that* B *is an event, such that for each* $x, y \in \mathbb{Z}$, $\mathbb{P}[X = x \mid B \cap (Y = y)] = \mathbb{P}[X = x \mid Y = y]$. *Then*

$$\mathbb{E}[X \mid B] \leqslant \max_{y \in \mathbb{Z}} \mathbb{E}[X \mid Y = y].$$

Proof. Note that for each $x \in \mathbb{Z}$,

$$\mathbb{P}[X = x \mid B] = \sum_{y \in \mathbb{Z}} \mathbb{P}[(X = x) \cap (Y = y) \mid B].$$

Further,

$$\begin{aligned}\mathbb{P}[(X = x) \cap (Y = y) \mid B] &= \frac{\mathbb{P}[(X = x) \cap (Y = y) \cap B]}{\mathbb{P}[B]} \\ &= \frac{\mathbb{P}[(X = x) \cap (Y = y) \cap B]}{\mathbb{P}[(Y = y) \cap B]} \cdot \frac{\mathbb{P}[(Y = y) \cap B]}{\mathbb{P}[B]} \\ &= \mathbb{P}[X = x \mid Y = y] \cdot \mathbb{P}[Y = y \mid B].\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[X \mid B] &= \sum_{x \in \mathbb{Z}} x\mathbb{P}[X = x \mid B] = \sum_{x \in \mathbb{Z}} x \sum_{y \in \mathbb{Z}} \mathbb{P}[X = x \mid Y = y] \cdot \mathbb{P}[Y = y \mid B] \\ &= \sum_{y \in \mathbb{Z}} \mathbb{P}[Y = y \mid B] \sum_{x \in \mathbb{Z}} x\mathbb{P}[X = x \mid Y = y] = \sum_{y \in \mathbb{Z}} \mathbb{P}[Y = y \mid B] \cdot \mathbb{E}[X \mid Y = y] \\ &\leqslant \max_{y \in \mathbb{Z}} \mathbb{E}[X \mid Y = y].\end{aligned}$$

□

6. The Mobility Lemma

In order to state our next result we first introduce a slight variant of the notion of a reduced graph. Let $\varepsilon, \varepsilon', d, d' > 0$. Suppose that G is a graph and $V_1, \dots, V_k$ is a partition of $V(G)$. We say that a graph R is an *(ε, d)-reduced graph of G on $V_1, \dots, V_k$* if the following holds:

- $V(R) = \{V_1, \dots, V_k\}$,
- if $V_iV_j \in E(R)$ then $(V_i, V_j)_G$ is an (ε, d)-regular pair (for all $1 \leqslant i \neq j \leqslant k$).

Suppose $V'_1, \dots, V'_k$ is another partition of $V(G)$ and R is as above (in particular, $V(R) = \{V_1, \dots, V_k\}$). We also say that R is an *(ε', d')-reduced graph of G on $V'_1, \dots, V'_k$* if the following holds:

- if $V_iV_j \in E(R)$ then $(V'_i, V'_j)_G$ is an (ε', d')-regular pair (for all $1 \leqslant i \neq j \leqslant k$).

Suppose that $V_1, \dots, V_k$ and $V'_1, \dots, V'_k$ are both partitions of the vertex set of a graph G. Given a cluster $V = V_i$ for some $1 \leqslant i \leqslant k$, we will often denote by V' the cluster V'_i.

We will apply the next result in the proof of the Lemma for G (Lemma 7.1) so that we can alter a particular partition of a graph G somewhat without destroying the structure of our reduced graph R.

Lemma 6.1 (Mobility Lemma). *Let $k \in \mathbb{N}$, and let $\xi, \varepsilon, \varepsilon', d', d$ be positive constants such that*

$$0 < \xi \ll 1/k \ll \varepsilon \ll \varepsilon' \ll d' \ll d \ll 1.$$

Suppose G is a graph on n vertices, $A_1, B_1, A_2, B_2, \dots, A_k, B_k$ is a partition of $V(G)$ such that $|A_i|, |B_i| \geqslant n/3k$ for all $1 \leqslant i \leqslant k$ and R is an (ε, d)-reduced graph on $A_1, B_1, \dots, A_k, B_k$. Let $(a_i)_{i=1}^k$ and $(b_i)_{i=1}^k$ be integers. Suppose that the following conditions hold:

(i) *R contains the Hamilton cycle $C = A_1B_1A_2B_2 \dots A_kB_kA_1$,*
(ii) *R contains an edge $A_{i_1}A_{j_1}$ for some $i_1 \neq j_1$,*
(iii) *R contains an edge $B_{i_2}B_{j_2}$ for some $i_2 \neq j_2$,*
(iv) *the pair $(A_i, B_i)_G$ is (ε, d)-super-regular for all $1 \leqslant i \leqslant k$,*
(v) *$|a_i|, |b_i| < \xi n$ for each $1 \leqslant i \leqslant k$,*
(vi) *$\sum_{i=1}^k a_i + \sum_{i=1}^k b_i = 0$,*
(vii) *$|\sum_{i=1}^k a_i| = |\sum_{i=1}^k b_i| \leqslant \xi n$.*

Then there exists a partition $A'_1, B'_1, A'_2, B'_2, \dots, A'_k, B'_k$ of $V(G)$ such that $|A'_i| = |A_i| + a_i$ and $|B'_i| = |B_i| + b_i$ for each $1 \leqslant i \leqslant k$, R is an (ε', d')-reduced graph of G on $A'_1, B'_1, A'_2, B'_2, \dots, A'_k, B'_k$, and $(A'_i, B'_i)_G$ is (ε', d')-super-regular for each $1 \leqslant i \leqslant k$.

Proof. Without loss of generality we may assume that $\sum_{i=1}^k a_i \geqslant 0$. (As a consequence of this assumption we will in fact only need the edge $B_{i_2}B_{j_2}$, and not the edge $A_{i_1}A_{j_1}$.) Note that by (iii) and Fact 4.1 there are at least $(1-\varepsilon)|B_{i_2}| \gg \xi n$ vertices in B_{i_2} with at least $(d-\varepsilon)|B_{j_2}|$ neighbours in B_{j_2}. Pick $\sum_{i=1}^k a_i \leqslant \xi n$ of these vertices and move them from B_{i_2} into A_{j_2}. Call the resulting sets $B^*_{i_2}$ and $A^*_{j_2}$ respectively.

We now perform an iterative procedure which will reassign vertices among the vertex classes $(A_i)_{i=1}^k$ and, separately, $(B_i)_{i=1}^k$. Initially we define the classes $A^*_i = A_i$ for each $i \neq j_2$ and

$B_i^* = B_i$ for each $i \neq i_2$. Roughly speaking, A_i^* (or B_i^*) will be the current version of A_i (or B_i). The choice of how we defined $B_{i_2}^*$ and $A_{j_2}^*$ is such that, initially,

$$\sum_{i=1}^{k} |A_i^*| = \sum_{i=1}^{k} |A_i| + \sum_{i=1}^{k} a_i \quad \text{and} \quad \sum_{i=1}^{k} |B_i^*| = \sum_{i=1}^{k} |B_i| + \sum_{i=1}^{k} b_i. \tag{6.1}$$

Throughout the procedure we will ensure that (6.1) holds. Furthermore, throughout we will ensure that

$$|A_i^* \Delta A_i|, |B_i^* \Delta B_i| \leqslant 5k\xi n \leqslant \varepsilon|A_i|, \varepsilon|B_i| \tag{6.2}$$

for every $1 \leqslant i \leqslant k$. We will also ensure that whenever a vertex v is moved to a cluster A_i^*, v has at least $(d-\varepsilon)|B_i|$ neighbours in B_i, and *vice versa*. We will terminate the procedure when $|A_i^*| = |A_i| + a_i$ and $|B_i^*| = |B_i| + b_i$, and then set $A_i' := A_i^*$ and $B_i' := B_i^*$ for each $1 \leqslant i \leqslant k$.

Each iteration proceeds as follows. Let $1 \leqslant i \leqslant k$ be such that $|A_i^*| < |A_i| + a_i$ and let $j \neq i$ be such that $|A_j^*| > |A_j| + a_j$. (Such i and j exist by (6.1).) Suppose that $i < j$. Note that (i) implies that $(B_{j-1}, A_j)_G$ is an (ε, d)-regular pair. So by (6.2) and Fact 4.1 there is a vertex v in A_j^* which has at least $(d-\varepsilon)|B_{j-1}|$ neighbours in B_{j-1}. We move v from A_j^* to A_{j-1}^*. Similarly we move one vertex (which need not be v) from A_{j-1}^* to A_{j-2}^*, and so on until we move one vertex from A_{i+1}^* to A_i^*. On the other hand, if $j < i$ we perform the same procedure moving vertices in the *same* direction as before. That is, we move a vertex from A_j^* to A_{j-1}^* and so on until we move a vertex A_2^* to A_1^*. Then we move a vertex A_1^* to A_k^* and continue until we move a vertex from A_{i+1}^* to A_i^*.

Since in each step of the process we only move vertices between the A_i^*, certainly (6.1) holds throughout. Now when the procedure terminates we have $|A_i^*| = |A_i| + a_i$ for all $1 \leqslant i \leqslant k$. It remains to show that (6.2) holds. Note that in each step of the iteration we add at most one vertex to each A_i^* and remove at most one vertex from each A_i^*. Further, in total we need to perform the iterative procedure at most

$$\sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} a_i \leqslant (k+1)\xi n \leqslant 2k\xi n$$

times. (The $\sum_{i=1}^{k} a_i$ here comes from the fact that, at the start, we moved $\sum_{i=1}^{k} a_i$ vertices from B_{i_2} to A_{j_2}.) Thus, at the end of the procedure $|A_{j_2}^* \Delta A_{j_2}| \leqslant 5k\xi n$ and $|A_i^* \Delta A_i| \leqslant 4k\xi n$ for all $i \neq j_2$. We now set $A_i' := A_i^*$ for each $1 \leqslant i \leqslant k$.

We apply an identical iterative procedure to the B_i^*. However, we now move vertices in the *opposite* direction to before (so vertices are moved from B_j^* to B_{j+1}^*, and so on). Therefore we obtain sets B_i' such that $|B_i'| = |B_i| + b_i$ and $|B_i' \Delta B_i| \leqslant 5k\xi n$ for all $1 \leqslant i \leqslant k$.

For any $VW \in E(R)$, $(V, W)_G$ is an (ε, d)-regular pair. By (6.2), $|V' \Delta V|$, $|W' \Delta W| \leqslant \varepsilon|V|$, $\varepsilon|W|$, so Proposition 4.2 implies that $(V', W')_G$ is an (ε', d')-regular pair. So indeed, R is an (ε', d')-reduced graph of G on $A_1', B_1', A_2', B_2', \dots, A_k', B_k'$. It remains to show that the pair $(A_i', B_i')_G$ is (ε', d')-super-regular for every $1 \leqslant i \leqslant k$. By (iv) and (6.2), every vertex $v \in A_i$ has at least $(d-\varepsilon)|B_i| \geqslant d'|B_i'|$ neighbours in B_i'. Further, during our iterative procedure we ensured that every vertex $v \in A_i' \backslash A_i$ has at least $(d-\varepsilon)|B_i|$ neighbours in B_i. Hence (6.2) implies that every

$v \in A_i'$ has at least

$$(d-\varepsilon)|B_i| - \varepsilon|B_i| \geqslant d'|B_i'|$$

neighbours in B_i'. Similarly, each $w \in B_i'$ has at least $d'|A_i'|$ neighbours in A_i'. So $(A_i', B_i')_G$ is an (ε', d')-super-regular pair for all $1 \leqslant i \leqslant k$, as desired. □

7. The Lemma for G

Lemma 7.1 (Lemma for G). *Let $n_0 \in \mathbb{N}$ and let $\lambda, \xi, \varepsilon, d, \nu, \tau, \eta$ be positive constants such that*

$$0 < 1/n_0 \ll \lambda \ll \xi \ll \varepsilon \ll d \ll \nu \leqslant \tau \ll \eta \ll 1.$$

Suppose G is a graph on $n \geqslant n_0$ vertices with $\delta(G) \geqslant \eta n$ which is a robust (ν, τ)-expander. Then there exists an integer k such that $\xi \ll 1/k \ll \varepsilon$, integers $1 \leqslant i_1 \neq j_1, i_2 \neq j_2 \leqslant k$ and a partition $(n_i)_{i=1}^{2k}$ of n with $n_i > n/3k$ for all $1 \leqslant i \leqslant 2k$ and $|n_{2i-1} - n_{2i}| \leqslant \lambda n$ for all $1 \leqslant i \leqslant k$ such that the following holds. For every partition $(n_i')_{i=1}^{2k}$ of n satisfying $n_i' \leqslant n_i + \xi n$ for all $1 \leqslant i \leqslant 2k$, there exists a partition $A_1', B_1', A_2', B_2', \dots, A_k', B_k'$ of $V(G)$ and a spanning subgraph G' of G such that the following properties are satisfied.

(α_1) *$|A_i'| = n_{2i-1}'$ and $|B_i'| = n_{2i}'$ for all $1 \leqslant i \leqslant k$,*
(α_2) *$(A_i', B_i')_{G'}$ is (ε, d)-super-regular for all $1 \leqslant i \leqslant k$,*
(α_3) *$(B_i', A_{i+1}')_{G'}$ is (ε, d)-regular for all $1 \leqslant i \leqslant k$ (where $A_{k+1}' := A_1'$),*
(α_4) *$(A_{i_1}', A_{j_1}')_{G'}$ is (ε, d)-regular,*
(α_5) *$(B_{i_2}', B_{j_2}')_{G'}$ is (ε, d)-regular.*

Proof. Choose additional constants ε' and d' such that

$$\xi \ll \varepsilon' \ll \varepsilon \ll d \ll d' \ll \nu.$$

Apply the Regularity Lemma (Lemma 4.3) with parameters ε', d' and $k_0 := 1/\varepsilon'$ to obtain clusters $V_1, \dots, V_{k'}$ of size m (where $(1-\varepsilon')n/k' \leqslant m \leqslant n/k'$), an exceptional set V_0, a pure graph $G' \subseteq G$ and the reduced graph R of G with parameters ε', d' and k_0. Since $\xi \ll \varepsilon'$ we may assume that

$$\xi \ll 1/k' \leqslant \varepsilon'.$$

If k' is odd then we delete $V_{k'}$ from R and add all of the vertices of $V_{k'}$ to V_0. So $|V_0| \leqslant \varepsilon' n + m \leqslant 2\varepsilon' n$. We now refer to this modified reduced graph as R and redefine $k' = |R|$. By Lemma 4.4, R originally had minimum degree at least $\eta k'/2$ and was a robust $(\nu/2, 2\tau)$-expander. So R still has minimum degree at least $\eta k'/3$ and by Lemma 5.4, R is still a robust $(\nu/3, 3\tau)$-expander.

Set $k := k'/2$. Since $1/k' \ll \nu \leqslant \tau \ll \eta < 1$, Theorem 1.7 implies that R contains a Hamilton cycle $C = A_1 B_1 \dots A_k B_k A_1$. Since $|C| = 2k$ is even, C contains a perfect matching $M = \{A_1 B_1, \dots, A_k B_k\}$. Notice that R contains an edge $A_{i_1} A_{j_1}$ for some $1 \leqslant i_1 \neq j_1 \leqslant k$ and an edge $B_{i_2} B_{j_2}$ for some $1 \leqslant i_2 \neq j_2 \leqslant k$. Indeed, let $\mathcal{A} := \{A_i\}_{i=1}^k$ and note that since R is a robust $(\nu/3, 3\tau)$-expander we have $|RN_{\nu,R}(\mathcal{A})| \geqslant k + \nu k'$. This implies that $\mathcal{A} \cap RN_{\nu,R}(\mathcal{A}) \neq \emptyset$ and hence that R contains some edge $A_{i_1} A_{j_1}$. Similarly R contains an edge $B_{i_2} B_{j_2}$.

Fact 4.1 implies that we can replace each cluster in $V(R)$ with a subcluster of size $m' := (1-\varepsilon')m$ such that for every edge $A_j B_j \in M$ the chosen subclusters of A_j and B_j form a

$(2\varepsilon', d'/2)$-super-regular pair in G'. We add all of the vertices not in these subclusters to V_0, and from now on we refer to the subclusters as the clusters of R. So $(V, W)_{G'}$ is still a $(2\varepsilon', d'/2)$-regular pair for all $VW \in E(R)$. Note that $|V_0| \leqslant 2\varepsilon' n + \varepsilon' n = 3\varepsilon' n$.

Our next task is to incorporate the vertices of V_0 into the clusters $V_1, \dots, V_{k'}$ such that the pairs $(A_j, B_j)_{G'}$ remain super-regular and such that the pairs $(V_i, V_j)_{G'}$ remain regular for all $V_iV_j \in E(R)$ (with somewhat weaker constants in each case). Let $V_0 = \{x_1, \dots, x_t\}$ where $t \leqslant 3\varepsilon' n$. We will assign the vertices of V_0 in such a way that:

(a) at most $8\varepsilon' m'/\eta$ vertices are assigned to each cluster $V \in V(R)$,
(b) whenever a vertex $x_i \in V_0$ is assigned to a cluster A_j, x_i has at least $\eta m'/4$ neighbours in B_j. Similarly any vertex from V_0 assigned to B_j has at least $\eta m'/4$ neighbours in A_j.

Suppose we have assigned $x_1, \dots, x_{i-1}$ to clusters in $V(R)$ such that (a) and (b) are satisfied. Call a cluster $V \in V(R)$ *full* if it has already been assigned $8\varepsilon' m'/\eta$ vertices of V_0. Let F be the set of full clusters. Since $|V_0| \leqslant 3\varepsilon' n$, we have $|F| \leqslant (3\varepsilon'\eta n)/(8\varepsilon' m') \leqslant \eta k$. Thus, as $\delta(G) \geqslant \eta n$,

$$\left| N_G(x_i) \backslash \left(V_0 \cup \bigcup_{V \in F} V \right) \right| \geqslant \eta n - 3\varepsilon' n - (\eta k)m' \geqslant \eta n/3.$$

Hence, by the pigeonhole principle there exists some $V \in V(R)\backslash F$ such that $|N_G(x_i) \cap V| \geqslant \eta n/3k' \geqslant \eta m'/4$. Now if $V = A_j$ for some $1 \leqslant j \leqslant k$ then we add x_i to B_j; otherwise, $V = B_j$ for some $1 \leqslant j \leqslant k$ and we add x_i to A_j. Repeating this process for each x_i, we indeed assign all of the vertices of V_0 to the clusters of R in such a way that (a) and (b) are satisfied. We now incorporate all of the assigned vertices into their respective clusters. Further, we add all those edges from G with endpoints in V_0 to G'. Note that

(c) $m' \leqslant |V| \leqslant m' + 8\varepsilon' m'/\eta \leqslant (1 + \sqrt{\varepsilon'})m'$ for all $V \in V(R)$,
(d) $(V, W)_{G'}$ is a $((\varepsilon')^{1/3}, d'/4)$-regular pair for every edge $VW \in E(R)$, and
(e) $(V, W)_{G'}$ is a $((\varepsilon')^{1/3}, d'/4)$-super-regular pair for every edge $VW \in E(M)$.

(Conditions (d) and (e) follow by Proposition 4.2.)

Next we will perform an algorithm which redistributes vertices among the clusters in R in such a way that $||A_i| - |B_i|| \leqslant \lambda n$ for each $1 \leqslant i \leqslant k$. We define $\{A_i^*, B_i^*\}_{i=1}^k$, R^* and M^* as follows. Initially we set $A_i^* := A_i$ and $B_i^* := B_i$ for all $1 \leqslant i \leqslant k$, $R^* := R$ and $M^* := M$. At each step we will redefine each A_i^* and B_i^*, R^* and M^* and reassign vertices so that the quantity

$$\Sigma^* = \sum_{1 \leqslant i \leqslant k, ||A_i^*| - |B_i^*|| > \lambda n} ||A_i^*| - |B_i^*||$$

decreases by at least λn. The algorithm will terminate when $\Sigma^* = 0$, that is, when $||A_i^*| - |B_i^*|| \leqslant \lambda n$ for all $1 \leqslant i \leqslant k$. Initially $\Sigma^* \leqslant 8\varepsilon' m' k/\eta \leqslant 4\varepsilon' n/\eta$ by (c), and hence we need at most $4\varepsilon'/\eta\lambda$ steps to complete the process. R^* will always be an *induced* subgraph of R and at each step we set M^* to be the submatching of M induced by $V(R^*)$. (Note that $V(R^*)$ is a subset of $V(R) = \{A_i, B_i\}_{i=1}^k$ throughout the algorithm.)

We will ensure that the inequality

$$|R^*| \geqslant \left(1 - \nu/12\right)k' \tag{7.1}$$

holds throughout, and that M^* is a perfect matching in R^*. Further, we will ensure that

$$|A_i^*\backslash A_i| \leqslant (\varepsilon')^{1/3}m' \quad \text{and} \quad |B_i^*\backslash B_i| \leqslant (\varepsilon')^{1/3}m', \tag{7.2}$$

$$|A_i\backslash A_i^*| \leqslant (\varepsilon')^{1/3}m' \quad \text{and} \quad |B_i\backslash B_i^*| \leqslant (\varepsilon')^{1/3}m' \tag{7.3}$$

for all $1 \leqslant i \leqslant k$.

Each step proceeds as follows. Call a vertex v *well-connected* to a cluster $V \in V(R)$ if v has at least $d'm'/8$ neighbours in V. Recall that if $VW \in E(R)$ then $(V,W)_{G'}$ is a $((\varepsilon')^{1/3}, d'/4)$-regular pair and so V contains at least $m'/2$ vertices v which are well-connected to W. In what follows we will ensure that every vertex we redistribute to a cluster A_i^* is well-connected to B_i and *vice versa*. Since (7.3) holds throughout the process, given any $VW \in E(R^*)$, V^* will always contain at least $m'/2-(\varepsilon')^{1/3}m' \geqslant m'/3 \gg \lambda n$ vertices that are well-connected to W (where $V^* := A_i^*$ if $V = A_i$ for some i and $V^* := B_i^*$ if $V = B_i$ for some i). Thus, at any point during the algorithm we may choose a set of λn well-connected vertices from any of the A_i^* and B_i^*. (When it is clear from the context, we will not explicitly specify which cluster a vertex v is well-connected to.)

Let S be the set of clusters $V^* \in V(R^*)$ such that either $V^* = A_i$ where $|A_i^*| > |B_i^*| + \lambda n$ or $V^* = B_i$ where $|B_i^*| > |A_i^*| + \lambda n$. If S is empty then the algorithm terminates. (We shall see later that in this case we must have that $\Sigma^* = 0$.) Otherwise, choose $V^* \in S$ arbitrarily. Since R is a robust $(\nu/3, 3\tau)$-expander and $\delta(R) \geqslant \eta k'/3$, (7.1) implies that $\delta(R^*) \geqslant \eta|R^*|/4$ and Lemma 5.4 implies that R^* is a robust $(\nu/4, 4\tau)$-expander. Hence Lemma 5.3 implies that R^* contains a shifted M^*-walk P' of length at most $12/\nu$ which starts and finishes at V^*. By Lemma 5.1, P' contains a simple shifted M^*-walk P'' which also starts and finishes at V^*. Now apply Lemma 5.2 to P'' to obtain a simple shifted M^*-walk P of length at most $12/\nu$, such that the endpoints of P both lie in S and no other vertices of P lie in S. We call P the *active walk* of this step of the algorithm.

Let

$$P = U_1W_2U_2\dots W_{\ell-1}U_{\ell-1}W_\ell$$

such that $W_iU_i \in E(M^*)$ for each $2 \leqslant i \leqslant \ell-1$. Let W_1 and U_ℓ denote the clusters such that $W_1U_1, W_\ell U_\ell \in E(M^*)$. Given any $1 \leqslant i \leqslant \ell$, if $U_i = A_j$ for some $1 \leqslant j \leqslant k$, set $U_i^* := A_j^*$; otherwise $U_i = B_j$ for some $1 \leqslant j \leqslant k$, so set $U_i^* := B_j^*$. Define W_i^* analogously for each $1 \leqslant i \leqslant \ell$. Move $\lambda n/2$ well-connected vertices from U_1^* into U_2^*, $\lambda n/2$ well-connected vertices from U_2^* into U_3^*, and so on until we have moved $\lambda n/2$ well-connected vertices from $U_{\ell-1}^*$ to U_ℓ^*. Then move $\lambda n/2$ well-connected vertices from W_2^* into W_1^*, $\lambda n/2$ well-connected vertices from W_3^* into W_2^*, and so on until we have moved $\lambda n/2$ well-connected vertices from W_ℓ^* to $W_{\ell-1}^*$. Note that since P is simple, each cluster loses at most λn vertices and gains at most λn vertices. Further, for each $1 < i < \ell$ the quantity $||W_i^*|-|U_i^*||$ remains unchanged (in fact, $|W_i^*|$ and $|U_i^*|$ remain unchanged). For $i = 1, \ell$, $||W_i^*|-|U_i^*||$ decreases by precisely λn (or $2\lambda n$ if $U_1 = W_\ell$).

In order to ensure that (7.2) holds, we remove from R^* every pair $\{A_i, B_i\}$ of clusters such that $|A_i^*\backslash A_i| \geqslant (\varepsilon')^{1/3}m' - \lambda n$ or $|B_i^*\backslash B_i| \geqslant (\varepsilon')^{1/3}m' - \lambda n$. Since each cluster gains at most λn vertices in each step, any clusters which are not removed will still satisfy (7.2) at the end of the next step. Similarly, to ensure that (7.3) holds, we remove from R^* every pair $\{A_i, B_i\}$ of clusters such that $|A_i\backslash A_i^*| \geqslant (\varepsilon')^{1/3}m' - \lambda n$ or $|B_i\backslash B_i^*| \geqslant (\varepsilon')^{1/3}m' - \lambda n$.

Claim 7.2. *For each pair $\{A_i, B_i\}$ of clusters which is removed from R^*, we have that $||A_i^*| - |B_i^*|| \leqslant \lambda n$.*

To prove the claim, suppose for a contradiction that some pair $\{A_i, B_i\}$ of clusters is removed from R^* and that $||A_i^*| - |B_i^*|| > \lambda n$. In order for $\{A_i, B_i\}$ to be removed we must have that $|A_i^* \backslash A_i| \geqslant (\varepsilon')^{1/3}m' - \lambda n$, $|B_i^* \backslash B_i| \geqslant (\varepsilon')^{1/3}m' - \lambda n$, $|A_i \backslash A_i^*| \geqslant (\varepsilon')^{1/3}m' - \lambda n$ or $|B_i \backslash B_i^*| \geqslant (\varepsilon')^{1/3}m' - \lambda n$. Without loss of generality assume that $|A_i^* \backslash A_i| \geqslant (\varepsilon')^{1/3}m' - \lambda n$. Since in each step we add at most λn vertices to A_i^*, there must have been at least $(\varepsilon')^{1/3}m'/2\lambda n$ steps in the algorithm so far, such that A_i is contained in the active walk P of each step. By the definition of P, either A_i or B_i must be an endpoint of P. So $||A_i^*| - |B_i^*||$ is reduced by at least λn during each such step, and hence by at least $(\varepsilon')^{1/3}m'/2$ during the algorithm so far. But this is a contradiction since initially $||A_i^*| - |B_i^*|| \leqslant \sqrt{\varepsilon'}m' \leqslant (\varepsilon')^{1/3}m'/2$.

It remains to show that (7.1) holds throughout the process. Suppose for a contradiction that at some point more than $\nu k'/12$ clusters have been removed from R^*. Then at least

$$((\varepsilon')^{1/3}m' - \lambda n)\left(\frac{\nu k'}{48}\right) > \sqrt{\varepsilon'}n \tag{7.4}$$

vertices of G must have been redistributed during the process so far. But at most $12\lambda n/\nu$ vertices were redistributed during each step, and at most $4\varepsilon'/\eta\lambda$ steps were performed during the process. Hence the number of redistributed vertices is at most

$$\frac{12\lambda n}{\nu} \cdot \frac{4\varepsilon'}{\eta\lambda} < \sqrt{\varepsilon'}n,$$

which contradicts (7.4). This proves that (7.1) holds throughout.

By construction, when the algorithm terminates we have that $V(R^*)$ does not contain any A_i and B_i such that $||A_i^*| - |B_i^*|| > \lambda n$. Further, by Claim 7.2, for those clusters $A_i, B_i \notin V(R^*)$ we have that $||A_i^*| - |B_i^*|| \leqslant \lambda n$. Hence, we indeed obtain clusters $\{A_i^*, B_i^*\}_{i=1}^k$ such that $\Sigma^* = 0$ and (7.2) and (7.3) hold.

We now set $n_{2i-1} := |A_i^*|$ and $n_{2i} := |B_i^*|$ for each $1 \leqslant i \leqslant k$. Notice that $n_j \geqslant (1 - (\varepsilon')^{1/3})m' > n/3k$ for each $1 \leqslant j \leqslant 2k$. We now relabel the clusters of R in the natural way so that $V(R) = \{A_i^*, B_i^*\}_{i=1}^k$. Note that by (7.2) and (7.3) we have

$$|A_i \Delta A_i^*| \leqslant 2(\varepsilon')^{1/3}m' \quad \text{and} \quad |B_i \Delta B_i^*| \leqslant 2(\varepsilon')^{1/3}m'$$

for each $1 \leqslant i \leqslant k$. Hence by Proposition 4.2 the pair $(V, W)_{G'}$ is $((\varepsilon')^{1/10}, d'/10)$-regular for every edge $VW \in E(R)$. Further, the pair $(A_i^*, B_i^*)_{G'}$ is $((\varepsilon')^{1/10}, d'/10)$-super-regular for every $1 \leqslant i \leqslant k$. Indeed, we ensured that every vertex v which was redistributed to A_i^* had at least $d'm'/8$ neighbours in B_i. Since $|B_i^* \Delta B_i| \leqslant 2(\varepsilon')^{1/3}m'$, v has at least $d'|B_i^*|/10$ neighbours in B_i^*. Similarly every vertex $w \in B_i^*$ has at least $d'|A_i^*|/10$ neighbours in A_i^*.

Now suppose we are given a partition $(n_i')_{i=1}^{2k}$ of n such that $n_i' \leqslant n_i + \xi n$ for each $1 \leqslant i \leqslant 2k$. Set $a_i := n_{2i-1}' - n_{2i-1}$ and $b_i := n_{2i}' - n_{2i}$ for all $1 \leqslant i \leqslant k$. Notice that $|a_i|, |b_i| \leqslant 2k\xi n$ for each $1 \leqslant i \leqslant k$ and $|\sum_{i=1}^k a_i| = |\sum_{i=1}^k b_i| \leqslant 2k\xi n$. Recall that R contains the edges $A_{i_1}^* A_{j_1}^*$ and $B_{i_2}^* B_{j_2}^*$. Thus, we can apply the Mobility Lemma (Lemma 6.1) with parameters $k, 2k\xi, (\varepsilon')^{1/10}, \varepsilon, d$ and $d'/10$ to obtain a partition $A_1', B_1', \ldots, A_k', B_k'$ of $V(G)$ which satisfies conditions (α_1)–(α_5). $\square$

8. The Lemma for H

Lemma 8.1 (Lemma for H). *For any $\Delta, k \in \mathbb{N}$ and $\xi > 0$, there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Let H be a bipartite graph on $n \geqslant n_0$ vertices with bandwidth at most βn and such that $\Delta(H) \leqslant \Delta$. Let $n_1, n_2, \dots, n_{2k}$ be an integer partition of n such that $n_i > n/(3k)$ for all $1 \leqslant i \leqslant 2k$ and $|n_{2i-1} - n_{2i}| \ll \xi n$ for all $1 \leqslant i \leqslant k$. Suppose C is the cycle $12\dots(2k)1$ on $[2k]$, and let $c = \{2i_1, 2i_2\}$ be a chord of C (for some distinct $1 \leqslant i_1, i_2 \leqslant k$). Then there exists a set $S \subseteq V(H)$ and a graph homomorphism $f : H \to C \cup \{c\}$, such that*

(β_1) $|S| \leqslant \xi n$,
(β_2) $|f^{-1}(i)| \leqslant n_i + \xi n$ *for all* $1 \leqslant i \leqslant 2k$,
(β_3) *every edge which is not in* $H[S]$ *is mapped to an edge* $\{2i-1, 2i\}$, *for some* $1 \leqslant i \leqslant k$.

Proof. Choose $\beta > 0$ and integers n_0, m_1, m_2 and k_1 such that

$$1/n_0 \ll \beta \ll 1/m_1 \ll 1/m_2 \ll 1/k_1 \ll 1/\Delta, 1/k, \xi.$$

Further, we may assume that m_2 divides m_1. We begin by defining a new cycle C' with chord c' which will act as an intermediate stage between C and H, that is, we will construct homomorphisms $f_1 : C' \cup \{c'\} \to C \cup \{c\}$ and $f_2 : H \to C' \cup \{c'\}$ such that $f = f_1 \circ f_2$ is our desired homomorphism. The homomorphism f_2 will be constructed to map roughly the same number of vertices to each vertex in C'. Notice, however, that our desired homomorphism f may not map vertices in an 'equal' way (since, in general, the n_i may be far from equal). Thus, the role of f_1 is to ensure f maps the 'correct' proportion of vertices to each vertex in C.

Let C' be the cycle $12\dots(2k')1$ on $[2k']$, where $k' := \sum_{i \in [k]} \lceil (n_{2i-1} + n_{2i}) k_1 / n \rceil$. Note that $k_1 \leqslant k' \leqslant k_1 + k$. We define f_1 as follows. For each $1 \leqslant j \leqslant k'$, let $g(j) \in \mathbb{N}$ be such that

$$\sum_{i=1}^{g(j)-1} \lceil (n_{2i-1} + n_{2i}) k_1 / n \rceil < j \leqslant \sum_{i=1}^{g(j)} \lceil (n_{2i-1} + n_{2i}) k_1 / n \rceil.$$

Then set $f_1(2j-1) = 2g(j) - 1$ and $f_1(2j) = 2g(j)$ for each $1 \leqslant j \leqslant k'$.

Recall that $c = \{2i_1, 2i_2\}$ is a chord of C. Suppose that i_1', i_2' are such that $f_1(2i_1') = 2i_1$ and $f_1(2i_2') = 2i_2$. Notice that as $i_1 \neq i_2$, we have that $i_1' \neq i_2'$. Thus, set $c' := \{2i_1', 2i_2'\}$ to be the chord of C'.

By construction $f_1(c') = c$. Given any edge $c_1 = \{2j-1, 2j\}$ on C', we have that $f_1(c_1) = \{2g(j)-1, 2g(j)\}$. Further, consider any edge $c_2 = \{2j, 2j+1\} = \{2j, 2(j+1)-1\}$ on C'. Then $f_1(2j) = 2g(j)$ and $f_1(2(j+1)-1) = 2g(j+1) - 1$. By definition of f_1, either $g(j+1) = g(j)$ or $g(j+1) = g(j) + 1$. But both $\{2g(j), 2g(j)-1\}$ and $\{2g(j), 2g(j)+1\}$ are edges of C. So in either case f_1 maps c_2 to an edge of C. Therefore, indeed f_1 is a graph homomorphism.

Roughly speaking, we will construct f_2 as follows. Initially we split H up into *small segments* $A_1, B_1, \dots, A_{m_1}, B_{m_1}$ in such a way that almost all of the edges of H lie in the pairs $(A_i, B_i)_{i=1}^{m_1}$ and the remainder lie in the pairs $(B_i, A_{i+1})_{i=1}^{m_1}$. Our ideal strategy would be to map all of the vertices of A_1 onto vertex 1 of C', the vertices of B_1 onto vertex 2, the vertices of A_2 onto vertex 3, and so on. This ensures that f_2 is a homomorphism and that almost all of the edges of H are mapped onto an edge of the form $\{2i-1, 2i\}$ for some i. However the number of vertices mapped onto each vertex of C' may vary widely. To solve this problem we introduce 'drunken' segments in which the assignment of the vertices is random, and use a probabilistic argument to show that with positive probability each vertex of C' receives approximately the same number of vertices of H. We also use the chord c' to 'turn around' at some point during the process, in order to eliminate the possible inequality between the number of vertices of H assigned to odd and even vertices of C'.

Chopping H up into segments. Since H has bandwidth at most βn, there exists an ordering $x_1, x_2, \dots, x_n$ of $V(H)$ such that for every edge $x_i x_j$ of H, $|i-j| \leqslant \beta n$. Let (A, B) be a bipartition of $V(H)$. We define $\{A_i, B_i\}_{i=1}^{m_1}$ as follows. For each vertex $x_s \in A$ there exists $1 \leqslant i \leqslant m_1$ such that $(i-1)n/m_1 - \beta n < s \leqslant in/m_1 - \beta n$ (unless $s > n - \beta n$). We assign x_s to A_i (or to A_{m_1} if $s > n - \beta n$). Similarly for each vertex $x_t \in B$ there exists $1 \leqslant j \leqslant m_1$ such that $(j-1)n/m_1 < t \leqslant jn/m_1$, and we assign x_t to B_j. Let S be the set of vertices x_s such that $in/m_1 - 2\beta n < s \leqslant in/m_1 + \beta n$ for some $1 \leqslant i \leqslant m_1$. Note that the following properties hold:

(a) $n/m_1 - \beta n \leqslant |A_i| + |B_i| \leqslant n/m_1 + \beta n$ for each $1 \leqslant i \leqslant m_1$,
(b) $|S| \leqslant 3m_1\beta n \ll \xi n$,
(c) every edge of H which is not in $H[S]$ lies in one of the pairs (A_i, B_i) for some $1 \leqslant i \leqslant m_1$,
(d) every edge of $H[S]$ lies in one of the pairs (A_i, B_i) or one of the pairs (B_i, A_{i+1}) for some $1 \leqslant i \leqslant m_1$ (where $A_{m_1+1} := A_1$).

Properties (c) and (d) follow from the fact that H has bandwidth at most βn and that $n/m_1 \gg \beta n$. We now modify the small segments so that properties (a)–(d) are still satisfied and so that every small segment has size at least $n/(4\Delta m_1)$. Suppose a small segment A_i has size smaller than $n/(4\Delta m_1)$. Note that $|N_H(A_i)| \leqslant n/(4m_1)$ and so

$$|B_i \backslash (S \cup N_H(A_i))| \overset{(a)}{\geqslant} (n/m_1 - \beta n - n/(4\Delta m_1)) - 3\beta n - n/(4m_1) \geqslant n/(4m_1).$$

But (c) implies that any vertex in $B_i \backslash (S \cup N_H(A_i))$ must be isolated in H and so may be reassigned to A_i without affecting properties (a)–(d). Hence we may reassign sufficiently many vertices so that $|A_i|, |B_i| \geqslant n/(4\Delta m_1)$. For any segment B_i which has size smaller than $n/(4\Delta m_1)$, we proceed in an identical way. From now on we denote by A the union of small segments $\bigcup_{i=1}^{m_1} A_i$ and by B the union $\bigcup_{i=1}^{m_1} B_i$.

We now group the small segments together to form *large segments* $\{L_j\}_{j=1}^{m_2}$, defined by

$$L_j := \bigcup_{\frac{(j-1)m_1}{m_2} < t \leqslant \frac{jm_1}{m_2}} (A_t \cup B_t).$$

Note that since $\beta \ll 1/m_1$, (a) implies that

$$\frac{n}{m_2} - \sqrt{\beta} n \leqslant |L_j| \leqslant \frac{n}{m_2} + \sqrt{\beta} n \tag{8.1}$$

for each $1 \leqslant j \leqslant m_2$. In order to eliminate any inequality between the number of vertices of H assigned to odd and even vertices of C', we need to partition $\{L_j\}_{j=1}^{m_2}$ into two parts. We will assign the vertices in each part separately. For each $1 \leqslant j \leqslant m_2$, set $s_j := |L_j \cap A| - |L_j \cap B|$. Note that $|s_j| \leqslant n/m_2 + \sqrt{\beta} n - n/(4\Delta m_2) \leqslant n/m_2$ for each $1 \leqslant j \leqslant m_2$, and that $\sum_{j=1}^{m_2} s_j = |A| - |B|$. Suppose without loss of generality that $|A| - |B| \geqslant 0$. Then, since $\beta, 1/m_2 \ll \xi$ there exists an integer m_3 so that $\xi m_2/20 \leqslant m_3 \leqslant (1-\xi/20)m_2$ and

$$\frac{(|A|-|B|)}{2} - \frac{\xi n}{20} \leqslant \sum_{i=1}^{m_3} s_j \leqslant \frac{(|A|-|B|)}{2} + \frac{\xi n}{20}. \tag{8.2}$$

We will embed separately the large segments $\{L_j\}_{j=1}^{m_3}$ and the segments $\{L_j\}_{j=m_3+1}^{m_2}$.

For each $1 \leqslant j \leqslant m_3$, let the *drunken segment* D_j be the union of the last $k_2 := \xi m_1/(6k'm_2)$ pairs of small segments in L_j and let the *sober segment* S_j be the union of the rest of the small segments in L_j.

Defining our algorithms. We now define three different algorithms for assigning the vertices of a segment to vertices of C', given an *initial vertex* $2i'_0 - 1 \in C'$. In each case we work mod $2k'$ when dealing with vertices of C'. The *sober algorithm* is a deterministic process which proceeds as follows. Let S_j be a sober segment whose first small segments are A_{i_0} and B_{i_0}. For every i such that A_i and B_i are small segments of S_j, assign every vertex of A_i to the vertex $2i'-1$ of C' and every vertex of B_i to the vertex $2i'$ of C' where $i' \equiv i'_0 + i - i_0 \pmod{k'}$. So whenever B_i is assigned to $2i'$, A_{i+1} is assigned to $2i'+1 \pmod{2k'}$. We call the vertex $2i^*$ of C' to which the vertices of the last small segment of S_j are assigned the *final vertex* of the algorithm and define this term in a similar way for the remaining two algorithms.

The *drunken algorithm* is a randomized algorithm which proceeds as follows. Given a drunken segment D_j whose first small segment is A_{i_0}, assign every vertex of A_{i_0} to the vertex $2i'_0 - 1$ of C' and every vertex of B_{i_0} to the vertex $2i'_0$ of C'. Then, for every pair A_{i+1}, B_{i+1} of small segments in D_j, let $2i'$ be the vertex to which the vertices of B_i were assigned and let

$$i'' = \begin{cases} i' & \text{with probability } \frac{1}{2}, \\ i'+1 & \text{with probability } \frac{1}{2}. \end{cases}$$

(All random choices are made independently.) Assign every vertex of A_{i+1} to $2i''-1$ and every vertex of B_{i+1} to $2i''$.

Claim 8.2. *Suppose that the vertices of D_j are assigned using the drunken algorithm with initial vertex $2i'_0 - 1$, and let $i'_1 \in [k']$ be arbitrary. Let the random variable I be the final vertex of the drunken algorithm. Then*

$$\mathbb{P}[I = 2i'_1 \mid i'_0] \leqslant \frac{1+\xi/20}{k'}.$$

To prove the claim, note that $I \sim 2(i'_0 + \mathrm{Bin}(k_2, 1/2))$, that $k_2 \gg (k')^3/6$ and that $1/k' \ll \xi/20$. So Lemma 5.5 with $\varepsilon := \xi/20$ implies that $\mathbb{P}[I - 2i'_0 = 2i'_1 - 2i'_0] \leqslant (1+\xi/20)/k'$, and Claim 8.2 follows immediately.

The $2i'_1$*-seeking algorithm* is a deterministic algorithm which proceeds as follows. Given a drunken segment D_j whose first small segment is A_{i_0}, assign every vertex of A_{i_0} to the vertex $2i'_0 - 1$ of C' and every vertex of B_{i_0} to the vertex $2i'_0$ of C'. Then, for every pair A_{i+1}, B_{i+1} of small segments in D_j, let $2i'$ be the vertex of C' to which the vertices of B_i were assigned and let

$$i'' = \begin{cases} i' & \text{if } i' = i_1, \\ i'+1 & \text{otherwise.} \end{cases}$$

Assign every vertex of A_{i+1} to $2i''-1$ and every vertex of B_{i+1} to $2i''$. Note that the final vertex of this algorithm is always $2i_1'$, since $k' \leqslant \xi m_1/(6k'm_2)$.

Applying the algorithms. We use these algorithms to assign small segments to vertices of C' as follows. Choose $i_0' \in [k']$ randomly and let $2i_0'-1$ be the initial vertex for S_1. For each $1 \leqslant j \leqslant m_3 - 1$, use the sober algorithm to assign the segments of S_j and then use the drunken algorithm to assign the vertices of D_j, where in each case the initial vertex of each segment is the successor of the final vertex of the previous segment. (So, for example, if the final vertex of the drunken algorithm, when applied to D_j, is $2i^*$, then the initial vertex of the sober algorithm, when applied to S_{j+1}, is $2i^*+1$.) Then assign the vertices of S_{m_3} using the sober algorithm and assign the vertices of D_{m_3} using the $2i_1'$-seeking algorithm. (Recall that $2i_1'$ was a vertex of the chord c'.) We explain later how we assign the small segments from $\bigcup_{j=m_3+1}^{m_2} L_j$.

Claim 8.3. *For each $1 \leqslant i \leqslant k'$, let X_i be the number of vertices of $\bigcup_{j=1}^{m_3} S_j$ assigned to the vertex $2i-1$ of C'. Then*

$$\mathbb{P}\left[X_i > \frac{1}{2k'}\left(\frac{m_3 n}{m_2} + \frac{|A|-|B|}{2}\right) + \frac{\xi n}{6k'}\right] \leqslant \frac{1}{3k'}.$$

For each $1 \leqslant j \leqslant m_3$, let $Y_j = |S_j \cap A|$ and let $X_{i,j}$ be the number of vertices of S_j which are assigned to $2i-1$. To prove the claim, we first use Claim 8.2 to bound $\mathbb{E}[X_{i,j} \mid X_{i,j-1},\dots,X_{i,1}]$. Let r_j be the initial vertex of S_j for each j. Let B be the event $X_{i,j-1} = x_{i,j-1},\dots,X_{i,1} = x_{i,1}$ for some $x_{i,1},\dots x_{i,j-1}$. Now for $1 \leqslant i' \leqslant k'$ and any integer x we have

$$\mathbb{P}[X_{i,j} = x \mid B \cap (r_{j-1} = 2i'-1)] = \mathbb{P}[X_{i,j} = x \mid r_{j-1} = 2i'-1].$$

Hence Lemma 5.7 implies that

$$\mathbb{E}[X_{i,j} \mid X_{i,j-1},\dots,X_{i,1}] \leqslant \max_{1\leqslant i'\leqslant k'} \mathbb{E}[X_{i,j} \mid r_{j-1} = 2i'-1].$$

But Claim 8.2 implies that

$$\begin{aligned}\mathbb{E}[X_{i,j} \mid r_{j-1} = 2i'-1] &= \sum_{i''=1}^{k'} \mathbb{E}[X_{i,j} \mid r_j = 2i''-1]\mathbb{P}[r_j = 2i''-1 \mid r_{j-1} = 2i'-1] \\ &\leqslant \frac{1+\xi/20}{k'} \sum_{i''=1}^{k'} \mathbb{E}[X_{i,j} \mid r_j = 2i''-1] = \frac{(1+\xi/20)Y_j}{k'}.\end{aligned}$$

Hence

$$\mathbb{E}[X_{i,j} \mid X_{i,j-1},\dots,X_{i,1}] \leqslant (1+\xi/20)Y_j/k'.$$

Set $X_{i,j}' := X_{i,j} m_2/n$. Since

$$|S_j| \overset{(8.1)}{\leqslant} \left(\frac{n}{m_2} + \sqrt{\beta} n\right) - \left(\frac{\xi m_1}{6k'm_2}\right)\left(\frac{n}{4\Delta m_1}\right) \leqslant \frac{n}{m_2},$$

we have that $X'_{i,j} \in [0,1]$, for each $1 \leqslant j \leqslant m_3$. Let

$$\mu = \sum_{j=1}^{m_3} \frac{(1+\xi/20)Y_j m_2}{k'n}, \tag{8.3}$$

and note that

$$\sum_{j=1}^{m_3} Y_j \leqslant \sum_{j=1}^{m_3} |A \cap L_j| = \frac{1}{2}\left(\sum_{j=1}^{m_3} |L_j| + s_j\right) \tag{8.4}$$

$$\overset{(8.1),(8.2)}{\leqslant} \left(\frac{m_3 n}{2m_2} + \frac{m_3\sqrt{\beta}n}{2}\right) + \left(\frac{|A|-|B|}{4} + \frac{\xi n}{40}\right) \leqslant \frac{m_3 n}{2m_2} + \frac{|A|-|B|}{4} + \frac{\xi n}{20}.$$

Note also that $Y_j \geqslant (n/(4\Delta m_1)) \times (m_1/2m_2) = n/(8\Delta m_2)$ for each $1 \leqslant j \leqslant m_1$. Thus we have $\mu \geqslant m_3/(8\Delta k') \gg (\log k')/\xi^2$. We now apply Lemma 5.6 with $\delta := \xi/20$ to obtain

$$\mathbb{P}\left[\sum_{j=1}^{m_3} X'_{i,j} > (1+\xi/20)\mu\right] \leqslant e^{-\frac{\xi^2\mu}{1200}} \leqslant \frac{1}{3k'}.$$

It follows that with probability at least $1 - 1/3k'$,

$$X_i \overset{(8.3)}{\leqslant} \frac{(1+\xi/20)^2}{k'} \sum_{i=1}^{m_3} Y_j \overset{(8.4)}{\leqslant} \frac{(1+\xi/20)^2}{k'}\left(\frac{m_3 n}{2m_2} + \frac{|A|-|B|}{4} + \frac{\xi n}{20}\right)$$

$$\leqslant \frac{1}{2k'}\left(\frac{m_3 n}{m_2} + \frac{|A|-|B|}{2}\right) + \frac{\xi n}{6k'},$$

which proves Claim 8.3.

By a similar argument we have that if X'_i is the number of vertices of $\bigcup_{j=1}^{m_3} S_j$ assigned to $2i$, then

$$\mathbb{P}\left[X'_i > \frac{1}{2k'}\left(\frac{m_3 n}{m_2} + \frac{|B|-|A|}{2}\right) + \frac{\xi n}{6k'}\right] \leqslant \frac{1}{3k'}$$

for every $1 \leqslant i \leqslant k'$. Taken together with Claim 8.3, this implies that with probability at least $1/3$,

$$X_i \leqslant \frac{1}{2k'}\left(\frac{m_3 n}{m_2} + \frac{|A|-|B|}{2}\right) + \frac{\xi n}{6k'} \quad \text{and} \quad X'_i \leqslant \frac{1}{2k'}\left(\frac{m_3 n}{m_2} + \frac{|B|-|A|}{2}\right) + \frac{\xi n}{6k'} \tag{8.5}$$

for every $1 \leqslant i \leqslant k'$, and hence there exists an assignment such that (8.5) holds.

For each $m_3 < j \leqslant m_2$, let D_j be the union of the *first* k_2 small segments of L_j and S_j the union of the remaining small segments. We now assign the vertices of $\bigcup_{j=m_3+1}^{m_2} L_j$ using an algorithm similar to that for $\bigcup_{j=1}^{m_3} L_j$, but in reverse order. That is, we first choose $1 \leqslant i''_0 \leqslant k'$ randomly and assign the vertices of S_{m_2} using the sober algorithm, but with the roles of A_i and B_i exchanged for each i. Thus we assign the vertices of B_{m_1} to $2i''_0 - 1$, the vertices of A_{m_1} to $2i''_0$, and so on. Similarly we use the drunken algorithm to assign the vertices of D_{m_2} (again with the roles of A_i and B_i exchanged for each i), and so on until we have assigned all the vertices up to S_{m_3+1}. (As before, the initial vertex of any application of an algorithm is the successor of the final vertex of the previous application of an algorithm.) Finally we use the $2i'_2$-seeking algorithm to assign the vertices of the last drunken segment D_{m_3+1}. (Recall that the final vertex of the $2i_2'$-seeking algorithm is always $2i_2'$.) Let $\overline{X_i}$ be the number of vertices of $\bigcup_{j=m_3+1}^{m_2} S_j$ assigned to to $2i-1$ and $\overline{X_i'}$ the number assigned to $2i$. By using a proof analogous to that of Claim 8.3, we can ensure that

$$\begin{aligned}\overline{X_i} &\leqslant \frac{1}{2k'}\left(\frac{(m_2-m_3)n}{m_2}+\frac{|B|-|A|}{2}\right)+\frac{\xi n}{6k'} \quad \text{and}\\ \overline{X_i'} &\leqslant \frac{1}{2k'}\left(\frac{(m_2-m_3)n}{m_2}+\frac{|A|-|B|}{2}\right)+\frac{\xi n}{6k'}\end{aligned} \tag{8.6}$$

for each $1 \leqslant i \leqslant k'$.

Note that

$$\left|\bigcup_{j=1}^{m_2} D_j\right| \leqslant m_2 \times \frac{\xi m_1}{6k'm_2} \times (n/m_1+\beta n) \leqslant \xi n/3k',$$

and hence in total we assign at most

$$\begin{aligned}&X_i + X_i' + |\bigcup_{j=1}^{m_2} D_j|\\ &\overset{(8.5),(8.6)}{\leqslant} \frac{1}{2k'}\left(\frac{m_3 n}{m_2}+\frac{|A|-|B|}{2}\right)+\frac{1}{2k'}\left(\frac{(m_2-m_3)n}{m_2}+\frac{|B|-|A|}{2}\right)+\frac{2\xi n}{3k'}\\ &= \frac{1}{2k'}(1+4\xi/3)n\end{aligned}$$

vertices to $2i-1$ and at most $(1+4\xi/3)n/2k'$ vertices to $2i$, for each $1 \leqslant i \leqslant k'$.

Completing the proof. This completes our definition of f_2. We now check that f_2 is a homomorphism. By properties (c) and (d) it suffices to show for each i that whenever A_i and B_i (or B_i and A_{i+1}) are assigned to vertices $1 \leqslant i', j' \leqslant 2k'$ of C', then $i'j'$ is an edge of $C' \cup \{c'\}$. Observe first that the sober, drunken and seeking algorithms all assign vertices in such a way that $i'j'$ is an edge of C'. Further, recall that the initial vertex of any application of an algorithm is the successor of the final vertex of the previous application of an algorithm. So if, for example, the vertices of B_i are assigned to the final vertex $2i^*$ where B_i is the final segment assigned in an application one of the algorithms, then the vertices of A_{i+1} will be assigned to the initial vertex $2i^*+1$ in the next application of an algorithm.

The only pair this argument does not deal with is the pair (B_j, A_{j+1}), where B_j is the last small segment of D_{m_3} (and hence A_{j+1} is the first small segment of D_{m_3+1}, and therefore the last to be assigned). Now by the definition of the seeking algorithm, the vertices of B_j are assigned to the vertex $2i_1'$ of C' and the vertices of A_{j+1} are assigned to the vertex $2i_2'$ of C'. Recalling that $c' = \{2i_1', 2i_2'\}$, we have that f_2 is indeed a homomorphism.

Now consider $f = f_1 \circ f_2$. Since f_1 and f_2 are both homomorphisms we have that f is a homomorphism. Property (b) implies that condition (β_1) holds. By (c), every edge xy not in $H[S]$ lies in a pair (A_i, B_i) for some i. Thus, by definition of our three algorithms, xy is mapped to an edge $\{2j-1, 2j\}$ by f_2 for some j. By definition of f_1, $\{2j-1, 2j\}$ is mapped to $\{2j'-1, 2j'\}$ by f_1 for some j'. Therefore, f satisfies (β_3).

To see that condition (β_2) also holds, recall that f_1 assigns to each vertex $2i-1$ of C (and also to $2i$) exactly $\lceil (n_{2i-1}+n_{2i})k_1/n \rceil$ vertices of C'. Hence f assigns at most

$$\begin{aligned}\frac{1}{2k'}(1+4\xi/3)n\lceil (n_{2i-1}+n_{2i})k_1/n\rceil &\leqslant (1+4\xi/3)(n_{2i-1}+\xi n/10)+\frac{1}{2k'}(1+4\xi/3)n\\ &\leqslant n_{2i-1}+\xi n\end{aligned}$$

vertices of H to the vertex $2i-1$ of C, for each $1 \leqslant i \leqslant k$. Similarly f assigns at most $n_{2i}+\xi n$ vertices of H to the vertex $2i$. $\square$

9. Completing the proof

In this section we use Lemmas 7.1 and 8.1 to prove Theorem 1.8. We use the following definition and lemma from [3]; these allow us to prove that H embeds into G by checking some relatively simple conditions.

Definition 9.1. Let H be a graph on n vertices, let R be a graph on $[k]$, and let $R' \subseteq R$. We say that a vertex partition $V(H) = (W_i)_{i\in[k]}$ of H is *ε-compatible* with an integer partition $(n_i)_{i\in[k]}$ of n and $R' \subseteq R$ if the following holds. For $i \in [k]$ let S_i be the set of vertices in W_i with neighbours in some W_j with $ij \notin E(R')$ and $i \neq j$. Set $S := \bigcup_{i\in[k]} S_i$ and $T_i := N_H(S) \cap (W_i \backslash S)$. Then, for all $i, j \in [k]$ we have that:

(γ_1) $|W_i| = n_i$,
(γ_2) $xy \in E(H)$ for $x \in W_i$ and $y \in W_j$ implies that $ij \in E(R)$,
(γ_3) $|S_i| \leqslant \varepsilon n_i$ and $|T_i| \leqslant \varepsilon \cdot \min\{n_j \mid i \text{ and } j \text{ are in the same component of } R'\}$.

The partition $V(H) = (W_i)_{i\in[k]}$ is *ε-compatible* with a partition $V(G) = (V_i)_{i\in[k]}$ of a graph G and $R' \subseteq R$ if $V(H) = (W_i)_{i\in[k]}$ is *ε-compatible* with $(|V_i|)_{i\in[k]}$ and $R' \subseteq R$.

Lemma 9.2 ([3], Lemma 3.12). *For all $d, \Delta, r > 0$ there is a constant $\varepsilon = \varepsilon(d, \Delta, r)$ such that the following holds. Let G be a graph on n vertices and suppose that $(V_i)_{i\in[k]}$ is a partition of $V(G)$. Suppose R is an (ε, d)-reduced graph of G on $V_1, \dots, V_k$ and that R' is a subgraph of R whose connected components have size at most r. Assume that $(V_i, V_j)_G$ is an (ε, d)-super-regular pair for every edge $V_iV_j \in E(R')$. Further, let H be a graph on n vertices with maximum degree $\Delta(H) \leqslant \Delta$ that has a vertex partition $V(H) = (W_i)_{i\in k}$ which is ε-compatible with $V(G) = (V_i)_{i\in[k]}$ and $R' \subseteq R$. Then $H \subseteq G$.*

Lemma 9.2 is a consequence of the Blow-Up Lemma of Komlós, Sárközy and Szemerédi [19]. We now prove Theorem 1.8.

Proof of Theorem 1.8. Firstly, note that it suffices to prove Theorem 1.8 under the additional assumption that $\eta \ll 1$. We choose β, n_0 as well as additional constants $d, \varepsilon, \xi, \lambda$, as follows. Choose $d \ll \nu$ as required by Lemma 7.1 also ensuring that $d \ll 1/\Delta$. Then take $\varepsilon \leqslant \varepsilon(d, \Delta, 2)$ as in Lemma 9.2, ensuring also that $\varepsilon \ll d$. Finally, choose

$$1/n_0 \ll \beta \ll \lambda \ll \xi \ll \varepsilon,$$

as required by Lemmas 7.1 and 8.1.

Apply Lemma 7.1 to G to obtain an integer k such that $\xi \ll 1/k \ll \varepsilon$, a partition $(n_i)_{i=1}^{2k}$ of n and integers $1 \leqslant i_1 \neq j_1, i_2 \neq j_2 \leqslant k$. Suppose C is the cycle $12\dots(2k)1$ with the chord $c = \{2i_2, 2j_2\}$. Next we apply Lemma 8.1 with the partition $(n_i)_{i=1}^{2k}$ of n as input to obtain a set $S \subseteq V(H)$ and a homomorphism $f : H \to C \cup \{c\}$, such that:

(i) $|S| \leqslant \xi n$,
(ii) $|f^{-1}(i)| \leqslant n_i + \xi n$ for all $1 \leqslant i \leqslant 2k$,
(iii) every edge which is not in $H[S]$ is mapped to the edge $\{2i-1, 2i\}$, for some $1 \leqslant i \leqslant k$.

Let $W_i := f^{-1}(i)$ and $n'_i := |f^{-1}(i)|$ for each i, and note that $(n'_i)_{i=1}^{k}$ is a partition of n. Condition (ii) together with Lemma 7.1 imply that there is a partition $A'_1, B'_1, A'_2, B'_2, \dots, A'_k, B'_k$ of $V(G)$ and a spanning subgraph G' of G which satisfy conditions (α_1)–(α_5).

Relabel these clusters $V_1, \dots, V_k$ such that $V_{2i-1} := A'_i$ and $V_{2i} := B'_i$ for all $1 \leqslant i \leqslant k$. So $|V_i| = n'_i$ for all $1 \leqslant i \leqslant 2k$. Let R be the (ε, d)-reduced graph of G' on $V_1, \dots, V_k$ with the maximal number of edges. Hence (α_2)–(α_5) imply that R contains the Hamilton cycle $C' = V_1V_2\dots V_{2k}V_1$ and the chord $c' := V_{2i_2}V_{2j_2}$ (we view $C' \cup \{c'\}$ as a copy of $C \cup \{c\}$ in R). Let R' be the spanning subgraph of R containing precisely the edges $V_{2i-1}V_{2i}$ for $1 \leqslant i \leqslant k$. Note that (α_2) implies that $(V_{2i-1}V_{2i})_{G'}$ is an (ε, d)-super-regular pair for all $1 \leqslant i \leqslant k$.

We now check that the partition $V(H) = (W_i)_{i=1}^{2k}$ is ε-compatible with the partition $(V_i)_{i=1}^{2k}$ and $R' \subseteq R$. We defined $(V_i)_{i=1}^{2k}$ so that $|V_i| = |W_i|$ for each $1 \leqslant i \leqslant 2k$ and hence condition (γ_1) of Definition 9.1 holds. Condition (γ_2) holds since $f : H \to C \cup \{c\}$ is a homomorphism and $C \cup \{c\}$ is a subgraph of R. Note that for all $1 \leqslant i \leqslant 2k$, Lemma 7.1 implies that

$$\varepsilon n'_i \geqslant \varepsilon(n_i - 2k\xi n) \geqslant \varepsilon(n/3k - 2k\xi n) \geqslant \varepsilon n/4k \gg \xi n \overset{(i)}{\geqslant} |S|.$$

Furthermore, $|N_H(S) \cap (W_i \backslash S)| \leqslant \Delta|S| \leqslant \Delta\xi n \ll \varepsilon n/4k \leqslant \varepsilon n'_j$ for all $1 \leqslant i, j \leqslant 2k$. Thus, condition (γ_3) holds. Hence, Lemma 9.2 implies that G' (and therefore G) contains H, as desired. □

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