ON THE DIAMETER OF A p-CYCLIC STRONGLY CONNECTED DIGRAPH

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1. Introduction. In this paper we follow the notation of **(2)**. In **(5)**, Luce showed, in other terminology, that if d is the diameter of a strongly connected digraph, D, on n vertices with m edges, then

$$(1.1) \qquad 2m \leqslant 2n^2 + 2n - 4 + d^2 - d - 2nd,$$

this inequality being sharp; from (1.1) one may immediately derive sharp upper bounds for d in terms of m and n, this being a generalization of the obvious and well-known inequality

$$(1.2) \qquad d \leqslant n - 1.$$

The purpose of this paper is to obtain a better inequality than (1.1) when D is in addition $2k$-cyclic for some k, that is (for our purposes) D contains directed cycles of lengths whose greatest common divisor is $2k$. We shall in fact show that, if D is $2k$-cyclic, then

$$(1.3) \qquad 4m \leqslant 2n^2 - 2nd + d^2 + 6n - 2d - 8 + \begin{cases} 0, & \text{if } d \text{ is even,} \\ 1, & \text{if } d \text{ is odd and } n \text{ is even,} \\ -1, & \text{if } d \text{ is odd and } n \text{ is odd,} \end{cases}$$

this inequality being the best possible if $k = 1$, that is, if D is 2-cyclic. From (1.3) one may obtain sharper bounds for the diameter of $2k$-cyclic digraphs than those obtainable from (1.1).

The appropriate inequality for general p-cyclic matrices is implicitly stated, in § 2, in the form of an integer quadratic-programming problem. This, as is implied above, is explicitly solved, however, only for $p = 1$ and $p = 2$ (the case $p = 1$ is the result of Luce, which is rederived in § 3, partly for the sake of completeness, but mainly because the method of proof is new and is illustrative of the proof for the case $p = 2$).

Knowledge about the diameter of a strong digraph is important in various applications of graph theory where structural questions are concerned (see, for example, any of the references at the end of this paper).

2. A fundamental lemma. In this section we prove:

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LEMMA 1. *Let d be the diameter of D, a p-cyclic, strongly connected digraph on n vertices and with m edges; then**

$$(2.1)\quad m \leqslant \max\left\{\sum_{r=0}^{d-1} n_r n_{r+1} + \sum_{r=p-1}^{d}\ \sum_{s=1}^{[(r+1)/p]} n_r n_{r-sp+1}\right\} - \begin{cases} 0, & \text{if } p > 1, \\ n, & \text{if } p = 1, \end{cases}$$

where $n_0 = 1$ and the maximum is taken over all positive integers $\{n_s\}$ $(s = 1, \ldots, d)$ such that

$$(2.2)\quad \sum_{s=1}^{d} n_s = n - 1.$$

Proof. Let u be the initial vertex of any diametral path, that is a directed path whose length equals d. Define disjoint sets of vertices $\{U_i\}$ $(i = 0, 1, \ldots, d)$ by

$$(2.3)\quad \begin{aligned} U_0 &= \{u\}, \\ U_{i+1} &= \{v \mid v \text{ is adjacent from some vertex in } U_i \text{ and } v \notin U_s \text{ for any } s < i\}. \end{aligned}$$

From the definition of diameter and diametral path, it is clear that every vertex of D belongs to some U_i. Let

$$(2.4)\quad n_i = |U_i|$$

be the number of vertices in U_i, so that $n_0 = 1$ and so that (2.2) also holds.

Now there are at most $n_r n_{r+1}$ $(r = 0, 1, \ldots, d - 1)$ edges connecting vertices of U_r to vertices of U_{r+1}. Furthermore, since u is the initial vertex of a diametral path, there can be no edges connecting vertices of U_r to any vertices of U_s if $s > r + 1$. Finally, vertices of U_r can only be connected to vertices of U_t with $t \leqslant r$ if $t = r - sp + 1$ for some $s > 0$; otherwise D could not be p-cyclic, and there are at most $n_r n_{r-sp+1}$ such edges. The inequality (2.1) now follows, the adjustment for $p = 1$ being present to ensure that we do not count loops in the previous sentence.

As stated in the Introduction, we can only solve the integer quadratic-programming problem defined by (2.1) and (2.2) for $p = 1$ or $p = 2$. However, from the construction used in proving the above lemma, it is clear the case $p = 1$ will provide a valid inequality for *all* strong digraphs, whereas the case $p = 2$ will provide a valid inequality for all strong digraphs which are cyclic of even order. These inequalities will only be sharp if $p = 1$ or 2, respectively.

3. Luce's theorem. In this section we establish the result of Luce **(5)**, viz.

THEOREM 1 (Luce). *If D is a strongly connected digraph on n vertices with m edges and diameter d, then*

*[ ] denotes the least integer function.

$$2m \leqslant 2n^2 + 2n - 4 + d^2 - d - 2nd, \tag{3.1}$$

this inequality being best possible.

Proof. Letting $p = 1$ in (2.1), we have

$$m \leqslant \max g_1(n_1, n_2, \ldots, n_d), \qquad n_0 = 1, \qquad n_1 + \ldots + n_d = n - 1, \tag{3.2}$$

where

$$\begin{aligned} 2g_1(n_1, \ldots, n_d) &= 2\sum_{r=0}^{d-1} n_r n_{r+1} + 2\sum_{r=0}^{d} \sum_{s=1}^{r+1} n_r n_{r-s+1} - 2n \\ &= (n-1)^2 + \sum_{r=1}^{d} n_r^{\,2} + 2\sum_{r=1}^{d} n_r n_{r-1} \\ &= (n-1)^2 + \theta \text{ (say)}. \end{aligned}$$

However, since $n_d \geqslant 1$,

$$\begin{aligned} \theta - 6(n-1) + 3d &\leqslant \theta - 4(n-1) - 2(n - n_d) + 3d \\ &= \sum_{r=1}^{d} (n_r - 1)(n_r + 2n_{r-1} - 3) \\ &= \sum_{r=1}^{d} (n_r - 1)[(n_r - 1) + 2(n_{r-1} - 1)] \\ &\leqslant \left[\sum_{r=1}^{d} (n_r - 1)\right]^2 \\ &= (n - 1 - d)^2, \end{aligned} \tag{3.3}$$

with equality throughout if $n_0 = n_2 = n_3 = \ldots = n_d = 1$ and $n_1 = n - d$. After some minor algebraic manipulation, the theorem follows.

COROLLARY 1. *Under the hypothesis of Theorem* 1
(i) $d \leqslant n - 1$ *if* $n \leqslant m \leqslant n(n + 1)/2 - 1$,
(ii) $d \leqslant n - \sigma$ *if* $(n + 1)/2 \leqslant m \leqslant n^2 - n - 2$,
(iii) $d = 2$ *if* $m = n^2 - n - 1$,
(iv) $d = 1$ *if* $m = n^2 - n$,
where σ *is the largest positive integer such that*

$$\sigma(\sigma - 1) < 2(m + 2) - n(n + 1).$$

Furthermore, these bounds are best possible.

Proof. This follows from (3.1). The fact that these bounds are best possible follows from the constructive proofs of Lemma 1 and Theorem 1. In fact, given m and n, the digraph for which the bounds in Corollary 1 are sharp is that illustrated symbolically in Figure 1, in which the left-pointing arrow implies that there is an edge from every vertex to every vertex to the left of it. Clearly we have equality in (3.1) for this digraph, and the sharpness of the bounds in the corollary then follows.

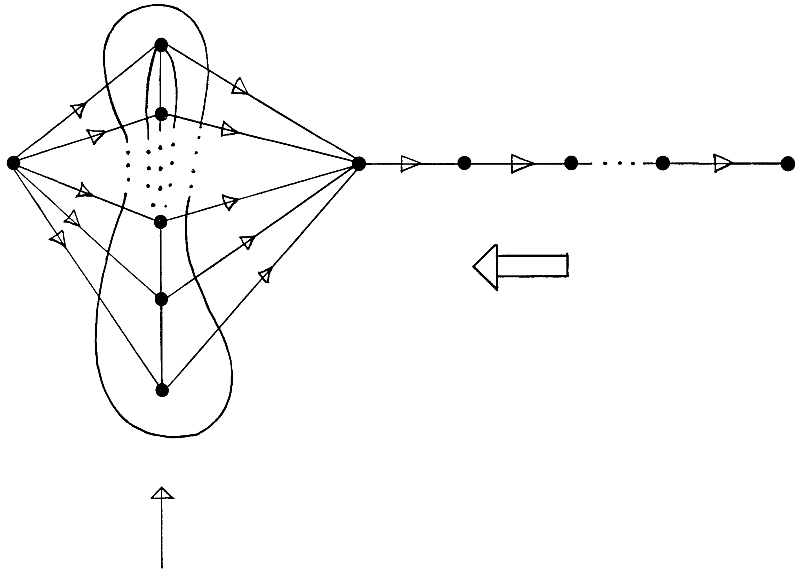

FIGURE 1

4. The $2k$-cyclic case. We now prove the main theorem of this paper.

THEOREM 2. *If d is the diameter of a $2k$-cyclic, strong digraph, D, with n vertices and m edges, then*

$$(4.1) \qquad 4m \leqslant 2n^2 - 2nd + d^2 + 6n - 2d - 8 + \begin{cases} 0, & \text{if } d \text{ is even,} \\ 1, & \text{if } d \text{ is odd and } n \text{ is even,} \\ -1, & \text{if } d \text{ is odd and } n \text{ is odd.} \end{cases}$$

Furthermore, this inequality is sharp, in the sense that, given m and n, there is a 2-cyclic strong digraph with diameter d satisfying (4.1) *with equality.*

Proof. From (2.1), with $p = 2$ we have that

$$(4.2) \quad m \leqslant \max g_2(n_1, n_2, \ldots, n_d), \qquad n_0 = 1, n_1 + n_2 + \ldots + n_d = n - 1,$$

where

$$(4.3) \qquad g_2(n_1, n_2, \ldots, n_d) = \sum_{r=0}^{d-1} n_r\, n_{r+1} + \sum_{r=2}^{d} \sum_{s=1}^{[\frac{1}{2}(r+1)]} n_r\, n_{r-2s+1}.$$

From the method of proof of Lemma 1, it is clear that (4.2) also holds for $2k$-cyclic strong digraphs for *all* $k \geqslant 1$, except that we may only achieve equality when $k = 1$.

We prove the following lemma.

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LEMMA 2.

$$g_2(n_1, n_2, \ldots, n_d) = XY + \sum_{r=0}^{d-1} n_r\, n_{r+1}$$

where

(4.4) $$X = n_0 + n_2 + n_4 + \ldots$$

and

$$Y = n_1 + n_3 + n_5 + \ldots .$$

We prove instead that

$$\sum_{r=1}^{d} \sum_{s=1}^{[\frac{1}{2}(r+1)]} n_r\, n_{r-2s+1} = XY,$$

from which the lemma follows at once.

Proof. Under the summation conditions we note that $r \geqslant r - 2s + 1 \geqslant 0$, and that r and $r - 2s + 1$ are of opposite parity. Indeed as s sweeps through the integers from 1 to $[\frac{1}{2}(r + 1)]$, $r - 2s + 1$ runs through the non-negative integers that are less than r and of opposite parity to r. The lemma follows at once.

Returning to the proof of Theorem 2, we must consider two cases:
Case 1, *d even*. We have

(4.5) $$\begin{aligned} g_2 &= XY + \sum_{s=0}^{\frac{1}{2}d-1} n_{2s+1}(n_{2s} + n_{2s+2}) \\ &= XY + \sum_{s=0}^{\frac{1}{2}d-1} (n_{2s+1} - 1)(n_{2s} - 1 + n_{2s+2} - 1) \\ &\qquad\qquad + 2Y + 2X - n_d - 1 - d \\ &\leqslant \left\{ \sum_{s=0}^{\frac{1}{2}d-1} (n_{2s+1} - 1) \right\} \left\{ \sum_{s=0}^{\frac{1}{2}d} (n_{2s} - 1) \right\} \\ &\qquad\qquad + XY + 2n - n_d - 1 - d \end{aligned}$$

(since $X + Y = n$)

(4.6) $$\begin{aligned} &= (Y - \tfrac{1}{2}d)(X - \tfrac{1}{2}d - 1) + XY + 2n - n_d - 1 - d \\ &\leqslant 2XY - Y + 2n - 2 - \tfrac{1}{2}d + \tfrac{1}{4}d^2 - \tfrac{1}{2}nd. \end{aligned}$$

(Note: We have equality throughout if

$$\begin{aligned} n_0 = n_2 = n_4 = \ldots = n_{d-4} = n_d &= 1, \\ n_1 = n_3 = n_5 = \ldots = n_{d-3} &= 1, \\ n_{d-2} = X - \tfrac{1}{2}d, \qquad n_{d-1} &= Y + 1 - \tfrac{1}{2}d). \end{aligned}$$

But, since $X + Y = N$,

(4.7) $$Y(2X - 1) = (2n + 1)X - 2X^2 - n \leqslant \tfrac{1}{2}n(n - 1)$$

with equality if $X = [\frac{1}{2}(n+1)]$. Substituting (4.7) into (4.6), the theorem follows in the first of the three cases of (4.1); the bound is assumed for the digraph constructed in Lemma 1, with $p = 2$ and

$$(4.8) \qquad n_0 = n_1 = n_2 = \ldots = n_{d-4} = n_{d-3} = n_d = 1,$$
$$n_{d-2} = [\tfrac{1}{2}(n+1)] - \tfrac{1}{2}d, \qquad n_{d-1} = [\tfrac{1}{2}n] + 1 - \tfrac{1}{2}d.$$

Case 2, *d odd*. We have

$$(4.9) \qquad g_2 = XY + n_1 + \sum_{r=1}^{\frac{1}{2}(d-1)} n_{2r}(n_{2r-1} + n_{2r+1})$$
$$= XY + \sum_{r=1}^{\frac{1}{2}(d-1)} (n_{2r} - 1)(\overline{n_{2r-1} - 1} + \overline{n_{2r+1} - 1})$$
$$+ 2X - 2 + 2Y - n_d - (d-1).$$
$$\leqslant XY + 2n - 1 - n_d - d + [X - \tfrac{1}{2}(d+1)][Y - \tfrac{1}{2}(d+1)]$$
$$\leqslant 2XY + \tfrac{1}{4}(d+1)^2 - \tfrac{1}{2}(d+1)n + 2(n-1) - d,$$

with equality throughout if

$$(4.10) \qquad n_0 = n_1 = n_2 = \ldots = n_{d-4} = n_{d-3} = n_d = 1,$$
$$n_{d-2} = Y - \tfrac{1}{2}(d-1), \qquad n_{d-1} = X - \tfrac{1}{2}(d-1).$$

However,

$$2XY = 2nX - 2X^2$$
$$\leqslant \begin{cases} \tfrac{1}{2}n^2, & \text{if } n \text{ is even, with equality if } X = Y = \tfrac{1}{2}n, \\ \tfrac{1}{2}(n^2 - 1), & \text{if } n \text{ is odd, with equality if } X = \tfrac{1}{2}(n-1),\ Y = \tfrac{1}{2}(n+1). \end{cases}$$

The remaining statements of Theorem 2 now follow. Once again equality holds for the digraph constructed in Lemma 1 with $p = 2$ and

(a) d odd, n even:

$$n_0 = n_1 = n_2 = \ldots = n_{d-4} = n_{d-3} = n_d = 1,$$
$$n_{d-2} = n_{d-1} = \tfrac{1}{2}(n - d + 1),$$

(b) d odd, n odd:

$$n_0 = n_1 = \ldots = n_{d-4} = n_{d-3} = n_d = 1,$$
$$n_{d-2} = \tfrac{1}{2}(n - d + 2), \qquad n_{d-1} = \tfrac{1}{2}(n - d).$$

One may use Theorem 2 to obtain bounds for the diameter of strong digraphs which only contain cycles of even length analogously to Corollary 1. We omit the details.

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