Adv. Appl. Prob. **48,** 832–847 (2016)
*doi:*10.1017/apr.2016.30

DYNAMIC PROGRAMMING FOR DISCRETE-TIME FINITE-HORIZON OPTIMAL SWITCHING PROBLEMS WITH NEGATIVE SWITCHING COSTS

R. MARTYR,* *The University of Manchester*

Abstract

In this paper we study a discrete-time optimal switching problem on a finite horizon. The underlying model has a running reward, terminal reward, and signed (positive and negative) switching costs. Using optimal stopping theory for discrete-parameter stochastic processes, we extend a well-known explicit dynamic programming method for computing the value function and the optimal strategy to the case of signed switching costs.

Keywords: Optimal switching; stopping time; optimal stopping problem; Snell envelope

2010 Mathematics Subject Classification: Primary 93E20
Secondary 60G40; 62P20

1. Introduction

The relatively recent papers [4], [10] have shown the connection between Dynkin games and optimal switching problems where negative switching costs are allowed. In particular, Guo and Tomecek [4] proved that the value of the Dynkin game is equal to the difference of the value functions for the two-mode optimal switching problem. However, there is no rigorous derivation of the dynamic programming algorithm for computing the value function in the case of signed (positive and negative) switching costs. In fact, in the literature there are relatively few theoretical results for optimal switching problems in discrete time.

1.1. Literature review

The discrete-time optimal switching problem with multiple modes was used in [1] and [3] as an approximation to the solution of a continuous-time problem. The dynamic programming algorithm advocated in those papers follows from the backward induction method for solving optimal stopping problems [7, Chapter I, Section 1.1], and requires the existence of a system of Snell envelope processes which solve the continuous-time optimal switching problem. However, their arguments for proving the existence of these processes assumed strictly positive switching costs. A backward induction formula for the value function of a discrete-time optimal switching problem with two modes and strictly positive, constant switching costs was obtained in [9] under general non-Markovian assumptions. In [10], the authors studied the discrete-time optimal switching problem with two modes in a Markovian model, and obtained a different type of dynamic programming equation for the value function—one which is more in the spirit of the Wald–Bellman equations [7, Chapter I, Section 1.2].

Received 16 December 2014; revision received 6 October 2015.
* Current address: School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK
Email address: r.martyr@qmul.ac.uk

1.2. Aim and layout of the paper

In this paper we look at optimal switching for a finite-horizon discrete-time model which has a running reward, terminal reward, and allows for negative switching costs. In Section 2 we define this problem along with the notation and assumptions. In Section 3 we use the martingale approach to optimal stopping problems to provide a discrete-time analogue of the verification theorem established in [2] for continuous-time optimal switching problems. In Section 4 we justify and extend the dynamic programming method in [1] and [3] to the case of signed switching costs. A numerical example which utilises these results is presented in Section 5. The conclusion and appendix then follow.

2. Discrete-time optimal switching

2.1. Definitions

2.1.1. *Probabilistic setup.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which is given. The expectation operator with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$, and the indicator function of a set or event A is written as $\mathbf{1}_A$. Let $\mathbb{T} = \{0, 1, \dots, T\}$ represent a sequence of integer-valued times with $0 < T < \infty$. The probability space is equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$, where it is assumed that $\mathcal{F}_0$ is the trivial σ-algebra, $\mathcal{F}_0 = \{\varnothing, \Omega\}$, and $\mathcal{F} = \mathcal{F}_T$. The notation a.s. stands for 'almost surely'. For a given $\mathbb{F}$-stopping time ν, the notation $\mathcal{T}_\nu$ is used for the set of $\mathbb{F}$-stopping times τ such that $\nu \leq \tau \leq T$ $\mathbb{P}$-a.s. Martingales, stopping times, and other relevant concepts are understood to be defined with respect to the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ unless stated otherwise. The usual convention to suppress the dependence on $\omega \in \Omega$ is used below.

2.1.2. *Optimal switching definitions.* The following data for the optimal switching problem are assumed.

(i) A discrete set of operational modes $\mathbb{I} = \{1, 2, \dots, m\}$, where $2 \leq m < \infty$.

(ii) A reward received at time T for being in mode $i \in \mathbb{I}$, which is modelled by an $\mathcal{F}_T$-measurable real-valued random variable Γ_i.

(iii) A running reward received while in mode $i \in \mathbb{I}$, which is represented by a real-valued adapted process $\Psi_i = (\Psi_i(t))_{t\in\mathbb{T}}$.

(iv) A cost for switching from mode $i \in \mathbb{I}$ to $j \in \mathbb{I}$, which is modelled by a real-valued adapted process $\gamma_{i,j} = (\gamma_{i,j}(t))_{t\in\mathbb{T}}$.

We next define a class of *admissible switching controls*.

Definition 2.1. Let $t \in \mathbb{T}$ and $i \in \mathbb{I}$ be given. An admissible switching control starting from time t in mode i is a sequence $\alpha = (\tau_n, \iota_n)_{n\geq 0}$ with the following properties.

(i) For $n \geq 0$, $\tau_n \in \mathcal{T}_t$ and satisfies $t = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq T$; if $n \geq 1$ then $\mathbb{P}(\{\tau_n < T\} \cap \{\tau_n = \tau_{n+1}\}) = 0$.

(ii) For $n \geq 0$, $\iota_n : \Omega \to \mathbb{I}$ is $\mathcal{F}_{\tau_n}$-measurable with $\iota_0 = i$ and $\iota_n \neq \iota_{n+1}$ $\mathbb{P}$-a.s.

Let $\mathcal{A}_{t,i}$ denote the class of admissible switching controls (also called strategies) for the initial condition $(t, i) \in \mathbb{T} \times \mathbb{I}$.

The switching control $\alpha = (\tau_n, \iota_n)_{n\geq 0}$ models the controller's decision to switch at time τ_n, $n \geq 1$, from the active mode ι_{n-1} to another one ι_n. The condition

$$\mathbb{P}(\{\tau_n < T\} \cap \{\tau_n = \tau_{n+1}\}) = 0, \qquad n \geq 1,$$

renders inadmissible those strategies with multiple switches at the same time. The above definition is similar to the one given in [10, p.145].

Definition 2.2. Associated with each $\alpha = (\tau_n, \iota_n)_{n\geq 0} \in \mathcal{A}_{t,i}$ are the following objects.

(i) A mode indicator function $\boldsymbol{u} \colon \Omega \times \mathbb{T} \to \mathbb{I}$ defined by

$$\boldsymbol{u}_s = \sum_{n\geq 0} \iota_n \mathbf{1}_{\{\tau_n \leq s < \tau_{n+1}\}}, \qquad t \leq s \leq T. \tag{2.1}$$

(ii) The (random) total number of switches before T is

$$N(\alpha) = \sum_{n\geq 1} \mathbf{1}_{\{\tau_n < T\}}. \tag{2.2}$$

2.2. The optimal switching problem

Define the following performance index for switching controls with initial mode $i \in \mathbb{I}$ at time $t \in \mathbb{T}$:

$$J(\alpha; t, i) = \mathbb{E}\Bigg[\sum_{s=t}^{T-1} \Psi_{\boldsymbol{u}_s}(s) + \Gamma_{\iota_{N(\alpha)}} - \sum_{n\geq 1} \gamma_{\iota_{n-1},\iota_n}(\tau_n)\mathbf{1}_{\{\tau_n < T\}} \;\Bigg|\; \mathcal{F}_t\Bigg], \qquad \alpha \in \mathcal{A}_{t,i}, \tag{2.3}$$

where $\iota_{N(\alpha)}$ is the last mode switched to before T. The optimisation problem is to maximise the objective function $J(\alpha; t, i)$ over all admissible controls $\alpha \in \mathcal{A}_{t,i}$. The value function V for the optimal switching problem is defined as a random function of the initial time and mode (t, i):

$$V(t, i) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,i}} J(\alpha; t, i). \tag{2.4}$$

A switching control $\alpha^* \in \mathcal{A}_{t,i}$ is said to be optimal if it achieves the essential supremum in (2.4):

$$V(t, i) = J(\alpha^*; t, i) \geq J(\alpha; t, i) \quad \text{for all } \alpha \in \mathcal{A}_{t,i}\ \mathbb{P}\text{-a.s.}$$

Remark 2.1. Processes or functions with super(sub)scripts in terms of the mode indicators $\{\iota_n\}$ are interpreted in the following way (for example):

$$\gamma_{\iota_{n-1},\iota_n}(\cdot) = \sum_{j\in\mathbb{I}} \sum_{k\in\mathbb{I}} \mathbf{1}_{\{\iota_{n-1}=j\}} \mathbf{1}_{\{\iota_n=k\}} \gamma_{j,k}(\cdot).$$

Note that the summations are finite.

2.3. Notation, conventions and assumptions

The convention that $\sum_{s=v}^{t}(\cdot) = 0$ for any integers t and v with $t < v$ is used. The following terminology is referred to in later developments. For a constant $p \geq 1$,

(i) let L^p denote the class of random variables Z satisfying $\mathbb{E}[|Z|^p] < \infty$;

(ii) let $\mathcal{S}^p$ denote the class of adapted processes X satisfying $\mathbb{E}[\max_{t\in\mathbb{T}} |X_t|^p] < \infty$.

Assumption 2.1. *For each $i \in \mathbb{I}$, $\Gamma_i \in L^2$ and is $\mathcal{F}_T$-measurable, $\Psi_i \in \mathcal{S}^2$, and $\gamma_{i,j} \in \mathcal{S}^2$ for every $j \in \mathbb{I}$.*

Assumption 2.1 ensures that the performance index (2.3) is well defined and allows the application of optimal stopping theory later. The following standard assumption on the switching costs [4], [10] is also imposed.

Assumption 2.2. *For every $i, j, k \in \mathbb{I}$, and for all $t \in \mathbb{T}$, $\mathbb{P}$-a.s.,*

(i) $\gamma_{i,i}(t) = 0$;

(ii) $\gamma_{i,k}(t) < \gamma_{i,j}(t) + \gamma_{j,k}(t)$ *if* $i \neq j$ *and* $j \neq k$

Assumption 2.2(i) says there is no additional cost for staying in the same mode. Assumption 2.2(ii) ensures that when going from one mode i to another mode k, it is never profitable to immediately visit an intermediate mode j.

3. The verification theorem

In this section we propose a probabilistic solution to the optimal switching problem. The approach follows that of [2] in continuous time, which postulates the existence of a particular system of m stochastic processes and verifies (see Theorem 3.1) that the components of this system solve the optimal switching problem. The existence of these processes is proved in the following section (see Theorem 4.1). Before we proceed, however, let us recall the following standard results from the theory of optimal stopping which will be used below (see [7]).

Proposition 3.1. *Let $U = (U_t)_{t\in\mathbb{T}} \in \mathcal{S}^1$. There exists an adapted, integrable process $Z = (Z_t)_{t\in\mathbb{T}}$ such that Z is the smallest supermartingale which dominates U. The process Z is called the Snell envelope of U and it enjoys the following properties.*

(i) *For any $t \in \mathbb{T}$, Z_t is defined by $Z_t = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}[U_\tau \mid \mathcal{F}_t]$. Moreover, Z can also be defined recursively as $Z_T := U_T$ and $Z_t := U_t \vee \mathbb{E}[Z_{t+1} \mid \mathcal{F}_t]$ for $t = T-1, \dots, 0$.*

(ii) *For any $\theta \in \mathcal{T}$, the stopping time $\tau_\theta^* = \inf\{t \geq \theta : Z_t = U_t\}$ is optimal after θ in the sense that $Z_\theta = \mathbb{E}[U_{\tau_\theta^*} \mid \mathcal{F}_\theta] = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[U_\tau \mid \mathcal{F}_\theta]$ $\mathbb{P}$-a.s.*

(iii) *For any $t \in \mathbb{T}$ given and fixed, the stopped process $(Z_{r\wedge\tau_t^*})_{t\leq r\leq T}$ is a martingale.*

3.1. An iterative optimal stopping problem

Suppose that there exist m real-valued adapted processes $Y^i = (Y_t^i)_{t\in\mathbb{T}}$, $i \in \mathbb{I}$, such that $Y^i \in \mathcal{S}^2$ and

$$Y_t^i = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\Bigg[\sum_{s=t}^{\tau-1} \Psi_i(s) + \Gamma_i \mathbf{1}_{\{\tau=T\}} + \max_{j\neq i}\{Y_\tau^j - \gamma_{i,j}(\tau)\}\mathbf{1}_{\{\tau<T\}} \;\Bigg|\; \mathcal{F}_t\Bigg]. \tag{3.1}$$

For $i \in \mathbb{I}$, define the *implicit gain process* $(U_t^i)_{t\in\mathbb{T}}$ by

$$U_t^i = \max_{j\neq i}\{Y_t^j - \gamma_{i,j}(t)\}\mathbf{1}_{\{t<T\}} + \Gamma_i \mathbf{1}_{\{t=T\}}. \tag{3.2}$$

Then (3.1) can be expressed as

$$Y_t^i = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\Bigg[\sum_{s=t}^{\tau-1} \Psi_i(s) + U_\tau^i \;\Bigg|\; \mathcal{F}_t\Bigg]. \tag{3.3}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

Note that the assumptions on Y^i, and the costs guarantee that $U^i \in \mathcal{S}^2$ for every $i \in \mathbb{I}$. Recalling Proposition 3.1, $(Y_t^i + \sum_{s=0}^{t-1} \Psi_i(s))_{t\in\mathbb{T}}$ can be identified as the Snell envelope of the process $(U_t^i + \sum_{s=0}^{t-1} \Psi_i(s))_{t\in\mathbb{T}}$ (also see Lemma A.1).

Theorem 3.1. (Verification theorem.) *Let $i \in \mathbb{I}$ be the active mode at some fixed initial time $t \in \mathbb{T}$ and suppose that $Y^1, \dots, Y^m$ as defined in (3.1) are in $\mathcal{S}^2$. Define sequences of random times $\{\tau_n^*\}_{n\geq 0}$ and mode indicators $\{\iota_n^*\}_{n\geq 0}$ as $\tau_0^* = t$, $\iota_0^* = i$, and, for $n \geq 1$,*

$$\tau_n^* = \inf\{\tau_{n-1}^* \leq s \leq T : Y_s^{\iota_{n-1}^*} = U_s^{\iota_{n-1}^*}\}, \qquad \iota_n^* = \sum_{j\in\mathbb{I}} j \mathbf{1}_{A_j^{\iota_{n-1}^*}}, \tag{3.4}$$

where $A_j^{\iota_{n-1}^*} (= A_j^{\iota_{n-1}^*}(\omega))$ is the event

$$A_j^{\iota_{n-1}^*} := \Big\{ Y_{\tau_n^*}^j - \gamma_{\iota_{n-1}^*, j}(\tau_n^*) = \max_{k\neq\iota_{n-1}^*} \{Y_{\tau_n^*}^k - \gamma_{\iota_{n-1}^*, k}(\tau_n^*)\} \Big\}.$$

Then $\alpha^* = (\tau_n^*, \iota_n^*)_{n\geq 0} \in \mathcal{A}_{t,i}$ and satisfies $Y_t^i = J(\alpha^*; t, i) = \operatorname{ess\,sup}_{\alpha\in\mathcal{A}_{t,i}} J(\alpha; t, i)$ a.s.

Proof. The proof is essentially the same as the proof of [2, Theorem 1]. Note that the infimum in (3.4) is always attained since $Y_T^i = U_T^i$ a.s. for every $i \in \mathbb{I}$.

In Lemma A.2 in the appendix we verify that $\alpha^* \in \mathcal{A}_{t,i}$. Using (3.2) and (3.3), it is simple to check that the second claim follows trivially when $t = T$ since $Y_T^i = \Gamma_i = J(\alpha; T, i) = V(T, i)$ a.s. for any $\alpha \in \mathcal{A}_{T,i}$. Suppose now that $t < T$.

The stopping time τ_1^* in (3.4) is optimal after t by Proposition 3.1. Using this together with the definition of ι_1^*, we have

$$\begin{aligned} Y_t^i &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\bigg[\sum_{s=t}^{\tau-1} \Psi_i(s) + U_\tau^i \;\bigg|\; \mathcal{F}_t \bigg] \\ &= \mathbb{E}\bigg[\sum_{s=t}^{\tau_1^*-1} \Psi_i(s) + \Gamma_i \mathbf{1}_{\{\tau_1^*=T\}} + \max_{j\neq i}\{Y_{\tau_1^*}^j - \gamma_{i,j}(\tau_1^*)\}\mathbf{1}_{\{\tau_1^*<T\}} \;\bigg|\; \mathcal{F}_t \bigg] \\ &= \mathbb{E}\bigg[\sum_{s=t}^{\tau_1^*-1} \Psi_i(s) + \Gamma_i \mathbf{1}_{\{\tau_1^*=T\}} + \{Y_{\tau_1^*}^{\iota_1^*} - \gamma_{i,\iota_1^*}(\tau_1^*)\}\mathbf{1}_{\{\tau_1^*<T\}} \;\bigg|\; \mathcal{F}_t \bigg] \quad \text{a.s.} \end{aligned} \tag{3.5}$$

Lemma A.1 in the appendix confirms that $Y^{\iota_1^*}$ satisfies

$$Y_s^{\iota_1^*} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}\bigg[\sum_{r=s}^{\tau_1^*-1} \Psi_{\iota_1^*}(r) + U_\tau^{\iota_1^*} \;\bigg|\; \mathcal{F}_s \bigg] \quad \text{on } [\tau_1^*, T]. \tag{3.6}$$

Using (3.6) together with the definition and optimality of τ_2^* and ι_2^*, we obtain

$$\begin{aligned} &\mathbf{1}_{\{\tau_1^*<T\}} Y_{\tau_1^*}^{\iota_1^*} \\ &\quad= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\tau_1^*}} \mathbb{E}\bigg[\sum_{r=\tau_1^*}^{\tau-1} \Psi_{\iota_1^*}(r) + U_\tau^{\iota_1^*} \;\bigg|\; \mathcal{F}_{\tau_1^*} \bigg] \mathbf{1}_{\{\tau_1^*<T\}} \\ &\quad= \mathbb{E}\bigg[\sum_{r=\tau_1^*}^{\tau_2^*-1} \Psi_{\iota_1^*}(r) + \{Y_{\tau_2^*}^{\iota_2^*} - \gamma_{\iota_1^*,\iota_2^*}(\tau_2^*)\}\mathbf{1}_{\{\tau_2^*<T\}} + \Gamma_{\iota_1^*}\mathbf{1}_{\{\tau_2^*=T\}} \;\bigg|\; \mathcal{F}_{\tau_1^*} \bigg] \mathbf{1}_{\{\tau_1^*<T\}} \quad \text{a.s.} \end{aligned} \tag{3.7}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

Combining (3.5) and (3.7), we obtain the following expression for Y_t^i:

$$\begin{aligned} Y_t^i &= \mathbb{E}\bigg[\sum_{s=t}^{\tau_1^*-1} \Psi_i(s) + \Gamma_i \mathbf{1}_{\{\tau_1^*=T\}} + \{Y_{\tau_1^*}^{\iota_1^*} - \gamma_{i,\iota_1^*}(\tau_1^*)\}\mathbf{1}_{\{\tau_1^*<T\}} \;\bigg|\; \mathcal{F}_t\bigg] \\ &= \mathbb{E}\bigg[\sum_{s=t}^{\tau_1^*-1} \Psi_i(s) + \mathbb{E}\bigg[\sum_{r=\tau_1^*}^{\tau_2^*-1} \Psi_{\iota_1^*}(r) \;\bigg|\; \mathcal{F}_{\tau_1^*}\bigg]\mathbf{1}_{\{\tau_1^*<T\}} \;\bigg|\; \mathcal{F}_t\bigg] \\ &\quad + \mathbb{E}[\Gamma_i \mathbf{1}_{\{\tau_1^*=T\}} + \mathbb{E}[\Gamma_{\iota_1^*}\mathbf{1}_{\{\tau_2^*=T\}} \mid \mathcal{F}_{\tau_1^*}]\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_t] \\ &\quad - \mathbb{E}[\gamma_{i,\iota_1^*}(\tau_1^*)\mathbf{1}_{\{\tau_1^*<T\}} + \mathbb{E}[\gamma_{\iota_1^*,\iota_2^*}(\tau_2^*)\mathbf{1}_{\{\tau_2^*<T\}} \mid \mathcal{F}_{\tau_1^*}]\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_t] \\ &\quad + \mathbb{E}[\mathbb{E}[Y_{\tau_2^*}^{\iota_2^*}\mathbf{1}_{\{\tau_2^*<T\}} \mid \mathcal{F}_{\tau_1^*}]\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_t] \quad \text{a.s.} \end{aligned} \tag{3.8}$$

Since $\mathbf{1}_{\{\tau_1^*=T\}}$, $\mathbf{1}_{\{\tau_1^*<T\}}$, and $\gamma_{i,\iota_1^*}(\tau_1^*)$ are all $\mathcal{F}_{\tau_1^*}$-measurable, they can be brought inside the conditional expectation with respect to $\mathcal{F}_{\tau_1^*}$ in (3.8); thus,

$$\begin{aligned} Y_t^i &= \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{s=t}^{\tau_1^*-1} \Psi_i(s) + \sum_{r=\tau_1^*}^{\tau_2^*-1} \Psi_{\iota_1^*}(r)\mathbf{1}_{\{\tau_1^*<T\}} \;\bigg|\; \mathcal{F}_{\tau_1^*}\bigg] \;\bigg|\; \mathcal{F}_t\bigg] \\ &\quad + \mathbb{E}[\mathbb{E}[\Gamma_i \mathbf{1}_{\{\tau_1^*=T\}} + \Gamma_{\iota_1^*}\mathbf{1}_{\{\tau_2^*=T\}}\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_{\tau_1^*}] \mid \mathcal{F}_t] \\ &\quad - \mathbb{E}[\mathbb{E}[\gamma_{i,\iota_1^*}(\tau_1^*)\mathbf{1}_{\{\tau_1^*<T\}} + \gamma_{\iota_1^*,\iota_2^*}(\tau_2^*)\mathbf{1}_{\{\tau_2^*<T\}}\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_{\tau_1^*}] \mid \mathcal{F}_t] \\ &\quad + \mathbb{E}[\mathbb{E}[Y_{\tau_2^*}^{\iota_2^*}\mathbf{1}_{\{\tau_2^*<T\}}\mathbf{1}_{\{\tau_1^*<T\}} \mid \mathcal{F}_{\tau_1^*}] \mid \mathcal{F}_t] \quad \text{a.s.} \end{aligned}$$

Using $\iota_0^* = i$ and the definition of the mode indicator $\boldsymbol{u}^*$ (cf. (2.1)), we have

$$\sum_{s=t}^{\tau_2^*-1} \Psi_{\boldsymbol{u}_s^*}(s) = \sum_{r=t}^{\tau_1^*-1} \Psi_i(r) + \sum_{r=\tau_1^*}^{\tau_2^*-1} \Psi_{\iota_1^*}(r)\mathbf{1}_{\{\tau_1^*<T\}}.$$

Since $\alpha^* \in \mathcal{A}_{t,i}$, $\mathbb{P}(\{\tau_n^* < T\} \cap \{\tau_n^* = \tau_{n+1}^*\}) = 0$ and the above expression is well defined. Since $\tau_1^* \le \tau_2^*$, it follows that $\{\tau_2^* < T\} \subset \{\tau_1^* < T\}$ and, therefore,

$$\mathbf{1}_{\{\tau_2^*<T\}}\mathbf{1}_{\{\tau_1^*<T\}} = \mathbf{1}_{\{\tau_2^*<T\}} \quad \text{a.s.}$$

Note that $\tau_0^* = t < T$ so $\mathbb{P}(\{\tau_0^* < T\}) = 1$, and $\mathcal{F}_t \subseteq \mathcal{F}_{\tau_1^*}$ since $t \le \tau_1^*$. These observations together with the tower property of conditional expectations shows that Y_t^i satisfies

$$\begin{aligned} Y_t^i &= \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{s=t}^{\tau_2^*-1} \Psi_{\boldsymbol{u}_s^*}(s) + \sum_{n=1}^{2} \Gamma_{\iota_{n-1}^*}\mathbf{1}_{\{\tau_n^*=T\}}\mathbf{1}_{\{\tau_{n-1}^*<T\}} \;\bigg|\; \mathcal{F}_{\tau_1^*}\bigg] \;\bigg|\; \mathcal{F}_t\bigg] \\ &\quad + \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{n=1}^{2} -\gamma_{\iota_{n-1}^*,\iota_n^*}(\tau_n^*)\mathbf{1}_{\{\tau_n^*<T\}} + Y_{\tau_2^*}^{\iota_2^*}\mathbf{1}_{\{\tau_2^*<T\}} \;\bigg|\; \mathcal{F}_{\tau_1^*}\bigg] \;\bigg|\; \mathcal{F}_t\bigg] \\ &= \mathbb{E}\bigg[\sum_{s=t}^{\tau_2^*-1} \Psi_{\boldsymbol{u}_s^*}(s) + \sum_{k=0}^{1} \Gamma_{\iota_k^*}\mathbf{1}_{\{\tau_k^*<T\}}\mathbf{1}_{\{\tau_{k+1}^*=T\}} - \sum_{n=1}^{2} \gamma_{\iota_{n-1}^*,\iota_n^*}(\tau_n^*)\mathbf{1}_{\{\tau_n^*<T\}} \;\bigg|\; \mathcal{F}_t\bigg] \\ &\quad + \mathbb{E}[Y_{\tau_2^*}^{\iota_2^*}\mathbf{1}_{\{\tau_2^*<T\}} \mid \mathcal{F}_t] \quad \text{a.s.} \end{aligned}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

Let $N(\alpha^*)$ be the total number of switches under α^* (cf. (2.2)). Repeating the procedure of substituting for $Y^{\iota_n^*}_{\tau_n^*}$ with $n = 2, 3, \dots$ and using $\alpha^* \in \mathcal{A}_{t,i}$ yields

$$Y_t^i = \mathbb{E}\bigg[\sum_{s=t}^{T-1} \Psi_{\boldsymbol{u}_s^*}(s) + \sum_{k=0}^{N(\alpha^*)} \Gamma_{\iota_k^*} \mathbf{1}_{\{\tau_k^* < T\}} \mathbf{1}_{\{\tau_{k+1}^* = T\}} - \sum_{n=1}^{N(\alpha^*)} \gamma_{\iota_{n-1}^*, \iota_n^*}(\tau_n^*) \;\bigg|\; \mathcal{F}_t\bigg]. \tag{3.9}$$

Under the assumption $t < T$, the sum of terminal reward terms collapses to a single term

$$\sum_{k=0}^{N(\alpha^*)} \Gamma_{\iota_k^*} \mathbf{1}_{\{\tau_k^* < T\}} \mathbf{1}_{\{\tau_{k+1}^* = T\}} = \Gamma_{\iota_{N(\alpha^*)}^*} \quad \mathbb{P}\text{-a.s.}, \tag{3.10}$$

and using (3.9) and (3.10) we arrive at the following representation for Y_t^i:

$$Y_t^i = \mathbb{E}\bigg[\sum_{s=t}^{T-1} \Psi_{\boldsymbol{u}_s^*}(s) + \Gamma_{\iota_{N(\alpha^*)}^*} - \sum_{n \ge 1} \gamma_{\iota_{n-1}^*, \iota_n^*}(\tau_n^*) \mathbf{1}_{\{\tau_n^* < T\}} \;\bigg|\; \mathcal{F}_t\bigg] = J(\alpha^*; t, i) \quad \mathbb{P}\text{-a.s.} \tag{3.11}$$

Now let $\alpha = (\tau_n, \iota_n)_{n \ge 0} \in \mathcal{A}_{t,i}$ be any admissible control. The verification theorem can be completed by showing that $J(\alpha^*; t, i) \ge J(\alpha; t, i)$ a.s. First, note that

$$J(\alpha^*; T, i) = J(\alpha; T, i) \quad \text{when } t = T,$$

so assume, henceforth, that $t < T$. Then, due to possible suboptimality of the pair (τ_1, ι_1), it holds that

$$\begin{aligned}
Y_t^i &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\bigg[\sum_{s=t}^{\tau-1} \Psi_i(s) + U_\tau^i \;\bigg|\; \mathcal{F}_t\bigg] \\
&\ge \mathbb{E}\bigg[\sum_{s=t}^{\tau_1-1} \Psi_i(s) + U_{\tau_1}^i \;\bigg|\; \mathcal{F}_t\bigg] \\
&\ge \mathbb{E}\bigg[\sum_{s=t}^{\tau_1-1} \Psi_i(s) + \Gamma_i \mathbf{1}_{\{\tau_1 = T\}} + \{Y_{\tau_1}^{\iota_1} - \gamma_{i,\iota_1}(\tau_1)\} \mathbf{1}_{\{\tau_1 < T\}} \;\bigg|\; \mathcal{F}_t\bigg].
\end{aligned}$$

Repeating the arguments leading to (3.11), replacing the equalities ($=$) with inequalities ($\ge$) due to possible suboptimality of (τ_n, ι_n) for $n \ge 2$, eventually leads to

$$\begin{aligned}
Y_t^i &\ge \mathbb{E}\bigg[\sum_{s=t}^{T-1} \Psi_{\boldsymbol{u}_s}(s) + \sum_{k=0}^{N(\alpha)} \Gamma_{\iota_k} \mathbf{1}_{\{\tau_k < T\}} \mathbf{1}_{\{\tau_{k+1} = T\}} - \sum_{n=1}^{N(\alpha)+1} \gamma_{\iota_{n-1}, \iota_n}(\tau_n) \mathbf{1}_{\{\tau_n < T\}} \;\bigg|\; \mathcal{F}_t\bigg] \\
&\quad + \mathbb{E}[Y_{\tau_{N(\alpha)+1}}^{\iota_{N(\alpha)+1}} \mathbf{1}_{\{\tau_{N(\alpha)+1} < T\}} \mid \mathcal{F}_t] \\
&= \mathbb{E}\bigg[\sum_{s=t}^{T-1} \Psi_{\boldsymbol{u}_s}(s) + \Gamma_{\iota_{N(\alpha)}} - \sum_{n \ge 1} \gamma_{\iota_{n-1}, \iota_n}(\tau_n) \mathbf{1}_{\{\tau_n < T\}} \;\bigg|\; \mathcal{F}_t\bigg] \\
&= J(\alpha; t, i)
\end{aligned}$$

and proves that the strategy α^* is optimal. □

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

4. Existence of the optimal processes

4.1. Backward dynamic programming

Lemma 4.1. (Backward induction.) *Define the processes* $\widetilde{Y}^i = (\widetilde{Y}^i_t)_{t\in\mathbb{T}}$, $i \in \mathbb{I}$, *recursively as* $\widetilde{Y}^i_T = \Gamma_i$, *and, for* $t = T-1, \dots, 0$,

$$\widetilde{Y}^i_t = \max_{j\neq i}\{-\gamma_{i,j}(t) + \Psi_j(t) + \mathbb{E}[\widetilde{Y}^j_{t+1} \mid \mathcal{F}_t]\} \vee \{\Psi_i(t) + \mathbb{E}[\widetilde{Y}^i_{t+1} \mid \mathcal{F}_t]\}. \tag{4.1}$$

Then $\widetilde{Y}^i$ *is* $\mathbb{F}$*-adapted and in* $\mathcal{S}^2$.

Proof. Both claims can be established by using (4.1) and proceeding recursively for $t = T, T-1, \dots, 0$, noting that the conditional expectations are well defined by the integrability conditions on the rewards and switching costs. The details are omitted. □

4.1.1. *An explicit Snell envelope system.* A connection between $\widetilde{Y}^i$ in Lemma 4.1 and the Snell envelope becomes apparent upon defining a new process $(\hat{Y}^i_t)_{t\in\mathbb{T}}$ for every $i \in \mathbb{I}$ by

$$\hat{Y}^i_t := \widetilde{Y}^i_t + \sum_{s=0}^{t-1} \Psi_i(s).$$

Let $(\hat{U}^i_t)_{t\in\mathbb{T}}$ be the *explicit gain process* defined by

$$\begin{aligned}\hat{U}^i_t &:= \max_{j\neq i}\left\{-\gamma_{i,j}(t) + \sum_{s=0}^{t-1}(\Psi_i(s) - \Psi_j(s)) + \mathbb{E}[\hat{Y}^j_{t+1} \mid \mathcal{F}_t]\right\}\mathbf{1}_{\{t<T\}} \\ &\quad + \left\{\sum_{s=0}^{T-1}\Psi_i(s) + \Gamma_i\right\}\mathbf{1}_{\{t=T\}} \\ &= \sum_{s=0}^{t-1}\Psi_i(s) + \max_{j\neq i}\left\{-\gamma_{i,j}(t) - \sum_{s=0}^{t-1}\Psi_j(s) + \mathbb{E}[\hat{Y}^j_{t+1} \mid \mathcal{F}_t]\right\}\mathbf{1}_{\{t<T\}} + \Gamma_i\mathbf{1}_{\{t=T\}}.\end{aligned} \tag{4.2}$$

The processes $(\hat{Y}^i_t)_{t\in\mathbb{T}}$ and $(\hat{U}^i_t)_{t\in\mathbb{T}}$, $i \in \mathbb{I}$, belong to $\mathcal{S}^2$ by properties of the rewards, switching costs, and as $\widetilde{Y}^i \in \mathcal{S}^2$. Proposition 3.1 and the backward induction formula show that $(\hat{Y}^i_t)_{t\in\mathbb{T}}$ is the Snell envelope of $(\hat{U}^i_t)_{t\in\mathbb{T}}$.

Theorem 4.1. (Existence.) *Let* $(\widetilde{Y}^i_t)_{t\in\mathbb{T}}$, $i \in \mathbb{I}$, *be the processes defined by (4.1). Then, for every* $t \in \mathbb{T}$,

$$\widetilde{Y}^i_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\sum_{s=t}^{\tau-1}\Psi_i(s) + \Gamma_i\mathbf{1}_{\{\tau=T\}} + \max_{j\neq i}\{\widetilde{Y}^j_\tau - \gamma_{i,j}(\tau)\}\mathbf{1}_{\{\tau<T\}} \,\middle|\, \mathcal{F}_t\right] \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

Therefore, $\widetilde{Y}^1, \dots, \widetilde{Y}^m$ *satisfy the verification theorem.*

Proof. For notational convenience, introduce a new process $(\hat{W}^i_t)_{t\in\mathbb{T}}$ defined by

$$\hat{W}^i_t := \sum_{s=0}^{t-1}\Psi_i(s) + \Gamma_i\mathbf{1}_{\{t=T\}} + \max_{j\neq i}\left\{-\gamma_{i,j}(t) - \sum_{s=0}^{t-1}\Psi_j(s) + \hat{Y}^j_t\right\}\mathbf{1}_{\{t<T\}}. \tag{4.4}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

Note that $\hat{W}^i \in \mathcal{S}^2$ by the properties of Γ_i, Ψ_i, $\gamma_{i,j}$, and $\hat{Y}^j$ for $i, j \in \mathbb{I}$. Equation (4.3) can be proved if, for all $t \in \mathbb{T}$,

$$\hat{Y}_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[\hat{W}_\tau^i \mid \mathcal{F}_t] \quad \mathbb{P}\text{-a.s.} \tag{4.5}$$

In order to prove (4.5), first note that by Proposition 3.1 the Snell envelope of $\hat{W}^i$ exists and, denoting it by $Z^i = (Z_t^i)_{t\in\mathbb{T}}$, it satisfies

$$Z_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[\hat{W}_\tau^i \mid \mathcal{F}_t]$$

and the backward induction formulae

$$Z_T^i = \hat{W}_T^i \quad \text{and} \quad Z_t^i = \hat{W}_t^i \vee \mathbb{E}[Z_{t+1}^i \mid \mathcal{F}_t], \qquad t = T-1, \dots, 0. \tag{4.6}$$

Thus, establishing (4.5) is equivalent to showing that $\hat{Y}^i$ is a modification of (and, by [8, Proposition II.36.5], indistinguishable from) Z^i defined in (4.6). This is established below.

Note that $Z_T^i = \hat{W}_T^i = \hat{U}_T^i = \hat{Y}_T^i$ a.s. for every $i \in \mathbb{I}$. Now suppose inductively for $t = T-1, \dots, 0$ that for all $i \in \mathbb{I}$, $Z_{t+1}^i = \hat{Y}_{t+1}^i$ $\mathbb{P}$-a.s. Next, for every $i \in \mathbb{I}$ define a stopping time θ_t^i by

$$\theta_t^i = \inf\{t \le s \le T : \hat{Y}_s^i = \hat{U}_s^i\}, \tag{4.7}$$

noting that $t \le \theta_t^i \le T$ a.s. The following lines will establish $Z_t^i = \hat{Y}_t^i$ on the events $\{\theta_t^i = t\}$ and $\{\theta_t^i > t\} \equiv \{\theta_t^i \ge t+1\}$. This leads to $Z_t^i = \hat{Y}_t^i$ a.s. and the induction argument will complete the proof that $\hat{Y}^i$ is a modification of Z^i.

Case 1: $Z_t^i = \hat{Y}_t^i$ on $\{\theta_t^i = t\}$. Since $\hat{Y}^j$ is the Snell envelope of $\hat{U}^j$, $j \in \mathbb{I}$, it is a supermartingale. Using this with the definitions of $\hat{U}^i$ (cf. (4.2)) and $\hat{W}^i$ (cf. (4.4)) leads to $\hat{U}^i \le \hat{W}^i$ for $i \in \mathbb{I}$. Then, using (4.7), the backward induction formula and the induction hypothesis, we obtain

$$\hat{W}_t^i \ge \hat{U}_t^i = \hat{Y}_t^i \ge \mathbb{E}[\hat{Y}_{t+1}^i \mid \mathcal{F}_t] = \mathbb{E}[Z_{t+1}^i \mid \mathcal{F}_t] \quad \text{on } \{\theta_t^i = t\}. \tag{4.8}$$

Using (4.6) for Z^i and (4.8) above, we also have

$$\hat{W}_t^i = Z_t^i \quad \text{on } \{\theta_t^i = t\}. \tag{4.9}$$

Equation (4.9) and the finiteness of $\mathbb{I}$ imply the existence of an $\mathcal{F}_{\theta_t^i}$-measurable mode j_* (that is, j_* is an $\mathcal{F}_{\theta_t^i}$-measurable $\mathbb{I}$-valued random variable) such that

$$Z_t^i = \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \hat{Y}_t^{j_*}, \tag{4.10}$$

$$j_* = \operatorname*{arg\,max}_{j \ne i} \bigg\{ -\gamma_{i,j}(t) - \sum_{s=0}^{t-1} \Psi_j(s) + \hat{Y}_t^j \bigg\} \quad \text{on } \{\theta_t^i = t\}. \tag{4.11}$$

Let us show that

$$\mathbf{1}_{\{\theta_t^i = t\}} \hat{Y}_t^{j_*} = \mathbf{1}_{\{\theta_t^i = t\}} \mathbb{E}[\hat{Y}_{t+1}^{j_*} \mid \mathcal{F}_t] \quad \mathbb{P}\text{-a.s.} \tag{4.12}$$

Since j_* is $\mathcal{F}_{\theta_t^i}$-measurable, we have, for $t \le r \le T$,

$$\mathbb{E}\bigg[\sum_{j\in\mathbb{I}} \mathbf{1}_{\{j_*=j\}} \hat{Y}_r^j \,\bigg|\, \mathcal{F}_t\bigg] = \sum_{j\in\mathbb{I}} \mathbf{1}_{\{j_*=j\}} \mathbb{E}[\hat{Y}_r^j \mid \mathcal{F}_t] \le \sum_{j\in\mathbb{I}} \mathbf{1}_{\{j_*=j\}} \hat{Y}_t^j \quad \text{on } \{\theta_t^i = t\}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

so that $\hat{Y}^{j_*} := \sum_{j\in\mathbb{I}} \hat{Y}^j \mathbf{1}_{\{j_*=j\}}$ is a supermartingale on $[\theta_t^i, T]$. Now if (4.12) does not hold, by the supermartingale property of $\hat{Y}^{j_*}$ the event A_t^i defined by

$$A_t^i := \{\hat{Y}_t^{j_*} > \mathbb{E}[\hat{Y}_{t+1}^{j_*} \mid \mathcal{F}_t]\} \cap \{\theta_t^i = t\}$$

has positive probability. If $\mathbb{P}(A_t^i) > 0$ then there exists an $\mathcal{F}_{\theta_t^i}$-measurable mode k_* such that

$$\hat{Y}_t^{j_*} = \sum_{s=0}^{t-1} \Psi_{j_*}(s) - \gamma_{j_*,k_*}(t) - \sum_{s=0}^{t-1} \Psi_{k_*}(s) + \mathbb{E}[\hat{Y}_{t+1}^{k_*} \mid \mathcal{F}_t] \quad \text{on } A_t^i.$$

This leads to

$$\begin{aligned} Z_t^i &= \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \hat{Y}_t^{j_*} \\ &= \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \sum_{s=0}^{t-1} \Psi_{j_*}(s) - \gamma_{j_*,k_*}(t) \\ &\quad - \sum_{s=0}^{t-1} \Psi_{k_*}(s) + \mathbb{E}[\hat{Y}_{t+1}^{k_*} \mid \mathcal{F}_t] \\ &< \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,k_*}(t) - \sum_{s=0}^{t-1} \Psi_{k_*}(s) + \hat{Y}_t^{k_*} \quad \text{on } A_t^i, \end{aligned} \tag{4.13}$$

where the inequality comes from the no-arbitrage condition (Assumption 2.2(ii)) and the supermartingale property of $\hat{Y}^{k_*}$ on $[\theta_t^i, T]$. However, (4.13) contradicts the optimality of j_* and, therefore, shows that (4.12) holds.

Using (4.10) and (4.11) together with (4.12) and the definition of $\hat{Y}^i$ yields

$$Z_t^i = \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \mathbb{E}[\hat{Y}_{t+1}^{j_*} \mid \mathcal{F}_t] \le \hat{Y}_t^i \quad \text{on } \{\theta_t^i = t\}.$$

Using this with (4.8) and (4.9), we obtain

$$\hat{W}_t^i = \hat{U}_t^i = \hat{Y}_t^i = Z_t^i \quad \text{on } \{\theta_t^i = t\}. \tag{4.14}$$

Case 2: $Z_t^i = \hat{Y}_t^i$ *on* $\{\theta_t^i \ge t+1\}$. Note that $\{\theta_t^i \ge t+1\} \equiv \{\theta_t^i > t\}$ and is, therefore, $\mathcal{F}_t$-measurable. Using the properties of the Snell envelopes $\hat{Y}^i$ and Z^i together with the induction hypothesis, we obtain

$$\hat{Y}_t^i = \mathbb{E}[\hat{Y}_{t+1}^i \mid \mathcal{F}_t] = \mathbb{E}[Z_{t+1}^i \mid \mathcal{F}_t] \le Z_t^i \quad \text{on } \{\theta_t^i \ge t+1\}. \tag{4.15}$$

Let B_t^i be the $\mathcal{F}_t$-measurable event that Z^i is a strict supermartingale at time t on $\{\theta_t^i \ge t+1\}$:

$$B_t^i := \{\mathbb{E}[Z_{t+1}^i \mid \mathcal{F}_t] < Z_t^i\} \cap \{\theta_t^i \ge t+1\}.$$

If $\mathbb{P}(B_t^i) > 0$ then this implies the existence of an $\mathcal{F}_t$-measurable mode j_* such that

$$\hat{Y}_t^i < Z_t^i = \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \hat{Y}_t^{j_*}, \tag{4.16}$$

$$j_* = \arg\max_{j\ne i} \left\{ -\gamma_{i,j}(t) - \sum_{s=0}^{t-1} \Psi_j(s) + \hat{Y}_t^j \right\} \quad \text{on } B_t^i. \tag{4.17}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

But by the definition of $\hat{Y}^i$, it holds that

$$\hat{Y}_t^i \geq \sum_{s=0}^{t-1} \Psi_i(s) - \gamma_{i,j_*}(t) - \sum_{s=0}^{t-1} \Psi_{j_*}(s) + \mathbb{E}[\hat{Y}_{t+1}^{j_*} \mid \mathcal{F}_t] \quad \text{on } B_t^i$$

and using this in (4.16) and (4.17) shows that $\mathbb{E}[\hat{Y}_{t+1}^{j_*} \mid \mathcal{F}_t] < \hat{Y}_t^{j_*}$ on B_t^i. Similar arguments as those used to establish (4.13) lead to a contradiction on the optimality of j_*, whence we conclude that $\mathbb{P}(B_t^i) = 0$. Therefore,

$$\mathbf{1}_{\{\theta_t^i \geq t+1\}} \mathbb{E}[Z_{t+1}^i \mid \mathcal{F}_t] = Z_t^i \mathbf{1}_{\{\theta_t^i \geq t+1\}} \quad \mathbb{P}\text{-a.s.} \tag{4.18}$$

and combining (4.14), (4.15), and (4.18) gives $\hat{Y}_t^i = Z_t^i$ $\mathbb{P}$-a.s. Since the $t = T-1$ case holds and $i \in \mathbb{I}$ was arbitrary, the proof by induction is complete. Therefore, it holds for all $i \in \mathbb{I}$, and for all $t \in \mathbb{T}$: $\hat{Y}_t^i = Z_t^i$ $\mathbb{P}$-a.s., which means $\hat{Y}^i$ is a modification of (and, therefore, indistinguishable from) Z^i, whence (4.5) follows. □

5. Numerical example: pricing a cancellable call option

A finite expiry cancellable call option is a financial contract on an underlying asset which gives its holder the right, but not the obligation, to purchase the asset at a fixed *strike* price. On the one hand, the holder is able to exercise this right at any time between the option's start and expiration. On the other hand, the writer is allowed to recall the option at any time before the holder exercises by paying the option's intrinsic value and a fixed penalty at the recall (cancellation) time. In this section we use discrete-time optimal switching to obtain a numerical approximation to the value of a cancellable call option in a continuous-time financial market model.

5.1. The model for the financial market and option

For $0 < S < \infty$, let the interval $[0, S]$ denote the lifetime of the option without exercise or cancellation. Suppose a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has been given on which is defined a standard Brownian motion $(W(r))_{r\in[0,S]}$. Let $\mathbb{F} = (\mathcal{F}_r)_{r\in[0,S]}$ be the completed natural filtration of the Brownian motion and suppose $\mathcal{F} = \mathcal{F}_T$.

The asset price is an $\mathbb{F}$-adapted stochastic process $X = (X(r))_{r\in[0,S]}$. We assume a Black–Scholes market where X is modelled by a geometric Brownian motion (GBM) and the risk-free interest rate is a constant $\rho > 0$. Let $\kappa > 0$ be the asset's constant volatility and suppose that dividends are paid at a constant rate $d > 0$. Assuming $\mathbb{P}$ to be a risk-neutral probability measure, the asset price X starting from an initial value $x_0 > 0$ evolves as

$$X(r) = x_0 \exp(\widetilde{\mu} r + \kappa W(r)), \qquad r \in [0, S]\, \mathbb{P}\text{-a.s.}$$

with $\widetilde{\mu} := \rho - d - \kappa^2/2$.

Let $K > 0$ denote the fixed strike price, $\delta > 0$ the cancellation penalty, and τ (respectively σ) be an $[0, S]$-valued $\mathbb{F}$-stopping time representing the holder's exercise (respectively writer's cancellation) time. The cost of the option with respect to $\mathbb{P}$ from the writer's perspective is given by

$$\mathcal{J}(\sigma, \tau) = \mathbb{E}[\mathrm{e}^{-\rho\sigma}[(X(\sigma) - K)^+ + \delta]\mathbf{1}_{\{\sigma<\tau\}} + \mathrm{e}^{-\rho\tau}(X(\tau) - K)^+ \mathbf{1}_{\{\tau\leq\sigma\}}],$$

where $a^+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. The writer (respectively holder) chooses σ (respectively τ) in order to minimise (respectively maximise) the expected payoff, given his/her counterpart's

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

decision. This leads to upper and lower values for the option which are defined respectively by

$$\mathcal{V}^{+} = \inf_{\sigma} \sup_{\tau} \mathcal{J}(\sigma, \tau), \qquad \mathcal{V}_{-} = \sup_{\tau} \inf_{\sigma} \mathcal{J}(\sigma, \tau).$$

If there is equality between these values, then there exists a fair price for the option which is given by the common value $\mathcal{V} = \mathcal{V}^{+} = \mathcal{V}_{-}$. In this sense, the pricing problem for the option can be viewed as an optimal stopping (Dynkin) game, which may be solved by an appropriately formulated continuous-time optimal switching problem [4]. The solution to this optimal switching problem will be approximated by a suitable discrete-time analogue.

5.2. The discretisation procedure and discrete-time switching problem

Let $0 < T < \infty$ be an integer and $h = S/T$ be the step size used to define an increasing sequence of times $\{r_t\}_{t=0}^{T}$ by $r_t = th$. The GBM X sampled at these times forms a sequence of random variables $\{X_{r_t}\}_{t=0}^{T}$ with respect to $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{T} = \{0, 1, \dots, T\}$ be the time parameter set as before, $\hat{X} = (\hat{X}_t)_{t\in\mathbb{T}}$ be a discrete-parameter stochastic process defined by $\hat{X}_t = X_{r_t}$, and $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t\in\mathbb{T}}$ be a filtration defined by $\hat{\mathcal{F}}_t = \mathcal{F}_{r_t}$. In this section we work with the stochastic basis $(\Omega, \mathcal{F}, \hat{\mathbb{F}}, \mathbb{P})$. All stopping times, adapted processes, and similar concepts are with respect to this basis unless stated otherwise.

We consider an optimal switching problem with $m = 2$ modes which are denoted by 0 and 1 in accordance with the literature. Define switching costs $\gamma_{ij} = (\gamma_{ij}(t))_{t\in\mathbb{T}}$, $i, j \in \{0, 1\}$, by

$$\gamma_{0,0} \equiv \gamma_{1,1} \equiv 0, \qquad \gamma_{0,1}(t) = (\hat{X}_t - K)^{+} + \delta, \qquad \gamma_{1,0}(t) = -(\hat{X}_t - K)^{+},$$

and terminal rewards $\Gamma_0 = 0$, $\Gamma_1 = -\gamma_{1,0}(T)$. Note that the switching cost $\gamma_{1,0}$ may become negative. The *discounted* discrete-time optimal switching problem starting in mode $i \in \{0, 1\}$ at time 0 takes a similar form as (2.3):

$$\begin{aligned} J(\alpha; 0, i) &= \mathbb{E}\Bigg[\mathrm{e}^{-\rho T h}\Gamma_{\iota_{N(\alpha)}} - \sum_{n\geq 1} \mathrm{e}^{-\rho\tau_n h}\gamma_{\iota_{n-1},\iota_n}(\tau_n)\mathbf{1}_{\{\tau_n<T\}}\Bigg], \qquad \alpha \in \mathcal{A}_{0,i}, \\ V(i) &= \sup_{\alpha\in\mathcal{A}_{0,i}} J(\alpha; 0, i). \end{aligned}$$

Note that $\mathbb{P}$-a.s., $\gamma_{1,0}(T) \leq \Gamma_1 \leq \gamma_{0,1}(T)$ and $\gamma_{0,1}(t) + \gamma_{1,0}(t) > 0$ for $t = 0, \dots, T$, relations which still hold when the terms are multiplied by the discount factor $\mathrm{e}^{-\rho th}$. Furthermore, the switching costs $\gamma_{0,1}$ and $\gamma_{1,0}$ are $\mathcal{S}^2$ processes. If $V(i)$ is a 'good' approximation to the value of its continuous-time analogue, we may argue that using the results of [4] that $V(1) - V(0)$ is also a 'good' approximation to the value $\mathcal{V}$ of the cancellable call option. In this case, one may use Theorems 3.1 and 4.1, taking into account discounting, to show that there exist $\mathcal{S}^2$ processes $\{\check{Y}_t^i\}_{t\in\mathbb{T}}$, $i \in \{0, 1\}$, defined by $\check{Y}_T^i = \Gamma_i$, and, for $t = T - 1, \dots, 0$,

$$\check{Y}_t^i = \max_{j\in\{0,1\}} \{-\gamma_{i,j}(t) + \mathbb{E}[\mathrm{e}^{-\rho h}\check{Y}_{t+1}^{j} \mid \hat{\mathcal{F}}_t]\} \tag{5.1}$$

such that $\check{Y}_0^i = V(i)$ and $\check{Y}_0^1 - \check{Y}_0^0 \approx \mathcal{V}$.

5.3. Numerical results

Least-squares Monte Carlo regression (LSMC) [1] provides a method for approximating the conditional expectations appearing in (5.1) and consequently obtaining numerical results for optimal switching problems in a Markovian setting. This procedure will be used to obtain the

TABLE 1: Mean and standard deviation of the cancellable call option values.

x_0	$\delta = 1$	$\delta = 5$	$\delta = 10$	$\delta = 15$
60	0.057 (0.029)	0.292 (0.036)	0.323 (0.050)	0.323 (0.055)
80	0.268 (0.087)	2.130 (0.065)	3.220 (0.125)	3.260 (0.167)
100	1.000 (0.000)	5.000 (0.000)	10.000 (0.000)	11.300 (0.248)
120	20.400 (0.108)	22.800 (0.256)	24.600 (0.209)	24.700 (0.212)
140	40.300 (0.103)	41.100 (0.259)	41.500 (0.258)	41.500 (0.254)

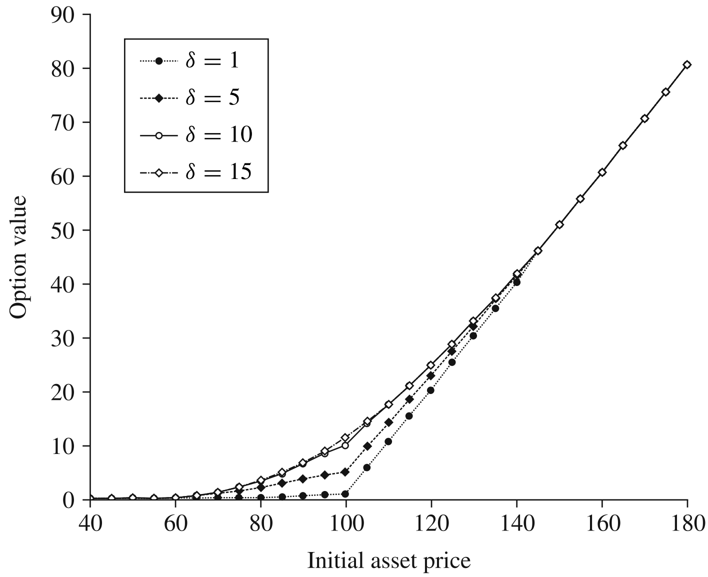

FIGURE 1: Mean values of the cancellable call option for different δ and x_0.

approximate value for the option. Sample paths for the GBM X at $\{r_t\}_{t=0}^{T}$ (and, therefore, $\hat{X}$), which are inputs to the algorithm, can be simulated exactly using the following relation:

$$\hat{X}_0 := X(r_0) = x_0, \qquad \hat{X}_{t+1} := X(r_{t+1}) = X(r_t)\exp(\widetilde{\mu}h + \kappa\sqrt{h}\xi_{t+1}), \quad t = 0, \dots, T-1, \tag{5.2}$$

where $\{\xi_t\}_{t=1}^{T}$ is a sequence of independent and identically distributed standard normal random variables.

Model parameters for the finite expiry cancellable call option were obtained from [5, p. 7] and are as follows: $S = 1$, $K = 100$, $\rho = 0.1$, $d = 0.09$, and $\kappa = 0.3$. We set $T = 200$ and generated 2500 sample paths for $\hat{X}$ using (5.2) and the well-known Monte Carlo variance reduction technique of *antithetic variates*. The mean and standard deviation of the option value were recorded after 100 runs of the LSMC algorithm. These results are recorded to three significant figures in Table 1 above different values of the initial asset price x_0 and the cancellation penalty δ. In Figure 1 we show the mean of the option price for these different values of δ with x_0 ranging from 40 to 180 in increments of 5.

It is beyond the scope of this paper to provide financial insight into these results—such details may be found in [5] and some of its references. However, the reader is invited to compare our

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

Figure 1 with [5, p. 7, Figure 1] to confirm that our results are in good agreement to theirs. Further examples can be found in [6].

6. Conclusion

In this paper we used a probabilistic approach to solve a finite-horizon discrete-time optimal switching problem for a model with signed switching costs. The approach, which works without Markovian assumptions, reduced the switching problem to iterated optimal stopping problems defined in terms of (coupled) Snell envelopes, just as in the verification theorem of [2] in the continuous-time case. We were able to define the Snell envelopes by an explicit backward induction scheme, thereby extending the numerical methods of [1] and [3] to problems with negative switching costs. Finally, we demonstrated these results numerically by approximating a game call option's value.

Appendix A. Supplementary proofs

Lemma A.1. *For each $i \in \mathbb{I}$, let $U^i \in \mathcal{S}^2$ and $Y^i \in \mathcal{S}^2$ be defined as in (3.2) and (3.3), respectively. Let $\tau_n \in \mathcal{T}$ and $\iota_n\colon \Omega \to \mathbb{I}$ be $\mathcal{F}_{\tau_n}$-measurable. Then*

$$Y_t^{\iota_n} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\bigg[\sum_{s=t}^{\tau-1} \Psi_{\iota_n}(s) + U_\tau^{\iota_n} \,\bigg|\, \mathcal{F}_t\bigg] \quad \text{on } [\tau_n, T]. \tag{A.1}$$

Proof. For notational simplicity define $(\check{U}_t^i)_{t\in\mathbb{T}}$ by $\check{U}_t^i = \sum_{s=0}^{t-1} \Psi_i(s) + U_t^i$. For any $i \in \mathbb{I}$ and any time $s \le t$, $\Psi_i(s)$ is $\mathcal{F}_t$-measurable and using this in (3.3), we obtain

$$\begin{aligned} Y_t^i &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\bigg[\sum_{s=t}^{\tau-1} \Psi_i(s) + U_\tau^i \,\bigg|\, \mathcal{F}_t\bigg] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\bigg[\sum_{s=0}^{\tau-1} \Psi_i(s) - \sum_{s=0}^{t-1} \Psi_i(s) + U_\tau^i \,\bigg|\, \mathcal{F}_t\bigg] \\ &= -\sum_{s=0}^{t-1} \Psi_i(s) + \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[\check{U}_\tau^i \mid \mathcal{F}_t]. \end{aligned} \tag{A.2}$$

Since $U^i, \Psi_i \in \mathcal{S}^2$, the Snell envelope of the process $(\sum_{s=0}^{t-1} \Psi_i(s) + U_t^i)_{t\in\mathbb{T}}$ exists (cf. Proposition 3.1), which we denote by $\check{Y}^i$. Using (A.2), $\check{Y}^i$ satisfies

$$\check{Y}_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[\check{U}_\tau^i \mid \mathcal{F}_t] = Y_t^i + \sum_{s=0}^{t-1} \Psi_i(s) \tag{A.3}$$

and is the smallest supermartingale which dominates $\check{U}^i$. Note that as $Y^i, \Psi_i \in \mathcal{S}^2$, the supermartingale property carries over to stopping times [8, Theorem II.59.1].

Consider the process $\sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n = i\}} \check{Y}^i$ on $[\tau_n, T]$, recalling that the sum over $\mathbb{I}$ is finite. Let $r, t \in \mathbb{T}$ be arbitrary times satisfying $r \le t$. Note that the indicator function $\mathbf{1}_{\{\iota_n = i\}}$ is nonnegative, and each $\mathbf{1}_{\{\iota_n = i\}}$ is $\mathcal{F}_{\tau_n}$-measurable and, therefore, $\mathcal{F}_r$-measurable on $\{\tau_n \le r\}$. Using these observations together with the supermartingale property yields

$$\mathbb{E}\bigg[\sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n = i\}} \check{Y}_t^i \,\bigg|\, \mathcal{F}_r\bigg] \mathbf{1}_{\{\tau_n \le r\}} = \sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n = i\}} \mathbb{E}[\check{Y}_t^i \mid \mathcal{F}_r] \mathbf{1}_{\{\tau_n \le r\}} \le \sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n = i\}} \mathbf{1}_{\{\tau_n \le r\}} \check{Y}_r^i \quad \text{a.s.}$$

https://doi.org/10.1017/apr.2016.30 Published online by Cambridge University Press

This shows that $\sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n=i\}}\check{Y}^i$ is a supermartingale on $[\tau_n, T]$. For each $i \in \mathbb{I}$, the dominating property of the Snell envelope and nonnegativity of $\mathbf{1}_{\{\iota_n=i\}}$ leads to

$$\mathbf{1}_{\{\tau_n\le t\}}\mathbf{1}_{\{\iota_n=i\}}\check{Y}^i_t \ge \mathbf{1}_{\{\tau_n\le t\}}\mathbf{1}_{\{\iota_n=i\}}\check{U}^i_t$$

and summing over $i \in \mathbb{I}$, we then have

$$\check{Y}^{\iota_n}_t := \sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n=i\}}\check{Y}^i_t \ge \sum_{i\in\mathbb{I}} \mathbf{1}_{\{\iota_n=i\}}\check{U}^i_t =: \check{U}^{\iota_n}_t \quad \text{on } \{\tau_n \le t\}.$$

The process $\check{Y}^{\iota_n}$ is, therefore, a supermartingale dominating $\check{U}^{\iota_n}$ on $[\tau_n, T]$. Similar arguments as above can be used to show that $\check{Y}^{\iota_n}$ is the smallest supermartingale with this property, and is, therefore, the Snell envelope of $\check{U}^{\iota_n}$. Proposition 3.1 leads to a representation for $\check{Y}^{\iota_n}$ similar to (A.3), and the $\mathcal{F}_t$-measurability of the summation term leads to (A.1). □

Lemma A.2. *Let $\alpha^* = (\tau^*_n, \iota^*_n)_{n\ge0}$ be the sequence defined in (3.4). Suppose that Assumption 2.2(i) holds for the switching costs. Then $\alpha^* \in \mathcal{A}_{t,i}$.*

Proof. The times $\{\tau^*_n\}_{n\ge0}$ are nondecreasing by definition, $\tau^*_0 = t$ and each $\tau^*_n \in \mathcal{T}_t$ since U^i and Y^i are adapted for every $i \in \mathbb{I}$. By [8, Lemma II.58.3], for any adapted process Z and stopping time τ, Z_τ is $\mathcal{F}_\tau$-measurable. The sets $A^{\iota^*_{n-1}}_j$ in (3.4) are, therefore, $\mathcal{F}_{\tau^*_n}$-measurable sets, which means the modes $\{\iota^*_n\}_{n\ge0}$ are also $\mathcal{F}_{\tau^*_n}$-measurable. Furthermore, $\iota^*_n \ne \iota^*_{n+1}$ a.s. for $n \ge 0$.

The last thing to verify is $\mathbb{P}(\{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\}) = 0$ for $n \ge 1$. Assume contrarily that for some $n \ge 1$, the event $\{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\}$ has positive probability (recall $\tau^*_{n+1} \ge \tau^*_n$). Using the definition for τ^*_n and τ^*_{n+1}, we have $Y^{\iota^*_{n-1}}_{\tau^*_n} = U^{\iota^*_{n-1}}_{\tau^*_n}$ and $Y^{\iota^*_n}_{\tau^*_{n+1}} = U^{\iota^*_n}_{\tau^*_{n+1}}$ $\mathbb{P}$-a.s. By the definition of ι^*_n and ι^*_{n+1}, it also holds that

$$Y^{\iota^*_{n-1}}_{\tau^*_n} = -\gamma_{\iota^*_{n-1},\iota^*_n}(\tau^*_n) + Y^{\iota^*_n}_{\tau^*_n} \quad \text{on } \{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\}, \tag{A.4}$$

$$Y^{\iota^*_n}_{\tau^*_n} = -\gamma_{\iota^*_n,\iota^*_{n+1}}(\tau^*_n) + Y^{\iota^*_{n+1}}_{\tau^*_n} \quad \text{on } \{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\}. \tag{A.5}$$

Let H be the event defined by

$$H = \{\iota^*_{n-1} = i,\ \iota^*_n = j,\ \iota^*_{n+1} = k\}\cap\{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\},$$

where $i, j, k \in \mathbb{I}$ are three modes satisfying $i \ne j$ and $j \ne k$. Substituting for $Y^{\iota^*_n}_{\tau^*_n}$ in (A.4) and (A.5), and using Assumption 2.2(ii) for the switching costs, we have

$$Y^i_{\tau^*_n} = -\gamma_{i,j}(\tau^*_n) - \gamma_{j,k}(\tau^*_n) + Y^k_{\tau^*_n} < -\gamma_{i,k}(\tau^*_n) + Y^k_{\tau^*_n} \quad \text{on } H.$$

In the previous arguments we have just shown that

$$-\gamma_{i,k}(\tau^*_n) + Y^k_{\tau^*_n} > -\gamma_{i,j}(\tau^*_n) + Y^j_{\tau^*_n} = \max_{l\ne i}\{-\gamma_{i,l}(\tau^*_n) + Y^l_{\tau^*_n}\} \quad \text{on } H,$$

which is a contradiction for every $k \in \mathbb{I}$. Since $i \ne j$ and $j \ne k$ were arbitrary modes, for $n \ge 1$ it holds that

$$\mathbb{P}(\{\tau^*_n < T\}\cap\{\tau^*_n = \tau^*_{n+1}\}) = 0.$$

□

Acknowledgements

This research was partially supported by the EPSRC (grant number EP/K00557X/1). The author would also like to acknowledge all those whose comments led to this improved version of the manuscript, including his PhD supervisor J. Moriarty and colleague T. De Angelis at the University of Manchester, and an anonymous referee.

References

[1] Carmona, R. and Ludkovski, M. (2008). Pricing asset scheduling flexibility using optimal switching. *Appl. Math. Finance* **15,** 405–447.

[2] Djehiche, B., Hamadène, S. and Popier, A. (2009). A finite horizon optimal multiple switching problem. *SIAM J. Control Optimization* **48,** 2751–2770.

[3] Gassiat, P., Kharroubi, I. and Pham, H. (2012). Time discretization and quantization methods for optimal multiple switching problem. *Stoch. Process. Appl.* **122,** 2019–2052.

[4] Guo, X. and Tomecek, P. (2008). Connections between singular control and optimal switching. *SIAM J. Control Optimization* **47,** 421–443.

[5] Kunita, H. and Seko, S. (2004). Game call options and their exercise regions. Tech. Rep., Nanzan Academic Society, Mathematical Sciences and Information Engineering.

[6] Martyr, R. (2016). Solving finite time horizon Dynkin games by optimal switching. To appear in *J. Appl. Prob.*

[7] Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.

[8] Rogers, L. C. G. and Williams, D. (2000). *Diffusions, Markov Processes, and Martingales*, Vol. 1, Foundations. Cambridge University Press.

[9] Tanaka, T. (1990). Two-parameter optimal stopping problem with switching costs. *Stoch. Process. Appl.* **36,** 153–163.

[10] Yushkevich, A. and Gordienko, E. (2002). Average optimal switching of a Markov chain with a Borel state space. *Math. Meth. Operat. Res.* **55,** 143–159.