Nagoya Math. J., **237** (2020), 188–208
DOI 10.1017/nmj.2018.19

GORENSTEIN HOMOLOGICAL PROPERTIES OF TENSOR RINGS

XIAO-WU CHEN AND MING LU

Abstract. Let R be a two-sided Noetherian ring, and let M be a nilpotent R-bimodule, which is finitely generated on both sides. We study Gorenstein homological properties of the tensor ring $T_R(M)$. Under certain conditions, the ring R is Gorenstein if and only if so is $T_R(M)$. We characterize Gorenstein projective $T_R(M)$-modules in terms of R-modules.

§1. Introduction

Let R be a two-sided Noetherian ring. In Gorenstein homological algebra, the following Gorenstein homological properties of R are the main concerns: the Gorensteinness of the ring R, the (stable) category of Gorenstein projective R-modules, and the Gorenstein projective dimensions and resolutions of R-modules.

Let M be an R-bimodule, which is finitely generated on both sides. The classical homological properties of the tensor ring $T_R(M)$ are studied in [7, 17, 19]. In general, the tensor ring $T_R(M)$ is not Noetherian. Hence, we require that M is *nilpotent*, that is, its nth tensor power vanishes for n large enough, in which case $T_R(M)$ is two-sided Noetherian.

We are concerned with the Gorenstein homological properties of the tensor ring $T_R(M)$ for a nilpotent R-bimodule M. The motivation is the example in [9], which is a tensor ring and is 1-Gorenstein, that is, the regular module has self-injective dimension at most one on each side. More precisely, the ring R considered in [9] is a certain self-injective algebra and the nilpotent R-bimodule M is projective on each side; see also [10]. Then the tensor ring $T_R(M)$ is 1-Gorenstein, whose modules yield a characteristic-free categorification of the root system. We mention other related examples in [15, 20, 21].

The main result of this paper is a vast generalization of the mentioned examples. An R-bimodule M is said to be *perfect* provided that it has finite

Received November 20, 2017. Revised May 16, 2018. Accepted May 16, 2018.
2010 Mathematics subject classification. 16E10, 16E65, 18G25.

projective dimension on each side and satisfies the following Tor-vanishing conditions: $\mathrm{Tor}_j^R(M, M^{\otimes_R i}) = 0$ for each $i, j \geqslant 1$, where $M^{\otimes_R i}$ is the ith tensor power of M.

THEOREM. *Let R be a two-sided Noetherian ring, and let M be a nilpotent perfect R-bimodule. Then the ring R is Gorenstein if and only if so is the tensor ring $T_R(M)$.*

We give an inequality between the Gorenstein dimensions of R and $T_R(M)$. For details, we refer the reader to Theorem 4.7.

We characterize Gorenstein projective $T_R(M)$-modules in terms of R-modules; see Theorem 3.9. This extends the corresponding description in [9, Theorem 10.9]. It is well known that the study of Gorenstein projective modules is intimately related to that of Frobenius categories; compare [4, 6]. The characterization of Gorenstein projective $T_R(M)$-modules relies on an explicit construction of a new Frobenius category; see Theorem 2.5. It seems that this construction might be of independent interest.

The paper is structured as follows. In Section 2, we construct a new Frobenius category, using the category of representations of a nilpotent endofunctor on an abelian category. In Section 3, we characterize Gorenstein projective $T_R(M)$-modules for a certain nilpotent R-bimodule M. We also study the Gorenstein projective dimensions of $T_R(M)$-modules. In Section 4, we introduce the notion of a perfect bimodule and prove Theorem 4.7. Some (non-)examples are studied in the end.

§2. The construction of a new Frobenius category

In this section, we construct a new Frobenius category, which is an exact subcategory in the category of representations of a certain nilpotent endofunctor on an abelian category. For exact categories, we refer the reader to [12, Appendix A].

2.1 The category of representations

Let $\mathcal{A}$ be an additive category with an additive endofunctor $F\colon \mathcal{A} \to \mathcal{A}$. By a *representation* of F, we mean a pair (X, u) with X an object and $u\colon F(X) \to X$ a morphism in $\mathcal{A}$. A morphism $f\colon (X, u) \to (Y, v)$ between two representations is a morphism $f\colon X \to Y$ in $\mathcal{A}$ satisfying $f \circ u = v \circ F(f)$. This defines the category $\mathrm{rep}(F)$ of representations of F. We have a forgetful functor

$$U\colon \mathrm{rep}(F) \longrightarrow \mathcal{A}$$

sending (X, u) to the underlying object X.

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

We assume that F is *nilpotent*, that is, $F^{N+1}=0$ for some $N \geqslant 0$. For each object A, we define $\mathrm{Ind}(A)=\bigoplus_{i=0}^{N} F^i(A)$ with $F^0=\mathrm{Id}_{\mathcal{A}}$, and a morphism $c_A\colon F\mathrm{Ind}(A)\to \mathrm{Ind}(A)$ such that its restriction to $F(F^i(A))=F^{i+1}(A)$ is the inclusion into $\mathrm{Ind}(A)$. This defines a representation $(\mathrm{Ind}(A), c_A)$ of F. Moreover, we have the *induction* functor

$$\mathrm{Ind}\colon \mathcal{A} \longrightarrow \mathrm{rep}(F)$$

sending A to $(\mathrm{Ind}(A), c_A)$, and a morphism f to $\bigoplus_{i=0}^{N} F^i(f)$.

Lemma 2.1. *Keep the notation as above. Then the pair* (Ind, U) *is adjoint.*

Proof. The natural isomorphism

$$\mathrm{Hom}_{\mathrm{rep}(F)}((\mathrm{Ind}(A), c_A), (X, u)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, X)$$

sends f to the restriction $f|_A$. The inverse map sends $g\colon A\to X$ to $g'\colon (\mathrm{Ind}(A), c_A)\to (X,u)$, such that the restriction of g' to $F^0(A)=A$ is g, and to $F^i(A)$ is given by $u\circ F(u)\circ\cdots\circ F^{i-1}(u)\circ F^i(g)$ for $i\geqslant 1$. ∎

We refer the reader to [14, Chapter VI] for monads and monadic adjoint pairs.

Remark 2.2. The nilpotent endofunctor F defines a monad M on $\mathcal{A}$, which, as a functor, equals $\bigoplus_{i=0}^{N} F^i$ and whose multiplication is induced by the composition of functors. There is an isomorphism of categories between $\mathrm{rep}(F)$ and $M\text{-}\mathrm{Mod}_{\mathcal{A}}$, the category of M-modules in $\mathcal{A}$. In other words, the adjoint pair (Ind, U) is strictly monadic.

We characterize the essential image of the induction functor.

Lemma 2.3. *A representation* (X, u) *is isomorphic to* $(\mathrm{Ind}(A), c_A)$ *for some object* A *if and only if* u *is a split monomorphism, which has a cokernel in* $\mathcal{A}$.

Proof. For the "only if" part, we just observe that the cokernel of c_A is the natural projection on A.

For the "if" part, we assume that the cokernel of u is $\pi\colon X\to A$. Take a section $\iota\colon A\to X$ of π. Then X is isomorphic to $A\oplus F(X)$, which is further isomorphic to $A\oplus F(A\oplus F(X))$. Using induction and the nilpotency of F, we infer that X is isomorphic to $\mathrm{Ind}(A)=A\oplus F(A)\oplus \cdots\oplus F^N(A)$; moreover, the corresponding inclusion map $F^i(A)\to X$ is

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

given by $u \circ F(u) \circ \cdots \circ F^{i-1}(u) \circ F^i(\iota)$. In other words, this gives rise to an isomorphism $(\mathrm{Ind}(A), c_A) \to (X, u)$, which corresponds to $\iota\colon A \to X$ in the adjoint pair (Ind, U); compare the proof of Lemma 2.1. ∎

We assume now that $\mathcal{A}$ is abelian and that the endofunctor F is right exact. Then $\mathrm{rep}(F)$ is an abelian category. Moreover, a sequence $(X, u) \xrightarrow{f} (Y, v) \xrightarrow{g} (Z, w)$ is exact in $\mathrm{rep}(F)$ if and only if the underlying sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact in $\mathcal{A}$. In particular, the forgetful functor $U\colon \mathrm{rep}(F) \to \mathcal{A}$ is exact.

For each representation (X, u), there is an exact sequence in $\mathrm{rep}(F)$

$$0 \longrightarrow (\mathrm{Ind}(FX), c_{FX}) \xrightarrow{\phi_{(X,u)}} (\mathrm{Ind}(X), c_X) \xrightarrow{\eta_{(X,u)}} (X, u) \longrightarrow 0. \tag{2.1}$$

Here, we view $\phi_{(X,u)}$ as a formal matrix; the only possibly nonzero entries are $\mathrm{Id}_{F^i(X)}\colon F^i(X) \to F^i(X)$ and $-F^{i-1}(u)\colon F^i(X) \to F^{i-1}(X)$ for $i \geqslant 1$. The morphism $\eta_{(X,u)}$ is given by the counit of the adjoint pair (Ind, U). More precisely, the restriction of $\eta_{(X,u)}$ on X is Id_X, and on $F^i(X)$ is $u \circ F(u) \circ \cdots \circ F^{i-1}(u)$ for $i \geqslant 1$. The above exact sequence is a categorical version of that in [19, Lemma].

2.2 A new Frobenius category

Let $\mathcal{A}$ be an abelian category. A full additive subcategory $\mathcal{E}$ is *exact* provided that it is closed under extensions. In this case, $\mathcal{E}$ becomes naturally an exact category in the sense of Quillen, whose conflations are given by short exact sequences with terms in $\mathcal{E}$.

Recall that an exact category $\mathcal{E}$ is *Frobenius* provided that it has enough projective objects and enough injective objects, and the class of projective objects coincides with the class of injective objects. Frobenius categories are important, since their stable categories modulo projective objects have natural triangulated structures; see [11, Section I.2].

We will consider the following conditions for the triple $(\mathcal{A}, \mathcal{A}', F)$.

- (F1) The category $\mathcal{A}$ is abelian, and $\mathcal{A}' \subseteq \mathcal{A}$ is an exact subcategory, which is Frobenius as an exact category. Denote by $\mathcal{P} \subseteq \mathcal{A}'$ the subcategory of projective objects.
- (F2) For every epimorphism $f\colon A \to X$ with $X \in \mathcal{A}'$, there is an epimorphism $g\colon Y \to A$ with $Y \in \mathcal{A}'$ and $\mathrm{Ker}(f \circ g) \in \mathcal{A}'$.
- (F3) The endofunctor $F\colon \mathcal{A} \to \mathcal{A}$ is right exact and nilpotent, satisfying that $\mathrm{Ext}^1_{\mathcal{A}}(X, F^i(P)) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(P, F^i(X))$ for any $X \in \mathcal{A}'$, $P \in \mathcal{P}$ and $i \geqslant 1$.

(F4) For any exact sequence $\eta\colon 0 \to A \to B \to X \to 0$ in $\mathcal{A}$, we have that $F(\eta)$ is exact, provided that X admits a monomorphism $u\colon F(X) \to X$ with $\operatorname{Cok} u \in \mathcal{A}'$.

LEMMA 2.4. *Assume that the triple $(\mathcal{A}, \mathcal{A}', F)$ satisfies (F1)–(F4). Then the following statements hold.*

(1) *For every epimorphism $f\colon A \to X$ with $X \in \mathcal{A}'$, there is an epimorphism $g'\colon P \to A$ with $P \in \mathcal{P}$ and $\operatorname{Ker}(f \circ g') \in \mathcal{A}'$.*
(2) *For an exact sequence $0 \to A \xrightarrow{f} B \to X \to 0$ with $X \in \mathcal{A}'$ and a morphism $a\colon A \to F^i(P)$ for some $P \in \mathcal{P}$ and $i \geqslant 0$, there is a morphism $b\colon B \to F^i(P)$ with $a = b \circ f$.*
(3) *For an exact sequence $\eta\colon 0 \to A \to B \to F^i(X) \to 0$ with $X \in \mathcal{A}'$ and $i \geqslant 0$, we have that $F(\eta)$ is exact.*
(4) *For any monomorphism $f\colon A \to B$ with cokernel in $\mathcal{A}'$, we have that $F^i(f)$ is mono for any $i \geqslant 1$.*
(5) *Assume that $u\colon F(X) \to X$ is a monomorphism with $\operatorname{Cok} u \in \mathcal{A}'$. Then $\operatorname{Ext}^1_{\mathcal{A}}(P, X) = 0$ for each $P \in \mathcal{P}'$.*

Proof. For (1), we just compose the morphism g in (F2) with an epimorphism $P \to Y$, whose kernel lies in $\mathcal{A}'$.

If $i = 0$ in (2), we use the fact that $\operatorname{Ext}^1_{\mathcal{A}}(X, P) = 0$, since P is also injective in $\mathcal{A}'$. If $i \geqslant 1$, we just apply $\operatorname{Ext}^1_{\mathcal{A}}(X, F^i(P)) = 0$ in (F3).

For (3), we may assume that $i \leqslant N$. By adding a trivial direct summand, we may replace η by $\eta'\colon 0 \to A \to B' \to \operatorname{Ind}(X) \to 0$ with $B' = B \oplus (\bigoplus_{j \neq i} F^j(X))$. By the exact sequence

$$0 \longrightarrow F(\operatorname{Ind}(X)) \xrightarrow{c_X} \operatorname{Ind}(X) \longrightarrow X \longrightarrow 0,$$

we might apply (F4) to η'. The exactness of $F(\eta')$ implies that for $F(\eta)$. Statement (4) follows from (3) and by induction.

For the last statement, we apply (4) to obtain that $F^i(u)$ is mono for each $i \geqslant 1$. The cokernel of $F^i(u)$ is isomorphic to $F^i(\operatorname{Cok} u)$. By the nilpotency of F, we infer that X is an iterated extension of the objects $F^i(\operatorname{Cok} u)$ for $0 \leqslant i \leqslant N$. By $\operatorname{Ext}^1_{\mathcal{A}}(P, F^i(\operatorname{Cok} u)) = 0$ in (F3), we deduce the required statement. □

The following consideration is inspired by [16, 21]. We consider the following full subcategory of $\operatorname{rep}(F)$:

$$\mathcal{B} = \{(X, u) \in \operatorname{rep}(F) \mid u \text{ is a monomorphism with } \operatorname{Cok} u \in \mathcal{A}'\}.$$

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

THEOREM 2.5. *Assume that the triple* $(\mathcal{A}, \mathcal{A}', F)$ *satisfies (F1)–(F4). Then* $\mathcal{B} \subseteq \mathrm{rep}(F)$ *is an exact subcategory, which is a Frobenius category. Moreover, its projective objects are precisely of the form* $(\mathrm{Ind}(P), c_P)$ *for* $P \in \mathcal{P}$.

Proof. Step 1. To show that $\mathcal{B} \subseteq \mathrm{rep}(F)$ is closed under extensions, we take an exact sequence $0 \to (X, u) \to (Y, v) \to (Z, w) \to 0$ with (X, u) and (Z, w) in $\mathcal{B}$. Since the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact by (F4), we infer by the five lemma that v is a monomorphism, whose cokernel is an extension of $\mathrm{Cok}\, w$ by $\mathrm{Cok}\, u$. Since $\mathcal{A}' \subseteq \mathcal{A}$ is closed under extensions, the cokernel of v lies in $\mathcal{A}'$. This proves that (Y, v) lies in $\mathcal{B}$.

Step 2. We claim that $(\mathrm{Ind}(P), c_P)$ is projective in $\mathcal{B}$ for $P \in \mathcal{P}$. Take an exact sequence $\xi\colon 0 \to (X, u) \to (Y, v) \to (Z, w) \to 0$ in $\mathcal{B}$. By the adjoint pair in Lemma 2.1, the sequence $\mathrm{Hom}_{\mathrm{rep}(F)}((\mathrm{Ind}(P), c_P), \xi)$ is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P, U(\xi))$. The latter is exact by $\mathrm{Ext}^1_{\mathcal{A}}(P, X) = 0$ in Lemma 2.4(5). This proves the claim.

Let (X, u) be an object in $\mathcal{B}$. Denote by $\pi\colon X \to \mathrm{Cok}\, u$ the projection. Then by Lemma 2.4(1), there is an epimorphism $f\colon P \to X$ with $P \in \mathcal{P}$ and $\mathrm{Ker}(\pi \circ f) \in \mathcal{A}'$. By the adjoint pair in Lemma 2.1, the morphism f corresponds to a morphism $f'\colon (\mathrm{Ind}(P), c_P) \to (X, u)$, which is clearly epic. Then we have an exact sequence in $\mathrm{rep}(F)$

$$0 \longrightarrow (Y, v) \longrightarrow (\mathrm{Ind}(P), c_P) \xrightarrow{f'} (X, u) \longrightarrow 0.$$

By (F4), we have that $0 \to F(Y) \to F\mathrm{Ind}(P) \to F(X) \to 0$ is exact. Using the five lemma, we infer that v is mono, whose cokernel is isomorphic to $\mathrm{Ker}(\pi \circ f)$ and thus lies in $\mathcal{A}'$. Hence (Y, v) lies in $\mathcal{B}$. The above exact sequence shows that $\mathcal{B}$ has enough projective objects. Moreover, each projective object is a direct summand of $(\mathrm{Ind}(P), c_P)$ for some $P \in \mathcal{P}$. Using Lemma 2.3, we infer that any projective object is of the form $(\mathrm{Ind}(Q), c_Q)$ for $Q \in \mathcal{P}$.

Step 3. We claim that $(\mathrm{Ind}(P), c_P)$ is injective for each $P \in \mathcal{P}$. For this, we take an arbitrary exact sequence $0 \to (X, u) \xrightarrow{f} (Y, v) \xrightarrow{g} (Z, w) \to 0$ in $\mathcal{B}$.

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Then we have the following exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\
0 & \longrightarrow & \operatorname{Cok} u & \xrightarrow{\bar{f}} & \operatorname{Cok} v & \xrightarrow{\bar{g}} & \operatorname{Cok} w & \longrightarrow & 0.
\end{array}$$

Take any morphism

$$(a_0, a_1, \ldots, a_N)^t\colon (X, u) \longrightarrow (\operatorname{Ind}(P), c_P),$$

where "t" denotes the transpose and $a_i\colon X \to F^i(P)$. We observe that $a_0 \circ u = 0$ and $a_i \circ u = F(a_{i-1})$ for $i \geqslant 1$. Then there exists a morphism $\bar{a}_0\colon \operatorname{Cok} u \to P$ with $a_0 = \bar{a}_0 \circ \pi_1$. By Lemma 2.4(2) we have a morphism $\bar{b}_0\colon \operatorname{Cok} v \to P$ with $\bar{a}_0 = \bar{b}_0 \circ \bar{f}$. We set $b_0 = \bar{b}_0 \circ \pi_2\colon Y \to P$. Hence, we have $a_0 = b_0 \circ f$ and $b_0 \circ v = 0$.

Since $\operatorname{Cok} v$ lies in $\mathcal{A}'$, by Lemma 2.4(2) we have a morphism $b_1'\colon Y \to F(P)$ satisfying $F(b_0) = b_1' \circ v$. We have

$$(a_1 - b_1' \circ f) \circ u = F(a_0) - b_1' \circ v \circ F(f) = F(a_0) - F(b_0) \circ F(f) = 0.$$

There exists $x\colon \operatorname{Cok} u \to F(P)$ satisfying $a_1 - b_1' \circ f = x \circ \pi_1$. Applying Lemma 2.4(2) again, we have a morphism $y\colon \operatorname{Cok} v \to F(P)$ with $x = y \circ \bar{f}$. Set $b_1 = b_1' + y \circ \pi_2$. Then we have $a_1 = b_1 \circ f$ and $b_1 \circ v = F(b_0)$.

We iterate the above argument to construct $b_i\colon Y \to F^i(P)$ such that $a_i = b_i \circ f$ and $b_i \circ v = F(b_{i-1})$ hold for $i \geqslant 2$. Then we have the morphism

$$(b_0, b_1, \ldots, b_N)^t\colon (Y, v) \longrightarrow (\operatorname{Ind}(P), c_P),$$

which makes $(a_0, a_1, \ldots, a_N)^t$ factor though $f\colon (X, u) \to (Y, v)$, as required.

Step 4. For the final step, we construct for each object (X, u) in $\mathcal{B}$ an exact sequence

$$0 \to (X, u) \longrightarrow (\operatorname{Ind}(P), c_P) \longrightarrow (Y, v) \longrightarrow 0$$

in $\mathcal{B}$ with $P \in \mathcal{P}$. Then we are done with the whole proof.

Denote by $\pi\colon X \to \operatorname{Cok} u$ the cokernel of u. We observe that for each $i \geqslant 1$, $F^{i-1}(u)$ is mono by Lemma 2.4(4). In what follows, we view $F^i(X)$ as a subobject of X.

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Since $\mathcal{A}'$ is Frobenius, we take a monomorphism $\iota\colon \mathrm{Cok}\, u \to P$ with its cokernel in $\mathcal{A}'$. Set $a_0 = \iota \circ \pi$. Then $\mathrm{Ker}\, a_0 = \mathrm{Im}\, u$. Since $F^i(\iota)$ is mono by Lemma 2.4(4), we infer that $\mathrm{Ker}\, F^i(a_0) = \mathrm{Im}\, F^i(u)$ for all $i \geqslant 1$.

By Lemma 2.4(2), we have a morphism $a_1\colon X \to F(P)$ with $F(a_0) = a_1 \circ u$. Then we have

$$\mathrm{Ker}\, a_1 \cap \mathrm{Im}\, u = \mathrm{Ker}\, F(a_0) = \mathrm{Im}\, F(u).$$

Similarly, we have a morphism $a_2\colon X \to F^2(P)$ with $F(a_1) = a_2 \circ u$. Then we have

$$\begin{aligned} \mathrm{Ker}\, a_2 \cap \mathrm{Im}\, u \cap \mathrm{Im}\, F(u) &= \mathrm{Ker}\, F(a_1) \cap \mathrm{Im}\, F(u) \\ &= \mathrm{Ker}\, F^2(a_0) \\ &= \mathrm{Im}\, F^2(u). \end{aligned}$$

Here, the first and second equalities follow from $F(a_1) = a_2 \circ u$ and $F(a_0) = a_1 \circ u$, respectively. We observe that the above identities actually prove the following ones:

$$\mathrm{Ker}\, a_1 \cap \mathrm{Ker}\, a_0 = \mathrm{Im}\, F(u), \qquad \text{and}$$
$$\mathrm{Ker}\, a_2 \cap \mathrm{Ker}\, a_1 \cap \mathrm{Ker}\, a_0 = \mathrm{Im}\, F^2(u).$$

We proceed in the same way to construct $a_i\colon X \to F^i(P)$. We observe that

$$\mathrm{Ker}\, a_N \cap \cdots \cap \mathrm{Ker}\, a_1 \cap \mathrm{Ker}\, a_0 = \mathrm{Im}\, F^N(u) = 0.$$

This gives rise to a monomorphism

$$(a_0, a_1, \ldots, a_N)^t\colon (X, u) \longrightarrow (\mathrm{Ind}(P), c_P),$$

which induces ι by taking the cokernels. Denote its cokernel by (Y, v). By the snake lemma and the injectivity of ι, we infer that v is mono, whose cokernel coincides with that of ι and thus lies in $\mathcal{A}'$. This proves the desired short exact sequence. $\square$

§3. Gorenstein projective modules and admissible bimodules

In this section, we study Gorenstein projective modules over a tensor ring. The main reference for Gorenstein homological algebra is [8].

Throughout R is a two-sided Noetherian ring. We denote by R-mod the abelian category of finitely generated left R-modules, and by R-proj the full subcategory of projective modules. We identify right R-modules with left R^{op}-modules, where R^{op} is the opposite ring. Hence, R^{op}-mod denotes the category of finitely generated right R-modules.

An unbounded complex $P^\bullet$ of projective R-modules is *totally acyclic* provided that it is acyclic and that its dual $(P^\bullet)^* = \text{Hom}_R(P^\bullet, R)$ is also acyclic. Recall that an R-module G is *Gorenstein projective* provided that there exists a totally acyclic complex $P^\bullet$ with its zeroth cocycle $Z^0(P^\bullet) \simeq G$. The complex $P^\bullet$ is called a *complete resolution* of G.

We denote by R-Gproj the full subcategory of Gorenstein projective modules. We observe that $R\text{-proj} \subseteq R\text{-Gproj}$. It is well known that $R\text{-Gproj} \subseteq R\text{-mod}$ is closed under extensions. Moreover, as an exact category, it is Frobenius, whose projective objects are precisely projective R-modules; compare [4, Proposition 3.8].

The following fact is standard; compare [6, Theorem 4.2].

PROPOSITION 3.1. *Let $\mathcal{C}$ be an exact subcategory of R-mod containing R-proj. Assume that $\mathcal{C}$ is Frobenius, whose projective objects are precisely projective R-modules. Then $\mathcal{C} \subseteq R$-Gproj.*

Proof. For a module $C \in \mathcal{C}$, we construct its complete resolution by gluing the projective resolution and injective resolution of C inside $\mathcal{C}$. □

For a left R-module X, we denote by $\text{pd}_R X$, $\text{id}_R X$, and $\text{Gpd}_R X$ the projective dimension, injective dimension, and Gorenstein projective dimension of X, respectively. Recall that $\text{Gpd}_R X \leqslant n$ if and only if there is an exact sequence $0 \to G^{-n} \to \cdots \to G^{-1} \to G^0 \to X \to 0$ with each $G^{-i} \in R$-Gproj.

Recall that a two-sided Noetherian ring R is *Gorenstein* provided that $\text{id}_R R < \infty$ and $\text{id}_{R^{\text{op}}} R < \infty$. In this case, we have $\text{id}_R R = \text{id}_{R^{\text{op}}} R$ by [22, Lemma A]. This common value is denoted by G.dim R. If G.dim $R \leqslant d$, we call R a *d-Gorenstein ring*. For example, 0-Gorenstein rings are precisely quasi-Frobenius rings.

LEMMA 3.2. *Let R be a two-sided Noetherian ring, and let d, d_1, d_2 be integers. Then the following statements hold.*

(1) *If R is d-Gorenstein, then $\text{Gpd}_R X \leqslant d$ and $\text{Gpd}_{R^{\text{op}}} Y \leqslant d$ for each left R-module X and right R-module Y.*
(2) *If $\text{Gpd}_R X \leqslant d_1$ and $\text{Gpd}_{R^{\text{op}}} Y \leqslant d_2$ for each left R-module X and right R-module Y, then R is $\min\{d_1, d_2\}$-Gorenstein.*

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Proof. We refer the reader to [8, Theorem 12.3.1] for a detailed proof. For (2), we just note that $\mathrm{Ext}^i_R(X, R) = 0$ for $i > \mathrm{Gpd}_R X$. Hence, the assumptions imply that $\mathrm{id}_R R \leqslant d_1$ and $\mathrm{id}_{R^{\mathrm{op}}} R \leqslant d_2$. Then we are done by using the fact $\mathrm{id}_R R = \mathrm{id}_{R^{\mathrm{op}}} R$ for any Gorenstein ring R. □

Let M be an R-bimodule, which is finitely generated on both sides. Write $M^{\otimes_R 0} = R$ and $M^{\otimes_R (j+1)} = M \otimes_R (M^{\otimes_R j})$ for $j \geqslant 0$. We say that M is *nilpotent*, if $M^{\otimes_R (N+1)} = 0$ for some $N \geqslant 0$. This is equivalent to the condition that the endofunctor $M \otimes_R -$ on R-mod is nilpotent.

Let M be a nilpotent bimodule. We denote by $T_R(M) = \bigoplus_{i=0}^{\infty} M^{\otimes_R i}$ the tensor ring, which is also two-sided Noetherian. There is an isomorphism of categories

$$\mathrm{rep}(M \otimes_R -) \xrightarrow{\sim} T_R(M)\text{-mod}, \tag{3.1}$$

which identifies a representation (X, u) of $M \otimes_R -$ with a left $T_R(M)$-module X such that $m \cdot x = u(m \otimes x)$ for $m \in M$ and $x \in X$. In particular, projective $T_R(M)$-modules correspond to the representations $(\mathrm{Ind}(P), c_P)$ for projective R-modules P, which might also be viewed as the scalar extension $T_R(M) \otimes_R P$. Indeed, for each left R-module Z, $(\mathrm{Ind}(Z), c_Z)$ corresponds to $T_R(M) \otimes_R Z$.

In what follows, we will identify the two categories in (3.1). The following results are standard.

LEMMA 3.3. *Keep the assumptions as above. Let Z be a left R-module. Then the following statements hold.*

(1) *The R-module Z is projective if and only if $T_R(M) \otimes_R Z$ is projective as a $T_R(M)$-module .*
(2) *If $\mathrm{Tor}^R_i(T_R(M), Z) = 0$ for each $i \geqslant 1$, then we have*

$$\mathrm{pd}_R Z = \mathrm{pd}_{T_R(M)} T_R(M) \otimes_R Z.$$

Proof. For (1), it suffices to show the "if" part. Recall the identification of $(\mathrm{Ind}(Z), c_Z)$ and $T_R(M) \otimes_R Z$. Assume that $T_R(M) \otimes_R Z$ is projective. Hence, $(\mathrm{Ind}(Z), c_Z)$ is a direct summand of $(\mathrm{Ind}(R^n), c_{R^n})$ for some $n \geqslant 1$. It follows that Z is projective, since it is isomorphic to the cokernel of c_Z.

For (2), we take an exact sequence

$$0 \to Y \to P^{-n} \to P^{1-n} \to \cdots \to P^{-1} \to P^0 \to Z \to 0$$

of R-modules with each P^{-i} projective. Applying $T_R(M)\otimes_R -$ to it, we get an exact sequence starting at $T_R(M)\otimes_R Y$, with middle terms projective $T_R(M)$-modules. We apply (1) to Y. Then we are done. ∎

We also consider the category $\mathrm{rep}(-\otimes_R M)$ of representations of the endofunctor $-\otimes_R M$ on R^{op}-mod. Similar to (3.1), we have the following isomorphism of categories

$$\mathrm{rep}(-\otimes_R M)\xrightarrow{\sim} T_R(M)^{\mathrm{op}}\text{-mod}. \tag{3.2}$$

We identify these categories.

Lemma 3.4. *We have a natural isomorphism*

$$(\mathrm{Ind}(P), c_P)^* \simeq (\mathrm{Ind}(P^*), c_{P^*})$$

of $T_R(M)^{\mathrm{op}}$-modules for each $P\in R$-proj.

Proof. We are done by the following canonical isomorphisms

$$\begin{aligned}(\mathrm{Ind}(P), c_P)^* &= \mathrm{Hom}_{T_R(M)}(T_R(M)\otimes_R P, T_R(M))\\ &\simeq \mathrm{Hom}_R(P, T_R(M))\\ &\simeq \mathrm{Hom}_R(P,R)\otimes_R T_R(M)\\ &\simeq (\mathrm{Ind}(P^*), c_{P^*}).\end{aligned}$$

As mentioned above, we identify $(\mathrm{Ind}(P), c_P)$ with $T_R(M)\otimes_R P$ via (3.1), and $(\mathrm{Ind}(P^*), c_{P^*})$ with $P^*\otimes_R T_R(M)$ via (3.2). ∎

Definition 3.5. An R-bimodule M is *left-admissible* provided that

$$\mathrm{Ext}^1_R(G, M^{\otimes_R i}) = 0 = \mathrm{Tor}^R_1(M, M^{\otimes_R i}\otimes_R G)$$

for each $G\in R$-Gproj and $i\geqslant 0$. Dually, it is *right-admissible* if M is left-admissible replacing R by its opposite. We say that M is *admissible* if it is both left- and right-admissible.

Lemma 3.6. *Let M be an R-bimodule. Then the triple*

$$(R\text{-mod}, R\text{-Gproj}, M\otimes_R -)$$

satisfies (F1)–(F4) if and only if M is nilpotent and left-admissible.

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Proof. For the "only if" part, it suffices to infer

$$\mathrm{Tor}_1^R(M, (M^{\otimes_R i}) \otimes_R G) = 0$$

from Lemma 2.4(3).

For the "if" part, we have that (F2) holds, since R-Gproj is closed under kernels of epimorphisms. It remains to verify (F4). Assume that an R-module X fits into a short exact sequence $\eta\colon 0 \to M \otimes_R X \to X \to G \to 0$ with G Gorenstein projective. By assumption, we infer that $(M^{\otimes_R i}) \otimes_R \eta$ is exact for each i. Recall that M is nilpotent. It follows that X is an iterated extension of the modules $M^{\otimes_R i} \otimes_R G$. By the Tor-vanishing assumption, we infer that $\mathrm{Tor}_1^R(M, X) = 0$. This proves (F4). ∎

Recall identifications (3.1) and (3.2). We introduce the following full subcategories of $T_R(M)$-mod and $T_R(M)^{\mathrm{op}}$-mod, respectively:

$$\begin{aligned}&\mathrm{Gmon}(M \otimes_R -)\\ &\quad := \{(X, u) \in \mathrm{rep}(M \otimes_R -) \mid u \text{ is mono with } \mathrm{Cok}\, u \in R\text{-Gproj}\}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Gmon}(- \otimes_R M)\\ &\quad := \{(Y, v) \in \mathrm{rep}(- \otimes_R M) \mid v \text{ is mono with } \mathrm{Cok}\, v \in R^{\mathrm{op}}\text{-Gproj}\}.\end{aligned}$$

LEMMA 3.7. *Let R be a two-sided Noetherian ring. Assume that M is a nilpotent left-admissible R-bimodule. Then we have* $\mathrm{Gmon}(M \otimes_R -) \subseteq T_R(M)\text{-Gproj}$.

Proof. Recall isomorphism (3.1), where projective $T_R(M)$-modules are identified with $(\mathrm{Ind}(P), c_P)$ for projective R-modules P. By Theorem 2.5 and Lemma 3.6, the category $\mathrm{Gmon}(M \otimes_R -)$ is Frobenius. Then the required inclusion follows from Proposition 3.1. ∎

PROPOSITION 3.8. *Let R be a two-sided Noetherian ring, and let M be a nilpotent R-bimodule. Then the following two statements are equivalent:*

(1) *We have* $T_R(M)\text{-Gproj} \subseteq \mathrm{Gmon}(M \otimes_R -)$ *and* $T_R(M)^{\mathrm{op}}\text{-Gproj} \subseteq \mathrm{Gmon}(- \otimes_R M)$.
(2) *For any totally acyclic complex $Q^\bullet$ of $T_R(M)$-modules and its dual $(Q^\bullet)^* = \mathrm{Hom}_{T_R(M)}(Q^\bullet, T_R(M))$, both the complexes $M \otimes_R Q^\bullet$ and $(Q^\bullet)^* \otimes_R M$ are acyclic.*

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Proof. For "(1) ⇒ (2)," we take a totally acyclic complex $Q^\bullet$ of $T_R(M)$-modules. We assume that $Q^i = (\mathrm{Ind}(P^i), c_{P^i})$ for $P^i \in R\text{-proj}$. The ith cocycle of $Q^\bullet$ is denoted by (Z^i, u^i), which is Gorenstein projective. In particular, u^i is mono by assumption. It follows that the upper row of the following commutative diagram is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M\otimes_R Z^i & \longrightarrow & M\otimes_R \mathrm{Ind}(P^i) & \longrightarrow & M\otimes_R Z^{i+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u^i} & & \downarrow{\scriptstyle c_{P^i}} & & \downarrow{\scriptstyle u^{i+1}} & & \\ 0 & \longrightarrow & Z^i & \longrightarrow & \mathrm{Ind}(P^i) & \longrightarrow & Z^{i+1} & \longrightarrow & 0 \end{array}$$

Then we infer that $M \otimes_R Q^\bullet$ is acyclic. Since $(Q^\bullet)^*$ is also totally acyclic, the same argument proves that $(Q^\bullet)^* \otimes_R M$ is acyclic.

To prove "(2) ⇒ (1)," we only prove the first inclusion. Take $(X, u) \in T_R(M)\text{-Gproj}$. Assume that its complete resolution is $Q^\bullet$, where $Q^i = (\mathrm{Ind}(P^i), c_{P^i})$ for projective R-modules P^i. We consider the monomorphism $\iota\colon (X, u) \to Q^0 = (\mathrm{Ind}(P^0), c_{P^0})$. By the acyclicity of $M \otimes_R Q^\bullet$, we infer that $M \otimes_R \iota$ is mono. By $c_{P^0} \circ (M \otimes_R \iota) = \iota \circ u$ and the monomorphism c_{P^0}, we infer that u is mono. We have the following commutative exact diagram.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M\otimes_R \mathrm{Ind}(P^{-1}) & \xrightarrow{M\otimes_R d^{-1}} & M\otimes_R \mathrm{Ind}(P^0) & \xrightarrow{M\otimes_R d^{0}} & M\otimes_R \mathrm{Ind}(P^1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle c_{P^{-1}}} & & \downarrow{\scriptstyle c_{P^0}} & & \downarrow{\scriptstyle c_{P^1}} & & \\ \cdots & \longrightarrow & \mathrm{Ind}(P^{-1}) & \xrightarrow{d^{-1}} & \mathrm{Ind}(P^0) & \xrightarrow{d^0} & \mathrm{Ind}(P^1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & P^1 & \longrightarrow & \cdots \end{array}$$

The bottom row $P^\bullet$ is acyclic, whose zeroth cocycle is isomorphic to $\mathrm{Cok}\, u$.

We consider the dual complex $(Q^\bullet)^*$, whose components are isomorphic to $(\mathrm{Ind}(P^i)^*, c_{(P^i)^*})$ by Lemma 3.4. Applying the same argument to $(Q^\bullet)^*$, we obtain a commutative exact diagram in R^{op}-mod, whose bottom row is exact. However, the bottom row is isomorphic to $(P^\bullet)^*$, proving the total acyclicity of $P^\bullet$. Consequently, $\mathrm{Cok}\, u$ is Gorenstein projective. It follows that $(X, u) \in \mathrm{Gmon}(M \otimes_R -)$, as required. □

We summarize the results in this section. Recall that a two-sided Noetherian ring R is *CM-free* provided that $R\text{-Gproj} = R\text{-proj}$. We mention that

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Gorenstein projective modules are also called maximal Cohen–Macaulay modules. Here, CM stands for Cohen–Macaulay.

THEOREM 3.9. *Let R be a two-sided Noetherian ring, and M be a nilpotent admissible R-bimodule satisfying the following condition: for any totally acyclic complex $Q^\bullet$ of $T_R(M)$-modules, the complexes $M \otimes_R Q^\bullet$ and* $\mathrm{Hom}_{T_R(M)}(Q^\bullet, T_R(M)) \otimes_R M$ *are both acyclic. Then we have*

$$T_R(M)\text{-Gproj} = \mathrm{Gmon}(M \otimes_R -) \qquad \text{and}$$
$$T_R(M)^{\mathrm{op}}\text{-Gproj} = \mathrm{Gmon}(- \otimes_R M).$$

Consequently, for any left R-module Z, the following statements hold.

(1) *The module Z lies in R-Gproj if and only if $T_R(M) \otimes_R Z$ lies in $T_R(M)$-Gproj.*
(2) *If $\mathrm{Tor}_i^R(T_R(M), Z) = 0$ for each $i \geqslant 1$, then we have*

$$\mathrm{Gpd}_R Z = \mathrm{Gpd}_{T_R(M)} T_R(M) \otimes_R Z.$$

(3) *The ring R is CM-free if and only if so is $T_R(M)$.*

Proof. We just prove the consequences. Recall isomorphism (3.1), which identifies $T_R(M) \otimes_R Z$ with the induced representation $(\mathrm{Ind}(Z), c_Z)$. The cokernel of c_Z is isomorphic to Z. Then (1) follows immediately.

For (2), we take an exact sequence

$$0 \to Y \to P^{-n} \to P^{1-n} \to \cdots \to P^{-1} \to P^0 \to Z \to 0$$

of R-modules with each P^{-i} projective. Applying $T_R(M) \otimes_R -$ to it, we get an exact sequence starting at $T_R(M) \otimes_R Y$, with middle terms projective $T_R(M)$-modules. Recall that $\mathrm{Gpd}_R Z \leqslant n+1$ if and only if Y is Gorenstein projective. A similar remark holds for $T_R(M) \otimes_R Z$. Then we are done by (1), applied to Y.

For the "only if" part of (3), we assume that R is CM-free. Gorenstein projective $T_R(M)$-modules are of the form (X, u) with u a monomorphism and $\mathrm{Cok}\, u \in R$-Gproj. Hence $\mathrm{Cok}\, u = P$ is projective. By Lemma 2.3 and its proof, we infer that (X, u) is isomorphic to $(\mathrm{Ind}(P), c_P)$, which is identified with a projective $T_R(M)$-module. For the "if" part, we take any Gorenstein projective R-module G. Then $T_R(M) \otimes_R G$ is Gorenstein projective and thus projective. Then we are done by Lemma 3.3(1). ∎

§4. Perfect bimodules

In this section, we study homological conditions, under which Theorem 3.9 applies. For this, we introduce the notion of a perfect bimodule; see Definition 4.4.

Let R be a two-sided Noetherian ring. We only consider finitely generated R-modules. The following result is well known.

LEMMA 4.1. *Let Y be a right R-module. Suppose that we are given an exact sequence in R-mod*

$$\cdots \to E^{-n} \to E^{-(n-1)} \to \cdots \to E^{-1} \to E^{0} \to X \to 0$$

with $\mathrm{Tor}_i^R(Y, E^{-j}) = 0$ for each $i \geqslant 1$ and $j \geqslant 0$. Denote its $(-i)$th cocycle by Z^{-i}. Then the following statements hold.

(1) *The complex $Y \otimes_R E^\bullet$ computes $\mathrm{Tor}_i^R(Y, X)$, that is, $H^{-i}(Y \otimes_R E^\bullet) \simeq \mathrm{Tor}_i^R(Y, X)$ for $i \geqslant 0$.*
(2) *There is an isomorphism $\mathrm{Tor}_i^R(Y, Z^{-j}) = \mathrm{Tor}_{i+j+1}^R(Y, X)$ for each $i \geqslant 1$ and $j \geqslant 0$.*
(3) *Assume that $\mathrm{pd}_{R^{\mathrm{op}}} Y < \infty$ and that $F^\bullet$ is an acyclic complex with $\mathrm{Tor}_i^R(Y, F^j) = 0$ for each $i \geqslant 1$ and $j \in \mathbb{Z}$. Then the complex $Y \otimes_R F^\bullet$ is acyclic.*

Proof. Statements (1) and (2) are classical. Statement (3) follows from (1) and (2). □

The following consideration is related to that in [18]. Let M be an R-bimodule, which is finitely generated on both sides. We will consider the following Tor-vanishing conditions:

$$\text{(P)} \qquad \mathrm{Tor}_i^R(M, M^{\otimes_R j}) = 0 \quad \text{for all } i, j \geqslant 1.$$

LEMMA 4.2. *We assume that the R-bimodule M satisfies condition (P). Then the following statements are equivalent for each left R-module Y:*

(1) *$\mathrm{Tor}_i^R(M, M^{\otimes_R j} \otimes_R Y) = 0$ for any $i \geqslant 1$ and $j \geqslant 0$;*
(2) *$\mathrm{Tor}_i^R(M^{\otimes_R s}, M^{\otimes_R j} \otimes_R Y) = 0$ for any $i, s \geqslant 1$ and $j \geqslant 0$;*
(3) *$\mathrm{Tor}_i^R(M^{\otimes_R s}, Y) = 0$ for any $i, s \geqslant 1$.*

We will say that an R-module Y is *M-flat*, provided that it satisfies one of the above equivalent conditions.

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

Proof. To show "(1) $\Rightarrow$ (2)," we fix $j \geqslant 0$ and $s \geqslant 2$. Take a projective resolution $P^\bullet$ of $M^{\otimes_R j} \otimes_R Y$. It follows from (1) that $M \otimes_R P^\bullet$ is quasi-isomorphic to $M^{\otimes_R (j+1)} \otimes_R Y$. By Lemma 4.1(1), the complex $M \otimes_R (M \otimes_R P^\bullet)$ is quasi-isomorphic to $M^{\otimes_R (j+2)} \otimes_R Y$, using the condition $\mathrm{Tor}_i^R(M, M^{\otimes_R (j+1)} \otimes_R Y) = 0$ for each $i \geqslant 1$. Iterating this argument, we infer that $M^{\otimes_R s} \otimes_R P^\bullet$ is quasi-isomorphic to $M^{\otimes_R (j+s)} \otimes_R Y$. This proves (2). The implications "(2) $\Rightarrow$ (1)" and "(2) $\Rightarrow$ (3)" are clear.

It remains to show "(3) $\Rightarrow$ (1)." For this, we fix $j \geqslant 1$. Take a projective resolution $P^\bullet$ of Y. Then by (3), the complex $M^{\otimes_R j} \otimes_R P^\bullet$ is quasi-isomorphic to $M^{\otimes_R j} \otimes_R Y$. Applying Lemma 4.1(1), we infer that $M \otimes_R (M^{\otimes_R j} \otimes_R P^\bullet)$ computes $\mathrm{Tor}_i^R(M, M^{\otimes_R j} \otimes_R Y)$. But the complex is isomorphic to $M^{\otimes_R (j+1)} \otimes_R P^\bullet$, which is quasi-isomorphic to $M^{\otimes_R (j+1)} \otimes_R Y$ by (3). It follows that $\mathrm{Tor}_i^R(M, M^{\otimes_R j} \otimes_R Y) = 0$ for $i \geqslant 1$. ∎

The following consequence implies that condition (P) is symmetric.

COROLLARY 4.3. *Let M be an R-bimodule. Then M satisfies condition (P) if and only if $\mathrm{Tor}_i^R(M^{\otimes_R s}, M^{\otimes_R j}) = 0$ for each $i, s, j \geqslant 1$, if and only if $\mathrm{Tor}_i^R(M^{\otimes_R s}, M) = 0$ for each $i, s \geqslant 1$.*

Proof. By duality, it suffices to show the first "if and only if." For this, we just take $Y = M$ in the previous lemma. ∎

DEFINITION 4.4. We call an R-bimodule M *perfect*, provided that it satisfies $\mathrm{pd}_R M < \infty$, $\mathrm{pd}_{R^{\mathrm{op}}} M < \infty$, and condition (P).

LEMMA 4.5. *Let M be a perfect bimodule, and $Y \in R$-mod. Then the following statements hold.*

(1) *If $\mathrm{Tor}_i^R(M, Y) = 0$ for each $i \geqslant 1$, then we have $\mathrm{pd}_R(M \otimes_R Y) \leqslant \mathrm{pd}_R M + \mathrm{pd}_R Y$.*
(2) *For each $i \geqslant 0$, we have $\mathrm{pd}_R M^{\otimes_R i} \leqslant i\,\mathrm{pd}_R M$ and $\mathrm{pd}_{R^{\mathrm{op}}} M^{\otimes_R i} \leqslant i\,\mathrm{pd}_{R^{\mathrm{op}}} M$.*
(3) *Assume further that $M^{\otimes_R (N+1)} = 0$ for $N \geqslant 0$. Then $\mathrm{pd}_R T_R(M) \leqslant N \mathrm{pd}_R M$ and $\mathrm{pd}_{R^{\mathrm{op}}} T_R(M) \leqslant N \mathrm{pd}_{R^{\mathrm{op}}} M$.*

Proof. For (1), we assume that $\mathrm{pd}_R Y = n < \infty$. Take a projective resolution $P^\bullet$ of Y, which has length n. Then by assumption, the complex $M \otimes_R P^\bullet$ is quasi-isomorphic to $M \otimes_R Y$. We observe that each component of $M \otimes_R P^\bullet$ has projective dimension at most $\mathrm{pd}_R M$. Then (1) follows immediately. (2) follows from (1) by induction, and (3) follows from (2). ∎

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

We observe that Theorem 3.9 applies to nilpotent perfect bimodules.

PROPOSITION 4.6. *Let R be a two-sided Noetherian ring, and let M be a nilpotent perfect R-bimodule. Then the conditions in Theorem 3.9 are fulfilled.*

Proof. Let G be a Gorenstein projective R-module. We recall that $\mathrm{Ext}_R^i(G, X) = 0 = \mathrm{Tor}_j^R(Y, G)$ for each $i, j \geqslant 1$, provided that $\mathrm{pd}_R X < \infty$ and $\mathrm{pd}_{R^{\mathrm{op}}} Y < \infty$. Hence, by Lemma 4.5(2), we have $\mathrm{Ext}_R^1(G, M^{\otimes_R i}) = 0$ for $i \geqslant 0$. Moreover, it follows that G is M-flat. Hence, by Lemma 4.2(2), $M^{\otimes_R i} \otimes_R G$ is M-flat for each $i \geqslant 0$. In particular, we have $\mathrm{Tor}_1^R(M, M^{\otimes_R i} \otimes_R G) = 0$. Hence, M is left-admissible. Similarly, it is right-admissible. The last condition follows from Lemma 4.1(3), since each projective $T_R(M)$-module, as an R-module, is M-flat. ∎

THEOREM 4.7. *Let R be a two-sided Noetherian ring, and let M be a nilpotent perfect R-bimodule. Then R is Gorenstein if and only if so is $T_R(M)$. In this case, we have the following equalities*

$$\mathrm{G.dim}\, R - \delta \leqslant \mathrm{G.dim}\, T_R(M) \leqslant \mathrm{G.dim}\, R + \delta + 1,$$

where $\delta = \min\{\mathrm{pd}_R T_R(M), \mathrm{pd}_{R^{\mathrm{op}}} T_R(M)\}$.

Proof. Set $l = \mathrm{pd}_R T_R(M)$ and $r = \mathrm{pd}_{R^{\mathrm{op}}} T_R(M)$. For the "only if" part, we assume that R is d-Gorenstein with $d = \mathrm{G.dim}\, R$. We claim that $\mathrm{Gpd}_{T_R(M)} X \leqslant r + d + 1$ for any left $T_R(M)$-module X.

For the claim, we take an exact sequence in $T_R(M)$-mod

$$0 \to Y \to Q^{1-r} \to \cdots \to Q^{-1} \to Q^0 \to X \to 0 \tag{4.1}$$

with each Q^{-i} projective. In case that $r = 0$, we set $Y = X$. Since M is perfect, we infer that $\mathrm{Tor}_j^R(T_R(M), Q^{-i}) = 0$ for $j \geqslant 1$ and $0 \leqslant i < r$. By Lemma 4.1(2), we infer that $\mathrm{Tor}_j^R(T_R(M), Y) = 0$ for $j \geqslant 1$. Hence, as R-modules, Y and thus $M \otimes_R Y$ are M-flat. Recall isomorphism (3.1). Then by (2.1), we have an exact sequence in $T_R(M)$-mod

$$0 \longrightarrow T_R(M) \otimes_R (M \otimes_R Y) \longrightarrow T_R(M) \otimes_R Y \longrightarrow Y \longrightarrow 0. \tag{4.2}$$

Since $\mathrm{Gpd}_R Y \leqslant d$ and $\mathrm{Gpd}_R(M \otimes_R Y) \leqslant d$, we infer from Theorem 3.9(2) that $\mathrm{Gpd}_{T_R(M)} T_R(M) \otimes_R Y \leqslant d$ and $\mathrm{Gpd}_{T_R(M)} T_R(M) \otimes_R (M \otimes_R Y) \leqslant d$. Therefore, the exact sequence implies that $\mathrm{Gpd}_{T_R(M)} Y \leqslant d + 1$, which implies the claim.

Similarly, we prove that $\mathrm{Gpd}_{T_R(M)^{\mathrm{op}}} X \leqslant l + d + 1$ for any right $T_R(M)$-module X. Then we are done by Lemma 3.2(2).

For the "if" part, we assume that $T_R(M)$ is d'-Gorenstein. For any left R-module Z, we consider the exact sequence of R-modules

$$0 \to K \to P^{1-r} \to \cdots \to P^{-1} \to P^0 \to Z \to 0$$

with each P^{-i} projective. If $r = 0$, we set $K = Z$. By a dimension shift, we infer that $\mathrm{Tor}_j^R(T_R(M), K) = 0$ for each $j \geqslant 1$. Hence K is M-flat. By Theorem 3.9(2), we have $\mathrm{Gpd}_R K = \mathrm{Gpd}_{T_R(M)} T_R(M) \otimes_R K \leqslant d'$. Hence, we have $\mathrm{Gpd}_R Z \leqslant r + d'$. By a similar statement for right R-modules and Lemma 3.2(2), we complete the proof. ∎

The global dimension of a ring S is denoted by $\mathrm{gl.dim}\, S$. Then we have the following immediate consequence. We refer the reader to [17, Theorem A.1] for a related result.

COROLLARY 4.8. *Keep the same assumptions as in Theorem 4.7. Then R has finite global dimension if and only if so does $T_R(M)$, in which case we have*

$$\mathrm{gl.dim}\, R - \delta \leqslant \mathrm{gl.dim}\, T_R(M) \leqslant \mathrm{gl.dim}\, R + \delta + 1,$$

where $\delta = \min\{\mathrm{pd}_R T_R(M), \mathrm{pd}_{R^{\mathrm{op}}} T_R(M)\}$.

Proof. Recall that a two-sided Noetherian ring S has finite global dimension if and only if it is Gorenstein and CM-free, in which case we have $\mathrm{G.dim}\, S = \mathrm{gl.dim}\, S$. Then we are done by Theorem 3.9(3). ∎

The following result is analogous to [9, Theorem 1.2]; compare Example 4.10(1).

PROPOSITION 4.9. *Keep the same assumptions as in Theorem 4.7. Let X be a left $T_R(M)$-module. Then $\mathrm{pd}_{T_R(M)} X < \infty$ if and only if $\mathrm{pd}_R X < \infty$.*

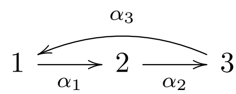
Proof. The "only if" part is trivial, since $\mathrm{pd}_R T_R(M) < \infty$.

For the "if" part, we assume that $\mathrm{pd}_R X < \infty$. Take the exact sequence (4.1) as above. Hence, as R-modules, Y and $M \otimes_R Y$ are M-flat. We observe that $\mathrm{pd}_R Y < \infty$ and by Lemma 4.5(1) $\mathrm{pd}_R(M \otimes_R Y) < \infty$. Hence, we obtain by Lemma 3.3(2) that the left two terms in (4.2) have finite projective dimension. Then so does Y. We deduce from (4.1) that the $T_R(M)$-module X has finite projective dimension. ∎

Let k be a field. We denote by $D = \mathrm{Hom}_k(-, k)$ the k-dual. For quivers, we refer the reader to [3].

EXAMPLE 4.10. (1) Let M be a nilpotent R-bimodule, which is finitely generated projective on both sides. Then M is perfect. Indeed, the tensor ring $T_R(M)$ is a projective left and right R-module. It follows from Theorem 4.7 that if R is d-Gorenstein, then $T_R(M)$ is $(d+1)$-Gorenstein. Taking $d=0$, we recover the examples in [9, Section 6] and [10, 15, 20].

Let R be the k-algebra given by the following quiver



subject to relations $\{\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_1\alpha_3\}$. Then it is a self-injective algebra. We denote by e_i the idempotent corresponding to the vertex i. Take $M = Re_1 \otimes_k e_3R$, which is an R-bimodule, projective on both sides. We observe that $M \otimes_R M = 0$. It follows that $T_R(M) = R \ltimes M$, the *trivial extension* of R by M. By Theorem 4.7, we infer that $T_R(M)$ is 1-Gorenstein. Indeed, the algebra $T_R(M)$ is a gentle algebra.

(2) Let R_1 and R_2 be two-sided Noetherian rings, and let M be an R_1-R_2-bimodule, which is finitely generated on both sides. Set $R = R_1 \times R_2$. Then M becomes naturally an R-module. Then the tensor ring $T_R(M)$ is isomorphic to the formal matrix ring $\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$. We observe that M is a perfect R-module if and only if M has finite projective dimension as a left R_1-module and a right R_2-module. Then Theorem 4.7 recovers [21, Theorem 2.2(iii)]; compare [5, Theorem 3.3]. It is of interest to compare the conditions in Theorem 3.9 with the compatible bimodule in [23, Definition 1.1].

(3) We mention that the perfectness assumption in Corollary 4.8 is necessary. Let Q be the quiver $1 \to 2$ of type A_2, and let $R = kQ$ be the path algebra. Let S be the nonprojective simple left R-module, let T be the nonprojective simple right R-module, and let $M = S \otimes_k T$ be the tensor product. Then we have $M \otimes_R M = 0$ and $T_R(M) = R \ltimes M$ is 1-Gorenstein of infinite global dimension. Indeed, the algebra $T_R(M)$ is self-injective and Nakayama. We observe an isomorphism $\mathrm{Tor}_1^R(M, M) \simeq M$. Hence, the R-bimodule M is not perfect.

(4) Let Q be a finite acyclic quiver, and let $\mathcal{C}_Q$ be the associated cluster category. Let T be a tilting module of the path algebra kQ. Denote by $R = \mathrm{End}_{kQ}(T)$ the tilted algebra, and by $B = \mathrm{End}_{\mathcal{C}_Q}(T)$ the cluster-tilted algebra, respectively. In particular, the global dimension of R is at most two. Denote $M = \mathrm{Ext}_R^2(DR, R)$, which is naturally an R-bimodule. Then

https://doi.org/10.1017/nmj.2018.19 Published online by Cambridge University Press

we have $B \cong T_R(M)$; see [1, Proposition 4.7]. On the other hand, we have $B \cong R \ltimes M$ by [2, Theorem 3.4]. Therefore, we infer that $M \otimes_R M = 0$. Recall from [13] that a cluster-tilted algebra is 1-Gorenstein, which usually has infinite global dimension. Hence, the R-bimodule M is not perfect in general.

We mention that Gorenstein homological properties of trivial extensions are studied in [18].

Acknowledgments. The project was initiated when both authors were visiting the University of Bielefeld. We thank Henning Krause for his hospitality. We thank the referee for helpful comments and Example 4.10(3). The authors are supported by the National Natural Science Foundation of China (Nos. 11522113, 11671245, 11401401, and 11601441).

References

[1] C. Amiot, *Cluster categories for algebras of global dimension* 2 *and quivers with potential*, Ann. Inst. Fourier (Grenoble) **59**(6) (2009), 2525–2590.
[2] I. Assem, T. Brustle and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. Lond. Math. Soc. **40** (2008), 151–162.
[3] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics **36**, Cambridge University Press, 1995.
[4] A. Beligiannis, *Cohen–Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras*, J. Algebra **288** (2005), 137–211.
[5] X. W. Chen, *Singularity categories, Schur functors and triangular matrix rings*, Algebr. Represent. Theory **12** (2009), 181–191.
[6] X. W. Chen, *Three results on Frobenius categories*, Math. Z. **270** (2012), 43–58.
[7] P. M. Cohn, *Algebra,* Vol. 3, 2nd ed., John Wiley Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1991.
[8] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Math. **30**, Walter de Gruyter, Berlin, New York, 2000.
[9] C. Geiss, B. Leclerc and J. Schroer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations*, Invent. Math. **209** (2017), 61–158.
[10] J. Geuenich, Quiver modulations and potentials, PhD Thesis, University of Bonn, 2017.
[11] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc., Lecture Notes Ser. **119**, Cambridge University Press, 1988.
[12] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), 379–417.
[13] B. Keller and I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi–Yau*, Adv. Math. **211** (2007), 123–151.
[14] S. Mac Lane, *Categories for the Working Mathematician*, Grad. Texts in Math. **5**, Springer-Verlag, New York, 1998.
[15] F. Li and C. Ye, *Representations of Frobenius-type triangular matrix algebras*, Acta Math. Sin. (Engl. Ser.) **33**(3) (2017), 341–361.

[16] X. Luo and P. Zhang, *Monic representations and Gorenstein-projective modules*, Pacific J. Math. **264**(1) (2013), 163–194.
[17] H. Minamoto, *Ampleness of two-sided tilting complexes*, Int. Math. Res. Not. IMRN **1** (2012), 67–101.
[18] H. Minamoto and K. Yamaura, *Homological dimension formulas for trivial extension algebras*, preprint, 2017, arXiv:1710.01469v1.
[19] Yu. V. Roganov, *The dimension of the tensor algebra of a projective bimodule*, Math. Notes **18**(5–6) (1975), 1119–1123.
[20] R. Wang, *Gorenstein triangular matrix rings and category algebras*, J. Pure Appl. Algebra **220**(2) (2016), 666–682.
[21] B. L. Xiong and P. Zhang, *Gorenstein-projective modules over triangular matrix Artin algebras*, J. Algebra Appl. **11**(4) (2012), 1250066.
[22] A. Zaks, *Injective dimensions of semiprimary rings*, J. Algebra **13** (1969), 73–89.
[23] P. Zhang, *Gorenstein-projective modules and symmetric recollements*, J. Algebra **388** (2013), 65–80.

Xiao-Wu Chen
Key Laboratory of Wu Wen-Tsun Mathematics
Chinese Academy of Sciences
School of Mathematical Sciences
University of Science and Technology of China
Hefei 230026
Anhui
PR China
xwchen@mail.ustc.edu.cn

Ming Lu
Department of Mathematics
Sichuan University
Chengdu 610064
PR China
luming@scu.edu.cn