



RESEARCH ARTICLE

Weyl chamber length compactification of the $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ maximal character variety

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Abstract

We study the vectorial length compactification of the space of conjugacy classes of maximal representations of the fundamental group Γ of a closed hyperbolic surface Σ in $\mathrm{PSL}(2, \mathbb{R})^n$. We identify the boundary with the sphere $\mathbb{P}((\mathcal{ML})^n)$, where \mathcal{ML} is the space of measured geodesic laminations on Σ . In the case $n = 2$, we give a geometric interpretation of the boundary as the space of homothety classes of \mathbb{R}^2 -mixed structures on Σ . We associate to such a structure a dual tree-graded space endowed with an \mathbb{R}^2 -valued metric, which we show to be universal with respect to actions on products of two \mathbb{R} -trees with the given length spectrum.

1. Introduction

Let Σ be a connected closed oriented surface of genus $g \geq 2$ and let $\Gamma = \pi_1(\Sigma)$ be its fundamental group. For a real reductive Lie group G , we denote by $\Xi(\Gamma, G) := \mathrm{Hom}_{\mathrm{red}}(\Gamma, G)/G$ the *character variety*, namely the space of conjugacy classes of completely reducible representations $\rho: \Gamma \rightarrow G$. The third named author used Weyl chamber valued length functions to construct the *Weyl chamber length compactification* of the character variety [25], a compactification that generalizes Thurston's compactification of the Teichmüller space $\mathcal{T} = \mathcal{T}(\Sigma)$, for which $G = \mathrm{PSL}(2, \mathbb{R})$. The boundary $\partial^{\mathrm{WL}}\Xi(\Gamma, G)$ of the character variety in the Weyl chamber length compactification is a compact subset of the set $\mathbb{P}(\bar{\alpha}^{+\Gamma})$ of homothety classes of nonzero functions $\vec{L}: \Gamma \rightarrow \bar{\alpha}^+$, where $\bar{\alpha}^+$ is a fixed closed Weyl chamber of G . Boundary points may be interpreted as projectivized $\bar{\alpha}^+$ -length functions of actions on \mathbb{R} -buildings [25].

In the case of $G = \mathrm{PSL}(2, \mathbb{R})$, we know much more: Thurston proved that the boundary of this compactification of Teichmüller space is the projectivization $\mathbb{P}(\mathcal{ML})$ of the space \mathcal{ML} of measured geodesic laminations on Σ , realized as the cone $\mathcal{ML} \subset \mathbb{R}_+^\Gamma$ of functions $\Gamma \rightarrow \mathbb{R}_+$ that are intersection functions of measured geodesic laminations (or equivalently, of measured foliations) on Σ [10, 2]. This allowed him to prove that this boundary is a sphere of dimension $6g - 7$ and the compactification is a closed ball [10]. Boundary points may also be interpreted as projectivized length functions of actions on real trees [1, 27, 17].

If G is of Hermitian type, that is, if the associated symmetric space admits a G -invariant complex structure, the character variety contains a generalization of the Teichmüller space, the space of *maximal representations*, which we denote by $\Xi_{\mathrm{Max}}(\Gamma, G)$. We refer the reader to [3, 6] for the theory of maximal representations. Maximal representations share many features with the subset of the character variety consisting of Hitchin representations, which are defined for G real split, and whose character variety we denote by $\Xi_{\mathrm{Hit}}(\Gamma, G)$. In [4, 5] we used geodesic currents to study the Weyl chamber length compactification $\mathcal{X}^{\mathrm{WL}}(\Gamma, G)$ of $\mathcal{X}(\Gamma, G)$, where $\mathcal{X}(\Gamma, G)$ denotes either the Hitchin or the maximal

character variety. We showed that, as soon as the group G has higher rank, the mapping class group admits a non-empty open domain of discontinuity for its action on the boundary $\partial^{\text{WL}}\mathcal{X}(\Gamma, G)$, the so-called *positive systole* subset; moreover, for a dense subset of boundary points, the associated length function can be computed as intersection with a weighted multicurve. Natural questions arise: to determine the topology of $\mathcal{X}^{\text{WL}}(\Gamma, G)$; to interpret boundary points in terms of geometric structures on the surface, in particular in the positive systole subset; and to relate such geometric structures with the associated actions on \mathbb{R} -buildings, for example, by finding a nice invariant subset following the third named author’s work in the case of $G = \text{PSL}(3, \mathbb{R})$ [26].

In this text, we address all the questions introduced above in the case of $G = \text{PSL}(2, \mathbb{R})^n$ for $n \geq 2$. The product structure of G lets us identify the space $\Xi_{\text{Max}}(\Gamma, G)$ with the product \mathcal{T}^n of n copies of the Teichmüller space $\mathcal{T} = \Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R}))$. A model Weyl chamber for G is $\bar{\mathfrak{a}}^+ = \mathbb{R}_+^n$, hence boundary points are homothety classes of nonzero functions $\vec{L} : \Gamma \rightarrow \mathbb{R}_+^n$. Such functions can be identified with n -tuples $(L_i)_{i=1, \dots, n}$ of functions $L_i : \Gamma \rightarrow \mathbb{R}_+$.

We identify the boundary of \mathcal{T}^n in its vectorial length compactification with the projectivization of $(\mathcal{ML})^n$; in particular, we show that, being a join of n spheres, it is topologically a sphere of dimension $n(6g - 6) - 1$. This also allows us to give a precise description of the open domain of discontinuity for the mapping class group alluded to before; for this we denote by $(\mathcal{ML})_{>0}^n$ the set consisting of n -tuples of measured laminations with *positive joint systole*, namely n -tuples for which the function $\sum_{i=1}^n L_i(\gamma)$ has a positive lower bound on Γ .

Theorem 1.1. *The boundary $\partial^{\text{WL}}\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$ is $\mathbb{P}((\mathcal{ML})^n)$, and $\text{MCG}(\Sigma)$ acts properly discontinuously on $\mathbb{P}((\mathcal{ML})_{>0}^n)$.*

We refer the reader to Section 4, and in particular to Corollary 4.4 for applications of this result to the study of compactifications induced by length functions on Γ associated to various $\text{PSL}(2, \mathbb{R})^n$ -invariant distance functions on the product $(\mathcal{H}^2)^n$ of n hyperbolic planes.

Inspired by the work of Duchin–Leiniger–Rafi [9] and of Morzadec [18], we give a geometric interpretation of boundary points in the case of $n = 2$. Duchin–Leiniger–Rafi introduced mixed structures to give a geometric compactification of the space of $\text{Flat}(\Sigma)$ of half translation structures on a compact surface Σ up to isometry. For the purposes of this paper, we define an \mathbb{R}^2 -mixed structure¹ on Σ as a triple $M = (\Sigma', K, \vec{\lambda})$ where Σ' is an open geodesic subsurface of Σ , K is a half-translation structure on Σ' , extendible at boundary components regarded as punctures, and $\vec{\lambda} = (\Lambda, \nu_1, \nu_2)$ is a 2-measured geodesic lamination, that is, a geodesic lamination Λ disjoint from Σ' and endowed with a pair of transverse measures (ν_1, ν_2) , with $\text{supp}(\nu_1) \cup \text{supp}(\nu_2) = \Lambda$. Note that the subsurface Σ' or the lamination Λ may be empty, so the space $\overrightarrow{\text{Mix}}(\Sigma)$ of mixed structures on Σ contains both the space $\overrightarrow{\text{Flat}}(\Sigma)$ of half-translation structures (up to isotopy) and the space of 2-measured geodesic laminations. The half-translation structure is equipped with a natural pair $(\mathcal{F}_1, \mathcal{F}_2)$ of transverse vertical/horizontal measured foliations. As a result, a mixed structure M defines a natural pair $(\lambda_{M,1}, \lambda_{M,2})$ of associated “vertical/horizontal” measured geodesic laminations on Σ . Taking intersection functions, we obtain a map

$$I : \overrightarrow{\text{Mix}}(\Sigma) \rightarrow \mathcal{ML} \times \mathcal{ML}$$

extending the natural map $\overrightarrow{\text{Flat}}(\Sigma) \rightarrow \mathcal{ML} \times \mathcal{ML}$. We show in Proposition 5.8 that I is a bijection, under which $\overrightarrow{\text{Flat}}(\Sigma)$ corresponds to $(\mathcal{ML})_{>0}^2$. Together with Theorem 1.1 this implies the following:

Theorem 1.2. *Let $\vec{L} : \Gamma \rightarrow \mathbb{R}_+^2$ represent a point in the Weyl chamber length boundary of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^2)$. Then*

¹ See Remark 5.6 in Section 5.2 for more details as well as a comparison with the structures considered in [9] and in [21, 20].

- (1) \vec{L} is the \mathbb{R}^2 -length function of a unique \mathbb{R}^2 -mixed structure $M = (\Sigma', K, \vec{\lambda})$ on Σ .
- (2) Every non-empty \mathbb{R}^2 -mixed structure on Σ arises in this way.

We now turn to the question of relating the geometric structures appearing in the boundary of $\Xi_{\text{Max}}(\Gamma, G)$ to the associated actions on \mathbb{R} -buildings and finding nice invariant subsets. In the case of $G = \text{PSL}(2, \mathbb{R})^n$, the \mathbb{R} -buildings appearing in the boundary of $\Xi_{\text{Max}}(\Gamma, G)$ are products of n \mathbb{R} -trees, and hence boundary points are homothety classes of $\bar{\alpha}^+$ -length functions of actions of Γ on a product of n \mathbb{R} -trees.

Notice that in his thesis [18] Morzadec used tree-graded spaces to obtain a geometric compactification of the space of flat structures and relate them with the mixed structures of [9]. Also, in the case of $G = \text{PSL}(3, \mathbb{R})$ and Γ a punctured surface group, the third named author associates in [26] to large families of boundary points of $\Xi_{\text{Hit}}^{\text{WL}}(\Gamma, \text{PSL}(3, \mathbb{R}))$ explicit finite $\bar{\alpha}^+$ -simplicial complexes whose universal cover, a tree-graded space with flat surface pieces, embeds equivariantly in the building preserving the natural $\bar{\alpha}^+$ -metric.

Our last result takes these perspectives in the study of the compactification of the space of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ -maximal representations. To be more precise, a tree-graded \mathbb{R}^2 -space (X, d_1, d_2) is for us a space X endowed with a pair (d_1, d_2) of pseudometrics such that $d := d_1 + d_2$ is a metric, the ℓ^1 -metric, with respect to which X is tree-graded in the sense of [8]; we furthermore require that X admits an action of Γ preserving each pseudometric. The \mathbb{R}^2 -length in X of $\gamma \in \Gamma$ is then the pair

$$\vec{L}_X(\gamma) = (L_{d_1}(\gamma), L_{d_2}(\gamma))$$

of translation lengths

$$L_{d_i}(\gamma) = \inf_{x \in X} d_i(x, \gamma x)$$

of γ for the pseudometric d_i .

We associate to each mixed structure M an \mathbb{R}^2 -tree-graded space (X_M, d_1, d_2) dual to M , whose pieces are either flat surfaces or \mathbb{R} -trees:

Theorem 1.3. *Let $M = (\Sigma', K, \vec{\lambda})$ be an \mathbb{R}^2 -mixed structure on Σ and $\vec{L} : \Gamma \rightarrow \mathbb{R}_+^2$ the associated \mathbb{R}^2 -length function. Then there is a tree-graded \mathbb{R}^2 -space X_M with length spectrum $\vec{L} : \Gamma \rightarrow \mathbb{R}_+^2$, and satisfying the following universal property: for any two actions of Γ on \mathbb{R} -trees T_1, T_2 with length function $(L_{T_1}, L_{T_2}) = \vec{L}$, there is an equivariant embedding*

$$f : X_M \mapsto T_1 \times T_2$$

preserving each pseudometric d_i (and isometric for the ℓ^1 -metric).

In particular, for any point in the boundary $\partial^{\text{WL}} \Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^2)$, with associated mixed structure M , the corresponding action on a \mathbb{R} -building (here a product of two trees) preserves an $\bar{\alpha}^+$ -isometrically equivariantly embedded copy of X_M .

We construct the space X_M by gluing the dual trees corresponding to the lamination components, endowed with the pair of pseudometrics induced by the pair of transverse measures, together with the completions of the flat components, endowed with the natural pair of horizontal and vertical pseudometrics (see Section 6 for more details). We show that the ℓ^1 -metric on X_M corresponds to the path metric induced by the ℓ^1 -norm on the flat components (Lemma 5.3).

A more analytic and independent approach to compactifications of the space of maximal representations in rank 2 Hermitian Lie groups has been independently pursued by Ouyang [19], Ouyang–Tamburelli [21, 22], and Martone–Ouyang–Tamburelli [16]. In their work, they consider the length spectrum of the negatively curved metric induced on the unique invariant minimal surface associated to a maximal representation. They obtain in this way a compactification that they denote $\text{Ind}(\bar{S})$.

In the case of $G = \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, Martone–Ouyang–Tamburelli [16] also consider a refined compactification dominating $\mathrm{Ind}(S)$, which they prove is a ball of dimension $12g - 12$. Their notion of $(A_1^+ \times A_1^+, 2)$ -mixed structure introduced in this work is equivalent to our of \mathbb{R}^2 -mixed structure defined above. Moreover, they prove that also the boundary of $\Xi_{\mathrm{Max}}(\Gamma, \mathrm{PSL}(2, \mathbb{R})^2)$ in their refined compactification can be identified with such structures.

Structure of the paper: The first two sections largely consist of preliminaries: we recall in Section 2 the construction of the Weyl chamber length compactification for $G = \mathrm{PSL}(2, \mathbb{R})^n$, and we discuss in Section 3 the relation between measured foliations, laminations, trees, and geodesic currents. The only new material in Section 3 is in Section 3.5 where we introduce the key notion of the \mathbb{R}^n -tree dual to an n -measured geodesic lamination. With these at hand, we prove Theorem 1.1 in Section 4. We introduce and study mixed structures in Section 5, where we also prove Theorem 1.2. We construct in Section 6 the \mathbb{R}^2 -tree-graded space (X_M, d_1, d_2) dual to a mixed structure M . We prove in Section 7 that this tree-graded space embeds isometrically in any product of trees (Proposition 7.1), which concludes the proof of Theorem 1.3.

2. The Weyl chamber length compactification

2.1. Weyl chamber-valued and \mathbb{R}^n -valued metrics

Given a finite reflection group (\mathfrak{a}, W) and a fixed model closed Weyl chamber $\bar{\mathfrak{a}}^+$, we define a $\bar{\mathfrak{a}}^+$ -valued pseudometric on a space X as a function $\vec{d} : X \times X \rightarrow \bar{\mathfrak{a}}^+$ satisfying $\vec{d}(x, x) = 0$ and

Triangular inequality: $\vec{d}(x, z) \leq \vec{d}(x, y) + \vec{d}(y, z);$
opp-symmetry: $\vec{d}(y, x) = \vec{d}(x, y)^{\mathrm{opp}},$

where opp is the opposition involution on \mathfrak{a} , and \mathfrak{a} is endowed with the partial order with positive cone the Euclidean dual $(\bar{\mathfrak{a}}^+)^*$ of $\bar{\mathfrak{a}}^+$. We will call \vec{d} a $\bar{\mathfrak{a}}^+$ -valued metric if in addition it is separated, namely $\vec{d}(x, y) = 0$ implies $x = y$.

In this article, we are interested only in the reflection group associated to the semisimple Lie group $G = \mathrm{PSL}(2, \mathbb{R})^n$, which is $\mathfrak{a} = \mathbb{R}^n$ with Weyl group $W = \{\pm \mathrm{id}_{\mathbb{R}}\}^n$. The model closed Weyl chamber is $\bar{\mathfrak{a}}^+ = \mathbb{R}_+^n$. In this case, the involution opposition is trivial, $(\bar{\mathfrak{a}}^+)^* = \bar{\mathfrak{a}}^+ = \mathbb{R}_+^n$ so a $\bar{\mathfrak{a}}^+$ -valued (pseudo)metric amounts simply to a n -tuple

$$\vec{d} = (d_i)_{i=1, \dots, n}$$

of usual \mathbb{R} -valued (pseudo)metric d_i . This gives rise to the equivalent notion of a \mathbb{R}^n -(pseudo)metric:

Definition 2.1. An \mathbb{R}^n -pseudometric space is a set X together with an n -tuple $\vec{d} = (d_i)_{i=1, \dots, n}$ of pseudometrics. We say that (X, \vec{d}) is an \mathbb{R}^n -metric space if $d := \sum_{i=1}^n d_i$ is a metric.

Observe that even if (X, \vec{d}) is an \mathbb{R}^n -metric space, it is possible that none of the pseudo-metrics d_i is separated.

Example 2.2. Examples of \mathbb{R}^n -metric spaces are products

$$X = X_1 \times \dots \times X_n$$

of ordinary metric spaces, as well as subspaces thereof with the induced metric. Another example that will play an important role in our paper are the \mathbb{R}^2 -metric naturally induced on the universal cover of a half translation surface, see Section 5.1 for details. Another relevant example is given by simplicial trees with \mathbb{R}^2 -valued edge lengths. This is discussed, and generalized, in Section 3.5.

In our case of interest,² it is easy to prove that, for any W -invariant norm $N: \mathfrak{a} = \mathbb{R}^n \rightarrow \mathbb{R}_+$, N is not decreasing on each variable on $\mathfrak{a}^+ = \mathbb{R}_+^n$, that is, for the partial order introduced above. As a result, a \mathfrak{a}^+ -valued (pseudo)metric \vec{d} on X induces an \mathbb{R} -valued (pseudo)metric by setting

$$d_N(x, y) := N(\vec{d}(x, y)).$$

Note that $\vec{d}(x, y) = 0$ if and only if $d_N(x, y) = 0$, in particular \vec{d} is separated if and only if d_N is separated.

Norms $N: \mathbb{R}^n \rightarrow \mathbb{R}$ of particular interest to us are the ℓ^2 and ℓ^1 norms, which we denote, respectively, by $\|\cdot\|_2$ and $\|\cdot\|_1$. They give the associated *Euclidean (pseudo)metric*

$$d_{\|\cdot\|_2}(x, y) := \sqrt{\sum_{i=1}^n d_i(x, y)^2}$$

and ℓ^1 -(pseudo)metric

$$d^1(x, y) := \sum_{i=1}^n d_i(x, y)$$

on X . We will also denote $d_{\|\cdot\|_2}$ by $\|\vec{d}\|_2$ and d^1 by $\|\vec{d}\|_1$.

Remark 2.3. If $X = X_1 \times \dots \times X_n$ is a product, then $d_{\|\cdot\|_2}$ is the usual product metric, in particular $(X, d_{\|\cdot\|_2})$ is CAT(0) if and only if each of the X_i are. However, in general, $(X, d_{\|\cdot\|_2})$ is not a geodesic metric space. For example, if X is a simplicial tree with \mathbb{R}^2 -valued edge lengths, the induced metric space $(X, d_{\|\cdot\|_2})$ is geodesic if and only if the two distance functions d_i are proportional.

2.2. The Weyl chamber valued length function on $\mathrm{PSL}(2, \mathbb{R})^n$

The symmetric space associated to $\mathrm{PSL}(2, \mathbb{R})^n$ is the product $X = (\mathcal{H}^2)^n$ of n copies of the hyperbolic plane \mathcal{H}^2 . On X the natural Weyl-chamber valued distance $\vec{d}: X \times X \rightarrow \mathbb{R}_+^n$ is simply the product distance

$$\vec{d}(x, y) = (d(x_i, y_i))_{i=1, \dots, n}.$$

The *Weyl chamber length* (or \mathbb{R}^n -length) of $g = (g_i)_{i=1, \dots, n} \in \mathrm{PSL}(2, \mathbb{R})^n$ can be defined as

$$\vec{L}(g) := \inf_{x \in X} \vec{d}(x, gx) \tag{1.1}$$

where the infimum is considered with respect to the partial order on \mathbb{R}^n with positive cone \mathbb{R}_+^n . As the metric space X is a product, it boils down to taking the list of the usual lengths in each factor:

$$\vec{L}(g) = (L(g_i))_{i=1, \dots, n}.$$

Here $L(g) := \inf_{x \in \mathcal{H}^2} d(x, gx)$ is the usual translation length of g on the hyperbolic plane \mathcal{H}^2 . This ensures in particular that the infimum in (1.1) exists.

The *length function* of a representation $\rho: \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R})$ is

$$L_\rho := L \circ \rho: \Gamma \rightarrow \mathbb{R}_+.$$

Similarly, the *Weyl chamber length function* (or \mathbb{R}^n -length function) of a representation $\rho: \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R})^n$ is

$$\vec{L}_\rho := \vec{L} \circ \rho: \Gamma \rightarrow \mathbb{R}_+^n.$$

Of course when $\rho = (\rho_i)_{i=1, \dots, n}$, it holds $\vec{L}_\rho = (L_{\rho_i})_{i=1, \dots, n}$.

²This is true for general finite reflection groups, but considerably harder to prove [23].

2.3. The Weyl chamber length compactification

Here we define the Weyl chamber length compactification constructed in [25], which has a simpler construction in our context. We endow the space $(\mathbb{R}_+^n)^\Gamma$ of functions $\vec{L} : \Gamma \rightarrow \mathbb{R}_+^n$ with the topology of pointwise convergence, and the space $\mathbb{P}((\mathbb{R}_+^n)^\Gamma)$ of homothety classes of nonzero functions \vec{L} with the quotient topology. The map

$$\begin{aligned} \vec{\mathcal{L}} : \text{Hom}(\Gamma, \text{PSL}(2, \mathbb{R})^n) &\rightarrow (\mathbb{R}_+^n)^\Gamma \\ \rho &\mapsto \vec{L} \circ \rho \end{aligned}$$

induces a continuous map

$$\mathbb{P} \vec{\mathcal{L}} : \mathcal{T}^n \rightarrow \mathbb{P}((\mathbb{R}_+^n)^\Gamma)$$

equivariant with respect to the mapping class group action. The injectivity of $\mathcal{T} \rightarrow \mathbb{P}(\mathbb{R}_+^\Gamma)$ implies that the map $\mathbb{P} \vec{\mathcal{L}}$ is injective as well. The *Weyl chamber length compactification* $\overline{\mathcal{T}^n}^{\text{WL}}$ of \mathcal{T}^n is by definition the closure of its image (which is a compact set). We denote

$$\partial^{\text{WL}}(\mathcal{T}^n) := \overline{\mathcal{T}^n}^{\text{WL}} - \mathbb{P} \vec{\mathcal{L}}(\mathcal{T}^n) \subset \mathbb{P}((\mathbb{R}_+^n)^\Gamma)$$

its boundary.

2.4. Other length compactifications

Recall from Section 2.1 that any W -invariant norm $N : \mathbb{R}^n \rightarrow \mathbb{R}$ induces a metric $d_N := N(\vec{d})$ on $X = (\mathcal{H}^2)^n$, and thus a corresponding length function, the N -length function L_{d_N} . We use this to construct the N -length compactification $\overline{\mathcal{T}^n}^N$ of \mathcal{T}^n , associated to L_{d_N} .

It is easy to verify that, since N is not decreasing on each variable on $\bar{\mathfrak{a}}^+ = \mathbb{R}_+^n$, and \mathcal{T}^n is a product, it holds $L_{d_N}(g) = N(\vec{L}(g))$. The Weyl chamber length compactification then dominates naturally $\overline{\mathcal{T}^n}^N$: the restriction of the natural map

$$\begin{aligned} (\mathbb{R}_+^n)^\Gamma &\rightarrow \mathbb{R}_+^\Gamma \\ \vec{L} &\mapsto N \circ \vec{L} \end{aligned}$$

induces a continuous $\text{Out}(\Gamma)$ -equivariant surjective map

$$\overline{\mathcal{T}^n}^{\text{WL}} \rightarrow \overline{\mathcal{T}^n}^N$$

restricting to identity on \mathcal{T}^n .

3. Trees dual to measured geodesic laminations

We refer the reader to [14, Chapter 11] for preliminaries on measured laminations, measured foliations, and the identification between equivalence classes of measured foliations and measured geodesic laminations on hyperbolic surfaces. The material in Sections 3.2, 3.3, and 3.4 is classical, while the viewpoint in Section 3.5 is new and important for our paper.

3.1. Measured foliations

Let \mathcal{F} be a transverse measured foliation on a topological surface Σ with fundamental group Γ . We denote by $i(\mathcal{F}, c)$ the measure of a path $c \subset \Sigma$ with respect to \mathcal{F} . The intersection of \mathcal{F} with $\gamma \in \Gamma$ is

defined as

$$i_{\mathcal{F}}(\gamma) := \inf_c i(\mathcal{F}, c)$$

where the infimum is taken over all closed loops c transverse to Λ and freely homotopic to γ . We denote by

$$I_{\mathcal{F}} := i(\mathcal{F}, \cdot): \Gamma \rightarrow \mathbb{R}_+$$

the corresponding intersection function.

This may be reformulated in terms of pseudometrics. We denote by $d_{\mathcal{F}}$ the pseudometric on the universal cover $\tilde{\Sigma}$ of Σ associated with the lift $\tilde{\mathcal{F}}$ of \mathcal{F} : it is defined by the formula

$$d_{\mathcal{F}}(x, y) := \inf_c i(\tilde{\mathcal{F}}, c)$$

where the infimum is taken on paths c joining x to y in $\tilde{\Sigma}$ and transverse to $\tilde{\mathcal{F}}$. The length function $L_d \in (\mathbb{R}_+)^{\Gamma}$ of a pseudometric d on a Γ -space X is defined as

$$L_d(\gamma) := \inf_{x \in X} d(x, \gamma x).$$

By definition, the length function of $d_{\mathcal{F}}$ on $\tilde{\Sigma}$ is the intersection function of \mathcal{F} :

$$L_{d_{\mathcal{F}}} = I_{\mathcal{F}}.$$

The dual tree of a measured foliation \mathcal{F} is defined as the quotient metric space $T(\mathcal{F}) := \tilde{\Sigma}/d_{\mathcal{F}}$ of $\tilde{\Sigma}$ by the pseudometric $d_{\mathcal{F}}$. The tree $T(\mathcal{F})$ inherits an action of Γ by isometries, with associated length function

$$L_{T(\mathcal{F})} = I_{\mathcal{F}}.$$

3.2. Measured geodesic laminations

Let $\lambda = (\Lambda, \nu)$ be a measured lamination on Σ . As for measured foliations, we denote by $i(\lambda, c)$ the λ -measure of a path $c: [0, 1] \rightarrow \Sigma$ transverse to Λ . The intersection of λ with $\gamma \in \Gamma$ is defined as

$$i(\lambda, \gamma) := \inf_c i(\lambda, c)$$

where the infimum is taken over all closed loops c transverse to Λ and freely homotopic to γ , and we denote by

$$I_{\lambda} := i(\lambda, \cdot): \Gamma \rightarrow \mathbb{R}_+$$

the corresponding intersection function.

We denote by \tilde{d}_{λ} the pseudometric on $\tilde{\Sigma}$ associated with λ : this is the pseudometric defined on $\tilde{\Sigma} - \tilde{\Lambda}_0$, where $\tilde{\Lambda}_0$ is the set of atomic³ leafs of the lift $\tilde{\lambda} = (\tilde{\Lambda}, \nu)$ of λ , by the formula

$$d_{\lambda}(x, y) := \inf_c i(\tilde{\lambda}, c)$$

where the infimum is taken on paths c joining x to y in $\tilde{\Sigma}$ and transverse to $\tilde{\Lambda}$. By definition, the length function of d_{λ} on $\tilde{\Sigma} - \tilde{\Lambda}_0$ is the intersection function of λ :

$$L_{d_{\lambda}} = I_{\lambda}.$$

We refer to [14] for the classical correspondence between measured foliations and measured geodesic laminations on hyperbolic surfaces.

³The leafs with positive weight.

3.3. Geodesic currents

We refer to [2] or [15, Section 8] for the background material in the case of closed geodesic surfaces, and for instance to [4] for the generalization to finite type surfaces.

A *geodesic current* on a finite type hyperbolic surface $\Sigma = \mathcal{H}^2 / \Gamma$ is a flip-invariant Γ -invariant positive Radon measure on the space of unoriented, unparametrized geodesics of \mathcal{H}^2 , that may be identified with the space

$$\mathcal{G}(\mathcal{H}^2) := \{(x, y) \in (\partial\mathcal{H}^2)^2 : x \neq y\}$$

of distinct pairs of points in the boundary at infinity $\partial\mathcal{H}^2$ of \mathcal{H}^2 . A basic example is the current δ_c associated to a closed geodesic c in Σ , which is defined as the sum of the Dirac masses on the lifts of c to \mathcal{H}^2 . Recall that the Bonahon intersection $i(\mu, \nu)$ of two geodesic currents μ, ν is defined as the $(\mu \times \nu)$ -measure of any Borel fundamental domain for the Γ -action on the space $\mathcal{DG}(\mathcal{H}^2) \subset \mathcal{G}(\mathcal{H}^2) \times \mathcal{G}(\mathcal{H}^2)$ of pairs of transverse geodesics (g, h) .

We refer for instance to [15, Section 8.3.4] for the bijective correspondence $\lambda \mapsto \mu_\lambda$ between measured geodesic laminations $\lambda = (\Lambda, \nu)$ of full support Λ and geodesic currents μ on Σ with $i(\mu, \mu) = 0$, equivalently currents μ such that no two geodesics in the support of μ intersect transversally. The geodesic current μ_λ has support Λ , and, for each geodesic arc c transverse to Λ , the restriction of μ_λ to the set of geodesics g of Λ intersecting c is the pullback of the λ -measure on c by the map $g \mapsto g \cap c$. The notions of intersection then coincide as

$$i(\mu_\lambda, \delta_c) = i(\lambda, c),$$

for all closed geodesic c in Σ . Note that the union of two measured geodesic laminations with disjoint support corresponds to the sum of the associated currents. From now on, we will freely identify λ with μ_λ whenever convenient.

Given a geodesic current μ on Σ and a geodesic subsurface Σ' of Σ , we denote by $\mu_{|\Sigma'}$ the restriction of μ to the subsurface Σ' , namely the geodesic current

$$\mu_{\Sigma'} := \chi_{\mathcal{G}(\Sigma')} \mu,$$

where $\chi_{\mathcal{G}(\Sigma')}$ is the characteristic function of the set $\mathcal{G}(\Sigma')$ of geodesics whose projection lies in Σ' . In general, a geodesic current of full support might restrict to the zero current on a proper subsurface (this is the case when μ is the Liouville current of a hyperbolic structure), but if, for every boundary component c of Σ' , $i(\mu, \delta_c) = 0$, then for every $\gamma \in \pi_1(\Sigma')$, $i(\mu, \delta_\gamma) = i(\mu_{|\Sigma'}, \delta_\gamma)$ [5, Proposition 4.13].

3.4. Dual tree of a measured geodesic lamination

Let $\lambda = (\Lambda, \nu)$ be a measured geodesic lamination on an hyperbolic surface Σ . We now recall the construction of the associated dual \mathbb{R} -tree. We follow the construction of [14, Section 11.12].

We first get rid of the atoms blowing-up along atomic leaves: for each isolated leaf c of Λ , cut Σ along c and insert an annulus $B(c) = c \times [0, 1]$, foliated by the parallel circles $c \times t$. We endow Σ^b with the locally CAT(0) metric m^b equal to the original metric m of Σ outside the annuli $B(c)$ and to the flat metric on $B(c)$. This gives a locally CAT(0) surface (Σ^b, m^b) homeomorphic to Σ , with a geodesic lamination Λ^b . We call (Σ^b, Λ^b) the *blow-up* of (Σ, Λ) . The *blow-up* of the transverse measure ν on Λ is the non-atomic transverse measure ν^b on Λ^b , obtained from ν by giving to each foliated annulus $B(c)$ the transverse measure $\nu(c)dt$ on $[0, 1]$. The pseudometric d_{λ^b} associated to the measured lamination $\lambda^b := (\Lambda^b, \nu^b)$ is then a continuous and everywhere-defined path pseudometric on $\widetilde{\Sigma}^b$. The \mathbb{R} -tree dual to λ is defined as the quotient metric space $T(\lambda) := \widetilde{\Sigma}^b / d_{\lambda^b}$, whose metric will be denoted by d_λ .

It is easy but crucial to see that the geodesics of $T(\lambda)$ are the projections of the m^b -geodesics of $\widetilde{\Sigma}^b$. In the following lemma, and in the rest of the article, when we write *geodesic* we mean a path $t \mapsto c(t)$ that is additive for the distance d , namely such that $d(c(t_1), c(t_3)) = d(c(t_1), c(t_2)) + d(c(t_2), c(t_3))$ for all

$t_1 \leq t_2 \leq t_3$. Observe that c may then be constant on some subset of its domain of definition and needs not be parametrized at constant speed.

Proposition 3.1. *Let $c: I \rightarrow \widetilde{\Sigma}^b$ be a m^b -geodesic. Then c is a minimizing curve for d_{λ^b} , and the projection of c on the tree $T(\lambda)$ is a geodesic.*

Proof. Let x, y be points of $\widetilde{\Sigma}^b$ and $c: [0, 1] \rightarrow \widetilde{\Sigma}^b$ be the constant speed geodesic segment from x to y . As the leaves of $\widetilde{\Lambda}^b$ are geodesics, and CAT(0) spaces are uniquely geodesic, the path c cannot cross twice the same leaf of $\widetilde{\Lambda}^b$, hence $i(\lambda^b, c)$ is minimal and $d_{\lambda^b}(x, y) = i(\lambda^b, c)$. Denote by $p: \widetilde{\Sigma}^b \rightarrow T(\lambda)$ the canonical projection. The projection $\bar{c} = p \circ c$ of c in the dual tree $T(\lambda)$ does not backtrack, hence is a geodesic. □

The tree $T(\lambda)$ inherits an action of Γ by isometries, with associated length function

$$L_{T(\lambda)} = I_\lambda.$$

Note that when Σ is closed the tree $T(\lambda)$ is then *minimal* for the action of Γ , namely there is no invariant proper subtree. Indeed an invariant proper subtree will lift as a proper closed invariant convex subset in $\widetilde{\Sigma}^b$, and taking the closure in the CAT(0) compactification we will then obtain a proper closed invariant subset of the boundary at infinity $\partial_\infty \widetilde{\Sigma}^b \simeq \partial \mathcal{H}^2$, which is impossible.

Note that the tree $T(\lambda)$ is essentially determined by its length function ; more generally, we will use the length rigidity of actions on minimal trees with length functions in \mathcal{ML} : It is easily seen that for $\lambda \neq 0$ the action of Γ on $T(\lambda)$ has no global fixed point in $\partial_\infty T$, where $\partial_\infty T$ denotes the boundary at infinity of T . This in fact depends only on the length function, since having a fixed point at infinity is equivalent to the length function being of the form $\gamma \mapsto |h(\gamma)|$, where $h: \Gamma \rightarrow \mathbb{R}$ is a morphism [7, Corollary 2.3]. In particular, all minimal Γ -trees with nonzero length function in \mathcal{ML} have no fixed point at infinity, in particular are semisimple and not shifts in the sense of [7]. It follows then from the length rigidity for minimal semisimple Γ -trees [7, Theorem 3.7] that if T and T' are any two minimal Γ -trees with the same length function in \mathcal{ML} , then there is a unique equivariant isometry $T \rightarrow T'$.

3.5. The \mathbb{R}^n -tree dual to a n -measured geodesic lamination

Let $n \in \mathbb{N}$.

Definition 3.2. A \mathbb{R}^n -tree is a \mathbb{R}^n -metric space $(X, \vec{d} = (d_i)_{i=1, \dots, n})$ which is a \mathbb{R} -tree for the associated l^1 -metric $d^1(x, y) := \sum_i d_i(x, y)$.

Observe that, while we only assume that d_i are pseudodistances, we require that their sum d^1 is a distance, namely it separates points.

Definition 3.3. A n -measured lamination $\vec{\lambda} = (\Lambda, \vec{v})$ on a surface Σ is a lamination Λ on Σ , endowed with a n -tuple $\vec{v} = (v_1, \dots, v_n)$ of transverse measures of full joint support, namely Λ is the union of the supports Λ_i of the measures v_i for $i = 1, \dots, n$.

It can equivalently be seen as a n -tuple of parallel measured laminations $(\lambda_1, \dots, \lambda_n)$, that is, such that any two leaves of λ_i and λ_j are either disjoint or equal.

When Σ is endowed with a locally CAT(0) metric, a n -measured geodesic lamination $\vec{\lambda} = (\Lambda, \vec{v})$ is called *geodesic* when Λ is geodesic.

When Σ is a closed hyperbolic surface, seeing measured geodesic laminations as geodesic currents, a n -tuple $(\lambda_1, \dots, \lambda_n)$ of measured geodesic laminations is parallel if and only if

$$i(\lambda_i, \lambda_j) = 0 \text{ for all } i, j$$

or equivalently that

$$\lambda := \sum_{i=1}^n \lambda_i \text{ is a measured lamination.}$$

Let $\vec{\lambda} = (\Lambda, \vec{\nu})$ be a n -measured lamination on a closed hyperbolic surface Σ . We now construct the associated dual \mathbb{R}^n -tree, adapting the construction of Section 3.4.

We take the \mathbb{R} -tree $T(\lambda) := \widetilde{\Sigma}^b/d_\lambda^b$ dual to the measured lamination $\lambda = (\Lambda, \nu)$ where $\nu = \sum_i \nu_i$.

Recall from Section 3.4 that (Σ^b, Λ^b) is the blow-up of (Σ, Λ) , which is a locally CAT(0) surface (Σ^b, m^b) homeomorphic to Σ , with a geodesic lamination Λ^b , and that $\lambda^b = (\Lambda^b, \nu^b)$ where ν^b is the non-atomic transverse measure on Λ^b obtained by blowing up ν , and that d_{λ^b} the associated continuous path pseudometric on $\widetilde{\Sigma}^b$.

Taking the blow-ups ν_i^b of the transverse measure ν_i , we obtained a n -tuple $\vec{\nu}^b := (\nu_i^b)_i$ of non-atomic transverse measures on λ^b (not necessarily of full support), namely a n -measured lamination $\vec{\lambda}^b = (\Lambda^b, \vec{\nu}^b)$ on Σ^b , which we will call the *blow-up* of $\vec{\lambda} = (\Lambda, \vec{\nu})$.

Each of the measured laminations $\lambda_i^b = (\Lambda^b, \nu_i^b)$ then induces a continuous everywhere-defined path pseudometric $d_{\lambda_i^b}$ on $\widetilde{\Sigma}^b$.

Definition 3.4. The \mathbb{R}^n -tree $T(\vec{\lambda})$ dual to $\vec{\lambda}$ is defined as the \mathbb{R} -tree $T(\lambda) := \Sigma^b/d_\lambda^b$ dual to the measured lamination λ , endowed with the \mathbb{R}^n -pseudometric given by the n -tuple

$$\vec{d}_\lambda := (d_{\lambda_i})_{i=1, \dots, n}$$

of quotient pseudometrics d_{λ_i} induced by $d_{\lambda_i^b}$.

It inherits an action of Γ preserving \vec{d}_λ , that is preserving each pseudometric d_{λ_i} .

Proposition 3.5. $(T(\lambda), \vec{d}_\lambda)$ is an \mathbb{R}^n -tree with associated ℓ^1 -metric $\|\vec{d}_\lambda\|_1 = d_\lambda$.

Proof. The crucial fact is that, by Proposition 3.1, the infima involved in the definition of the pseudometrics d_{λ_i} are in fact all realized simultaneously for a same path c in $\widetilde{\Sigma}^b$, the m^b -geodesic. In particular, c is a minimizing curve for d_{λ^b} and its projection on $T(\lambda)$ is a minimizing path for \vec{d}_λ . This ensures that $d_\lambda = \sum_i d_{\lambda_i}$ on $T(\lambda)$.

The separation of \vec{d}_λ on $T(\lambda)$ follows: if $\vec{d}_\lambda(x, y) = 0$ then $d_\lambda(x, y) = \sum_i d_{\lambda_i}(x, y) = 0$, hence $x = y$. □

Remark 3.6. Proposition 3.5 is special for the ℓ^1 -metric, and does not work for other distances, not even locally. For example, consider a minimal lamination λ that supports two mutually singular measured laminations. Then the distance function $\|\vec{d}_\lambda\|_2$ is not geodesic, not even locally.

4. The boundary of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$ is $\mathbb{P}((\mathcal{ML})^n)$

In this section, we prove Theorem 1.1, that identifies $\partial^{\text{WL}}\mathcal{T}^n$ with $\mathbb{P}((\mathcal{ML})^n)$, and deduce some consequences for other length spectra compactifications.

4.1. Proof of Theorem 1.1

Theorem 1.1 is classical when $n = 1$ and follows from the work of Thurston. Thurston furthermore proved [10, Exposé 8] that \mathcal{ML} is homeomorphic to \mathbb{R}^{6g-6} , that its projectivization $\mathbb{P}\mathcal{ML}$ is homeomorphic to \mathbb{S}^{6g-7} , and that the resulting compactification is homeomorphic to a closed ball [15, 8.3.13].

Using the latter fact, we can prove the following lemma, which ensures that given L in \mathcal{ML} , one can choose a sequence in \mathcal{T} converging to $[L]$ with a fixed scale sequence. It will be crucial in the proof of Theorem 1.1.

Lemma 4.1. *Let $L \in \mathcal{ML}$. Let (λ_k) be any increasing diverging sequence of positive real numbers. Then there exists a sequence of maximal representations $\rho_k: \Gamma \rightarrow \text{PSL}(2, \mathbb{R})$ such that $\frac{1}{\lambda_k} L_{\rho_k}$ converges to L in $(\mathbb{R}_+)^{\Gamma}$.*

Proof. Given a finite generating set $S \subset \Gamma$ and an isometric action ρ of Γ on a metric space X , the minimal displacement of ρ with respect to the generating set S is defined by

$$\lambda_S(\rho) := \inf_{x \in X} \sqrt{\sum_s d(x, \rho(s)x)^2}.$$

This defines a nonzero proper and continuous function $\lambda_S: \mathcal{T} \rightarrow \mathbb{R}_+$ (this is for example proven—in a much more general context—in [24, Prop. 25]).

We may suppose that L is nonzero (otherwise we can take a constant sequence ρ_k). Let T be a \mathbb{R} -tree with minimal Γ -action, with length function L . Let D be the minimal displacement of Γ on T with respect to the generating set S .

Fix a point $[\rho_0]$ in the Teichmüller space \mathcal{T} . As the compactification $\overline{\mathcal{T}}$ of \mathcal{T} is a closed ball, there exists a path $r(t)$, $t \in [0, 1]$ from $[\rho_0]$ to $[L]$ in $\overline{\mathcal{T}}$ such that $r(t)$ belongs to \mathcal{T} for $t \in [0, 1[$. Let ρ_t be a representation with $[\rho_t] = r(t)$. Then, as the map

$$\begin{aligned} [0, 1[&\rightarrow \mathbb{R}_+ \\ t &\mapsto \lambda_S(r(t)) \end{aligned}$$

is continuous and diverges as t goes to 1, there exists an increasing sequence $(t_k)_{k \geq K}$ in $[0, 1[$ with limit 1 such that $\lambda_S(r(t_k)) = D\lambda_k$. Since $[\rho_k] = r(t_k)$ converges to $[L]$ in $\overline{\mathcal{T}}$, we have that $\frac{1}{\lambda_S(\rho_k)} L_{\rho_k}$ converges to sL for some $s \in \mathbb{R}_+$. Taking the asymptotic cone of this sequence (see for example [25]), we also have that sL is the length function of an action of Γ on a real tree T_ω with minimal displacement 1 with respect to S .⁴ Let $T' \subset T_\omega$ be the minimal invariant subtree. As T' is a convex subset of T_ω , the minimal displacement of Γ in T' is the same as in T_ω . By length rigidity of actions on minimal trees with length function in \mathcal{ML} , the trees T and $\frac{1}{s}T'$ are equivariantly isometric, hence have same minimal displacement $s = \frac{1}{D}$. So we have that $\frac{1}{\lambda_k} L_{\rho_k} \rightarrow L$ as wanted. \square

We now have the ingredients needed to prove Theorem 1.1, which we recall for the reader's convenience:

Proposition 4.2. *The boundary of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n) = \mathcal{T}^n$ in the Weyl chamber length compactification is $\mathbb{P}((\mathcal{ML})^n)$.*

Proof. It is easy to see that $\partial^{\text{WL}}(\mathcal{T}^n) \subset \mathbb{P}((\mathcal{ML})^n)$. Let indeed $\rho_k = (\rho_{k,i})_{i=1, \dots, n}$, $k \in \mathbb{N}$, be a sequence in $\text{Hom}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$, which we identify with $\text{Hom}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$. We write a nonzero function $\vec{L}: \Gamma \rightarrow \mathbb{R}^n$ as $\vec{L} = (L_i)_{i=1, \dots, n}$ with $L_i: \Gamma \rightarrow \mathbb{R}$. The sequence of conjugacy classes $[\rho_k]$ converges to the homothety class $[\vec{L}]$ in the Weyl chamber length compactification if and only if there exists a sequence of positive real numbers $\lambda_k \rightarrow \infty$ (scale sequence) such that the renormalized Weyl chamber length function $\frac{1}{\lambda_k} \vec{L}_{\rho_k}$ converges to \vec{L} , that is, if $\frac{1}{\lambda_k} L_{\rho_{k,i}}$ converges to L_i for all $i = 1, \dots, n$ [25, Theorem 5.6]. Then either $L_i = 0$ or $[\rho_{k,i}]$ converges to $[L_i]$ in the length compactification $\overline{\mathcal{T}}$ of \mathcal{T} . As $\partial\mathcal{T} = \mathbb{P}(\mathcal{ML})$, we have that each L_i belongs to \mathcal{ML} , hence $[\vec{L}]$ belongs to $\mathbb{P}((\mathcal{ML})^n)$.

⁴See proof of Thm 5.6 in [25].

The converse implication is a consequence of Lemma 4.1. Let $\vec{L} = (L_i)_{i=1,\dots,n} \in (\mathcal{ML})^n$. Using Lemma 4.1 with scale sequence $\lambda_k = k$, we can construct for each $i = 1, \dots, n$ a sequence of maximal representations $\rho_{k,i}: \Gamma \rightarrow \text{PSL}(2, \mathbb{R})$, $k \in \mathbb{N}$, whose renormalized length function $\frac{1}{k}L_{\rho_{k,i}}$ converges to the length function L_i in $(\mathbb{R}_+)^{\Gamma}$ as $k \rightarrow \infty$. We now consider for $k \in \mathbb{N}$ the product representation $\rho_k := (\rho_{k,i})_{i=1,\dots,n}: \Gamma \rightarrow \text{PSL}(2, \mathbb{R})^n$, which, being a product of maximal representations, is a maximal representation. Then $\frac{1}{k}\vec{L}_{\rho_k} = (\frac{1}{k}L_{\rho_{k,i}})_{i=1,\dots,n}$ converges to $(L_i)_{i=1,\dots,n} = \vec{L}$ as $k \rightarrow \infty$. Hence, $[\vec{L}]$ belongs to $\partial^{\text{WL}}(\mathcal{T}^n)$. □

As a result, we deduce that $\mathbb{P}((\mathcal{ML})^n) \subset \mathbb{P}((\mathbb{R}_+^{\Gamma})^n)$ is a sphere of dimension $n(6g - 6) - 1$, being the topological join of n spheres:

Corollary 4.3. *The boundary of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n) = \mathcal{T}^n$ in the Weyl chamber length compactification is homeomorphic to $\mathbb{S}^{n(6g-6)-1}$.*

4.2. Applications to other length compactifications

As an application of Proposition 4.2, we can also understand various other compactifications of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$:

Corollary 4.4. *For any W -invariant norm $N: \mathbb{R}^n \rightarrow \mathbb{R}_+^n$, the boundary of $\Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$ in the N -length compactification is the projectivization $\mathbb{P}(N((\mathcal{ML})^n))$ of the image $N((\mathcal{ML})^n)$ of $(\mathcal{ML})^n$ by the map sending $\vec{L}: \Gamma \rightarrow \mathbb{R}^n$ to $N \circ \vec{L}: \Gamma \rightarrow \mathbb{R}$.*

In particular, the boundary of \mathcal{T}^n in the ℓ^1 -length compactification is the projectivization $\mathbb{P}(\sum_{i=1}^n \mathcal{ML})$ of the space of geodesic currents that can be decomposed as the sum of n measured laminations.

Remark 4.5. *While the map $(\mathcal{ML})^2 \rightarrow (\sum_{i=1}^2 \mathcal{ML})$ has fibers of cardinality 2 on the set of geodesic currents with positive systole, the fiber over minimal measured laminations (and more generally over geodesic currents that admit a Bonahon-orthogonal decomposition $\mu = \mu_1 + \mu_2$ with μ_1 a minimal measured lamination) is higher dimensional. As a result it is, in general, not clear how to determine the topology of $\mathbb{P}(\sum_{i=1}^2 \mathcal{ML})$. Indeed it follows from the arguments in [9, Theorem 2] that no finite set of simple closed curves is sufficient to separate points in $\mathbb{P}(\sum_{i=1}^2 \mathcal{ML})$. This is in strong contrast to $\mathbb{P}((\mathcal{ML})^2)$: there is a collection of $9g - 9$ simple closed curves on Σ whose \mathbb{R}^2 -length function already separate points in $\mathbb{P}((\mathcal{ML})^2)$.*

5. \mathbb{R}^2 -mixed structures

The purpose of this section is to prove Theorem 1.2 from the introduction, which interprets points in $\mathcal{ML} \times \mathcal{ML}$ as length functions of mixed structures. For this we introduce the notion of flat structures in Section 5.1, mixed structures in Section 5.2, and their associated length functions in Section 5.3, and prove the result in Section 5.4.

5.1. Flat structures

We will consider flat structures on a punctured finite type surfaces, namely the complement of a finite set of marked points considered as punctures in a compact topological surfaces. When dealing with mixed structures, the finite type surfaces will typically be obtained from geodesic subsurfaces of the original surface Σ by collapsing each boundary components to a cusp point.

Definition 5.1. A half-translation structure K on a finite type surface Σ is a $(\mathbb{R}^2 \rtimes \mathbb{Z}/2\mathbb{Z}, \mathbb{R}^2)$ -structure on Σ , with conical singularities of angle $k\pi$, $k \geq 2$, extendible at punctures, with possibly angle π singularities at the punctures.

With a slight abuse of notation we denote by $\overrightarrow{\text{Flat}}(\Sigma)$ the moduli space of half-translation structures on Σ , where two such structures are identified if they are isotopic. Note that the flat structures we consider here are *directed* (i.e., with a preferred vertical direction).

Remark 5.2. Duchin–Leiniger–Rafi [9], as well as Ouyang–Tamburelli [21] consider, instead, the space of flat structures on Σ , where they identify isometric marked structures. As a result, we have fibrations

$$\overrightarrow{\text{Flat}}(\Sigma) \rightarrow \text{Flat}(\Sigma) \rightarrow \mathcal{T}(\Sigma).$$

The fiber of $\overrightarrow{\text{Flat}}(\Sigma) \rightarrow \text{Flat}(\Sigma)$ is the circle \mathbb{S}^1 , which acts on a half-translation surface by rotation. Since our structures are marked, and any half-translation structure induces a conformal structure on the surface, the space $\overrightarrow{\text{Flat}}(\Sigma)$ fibers over the Teichmüller space. We will never need this fact, but it is well known that the fiber over X in this fibration identifies with the space of holomorphic quadratic differentials over the Riemann surface X .

Let K be a flat structure on Σ , and let $\mathcal{F}_1, \mathcal{F}_2$ be the vertical and horizontal measured foliations of K . This is a pair of transverse measured foliations. For $i = 1, 2$, we denote by

$$\ell_{K,i}(c) = i(\mathcal{F}_i, c) = \int_c |dx_i|$$

and by $d_{K,i} := d_{\mathcal{F}_i}$ the associated pseudometric on $\tilde{\Sigma}$. This defines the natural \mathbb{R}^2 -metric

$$\overrightarrow{d}_K = (d_{K,i})_i$$

on $\tilde{\Sigma}$. We denote $L_{K,i} := I_{\mathcal{F}_i} : \Gamma \rightarrow \mathbb{R}$ the corresponding length function. Let $T(\mathcal{F}_i) := \tilde{\Sigma}/d_{K,i}$ be the dual tree of \mathcal{F}_i , and $p_i : \tilde{\Sigma} \rightarrow T(\mathcal{F}_i)$ the corresponding projection.

The universal cover $\tilde{\Sigma}$ of Σ is a CAT(0) metric space for the flat metric associated to K , which we will denote by $d_{\text{CAT}(0)}$. Note that $d_{\text{CAT}(0)}$ is not in general equal to the metric $\|\overrightarrow{d}_K\|_2$ induced from the \mathbb{R}^2 -metric \overrightarrow{d} by taking the ℓ^2 -norm, nevertheless it is the associated length metric. We denote $\tilde{\Sigma}^c$ the completion of $\tilde{\Sigma}$ with respect to $d_{\text{CAT}(0)}$, which consists in adding one fixed point x_c for each parabolic subgroup Γ_c of $\Gamma = \pi_1(\Sigma)$ (corresponding to lifts \tilde{c} of punctures c of Σ), see for example [18, Lemma 7.2]. The flat structure extends on $\tilde{\Sigma}^c$. It is easy to see that $\mathcal{F}_i, d_{K,i}$, and p_i extend naturally to the completion $\tilde{\Sigma}^c$ of $\tilde{\Sigma}$.

We define the ℓ^1 -length metric on $\tilde{\Sigma}$, as the Finsler metric d_K induced by the ℓ^1 -norm $\|x\|_1 = |x_1| + |x_2|$ on \mathbb{R}^2 . This metric is clearly equivalent to the CAT(0) metric $d_{\text{CAT}(0)}$, in particular extends to $\tilde{\Sigma}^c$. We denote by

$$L_K(\gamma) := L_{d_K}(\gamma)$$

the corresponding length of $\gamma \in \Gamma$. We now establish some basic properties of the ℓ^1 -metric that we will need. Recall that in the following lemma, a geodesic is an additive path $t \mapsto c(t)$ for the distance d .

Lemma 5.3. Let K be a flat structure on a finite type surface Σ . The following properties hold:

- (1) For any CAT(0) geodesic $c : I \rightarrow \tilde{\Sigma}^c$, the projections $c_i = p_i \circ c$ of c in the dual trees $T(\mathcal{F}_i)$ are geodesics;
- (2) We have $d_K = d_{K,1} + d_{K,2} = \|\overrightarrow{d}_K\|_1$ on $\tilde{\Sigma}^c$. In particular, the CAT(0) geodesics are geodesics for d_K ;
- (3) We have $L_K(\gamma) = L_{K,1}(\gamma) + L_{K,2}(\gamma)$ for all $\gamma \in \Gamma$.

Proof. We prove (1). Let $i \in \{1, 2\}$. As $T(\mathcal{F}_i)$ is a tree, we only need to prove that the path c_i does not backtrack. Let $t_1 \leq t_2 \leq t_3$ be real parameters in I such that $c_i(t_1) = c_i(t_3)$. Then the points $c(t_1)$ and $c(t_3)$ are in a common leaf of the foliation \mathcal{F}_i . As the leaves of the foliation are geodesic for the CAT(0) metric d on $\tilde{\Sigma}$, we can deduce, by uniqueness of geodesics in CAT(0) metric spaces, that the point $c(t_2)$ is on the same leaf, hence that $c_i(t_1) = c_i(t_2) = c_i(t_3)$. This proves that c_i is a geodesic in the tree $T(\mathcal{F}_i)$. We now prove (2). The ℓ^1 -length of a path c is, by definition, $\ell^1(c) = \int_c |dx_1| + \int_c |dx_2| = \ell_1(c) + \ell_2(c)$. We then clearly have that $d(x, y) \geq d_1(x, y) + d_2(x, y)$. On the other hand, if c is the CAT(0) geodesic in $\tilde{\Sigma}^c$ from x to y , we know by (2) that both projections $c_i = p_i \circ c$ of c are geodesic. So for $i = 1, 2$ we have $d_i(x, y) = d(p_i(x), p_i(y)) = l(c_i) = \ell_i(c)$ and hence $\ell(c) = d_1(x, y) + d_2(x, y)$.

We finally prove (3). As $d = d_1 + d_2$ we clearly have $L_K(\gamma) \geq L_{K,1}(\gamma) + L_{K,2}(\gamma)$. If $L_K(\gamma) = 0$, then we are done. Otherwise, γ has no fixed point in $\tilde{\Sigma}^c$ and, since $\tilde{\Sigma}^c / \Gamma$ is compact, there is a CAT(0) geodesic c in $\tilde{\Sigma}^c$ translated by γ . By (2) both projections $c_i = p_i \circ c$ of c are geodesics translated by Γ . So, for any x on c , and $i = 1, 2$

$$d_i(x, \gamma x) = d_i(p_i(x), \gamma p_i(x)) = L_{T(\mathcal{F}_i)}(\gamma) = L_{K,i}(\gamma).$$

Then $d(x, \gamma x) = L_{K,1}(\gamma) + L_{K,2}(\gamma)$. □

5.2. \mathbb{R}^2 -mixed structures on a surface

In this section, we introduce a natural notion of \mathbb{R}^2 -mixed structure on a surface. This generalizes flat structures and refines the notion of mixed structure introduced by Duchin–Leininger–Rafi and Morzadec [9, 18]. The definition follows the point of view of [9], see Section 6 for the metric viewpoint analogous to [18]. Note that an equivalent notion has been independently introduced in [16, Definition 6.4].

Definition 5.4. A \mathbb{R}^2 -mixed structure on a compact hyperbolic surface Σ with boundary is a triple $M = (\Sigma', K, \vec{\lambda})$ where

- Σ' is a (possibly disconnected, possibly empty) open geodesic subsurface of Σ , namely a union of complementary components of a collection of disjoint simple closed geodesics.
- K is a flat structure on Σ' extendible at punctures (when Σ' is compactified as a punctured surface),
- $\vec{\lambda} = (\Lambda, \vec{\nu})$ is a 2-measured geodesic lamination on $\Sigma - \Sigma'$, that is, Λ is a geodesic lamination on Σ , included in $\Sigma - \Sigma'$, and $\vec{\nu} = (\nu_1, \nu_2)$ is a pair of transverse measures on Λ of full support, that is, $\Lambda = \text{supp}(\nu_1) \cup \text{supp}(\nu_2)$.⁵

This imposes topological restrictions on the subsurface Σ' : no connected component of Σ' can be a pair of pants, since a pair of pants does not support any non-trivial flat structure.

Example 5.5. We illustrate an example of an \mathbb{R}^2 -mixed structure on a surface of genus 3 in Figure 1. In this case, the support of the lamination Λ consists of the three colored curves c_1, c_2, c_3 , each with the given pair of weights $(x_i, y_i) \in \mathbb{R}_+^2 - \{(0, 0)\}$. Observe that in this example $\text{supp}(\nu_1) = \{c_1, c_2\}$ and $\text{supp}(\nu_2) = \{c_2, c_3\}$. The subsurface Σ' is the disjoint union of a thrice punctured torus Σ'_1 , bounded by the curve c_0 and the curve c_1 , and a twice punctured torus Σ'_2 , endowed with the flat structures K_1, K_2 illustrated above in the picture. In the flat pictures, the parallel sides are identified, and the colored points correspond to punctures (corresponding to the curves, in Σ , of the same color). In particular, the two black punctures p_0, p_0' in K_1 come from the same curve. Observe that, while c_0 is a boundary

⁵We allow the supports of ν_i to overlap, but don't require that one of the ν_i is fully supported.

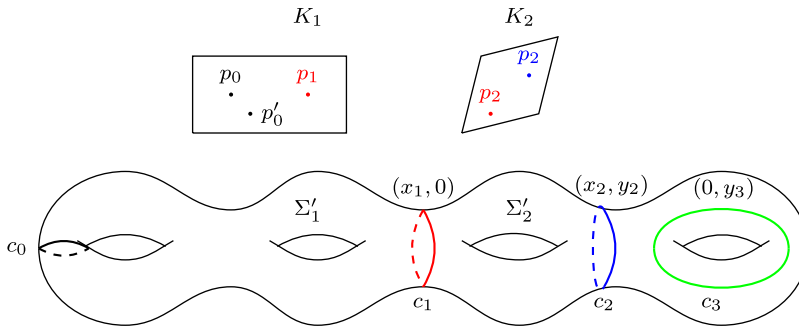


Figure 1. An \mathbb{R}^2 -mixed structure on a surface of genus 4.

component of Σ' , it does not belong to the lamination Λ , since the curve is not contained in the support of neither transverse measure.

We denote by $\overrightarrow{\text{Mix}}(\Sigma)$ the moduli space of \mathbb{R}^2 -mixed structures, where two \mathbb{R}^2 -mixed structures are identified if the subsurfaces and the laminations agree and the flat structures are equivalent, namely isotopic to each other.

Remark 5.6. Following up on Remark 5.2, observe that in the \mathbb{R}^2 -mixed structures we consider here, the flat part is always directed. Forgetting the vertical direction we obtain a map $\overrightarrow{\text{Mix}}(\Sigma) \rightarrow \text{Mix}(\Sigma)$, where $\text{Mix}(\Sigma)$ denotes the mixed structures studied in [9]. In general, the map $\overrightarrow{\text{Mix}}(\Sigma) \rightarrow \text{Mix}(\Sigma)$ has fibers $(\mathbb{S}^1)^k$ where k denotes the number of connected components of Σ' . In [9], the set $\text{Mix}(\Sigma)$ is identified with the corresponding set of geodesic currents, a perspective that is generalized in [21, 20], where the flat metrics associated to cubic (resp. quartic) differentials are considered.

5.3. The \mathbb{R}^2 -length function of a mixed structure

Let $M = (\Sigma', K, \vec{\lambda})$ be a \mathbb{R}^2 -mixed structure on Σ . Denote by $(\mathcal{F}_1, \mathcal{F}_2)$ the vertical and horizontal measured foliations on Σ' associated with the flat structure K . Let $\lambda_{K,i}$ be the measured geodesic lamination on Σ' corresponding to \mathcal{F}_i , namely, with the same intersection function. It can be seen as a measured geodesic lamination on Σ . Let $\lambda_{M,i}$ be the measured geodesic lamination on Σ obtained by taking the union of $\lambda_{K,i}$ and of the measured geodesic lamination $\lambda_i = (\Lambda_i, \nu_i)$ on $\Sigma - \Sigma'$, where $\Lambda_i := \text{supp}(\nu_i)$. Regarding measured geodesic laminations as geodesic currents on Σ , we have

$$\lambda_{M,i} := \lambda_{K,i} + \lambda_i$$

The \mathbb{R}^2 -length of $\gamma \in \Gamma$ with respect to the mixed structure M is then defined as the pair:

$$\vec{L}_M(\gamma) := (i(\lambda_{M,i}, \gamma))_{i=1,2}.$$

We denote $L_{M,i}(\gamma) := i(\lambda_{M,i}, \gamma)$ the factors. Note that if $\Sigma' = \Sigma$, then the length $L_{M,i}$ agrees with $L_{K,i}$.

This gives a map

$$\vec{L} : \overrightarrow{\text{Mix}}(\Sigma) \rightarrow \mathcal{ML} \times \mathcal{ML}.$$

We will show in the next subsection that this is indeed a bijection.

5.4. Interpretation of $\mathcal{ML} \times \mathcal{ML}$ as $\overrightarrow{\text{Mix}}(\Sigma)$

The goal of the section is to provide an identification of $\mathcal{ML} \times \mathcal{ML}$ with $\overrightarrow{\text{Mix}}(\Sigma)$, thus obtaining a geometric interpretation of the boundary $\partial^{\text{WL}} \Xi_{\text{Max}}(\Gamma, \text{PSL}(2, \mathbb{R})^n)$ for the case $n = 2$. The main ingredient for this is the following application of the decomposition result of [4] to a sum of measured geodesic laminations (seen as geodesic currents), a result that works for general n :

Proposition 5.7. *Let $\vec{L} = (L_i)_{i=1, \dots, n} \in (\mathcal{ML})^n$, and denote by μ_i the measured lamination with intersection function L_i . Then there exists a canonical open geodesic subsurface Σ' , such that each μ_i decomposes as a disjoint union $\mu_i = \mu_{\Sigma'}^i \cup \mu_{\Lambda}^i$ of measured laminations, and*

- (1) $\mu_{\Sigma'}^i$ is supported in Σ' , μ_{Λ}^i in its complement;
- (2) $(\mu_{\Sigma'}^i)_i$ have positive joint systole;
- (3) $(\mu_{\Lambda}^i)_i$ are parallel, namely the union of their support is a geodesic lamination Λ (contained in $\Sigma \setminus \Sigma'$).

Proof. Let $\mu_i = (\Lambda_i, \nu_i)$ be the measured lamination on Σ corresponding to L_i , that is, such that $i(\mu_i, \cdot) = L_i$ on Γ .

As in [4], we consider the collection $\mathcal{E} = \mathcal{E}_{\mu}$ of closed μ -short solitary geodesics. This is the collection, canonically associated to μ , of simple closed geodesics in Σ that have 0-intersection with μ , and that do not intersect any geodesic that does not intersect the support of μ . Applying the decomposition theorem [4, Corollary 1.9] to the geodesic current $\mu = \sum_{i=1}^n \mu_i$, we get that the surface decomposes along the collection $\mathcal{E} = \mathcal{E}_{\mu}$ in a finite number of open connected subsurfaces with geodesic boundary $\Sigma - \mathcal{E} = \bigcup_{v \in \mathcal{V}} \Sigma_v$, and the current μ decomposes as a sum

$$\mu = \sum_{v \in \mathcal{V}} \mu_v + \sum_{c \in \mathcal{E}} t_c \delta_c$$

where $\mu_v = \mu|_{\Sigma_v}$ is the restriction of μ to Σ_v (recall Section 3.3).

Furthermore, for every $v \in \mathcal{E}$ for which $\mu_v \neq 0$ precisely one of the following holds:

- (1) either $\text{Syst}_{\Sigma_v}(\mu_v) > 0$,
- (2) or μ_v is a measured lamination.

where

$$\text{Syst}_{\Sigma_v}(\mu_v) := \inf\{i(\mu_v, c) \mid c \subset \Sigma_v \text{ closed geodesic}\}.$$

Let $i \in \{1, \dots, n\}$. We first see that no closed geodesic $c \in \mathcal{E}$ intersects transversally the support of μ_i : indeed, since $i(\mu, c) = i(\mu_1, c) + i(\mu_2, c) = 0$, we have $i(\mu_i, c) = 0$. So the measured geodesic lamination μ_i decomposes as the union of measured geodesic laminations $\mu_{v,i}$ included in the open subsurface Σ_v , and possibly closed leafs $c \in \mathcal{E}$ with transverse measure $t_{c,i} \in \mathbb{R}_+$. That is, seeing all those measured geodesic laminations as geodesic currents on Σ :

$$\mu_i = \sum_{v \in \mathcal{V}} \mu_{v,i} + \sum_{c \in \mathcal{E}} t_{c,i} \delta_c$$

By uniqueness of the decomposition of μ along \mathcal{E} , for each $v \in \mathcal{V}$ we have

$$\mu_v = \sum_i \mu_{v,i}$$

and for each $c \in \mathcal{E}$

$$t_c = \sum_i t_{c,i}$$

Let $v \in \mathcal{V}$. If μ_v is a measured lamination, then the measured laminations $\mu_{v,i}, i = 1, \dots, n$ are parallel: indeed, as $i(\mu_v, \mu_v) = \sum_{1 \leq i, j \leq n} i(\mu_{v,i}, \mu_{v,j}) = 0$, and $i(\mu_{v,i}, \mu_{v,j}) \geq 0$, we get that $i(\mu_{v,i}, \mu_{v,j}) = 0$ for all $i, j \in \{1, \dots, n\}$. \square

We can now prove the main result of the section.

Proposition 5.8. Any $\vec{L} = (L_1, L_2)$ in $\mathcal{ML} \times \mathcal{ML}$ is the \mathbb{R}^2 -length function of a unique \mathbb{R}^2 -mixed structure $(\Sigma', K, \vec{\lambda})$ on Σ .

Note that a similar identification has been independently established in [16], see for instance Theorem 6.8 therein.

Proof. As above we denote by μ_i the geodesic current corresponding to the lamination (Λ_i, ν_i) with length function L_i . We use Proposition 5.7 to decompose $\mu_i := \mu_{\Sigma'}^i + \mu_{\Lambda}^i$.

For any connected component Σ_v of the positive systole subsurface Σ' , we denote by $\Gamma_v = \pi_1(\Sigma_v)$ and $L_{v,i} : \Gamma_v \rightarrow \mathbb{R}_+$ the intersection function of $\mu_{v,i}$ on Σ_v . Then $L_{v,i}$ is the intersection function $I_{\mathcal{F}_i}$ of a measured foliation \mathcal{F}_i on the surface Σ_v . Denote $C = \text{Syst}_{\Sigma_v}(\mu_v)$. We have that, for all non parabolic γ in Γ_v ,

$$\max(i(\mathcal{F}_1, \gamma), i(\mathcal{F}_2, \gamma)) \geq \frac{C}{2} > 0.$$

It is known (see for example [12, Theorem 7]) that the two measured foliations $\mathcal{F}_1, \mathcal{F}_2$ are then transversely realisable, that is, up to replacing \mathcal{F}_i by an equivalent measured foliation (an operation that does not change the length function), they arise as the vertical and horizontal measured foliation of a flat structure K_v on the surface Σ_v . In particular, the measured geodesic laminations on Σ_v associated with K_v are

$$\lambda_{K_v,i} = \mu_{v,i}$$

for $i = 1, 2$. We denote by K the flat structure on Σ' equal to K_v on each Σ_v .

Let λ_i be the measured geodesic lamination obtained by taking the union of $\mu_{v,i}$, for the v such that μ_v is a lamination, and of the closed geodesics $c \in \mathcal{E}$ with weight $t_{c,i}$. Then $\vec{\lambda} = (\lambda_1, \lambda_2)$ is a pair of parallel measured geodesic laminations on Σ , and $M = (\Sigma', K, \vec{\lambda})$ is a mixed structure on Σ , with associated pair of laminations

$$\lambda_{M,i} = \lambda_{K,i} + \lambda_i = \mu_i.$$

In particular, taking intersection functions on Γ we have $L_i = L_{M,i}$ for each i .

The uniqueness of M (up to isotopy of the flat part) is given by the uniqueness of the decomposition in Corollary 1.9 of [4] and by the injectivity of the natural map from quadratic differentials to pairs of equivalence classes of measured foliations [11, Theorem 3.1]. \square

6. The \mathbb{R}^2 -tree-graded space dual to a mixed structure

In this section, we construct the \mathbb{R}^2 -tree-graded space X_M dual to a \mathbb{R}^2 -mixed structure $M = (\Sigma', K, \vec{\lambda})$ on a closed hyperbolic surface Σ . In Section 6.1, we recall the definition of tree-graded space and discuss a general construction to glue pseudometrics which we use in Section 6.2 to construct the tree-graded space X_M dual to a mixed structure M . In Section 6.3, we prove that space is indeed tree-graded and discuss its relevant geometric properties.

6.1. Generalities on \mathbb{R}^n -tree-graded spaces.

Recall from [8] the notion of tree-graded metric space:⁶

Definition 6.1. A geodesic metric space (X, d) is tree-graded with respect to a collection \mathcal{P} of geodesic subsets called pieces if

(TG1) Any two different pieces have at most one common point.

(TG2) Any simple geodesic triangle in X is contained in one piece.

Here a geodesic triangle is simple if its sides meet only in the respective endpoints.

We now adapt this to define an tree-graded \mathbb{R}^n -space, recall from Definition 2.1 the notion of an \mathbb{R}^n -metric space:

Definition 6.2. A \mathbb{R}^n -metric space $(X, \vec{d} = (d_i)_{i=1, \dots, n})$ is tree-graded with respect to a collection \mathcal{P} of subsets if X is tree-graded (with respect to \mathcal{P}) for the associated ℓ^1 -metric $d := \sum_{i=1}^n d_i$.

In the next subsection, we will construct the tree-graded space associated with a mixed structure on a surface Σ as a quotient of a blowup $\widetilde{\Sigma}^b$ of Σ by pseudometrics defined by gluing. We now recall the general construction of a global pseudometric on a CAT(0) surface $\widetilde{\Sigma}$ obtained by gluing pseudometrics defined on geodesic pieces, the initial ingredient for the construction in the next section.

Let Σ be a locally CAT(0) surface, and \mathcal{E} be a set of disjoint simple closed geodesics on Σ . We denote by $\widetilde{\mathcal{E}}$ the set of their lifts to the universal cover $\widetilde{\Sigma}$. A piece of $\widetilde{\Sigma}$ is defined as the closure \widetilde{P} of a complementary component \widetilde{W} of $\widetilde{\mathcal{E}}$ in $\widetilde{\Sigma}$. Two different pieces are adjacent if they have non-empty intersection (which is then a geodesic in $\widetilde{\mathcal{E}}$ bounding each of the pieces). We denote by $\mathcal{P}(\widetilde{\Sigma})$ the set of pieces of $\widetilde{\Sigma}$.

Definition 6.3. A chain between two points x, y in $\widetilde{\Sigma}$ is a sequence $C = (x_0 = x, x_1, \dots, x_{k+1} = y)$ in $\widetilde{\Sigma}$ such that any two consecutive points x_j and x_{j+1} are in a same piece \widetilde{P}_j , with $\widetilde{P}_j \neq \widetilde{P}_{j+1}$.

For $j = 1, \dots, k$ the pieces \widetilde{P}_{j-1} and \widetilde{P}_j are then adjacent and x_j is on their common boundary geodesic \widetilde{c}_j . Such a chain defines a path in the simplicial tree dual to $\widetilde{\mathcal{E}}$. We call the chain straight if this path is geodesic, that is, if and only if $\widetilde{c}_{j-1} \neq \widetilde{c}_j$ for $j = 1, \dots, k$. Then $\widetilde{c}_1, \dots, \widetilde{c}_k$ is the ordered sequence of geodesics in $\widetilde{\mathcal{E}}$ separating x and y (going from x to y).

Given a pseudometric $d^{\widetilde{P}}$ on each piece \widetilde{P} of $\widetilde{\Sigma}$, we define the d -length of a chain C as

$$\ell_d(C) = \sum_{j=0}^{k+1} d^{\widetilde{P}_j}(x_j, x_{j+1})$$

The induced pseudometric on $\widetilde{\Sigma}$ is then defined by

$$d(x, y) = \inf_C \ell_d(C)$$

where the infimum is taken over all chains C joining x to y . It is easy to see that we may restrict to straight chains C . If the restriction of $d^{\widetilde{P}}$ to the geodesics \widetilde{c} of $\widetilde{\mathcal{E}}$ is 0 for each piece \widetilde{P} , then the pseudo-distance $d(x_j, x_{j+1})$ does not depend on the choice of x_j on \widetilde{c}_j , and thus all straight chains have the same length. In this case, we may alternatively define $d(x, y)$ as the d -length of any straight chain, thus in particular d restricts to the original pseudometric $d^{\widetilde{P}}$ on each piece \widetilde{P} .

⁶Observe, however, that in [8] the tree-graded spaces are additionally assumed to be complete. We relax this assumption because the tree-graded space associated to a mixed structure will, in general, not be complete: it is well known that the \mathbb{R} -tree dual to a measured lamination λ is only complete when λ has no minimal component, and these \mathbb{R} -trees are examples of tree-graded spaces associated to mixed structures.

6.2. Construction of the \mathbb{R}^2 -space X_M

Let $M = (\Sigma', K, \vec{\lambda})$ be a \mathbb{R}^2 -mixed structure on a closed hyperbolic surface Σ . Here, as always, $\vec{\lambda} = (\Lambda, \nu_1, \nu_2)$.

We first resolve the atoms in the lamination part, by taking the blowup $(\Sigma^b, \vec{\lambda}^b)$ of $(\Sigma, \vec{\lambda})$ as in Section 3.5. Recall from Sections 3.4 and 3.5 that Σ^b denotes the CAT(0) surface obtained inserting in Σ a flat foliated annulus $B(c) = c \times [0, 1]$ at each isolated leaf c of Λ , and Λ^b is the associated lamination whose non-atomic transverse measure ν_i^b is obtained by extending ν_i with the transverse measure $\nu_i(c)dt$ on $[0, 1]$ for each foliated annulus $B(c)$.

Our next goal is to define a \mathbb{R}^2 -pseudometric \vec{d}_M on $\widetilde{\Sigma}^b$ associated with M , that is, a pair of pseudometrics $(d_{M,i})_i$ on $\widetilde{\Sigma}^b$. We first define the pieces which we will glue as outlined in Section 6.1. We denote by Σ'^b the open subsurface of Σ^b corresponding to Σ' , and denote by \mathcal{E}^b its set of boundary geodesics in Σ^b . We denote by $\widetilde{\mathcal{E}}^b$ the set of their lifts to $\widetilde{\Sigma}^b$.

Definition 6.4. A piece of $\widetilde{\Sigma}^b$ is the closure \widetilde{P} of a complementary component \widetilde{W} of $\widetilde{\mathcal{E}}^b$ in $\widetilde{\Sigma}^b$. We will call such a piece

- a flat piece if \widetilde{W} projects in Σ'^b , and
- a lamination piece otherwise.⁷

Example 6.5. If M is the mixed structure described in Example 5.5, the locally CAT(0) surface (Σ^b, m^b) is obtained by endowing Σ with an hyperbolic metric and then gluing three flat cylinders $C_i = [0, 1] \times c_i$ to the hyperbolic surface $\Sigma \setminus \{c_1, c_2, c_3\}$. The pieces in its universal cover $\widetilde{\Sigma}^b$ have four types: flat pieces isometric to the universal covers of the completion of Σ'_1 and Σ'_2 (up to now these pieces are endowed with a complete hyperbolic metric with geodesic boundary), lamination pieces of annular type, isometric to the universal cover of C_1 , namely to the Euclidean strip $[0, 1] \times \mathbb{R} \subset \mathbb{R}^2$, with measured geodesic lamination equal to the vertical foliation with transverse measures $\nu_1 = x_1 dx$, $\nu_2 = y_1 dx$, and lamination pieces isometric the universal cover of $(\Sigma \setminus (\overline{\Sigma}' \cup c_3)) \cup C_2 \cup C_3$.

We now define a new pseudometric on each piece. Let $i \in \{1, 2\}$. On each lamination piece \widetilde{P} of $\widetilde{\Sigma}^b$, we define $d_{M,i}^{\widetilde{P}}$ as the restriction to \widetilde{P} of the pseudometric $d_{\lambda_i^b}$ associated to the non-atomic measured lamination $\lambda_i^b := (\Lambda^b, \nu_i^b)$. On each flat piece \widetilde{P} of $\widetilde{\Sigma}^b$ we, instead, define the pseudometric $d_{M,i}^{\widetilde{P}}$ as intersection with the horizontal (resp. vertical) measured foliation of K . More precisely we consider the canonical projection $\widetilde{P} = \widetilde{W} \rightarrow \widetilde{W}^c$ to the completion \widetilde{W}^c of \widetilde{W} with respect to the CAT(0) metric given by the flat structure K , and we define the pseudometric $d_{M,i}^{\widetilde{P}}$ as the pullback of the pseudo-distance $d_{K,i}$ introduced in Section 5.1 through this projection. Note that the restriction of $d_{M,i}^{\widetilde{P}}$ to any boundary geodesic of a piece \widetilde{P} is always 0.

We define the pseudometric $d_{M,i}$ on $\widetilde{\Sigma}^b$ as the gluing of the pseudometrics $d_{M,i}^{\widetilde{P}}$ on the pieces $\widetilde{P} \in \mathcal{P}(\widetilde{\Sigma}^b)$ as in Section 6.1. We denote by

$$\vec{d}_M = (d_{M,i})_i$$

the corresponding \mathbb{R}^2 -pseudometric on $\widetilde{\Sigma}^b$, and by

$$d_M := d_{M,1} + d_{M,2}$$

the associated ℓ^1 -pseudometric on $\widetilde{\Sigma}^b$. It follows from the construction that d_M is the gluing of the ℓ^1 -pseudometrics $d_M^{\widetilde{P}} = \sum_i d_{M,i}^{\widetilde{P}}$ on the pieces $\widetilde{P} \in \mathcal{P}(\widetilde{\Sigma}^b)$.

⁷Beware that the metric m^b is hyperbolic on flat pieces.

Definition 6.6. The \mathbb{R}^2 -metric space X_M associated with M is the quotient

$$X_M := \widetilde{\Sigma}^b / d_M$$

of $\widetilde{\Sigma}^b$ by the ℓ^1 -pseudometric d_M , endowed with the \mathbb{R}^2 -metric induced by \vec{d}_M , that is, the pair of pseudometrics induced by $d_{M,i}$, $i = 1, 2$.

The action of Γ on $\widetilde{\Sigma}^b$ induces an action of Γ on X_M preserving the \mathbb{R}^2 -metric \vec{d}_M .

Example 6.7. The pieces in the tree-graded \mathbb{R}^2 -space X_M associated to the mixed structure M described in Example 5.5 have 4 isometry types:

- flat pieces isometric to the completion of the universal cover of $K_1 \setminus \{p_1, p_0, p_0'\}$.
- flat pieces isometric to the completion of the universal cover of $K_2 \setminus \{p_1, p_2\}$,
- closed segments of \mathbb{R}^2 -length $(x_1, 0)$,
- \mathbb{R}^2 -simplicial trees of infinite valence, obtain by attaching, to the dual tree to the curve c_3 in the subsurface $\Sigma \setminus \Sigma'$, a segment of \mathbb{R}^2 -length (x_2, y_2) to the fixed points of the elements of Γ corresponding to the curve c_2 .

Observe that, in particular, the tree associated to the lamination pieces are, in this example, not minimal.

6.3. Basic properties of X_M

In this section, we prove that X_M is indeed tree-graded, that its induced \mathbb{R}^2 -length function corresponds to the pair $(\lambda_{M,1}, \lambda_{M,2})$ of measured geodesic laminations associated with the mixed structure M , and we discuss the isometry types of the pieces of X_M .

Recall that we defined the laminations $\lambda_{M,i}$ as the sum of two measured geodesic laminations on Σ

$$\lambda_{M,i} := \lambda_{K,i} + \lambda_i$$

where $(\lambda_{K,i})_{i=1,2}$ are the measured geodesic laminations supported in Σ' induced by the horizontal (resp. vertical) foliation of the flat structure K and $\lambda_i = (\Lambda, \nu_i)$.

We denote by $L_{X_{M,i}}$, $i = 1, 2$ the length function of $d_{M,i}$ on X_M , namely

$$L_{X_{M,i}}(\gamma) := \inf_{x \in X_M} d_{M,i}(x, \gamma x).$$

We denote by $\pi : \widetilde{\Sigma}^b \rightarrow X_M$ the canonical projection. The pieces of X_M are the images $X_{\tilde{P}} := \pi(\tilde{P})$ in X_M of the pieces \tilde{P} of $\widetilde{\Sigma}^b$. We denote by $\mathcal{P}(X_M)$ the set of pieces of X_M . When a lamination piece $\tilde{P} \subset \widetilde{\Sigma}^b$ does not meet the lamination Λ^b , then the corresponding piece $X_{\tilde{P}}$ is reduced to a point. We will call the space $(X_M, \vec{d}_M = (d_{M,1}, d_{M,2}))$ the tree-graded \mathbb{R}^2 -space dual to M .

Proposition 6.8.

- (1) The \mathbb{R}^2 -metric space (X_M, \vec{d}_M) is tree-graded with respect to $\mathcal{P}(X_M)$.
- (2) If \tilde{P} is a lamination piece, $(X_{\tilde{P}}, \vec{d}_M)$ is isometric to the image of \tilde{P} in the \mathbb{R}^2 -tree $T(\vec{\lambda}) := \Sigma^b / d_{\vec{\lambda}}$ dual to the 2-measured geodesic lamination $\vec{\lambda}$. In particular $X_{\tilde{P}}$ is then a \mathbb{R}^2 -tree.
- (3) If \tilde{P} is a flat piece, then $X_{\tilde{P}}$ is isometric to the complete flat surface \widetilde{W}^c , the pseudometrics $d_{M,i} = d_{K,i}$ are the vertical/horizontal pseudometrics, and d_M is the ℓ^1 -metric d_K (see Section 5.1).
- (4) For all $\gamma \in \Gamma$,

$$L_{X_{M,i}}(\gamma) = i(\lambda_{M,i}, \gamma) = L_{M,i}(\gamma)$$

Proof. The projection $\pi : \widetilde{\Sigma}^b \rightarrow X_M$ sends each geodesic $\widetilde{c} \in \widetilde{\mathcal{E}}^b$ to a point $x_{\widetilde{c}}$ of X_M .

We first prove (2) and (3). By construction, the piece $X_{\widetilde{P}}$ is the biggest Hausdorff quotient of \widetilde{P} with respect to the ℓ^1 -pseudometric d_M , endowed with the quotient pseudometrics $d_{M,i}$. As a result (3) holds.

To check (2) observe that if \widetilde{P} is a lamination piece, since $d_{M,i}$ is the restriction of $d_{\lambda_i^b}$, then d_M is the restriction of the pseudometric d_λ^b induced by the measured lamination $\lambda^b = (\Lambda^b, \nu_1^b + \nu_2^b)$ on Σ^b , and $(X_{\widetilde{P}}, d_{M,1}, d_{M,2})$ identifies with the image of \widetilde{P} in the \mathbb{R}^2 -tree $T(\widetilde{\lambda}) := \Sigma^b / d_\lambda^b$ dual to the 2-measured geodesic lamination $\widetilde{\lambda}$ (see Section 3.5).

We now prove (1). Since all pieces $X_{\widetilde{P}}$ are geodesic for the ℓ^1 -metric, it follows easily that (X_M, d_M) is geodesic (a geodesic between x and y is obtained by concatenating geodesics in the separating pieces).

We first prove (TG1), by showing that if two pieces $X_{\widetilde{P}}, X_{\widetilde{P}'}$ meet, then $X_{\widetilde{P}} \cap X_{\widetilde{P}'} = x_{\widetilde{c}}$ where \widetilde{c} is the boundary component of \widetilde{P} separating \widetilde{P} from \widetilde{P}' : we denote by \bar{x} the image in $X_{\widetilde{P}}$ of a point x in \widetilde{P} . Suppose that $\bar{x} = \bar{y}$ with $y \in \widetilde{P}'$. Then

$$d_M(x, y) = 0 = \sum_{j=0}^{k+1} d_M^{\widetilde{P}_j}(x_j, x_{j+1}) \tag{6.1}$$

where (x_0, \dots, x_{k+1}) is a straight chain from x to y . Recall that this means that $x_0 = x, x_{k+1} = y$ and, for all $j = 1, \dots, k$, the point x_j belongs to the geodesic \widetilde{c}_j , where $(\widetilde{c}_j)_{j=1, \dots, k}$ is the ordered sequence of geodesics in $\widetilde{\mathcal{E}}^b$ separating x and y , and we denote by \widetilde{P}_j the piece containing x_j and x_{j+1} . Then $\widetilde{c}_1 = \widetilde{c}$ and equation (6.1) implies that $d_M^{\widetilde{P}_0}(x, x_1) = 0$ proving that $\bar{x} = x_{\widetilde{c}} = \bar{y}$.

We now verify (TG2). Consider a simple nontrivial geodesic triangle with vertices $\bar{x}, \bar{y}, \bar{z}$ in X_M , and lift its vertices to three points x, y, z in $\widetilde{\Sigma}^b$. If x, y, z are not in a common piece of $\widetilde{\Sigma}^b$, then there is a geodesic \widetilde{c} in $\widetilde{\mathcal{E}}^b$ separating one of the three points, say x , from the others. Then the corresponding point $x_{\widetilde{c}}$ in X_M lies on each geodesic from \bar{x} to \bar{y} and on each geodesic from \bar{x} to \bar{z} in X_M . Hence, either the geodesic triangle is not simple or $x_{\widetilde{c}} = \bar{x}$. In the second case, we can change representative and obtain a geodesic triangle in $\widetilde{\Sigma}^b$ with shorter lengths (for the CAT(0) distance). Since the elements in $\widetilde{\mathcal{E}}^b$ are uniformly separated, the process terminates and shows that we can find preimages in the same piece. We conclude proving (4). We have that $L_{d_{M,i}}(\gamma) = \inf_{x \in \widetilde{\Sigma}^b} d_{M,i}(x, \gamma x)$. Let $x \in \widetilde{\Sigma}^b$. Let c be the geodesic segment from x to γx in $\widetilde{\Sigma}^b$ for the CAT(0) metric m^b . It crosses the decomposing geodesics $\widetilde{\mathcal{E}}^b$ in a straight chain $(x_0 = x, x_1, \dots, x_{k+1} = \gamma x)$, hence

$$d_{M,i}(x, \gamma x) = \sum_{j=0}^k d_{M,i}^{\widetilde{P}_j}(x_j, x_{j+1}).$$

(see Section 6.1). Let J_F be the set of $j \in J = \{0, \dots, k\}$ such that \widetilde{P}_j is a flat piece and J_L be the set of $j \in J$ such that \widetilde{P}_j is a lamination piece. Recall that the measured geodesic lamination on (Σ^b, m^b) corresponding to λ_i is $\lambda_i^b = (\Lambda^b, \nu_i^b)$. Denote by $\lambda_{K,i}^b$ the measured geodesic lamination on (Σ^b, m^b) induced by $\lambda_{K,i}$ (which is included in the flat pieces).

As the geodesic lamination λ_i^b is supported on lamination pieces, we have

$$\begin{aligned} \sum_{j \in J_L} d_{M,i}^{\widetilde{P}_j}(x_j, x_{j+1}) &= \sum_{j \in J_L} d_{\lambda_i^b}(x_j, x_{j+1}) \\ &= \sum_{j \in J} d_{\lambda_i^b}(x_j, x_{j+1}) \\ &= d_{\lambda_i^b}(x, \gamma x) \\ &\geq i(\lambda_i^b, \gamma). \end{aligned}$$

Similarly, since for any x, y in (possibly different components of) the boundary $\partial\tilde{P}$ of a flat piece \tilde{P} it holds $d_{M,i}^{\tilde{P}}(x, y) = d_{\lambda_{K,i}}(x, y)$, and the geodesic lamination $\lambda_{K,i}^b$ is supported on flat pieces, we have

$$\begin{aligned} \sum_{j \in J_F} d_{M,i}^{\tilde{P}_j}(x_j, x_{j+1}) &= \sum_{j \in J} d_{\lambda_{K,i}^b}(x_j, x_{j+1}) \\ &= d_{\lambda_{K,i}^b}(x, \gamma x) \\ &\geq i(\lambda_{K,i}^b, \gamma). \end{aligned}$$

Hence:

$$\begin{aligned} d_{M,i}(x, \gamma x) &\geq i(\lambda_{K,i}^b, \gamma) + i(\lambda_i^b, \gamma) \\ &= i(\lambda_{K,i}, \gamma) + i(\lambda_i, \gamma) \\ &= i(\lambda_{M,i}, \gamma). \end{aligned}$$

If x is on an axis of γ in the CAT(0) surface $(\tilde{\Sigma}^b, m^b)$, then there is equality. □

7. Embeddings in products of trees

If X is a product $X = X_1 \times \dots \times X_n$ of metric spaces (X_i, d_i) , the i th-factor pseudometric is the pseudometric $d_i(x, y) = d_i(x_i, y_i)$ obtained by pulling back the metric on X_i via the canonical projection. The ℓ^1 -metric on X is the metric given by $d = \sum_i d_i$.

Proposition 7.1. *Consider a \mathbb{R}^2 -mixed structure M on Σ with \mathbb{R}^2 -length function \vec{L} , and a pair of isometric actions of Γ on \mathbb{R} -trees T_1, T_2 with length functions $(L_{T_1}, L_{T_2}) = \vec{L}$. Let $(X_M, (d_1, d_2))$ be the tree-graded \mathbb{R}^2 -space associated with M . There is an equivariant embedding*

$$f: X_M \hookrightarrow T_1 \times T_2$$

preserving each factor pseudometric d_i . In particular, f is isometric for the ℓ^1 -metric $d = d_1 + d_2$.

Example 7.2. *If M is a flat surface, equivalently $\Sigma' = \Sigma$, the associated tree-graded \mathbb{R}^2 -space is the flat surface given by the universal cover \tilde{M} of M . The map f gives an embedding of \tilde{M} in any product of trees with the correct length functions. Such embedding is isometric for the ℓ^1 -metric, and bilipschitz for the CAT(0) metric (compare [13, Example 4, Section 2.3]). Such embedding is never isometric for the CAT(0) metric.*

If, instead, M is a 2-lamination, namely in the cases in which $\Sigma' = \emptyset$, (X_M, d) is a tree. The map f gives an equivariant embedding of this tree in any product of trees that induce the correct length function. This map is isometric if the product of trees is endowed with the ℓ^1 -metric. Observe, however, that there are laminations Λ that support mutually singular transverse measures. For these the image of the \mathbb{R}^2 -tree will not be a geodesic subset of $T_1 \times T_2$ if $T_1 \times T_2$ is endowed with the CAT(0) metric. In general, when $T_1 \times T_2$ is endowed with the CAT(0) metric, it is possible to show that the embedding is isometric if and only if all the laminations are homothetic.

Proof of Proposition 7.1. We first use rigidity of lengths in \mathbb{R} -trees to reduce to the case where each T_i is the tree $T(\lambda_{M,i})$ dual to the measured geodesic lamination $\lambda_{M,i}$ on Σ associated with M : Let T'_i denote the minimal subtree of T_i invariant by Γ . The actions of Γ on the trees T'_i and $T(\lambda_{M,i})$ are minimal and have same length function, which belongs to \mathcal{ML} . Hence there is an equivariant isometry $h_i: T(\lambda_{M,i}) \xrightarrow{\sim} T'_i \subset T_i$. Then the diagonal map $h = (h_i)_i$ is an equivariant embedding $\prod_i T(\lambda_{M,i}) \rightarrow \prod_i T_i$ preserving each factor pseudometric d_i .

We now construct the canonical map $p_i: X_M \rightarrow T(\lambda_{M,i})$. We denote $T_{M,i} := X_M/d_{M,i}$ the biggest Hausdorff quotient of X_M with respect to the pseudometric $d_{M,i}$ and by $p_i: X_M \rightarrow T_{M,i}$ the corresponding projection. The tree $T_{M,i}$ can be identified with $T(\lambda_{M,i})$: since $T_{M,i}$ is tree-graded with \mathbb{R} -trees pieces

$p_i(X_{\tilde{P}}) = \tilde{P}/d_{M,i}$, it is itself an \mathbb{R} -tree. Since the action of Γ on $\tilde{\Sigma}^b$ is minimal, the action on $T_{M,i}$ is minimal as well. The length function of this action is

$$L_{X_{M,i}} = L_{d_{M,i}} = i(\lambda_{M,i}, \cdot)$$

In particular, by rigidity of length functions for actions on minimal trees, there is an equivariant isometry $T_{M,i} \xrightarrow{\sim} T(\lambda_{M,i})$. Then the diagonal map $p = (p_i)_i$ from X_M to $\prod_i T(\lambda_{M,i})$ sends each pseudometric $d_{M,i}$ on the factor pseudometric d_i , hence it takes the metric $d_M = \sum_i d_{M,i}$ to the ℓ^1 -metric $d = \sum_i d_i$ on $\prod_i T(\lambda_{M,i})$. The results follow taking the map $f = h \circ p$. \square

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