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On Diophantine approximations of the solutions of q-functional equations

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Given a sequence of linear forms

$$R_n = P_{n,1}\alpha_1 + \cdots + P_{n,m}\alpha_m, \quad P_{n,1}, \dots, P_{n,m} \in \mathbb{K}, \quad n \in \mathbb{N},$$

in $m \geqslant 2$ complex or p-adic numbers $\alpha_1, \dots, \alpha_m \in \mathbb{K}_v$ with appropriate growth conditions, Nesterenko proved a lower bound for the dimension d of the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ over $\mathbb{K}$, when $\mathbb{K} = \mathbb{Q}$ and v is the infinite place. We shall generalize Nesterenko's dimension estimate over number fields $\mathbb{K}$ with appropriate places v, if the lower bound condition for $|R_n|$ is replaced by the determinant condition. For the q-series approximations also a linear independence measure is given for the d linearly independent numbers. As an application we prove that the initial values $F(t), F(qt), \dots, F(q^{m-1}t)$ of the linear homogeneous q-functional equation

$$NF(q^m t) = P_1F(q^{m-1}t) + P_2F(q^{m-2}t) + \cdots + P_mF(t),$$

where $N = N(q,t)$, $P_i = P_i(q,t) \in \mathbb{K}[q,t]$ $(i = 1, \dots, m)$, generate a vector space of dimension $d \geqslant 2$ over $\mathbb{K}$ under some conditions for the coefficient polynomials, the solution $F(t)$ and $t, q \in \mathbb{K}^*$.

1. Introduction

Let $F(t)$ be a non-zero solution of the q-functional equation

$$NF(q^m t) = P_1F(q^{m-1}t) + P_2F(q^{m-2}t) + \cdots + P_mF(t), \quad m \geqslant 2, \tag{1.1}$$

where $N = N(q,t)$, $P_i = P_i(q,t) \in \mathbb{K}[q,t]$ $(i = 1, \dots, m)$ are polynomials in q and t with coefficients from the field $\mathbb{K}$. Here we suppose that equation (1.1) satisfied by $F(t)$ is of the lowest order m. The analytic solutions of (1.1) include *inter alia* generalized q-hypergeometric (basic) series

$${}_k\Phi_l(t) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_k)_n}{(q)_n (b_1)_n \dots (b_l)_n} t^n,$$

where $(a)_0 = 1$ and $(a)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ for $n \in \mathbb{Z}^+$.

There are few works considering linear independence properties of the solutions within the general framework of equation (1.1). Osgood [15] studied quantitative irrationality for the Frobenius-type series solutions of (1.1) over the Gaussian field $\mathbb{Q}(i)$ when $q = 1/d$, $d \in \mathbb{Z}[i]$, $|d| > 1$. Bézivin [2, 3] proved linear independence results for the series

$$F(t) = \sum_{n=0}^{\infty} \frac{t^n}{\prod_{i=1}^{n} A(q^i)}, \quad A(x) \in \mathbb{K}[x] \tag{1.2}$$

and its derivatives over imaginary quadratic field $\mathbb{K}$, when $q \in \mathbb{Z}_{\mathbb{K}}$, $|q| > 1$. Töpfer [21] considered linear independence of the solutions and their derivatives in some special cases of (1.1). In the special case $qtF(q^2t) = -F(qt) + F(t)$ attached to the Rogers–Ramanujan continued fraction sharp irrationality measures have been obtained by Bundschuh [4] and Shiokawa [17] in the imaginary quadratic field case and by Matala-aho [10] when $\mathbb{K}$ is an algebraic number field.

On the other hand, there is much work considering the arithmetic nature of the q-hypergeometric series ${}_k\Phi_l(t)$ (see Stihl [19] and Katsudara [9]) and of the analytic solutions $F(t)$ of the first degree q-functional equation

$$A(q,t)F(qt) = B(q,t)F(t) + C(q,t),$$

see [1, 4–7, 13, 14, 16, 18, 20–22].

In the first part we shall state and prove linear independence results. Theorem 3.3 is especially designed to study quantitative aspects of q-functions and it is written in such a way that we do not need to multiply any denominators in the approximation formulae, all information is now included in the heights via the product formula (2.1).

Then we shall tackle equation (1.1) with $N(q,t) = t^s M(q,t)$ when the positive integer s and the degrees $r_0 = \deg_t M(q,t)$, $r_i = \deg_t P_i(q,t)$ in t of the coefficient polynomials satisfy the condition (4.2). The new phenomenon coming from the use of the functional equation method is that we do not need to know *a priori* any explicit forms for the solutions of equation (1.1). Only property of the solutions needed is a slight upper bound condition near zero, we shall use the condition (4.3). The methods using rational function approximations (Padé approximations) or Thue-Siegel's lemma usually need explicit knowledge of the behaviour of the q-series expansions. The crucial thing in studying equation (1.1) is to use matrix formalism for the functional equation method, which enables us to achieve transparent estimations for the approximation polynomials (3.2) and for the remainder term (3.3) and an easy determination of the determinant condition (3.4).

As a consequence of using the functional equation method, we are able to apply theorems 3.1 and 3.3 not only for the analytic solutions including the class of q-hypergeometric series studied by Stihl [19] and partly for the class of functions (1.2) studied by Bézivin [2] and recently by Amou *et al*. [1], but also for other—even non-continuous—solutions of (1.1). Our main result, theorem 4.1, for the non-zero solutions of equation (1.1) is that under the conditions (4.2)–(4.4) at least two of the numbers

$$F(t), F(qt), \ldots, F(q^{m-1}t) \tag{1.3}$$

are linearly independent over $\mathbb{K}$ (an algebraic number field) having a linear independence measure depending on the degrees s, r_i $(i = 1, \ldots, m)$ and the dimension of the vector space generated by the numbers (1.3).

The case $m = 2$ has interesting implications for the values of certain q-continued fractions (see [10, 12]).

2. Notation

Let $\mathbb{K}$ be an algebraic number field of degree κ over $\mathbb{Q}$. If the finite place v of $\mathbb{K}$ lies over the prime p, we write $v|p$, for an infinite place v of $\mathbb{K}$ we write $v|\infty$. We normalize the absolute value $\|_v$ of $\mathbb{K}$ so that

$$\begin{aligned} &\text{if } v|p, \quad \text{then } |p|_v = p^{-1}, \\ &\text{if } v|\infty, \quad \text{then } |x|_v = |x|, \end{aligned}$$

where $|\cdot|$ denotes the ordinary absolute value in $\mathbb{Q}$. By using the normalized valuations

$$\|\alpha\|_v = |\alpha|_v^{\kappa_v/\kappa}, \qquad \kappa_v = [\mathbb{K}_v : \mathbb{Q}_v],$$

the product formula has the form

$$\prod_v \|\alpha\|_v = 1 \quad \forall \alpha \in \mathbb{K}^*. \tag{2.1}$$

The Height $H(\alpha)$ of α is defined by the formula

$$H(\alpha) = \prod_v \|\alpha\|_v^*, \qquad \|\alpha\|_v^* = \max\{1, \|\alpha\|_v\}$$

and the height $H(\underline{\alpha})$ of vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{K}^m$ is given by

$$H(\underline{\alpha}) = \prod_v \|\underline{\alpha}\|_v^*, \qquad \|\underline{\alpha}\|_v^* = \max_{i=1,\ldots,m}\{1, \|\alpha_i\|_v\}.$$

We shall also use the notation

$$\begin{aligned} K_v(\underline{\beta}) &= \prod_{w \neq v} \max_{i=1,\ldots,d} \|\beta_i\|_w, \\ L_v(\underline{\beta}) &= \min_{i=1,\ldots,d} \max_{j \neq i} \|\beta_j\|_v K_v(\underline{\beta}) \end{aligned}$$

for any vector $\underline{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{K}^d$ and

$$K(\bar{B}) = \prod_v \max_{i,j} \|b_{i,j}\|_v$$

for any matrix $\bar{B} = (b_{i,j})$. Here the quantity $\min_{i=1,\ldots,d} \max_{j\neq i} \|\beta_j\|_v$ gives a second largest number of $\|b_j\|$, $j = 1, \ldots, d$, and the introduction of $L_v(\underline{\beta})$ is essential in the following argument. For any place v of $\mathbb{K}$, $q \in \mathbb{K}^*$ and $\|q\|_v \neq 1$, we define the number

$$\lambda = \lambda_q = \frac{\log H(q)}{\log \|q\|_v}$$

having the following properties $\lambda_{1/q} = -\lambda_q$, $|\lambda_q| \geqslant 1$. Also $\lambda_q \leqslant -1$ for all $|q|_v < 1$, and $\lambda_q = -1$, if moreover $|q|_w \geqslant 1$ for all $w \neq v$.

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3. Theorems for linear independence and measures

Let us have a sequence of linear forms

$$R_n = P_{n,1}\alpha_1 + \cdots + P_{n,m}\alpha_m, \quad P_{n,1}, \dots, P_{n,m} \in \mathbb{K}, \quad n \in \mathbb{N}, \tag{3.1}$$

in m complex or p-adic numbers $\alpha_1, \dots, \alpha_m \in \mathbb{K}_v$. First we shall study the dimension of the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ over $\mathbb{K}$ under the following assumptions:

$$\max\{\|P_{n,1}\|_w^*, \dots, \|P_{n,m}\|_w^*\} \leqslant P_w(n) \quad \forall w, \tag{3.2}$$

$$\|R_n\|_v \leqslant R_v(n) \tag{3.3}$$

and

$$\Delta(n) = \det \begin{pmatrix} P_{n,1} & \cdots & P_{n,m} \\ P_{n-1,1} & \cdots & P_{n-1,m} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ P_{n-m+1,1} & \cdots & P_{n-m+1,m} \end{pmatrix} \neq 0 \tag{3.4}$$

for all $n \geqslant n_0$.

THEOREM 3.1. *Let v be given and let $q \in \mathbb{K}^*$ satisfy $\|q\|_v < 1$. Let $P_{n,1}, \dots, P_{n,m} \in \mathbb{K}$ be such that the assumptions (3.2), (3.3) and (3.4) are valid with*

$$P_w(n) = c_w^{a(n)} \|q\|_w^{*\,p(n)}, \quad c_w \geqslant 1 \quad \forall w,$$
$$R_v(n) = c_1^{a(n)} \|q\|_v^{\,r(n)}, \quad c_1 > 0,$$

where c_γs are positive constants not depending on n (γ runs through all places and positive integer indices) with $\prod_w c_w = c_2 < \infty$, and $r(n) \to \infty$ such that $r(n)/a(n) \to \infty$ and $r(n)/p(n) \to l > 1$. If at least one of α_is is non-zero, say $\alpha_1 \neq 0$, then the dimension d of the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ over $\mathbb{K}$ satisfies

$$d \geqslant \frac{l}{-\lambda_q}. \tag{3.5}$$

Let us recall Nesterenko's result [8], which uses the lower bound condition

$$S(n) \leqslant |R_n| \tag{3.6}$$

for the remainder R_n instead of the determinant condition (3.4). Now $\mathbb{K} = \mathbb{Q}$ and $\mathbb{K}_v = \mathbb{R}$.

THEOREM 3.2 (see [8]). *Let $P_{n,1}, \dots, P_{n,m} \in \mathbb{Z}$ be such that the assumptions (3.2), (3.3) and (3.6) are valid with*

$$P(n) = \mathrm{e}^{\sigma(n)},$$
$$R(n) = \mathrm{e}^{-\tau_2\sigma(n)},$$
$$S(n) = \mathrm{e}^{-\tau_1\sigma(n)},$$

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$\tau_1 \geqslant \tau_2 > 0$ and $\sigma(n)$ is a monotonically increasing function on $\mathbb{N}$ such that $\sigma(n) \to \infty$ and $\sigma(n+1)/\sigma(n) \to 1$. Then the dimension d of the vector space $\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_m$ over $\mathbb{Q}$ satisfies

$$d \geqslant \frac{1+\tau_1}{1+\tau_1-\tau_2}. \tag{3.7}$$

In order to compare the above theorems, we put $\mathbb{K} = \mathbb{Q}$ and $\tau_1 = \tau_2$. Now, if the assumptions of theorem 3.2 are valid, then $d \geqslant 1 + \tau_2$. To apply theorem 3.1 we choose an integer $s \in \mathbb{Z}^+$ and a corresponding sequence $p(n)$ such that

$$s^{p(n)}/\mathrm{e}^{\sigma(n)} \to 1.$$

Set also $q = 1/s$, then

$$P_{n,1}/s^{p(n)}, \ldots, P_{n,m}/s^{p(n)}$$

and

$$R_n/s^{p(n)}$$

satisfy the conditions (3.2) and (3.3) respectively with $r(n)/p(n) \to \tau_2 + 1$. If also the determinant condition (3.4) is fulfilled, then by theorem 3.1

$$d \geqslant \frac{\tau_2+1}{-\lambda_{1/s}} = \tau_2 + 1.$$

From now on we shall suppose that the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ of dimension d has the base, say, $\{\alpha_1, \ldots, \alpha_d\}$.

In the applications for q-series the approximation forms (3.1) are usually such that $P_{n,k} = P_{n,k}(q,z)$ are polynomials of degree An^2 in q and of degree an in z and the remainder $R_n = R_n(q,z)$ has order Bn^2 in zero with respect to variable q, i.e.

$$R_n(q,z) = q^{Bn^2} S_n(q,z), \quad S_n(q,z) \in \mathbb{K}[[q,z]].$$

For this reason we prove the following results corresponding to the assumptions (3.2) and (3.3) with

$$a(n) = an, \qquad p(n) = An^2, \qquad r(n) = Bn^2, \tag{3.8}$$

where a, A and B are constants not depending on n. Further, we shall use the notation

$$\mu = \frac{B}{B + (d-1)\lambda A}$$

and $\omega = \mu - 1$ when $B + (d-1)\lambda A > 0$.

Theorem 3.3. *Let the assumptions of theorem 3.1 be valid with (3.8) and suppose that*

$$B + (d-1)\lambda A > 0. \tag{3.9}$$

Then there exist positive constants C, D and L_0 such that

$$|\beta_1\alpha_1 + \cdots + \beta_d\alpha_d|_v > \frac{C}{(KL^{\omega})^{\kappa/\kappa_v} L^{D(\log L)^{-1/2}}}. \tag{3.10}$$

for all $\underline{\beta} = (\beta_i) \in \mathbb{K}^d \setminus \{\underline{0}\}$ with $L = \max\{L_v(\underline{\beta}), L_0\}$ and $K = K_v(\underline{\beta})$.

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We call ω *a linear independence measure (exponent)* of the numbers $\alpha_1, \ldots, \alpha_d$.

COROLLARY 3.4. *Let $\mathbb{K} = \mathbb{Q}$ and let the assumptions of theorems 3.1 and 3.3 be valid. Then there exist positive constants C, D and L_0 such that*

$$|\beta_1\alpha_1 + \cdots + \beta_d\alpha_d| > \frac{C}{L^{\omega + D(\log L)^{-1/2}}} \tag{3.11}$$

for all $\underline{\beta} = (\beta_i) \in \mathbb{Z}^d \setminus \{\underline{0}\}$ with

$$\gcd(\beta_1, \ldots, \beta_d) = 1 \quad \textit{and} \quad L = \max\Big\{\min_{i=1,\ldots,d} \max_{j\neq i} |\beta_j|, L_0\Big\}$$

(depends on the second largest coefficient).

If we suppose $\alpha_1 = 1$ and $d \geqslant 2$, then by the general theory we have $\omega \geqslant d-1$ in (3.11) (see [8]). Let $\lambda_q = -1$. In order to get the best possible measure $\mu = d$ we should to construct such an approximations that $B/A = d$.

COROLLARY 3.5. *Let $\alpha_1 = 1$ and let the assumptions of theorems 3.1 and 3.3 be valid. Then there exist positive constants C, D and L_0 such that*

$$\max_{i=2,\ldots,m}\{|\alpha_i - \beta_i|_v\} > \frac{C}{(KL^{\omega})^{\kappa/\kappa_v} L^{D(\log L)^{-1/2}}}, \qquad \mu = \frac{B}{B + (d-1)\lambda A} \tag{3.12}$$

for all $\underline{\beta} = (1, \beta_2, \ldots, \beta_m) \in \mathbb{K}^m$ with $L = \max\{L_v(\underline{\beta}), L_0\}$ and $K = K_v(\underline{\beta})$.

COROLLARY 3.6. *Let $m = 2$, $\alpha_1 = 1$, $\alpha_2 = \alpha$ and let the assumptions of theorems 3.1 and 3.3 be valid. Suppose that $B + \lambda A > 0$, then there exist positive constants C, D and L_0 such that*

$$|\alpha - \beta|_v > \frac{C}{(KL^{\omega})^{\kappa/\kappa_v} L^{D(\log L)^{-1/2}}}, \qquad \mu = \frac{B}{B + \lambda A} \tag{3.13}$$

for all $\beta \in \mathbb{K}$ with $L = \max\{L_v(1, \beta), L_0\}$ and $K = K_v(1, \beta)$.

COROLLARY 3.7. *Let $\mathbb{K} = \mathbb{Q}$, $m = 2$, $\alpha_1 = 1$, $\alpha_2 = \alpha$ and let the assumptions of theorems 3.1 and 3.3 be valid. Suppose that $B + \lambda A > 0$, then there exist positive constants C, D and L_0 such that*

$$\left|\alpha - \frac{M}{N}\right| > \frac{C}{NL^{\omega + D(\log L)^{-1/2}}}, \qquad \mu = \frac{B}{B + \lambda A}. \tag{3.14}$$

for all $M/N \in \mathbb{Q}$ with $N > 0$ and $L = \max\{\min\{|M|, N\}, L_0\}$.

In the case of approximating only one number α like in corollaries 3.6 and 3.7 we shall call μ *an irrationality measure* of α.

Also we note that the lower bounds used in corollaries 3.5–3.7 are usually replaced by

$$|\alpha - \beta|_v > \frac{C'}{H^{\kappa\mu/\kappa_v + D(\log H)^{-1/2}}}, \qquad \mu = \frac{B}{B + \lambda A}, \tag{3.15}$$

where $H = \max\{H(\beta), H_0\}$.

Proof of theorems 3.1 and 3.3 and the corollaries. Let the base of the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ be (without loss of generality) $\alpha_1, \ldots, \alpha_d$. Thus there exist $\mu = m - d$ linearly independent linear forms

$$L_j \underline{x} = \beta_{j,1} x_1 + \cdots + \beta_{j,m} x_m, \quad \beta_{j,i} \in \mathbb{K}, \quad j = 1, \ldots, \mu, \ i = 1, \ldots, m$$

such that

$$L_j \underline{\alpha} = \beta_{j,1}\alpha_1 + \cdots + \beta_{j,m}\alpha_m = 0, \quad j = 1, \ldots, \mu. \tag{3.16}$$

We set

$$\Lambda_d \underline{x} = \beta_1 x_1 + \cdots + \beta_d x_d, \quad \beta_j \in \mathbb{K}, \quad j = 1, \ldots, d,$$

where at least one of $\beta_j \neq 0$. If now

$$a_1 L_1 + \cdots + a_\mu L_\mu + a_{\mu+1}\Lambda_d = 0$$

for some $a_1, \ldots, a_{\mu+1} \in \mathbb{K}$, then

$$a_1 L_1 \underline{\alpha} + \cdots + a_\mu L_\mu \underline{\alpha} + a_{\mu+1}\Lambda_d \underline{\alpha} = 0.$$

Using (3.16) we get

$$a_{\mu+1}\beta_1\alpha_1 + \cdots + a_{\mu+1}\beta_d\alpha_d = 0,$$

which implies $a_{\mu+1}\beta_1 = \cdots = a_{\mu+1}\beta_d = 0$ by the linear independence of $\alpha_1, \ldots, \alpha_d$. Thus $a_{\mu+1} = 0$ giving

$$a_1 L_1 + \cdots + a_\mu L_\mu = 0,$$

which by the linear independence of $L_1, \ldots, L_\mu$ implies that $a_1 = \cdots = a_\mu = a_{\mu+1} = 0$. Hence the linear forms $L_1, \ldots, L_\mu, \Lambda_d$ are linearly independent and so there exist $n_1, \ldots, n_{d-1} \in \{n, n-1, \ldots, n-m+1\}$ such that

$$\Delta = \det \begin{pmatrix} P_{n_1,1} & \cdots & \cdot & \cdot & \cdots & P_{n_1,m} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ P_{n_{d-1},1} & \cdots & \cdot & \cdot & \cdots & P_{n_{d-1},m} \\ \beta_1 & \cdots & \beta_d & 0 & \cdots & 0 \\ \beta_{1,1} & \cdots & \cdot & \cdot & \cdots & \beta_{1,m} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \\ \beta_{\mu,1} & \cdots & \cdot & \cdot & \cdots & \beta_{\mu,m} \end{pmatrix} \neq 0 \tag{3.17}$$

for all $n \geqslant n_0$ by the assumption (3.4). Further,

$$\begin{aligned} \alpha_1 \Delta = \det & \begin{pmatrix} R_{n_1} & P_{n_1,2} & \cdots & \cdot & \cdot & \cdots & P_{n_1,m} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \\ R_{n_{d-1}} & P_{n_{d-1},2} & \cdots & \cdot & \cdot & \cdots & P_{n_{d-1},m} \\ \Lambda_d & \beta_2 & \cdots & \beta_d & 0 & \cdots & 0 \\ 0 & \beta_{1,2} & \cdots & \cdot & \cdot & \cdots & \beta_{1,m} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \\ 0 & \beta_{\mu,2} & \cdots & \cdot & \cdot & \cdots & \beta_{\mu,m} \end{pmatrix} \\ &= R_{n_1}\Delta_1 + \cdots + R_{n_{d-1}}\Delta_{d-1} + \Lambda_d \Delta_d. \end{aligned}$$

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Thus we may use the product formula (2.1) to get

$$\begin{aligned} 1 &= \|\Delta\|_v \prod_{w\neq v} \|\Delta\|_w \\ &\leqslant c_3\Big(\max_{j=2,\dots,d}\|\beta_j\|_v R_v(n)P_v(n)^{d-2} + \|\Lambda_d\|_v P_v(n)^{d-1}\Big)\max_{i,j}\|\beta_{i,j}\|_v^{\mu} \\ &\qquad \cdot \prod_{w\neq v} P_w(n)^{d-1}\max_{i=1,\dots,d}\|\beta_i\|_w \max_{i,j}\|\beta_{i,j}\|_w^{\mu} = S(n)+W(n), \end{aligned} \tag{3.18}$$

where

$$S(n) = c_3 L_v(\underline{\beta})K(\bar{B})^{\mu} c_1^{a(n)}\|q\|_v^{r(n)} c_v^{a(n)(d-2)} \prod_{w\neq v} c_w^{(d-1)a(n)}\|q\|_w^{*\,(d-1)p(n)}$$

and

$$W(n) = c_3 K_v(\underline{\beta})K(\bar{B})^{\mu}\|\Lambda_d\|_v c_v^{(d-1)a(n)} \prod_{w\neq v} c_w^{(d-1)a(n)}\|q\|_w^{*\,(d-1)p(n)}$$

with $\bar{B} = (\beta_{i,j})$. The term

$$L_v(\underline{\beta}) = \min_{i=1,\dots,d}\max_{j\neq i}\|\beta_j\|_v K_v(\underline{\beta})$$

in $S(n)$ comes from the fact that no α_i $(i=1,\dots,d)$ is zero and so α_1 can be replaced by any α_i $(i=1,\dots,d)$.

We note that theorem 3.1 can be derived by a slight modification of the above argument, namely, if we use only the linear forms $L_1,\dots,L_\mu$ in (3.17), then there exist $n_1,\dots,n_d \in \{n, n-1,\dots,n-m+1\}$ such that

$$\det\begin{pmatrix} P_{n_1,1} & \cdots & \cdot & \cdot & \cdots & P_{n_1,m} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ P_{n_d,1} & \cdots & \cdot & \cdot & \cdots & P_{n_d,m} \\ \beta_{1,1} & \cdots & \cdot & \cdot & \cdots & \beta_{1,m} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \\ \beta_{\mu,1} & \cdots & \cdot & \cdot & \cdots & \beta_{\mu,m} \end{pmatrix} \neq 0 \quad \forall n \geqslant n_0.$$

So we can replace the upper bound $S(n)+W(n)$ in (3.18) by

$$U(n) = c_4 c_5^{a(n)}\|q\|_v^{r(n)} H(q)^{dp(n)} = c_4 c_5^{a(n)}\|q\|_v^{r(n)+d\lambda p(n)}. \tag{3.19}$$

However, $U(n)\to 0$ holds for every d satisfying

$$r(n) + d\lambda p(n) > 0 \quad \forall n \geqslant \bar{n}_1 \tag{3.20}$$

from some $\bar{n}_1 \geqslant n_0$ on. Hence we necessarily have

$$l + d\lambda \leqslant 0. \tag{3.21}$$

This proves theorem 3.1.

We next prove theorem 3.3.

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From (3.18) we get the Diophantine seesaw

$$1 \leqslant S(n) + W(n) \leqslant S'(n) + W'(n), \tag{3.22}$$

where

$$S'(n) = L_v c_6^{a(n)} \|q\|_v^{r(n)+(d-1)\lambda p(n)}, \quad L_v = L_v(\underline{\beta})$$

and

$$W'(n) = \|\Lambda_d\|_v K_v c_6^{a(n)} H(q)^{(d-1)p(n)}, \quad K_v = K_v(\underline{\beta}).$$

Choosing L_v big enough, say $L_v \geqslant L_0$ (if necessary), and using (3.9), we can find a largest $\bar{n}_2 \geqslant \bar{n}_1$ such that

$$S'(\bar{n}_2) \geqslant \tfrac{1}{2}. \tag{3.23}$$

Consequently, the Diophantine seesaw (3.22) implies

$$W'(\bar{n}_2 + 1) > \tfrac{1}{2}. \tag{3.24}$$

First we shall use the inequality (3.23) giving

$$(B + (d-1)\lambda A)\bar{n}_2^2 + a\frac{\log c_6}{\log \|q\|_v}\bar{n}_2 + \frac{\log 2L_v}{\log \|q\|_v} \leqslant 0, \tag{3.25}$$

which implies the bounds

$$\bar{n}_2 \leqslant c_7 + \sqrt{\frac{-\log 2L_v}{\log \|q\|_v (B + (d-1)\lambda A)}} \tag{3.26}$$

and

$$\bar{n}_2^2 \leqslant c_7^2 + 2c_7\sqrt{\frac{-\log 2L_v}{\log \|q\|_v (B + (d-1)\lambda A)}} - \frac{\log 2L_v}{\log \|q\|_v (B + (d-1)\lambda A)}. \tag{3.27}$$

Then using (3.24) we get

$$\begin{aligned} \tfrac{1}{2} &< \|\Lambda_d\|_v K_v c_6^{a(\bar{n}_2+1)} H(q)^{(d-1)p(\bar{n}_2+1)} \\ &< \|\Lambda_d\|_v K_v c_8^{\bar{n}_2} H(q)^{(d-1)A\bar{n}_2^2} \\ &< \|\Lambda_d\|_v K_v c_9 L_v^{-\lambda(d-1)A/(B+(d-1)\lambda A)+c_{10}/\sqrt{\log L_v}} \end{aligned} \tag{3.28}$$

proving the estimate (3.10).

In corollary 3.4 we have $\kappa = 1$ and $v = \infty$ giving

$$K = K_\infty(\underline{\beta}) = \prod_{p \neq \infty} \max_{i=1,\dots,d} |\beta_i|_p,$$

where $\max_{i=1,\dots,d} |\beta_i|_p = 1$ for any $p \neq \infty$ because $\gcd(\beta_1, \dots, \beta_d) = 1$. Thus $K = 1$ and

$$L = L_\infty(\underline{\beta}) = \min_{i=1,\dots,d} \max_{j \neq i} |\beta_j|$$

is a second largest coordinate of $\underline{\beta}$.

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In corollary 3.5 the dimension d of the vector space $\mathbb{K}\alpha_1 + \cdots + \mathbb{K}\alpha_m$ is at least 2 having a base $\{1, \dots, \alpha_j, \dots\}$. From (3.10) we get

$$|\alpha_j - \beta_j|_v > \frac{C}{(K'L'^{\omega})^{\kappa/\kappa_v} L'^{D(\log L')^{-1/2}}}, \qquad \mu = \frac{B}{B + \lambda A} \tag{3.29}$$

for any $\beta_j \in \mathbb{K}$ with $\underline{\beta'} = (-\beta_j, 0, \dots, 0, 1, 0, \dots) \in \mathbb{K}^m$, $L' = \max\{L_v(\underline{\beta'}), L'_0\}$ and $K' = K_v(\underline{\beta'})$. This immediately gives (3.12).

Corollary 3.6 follows directly from corollary 3.5.

In corollary 3.7

$$K = K_\infty = \prod_{p \neq \infty} \max\left\{1, \left|\frac{M}{N}\right|_p\right\} = N$$

and

$$L = L_\infty = \min\left\{1, \left|\frac{M}{N}\right|\right\} N = \min\{N, |M|\}$$

giving (3.14). □

4. q-series

Let $F(t)$ be a non-zero solution of the q-functional equation

$$t^s M F(q^m t) = P_{0,1} F(q^{m-1} t) + P_{0,2} F(q^{m-2} t) + \cdots + P_{0,m} F(t) \tag{4.1}$$

of lowest order m, where $m \geqslant 2$, $s \geqslant 1$ and $M = M(q,t)$, $P_{0,i} = P_{0,i}(q,t) \in \mathbb{K}[q,t]$ are of degree

$$r = \deg_t M, \qquad r_i = \deg_t P_{0,i}.$$

In the following we set

$$A = \max_{j=2,\dots,m} \left\{ r_1, \frac{(j-1)(s+r) + r_j}{2j} \right\}, \qquad B = \tfrac{1}{2} s$$

and we suppose

$$B > A. \tag{4.2}$$

THEOREM 4.1. *Let v be a place of $\mathbb{K}$ and let $q, t \in \mathbb{K}^*$ satisfy $B + \lambda A > 0$, $|q|_v < 1$, $M(q, q^k t) \neq 0$ and $P_{0,m}(q, q^k t) \neq 0$ for all $k \in \mathbb{N}$. Let $F(t)$ be a solution of the functional equation (4.1) such that*

$$|F(q^n t)|_v < c_{11}^n \quad \forall n \in \mathbb{N} \tag{4.3}$$

for some positive constant c_{11} and the numbers

$$F(t), F(qt), \dots, F(q^{m-1} t) \tag{4.4}$$

are not all zero, then

$$d = \dim_{\mathbb{K}} \{\mathbb{K}F(t) + \mathbb{K}F(qt) + \cdots + \mathbb{K}F(q^{m-1} t)\} \geqslant 2.$$

If $d = 2$, then any two of the numbers (4.4) being linearly independent over $\mathbb{K}$ have a linear independence measure $\omega = \mu - 1$, where $\mu = B/(B + \lambda A)$.

QUESTIONS 4.2. Let equation (4.1) be of the lowest order $m \geqslant 3$.

(i) Is the independence result in theorem 4.1 best possible for the general solution of the functional equation (4.1), i.e. does there exist solutions with $d = 2$?

(ii) If $F(t)$ is an analytic solution in theorem 4.1, is then always

$$\dim_{\mathbb{K}}\{\mathbb{K}F(t) + \mathbb{K}F(qt) + \cdots + \mathbb{K}F(q^{m-1}t)\} = m?$$

EXAMPLE 4.3. Let $q, t \in \mathbb{K}^*$ satisfy $-\frac{4}{3} < \lambda \leqslant -1$, $|q|_v < 1$. If $F(t)$ is a non-zero solution of the functional equation

$$t^2F(q^3t) = -F(q^2t) + tF(qt) + F(t), \tag{4.5}$$

satisfying the condition (4.3) and the numbers

$$F(t), \quad F(qt), \quad F(q^2t) \tag{4.6}$$

are not all zero, then at least two of them are linearly independent over $\mathbb{K}$.

If $d = 2$, then any two of the numbers (4.6) being linearly independent over $\mathbb{K}$ have a linear independence measure $\omega = \mu - 1$, where $\mu = 4/(4 + 3\lambda)$. If $\mathbb{K} = \mathbb{Q}$ and $q = d^{-1}$ ($d \in \mathbb{Z} \setminus \{0, \pm 1\}$) or $q = p^l$ (p is a prime, $l \in \mathbb{Z}^+$), then $\lambda = -1$ and $\omega = 3$ for the real or p-adic numbers (4.6).

Note that equation (4.5) has an entire solution

$$F_1(t) = \sum_{n=0}^{\infty} f_n t^n, \quad f_0 = 1, \quad f_1 = \frac{-1}{1 - q^2},$$

$$f_{n+2} = \frac{q^{n+1}}{1 - q^{2n+4}}(-f_{n+1} + q^{2n-1}f_n) \quad \forall n \in \mathbb{N}. \tag{4.7}$$

Let us define the orbits

$$H(t) = \{tq^n \mid n \in \mathbb{Z}\}, \quad t \in \mathbb{C}_p,$$

and the index set $\Omega = \{0\} \cup \{t \in \mathbb{C}_p \mid |q|_p < |t|_p \leqslant 1\}$, which make up a partition of $\mathbb{C}_p$ that is $\amalg_{\omega \in \Omega} H(\omega) = \mathbb{C}_p$. Given $q, t \in \mathbb{C}_p$ and any initial values

$$F(t), F(qt), \dots, F(q^{m-1}t) \in \mathbb{C}_p,$$

then the functional equation (4.1) has a unique solution $F(t)$ on the orbit $H(t)$, if $M(q, q^k t) \neq 0$ for all $k \in \mathbb{Z}$. So, *a priori* we do not even need to study continuous solutions of (4.1). In the following example we shall construct a non-continuous solution of (4.5) satisfying the conditions of theorem 4.1.

EXAMPLE 4.4. Let $v = \infty$, $\mathbb{K} = \mathbb{Q}$, $q = \frac{1}{2}$ and

$$h(t) = \begin{cases} -1, & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}, \\ 1, & \text{if } t \in \mathbb{Q}. \end{cases}$$

Then $F_2(t) = h(t)F_1(t)$ is a non-continuous solution of the functional equation (4.5) satisfying the condition (4.3) and thus

$$\dim_{\mathbb{Q}}\{\mathbb{Q}F_2(t) + \mathbb{Q}F_2(qt) + \mathbb{Q}F_2(q^2t)\} \geqslant 2$$

for every $t \in \mathbb{Q}^*$.

Osgood [15] studied the Frobenius-type series solutions

$$F(t) = t^c \sum_{n=0}^{\infty} f_n t^n, \quad c \in \mathbb{C}, \tag{4.8}$$

of the functional equation (4.1) giving the following kind of approximations. Let $\mathbb{K} = \mathbb{Q}(i)$ and let $q = 1/d$, $d \in \mathbb{Z}[i], |d| > 1$. Then there exists $\gamma > 0$ (depending on the degrees s, r and r_i) such that for every $\epsilon > 0$

$$\max_{i=0,\dots,m-1} \left\{ \left| F(q^i t) - \frac{M}{N} \right| \right\} > |N|^{\gamma+\epsilon}$$

when $M, N \in \mathbb{Z}[i]$ and $|N| > N(\epsilon)$ for some positive constant $N(\epsilon)$.

Corollary 4.5. *Let v be a place of $\mathbb{K}$ and let $q, t \in \mathbb{K}^*$ satisfy $|q|_v < 1$, $-1 - 1/(m-1) < \lambda \leqslant -1$. Let $F(t)$ be a solution of the functional equation*

$$tF(q^m t) = PF(q^{m-1} t) + \dots + SF(t), \quad m \geqslant 2, \tag{4.9}$$

where $P, \dots, S \in \mathbb{K}, S \in \mathbb{K}^*$, such that

$$|F(q^n t)|_v < c_{11}^n \quad \forall n \in \mathbb{N}$$

for some positive constant c_{11} and the numbers

$$F(t), F(qt), \dots, F(q^{m-1} t) \tag{4.10}$$

are not all zero, then

$$d = \dim_{\mathbb{K}} \{\mathbb{K}F(t) + \mathbb{K}F(qt) + \dots + \mathbb{K}F(q^{m-1} t)\} \geqslant 2.$$

If $d = 2$, then any two of the numbers (4.10) being linearly independent over $\mathbb{K}$ have a linear independence measure $\omega = \mu - 1$, where $\mu = m/(m + (m-1)\lambda)$.

Let

$$T(x) = Px^{m-1} + \dots + S.$$

If $T(1) = 0$, then (4.9) has an entire solution

$$F(t) = \sum_{n=0}^{\infty} \frac{q^{m\binom{n}{2}}}{\prod_{i=1}^{n} T(q^i)} t^n. \tag{4.11}$$

When we set $q = 1/s$ and note $\lambda_s = -\lambda_q$, then we shall get the following consequence, where we denote $Q(x) = x^k T(1/x)$.

Example 4.6. Let $Q(x) \in \mathbb{K}[x]$, $\deg_x Q(x) = k \leqslant m$ and

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{s^{(m-k)\binom{n}{2}} \prod_{i=1}^{n} Q(s^i)}, \quad s, z \in \mathbb{K}^*, \quad |s|_v > 1. \tag{4.12}$$

If $k \leqslant m - 1$ and $1 \leqslant \lambda_s < 1 + 1/(m-1)$, then

$$d = \dim_{\mathbb{K}} \{\mathbb{K}F(t) + \mathbb{K}F(qt) + \dots + \mathbb{K}F(q^{m-1} t)\} \geqslant 2. \tag{4.13}$$

If $d = 2$, then any two of the numbers (4.13) being linearly independent over $\mathbb{K}$ have a linear independence measure $\omega = \mu - 1$, where $\mu = m/(m - (m-1)\lambda_s)$.

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The results of Bézivin [2] imply qualitative linear independence results over the imaginary quadratic field $\mathbb{K}$ for the series (4.12) and its derivatives, if $s, t \in \mathbb{K}^*, s \in \mathbb{Z}_{\mathbb{K}}, |s| > 1$, where $\mathbb{Z}_{\mathbb{K}}$ denotes the ring of integers in $\mathbb{K}$.

Let

$$B_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q)_n (q)_n}$$

be the q-analogue of the Bessel function. Matala-aho [11] proved an irrationality measure $\mu = 7/(7 - 4\lambda_s)$ over $\mathbb{K}$ for $B_s(s)$ with $s \in \mathbb{K}$, $|s|_v > 1$ and $1 \leqslant \lambda_s < 7/4$.

Duverney [7] used Thue-Siegel lemma to construct approximations for the solutions in the form (4.12) of the equation

$$z^s f(z) = P(z) f(dz) + Q(z). \tag{4.14}$$

In Amov *et al.* [1] Thue-Siegel's method is developed in a significant manner to get irrationality measures for the functions (4.12) over algebraic number fields $\mathbb{K}$.

Based on the Padé approximations of q-hypergeometric series Stihl [19] has proved linear independence results over $\mathbb{Q}$ in the archimedean case with measures for a class of q-hypergeometric series which interlace partly with our analytic solutions of the functional equation (4.9).

In the following example a comparison is made in certain common cases between the results coming from our corollary 4.5, Theorem 2 of Amou *et al.* [1] and Satz 1-2 of Stihl [19].

EXAMPLE 4.7. Now we shall restrict to the archimedean case in $\mathbb{K} = \mathbb{Q}$. The q-hypergeometric series

$$F(t) = \sum_{n=0}^{\infty} \frac{q^{m\binom{n}{2}}}{(q)_n (a_1)_n \dots (a_l)_n} t^n, \quad m \geqslant 2, \quad l \leqslant m-2, \tag{4.15}$$

satisfies the functional equation

$$tF(q^m t) = d_{l+1} F(q^{l+1} t) + d_l F(q^l t) + \cdots + d_0 F(t), \tag{4.16}$$

where

$$d_{l+1} x^{l+1} + d_l x^l + \cdots + d_0 = (1 - qx)(1 - a_1 x) \dots (1 - a_l x). \tag{4.17}$$

If $d_i = d_i(a_1, \dots, a_l) \in \mathbb{Q}$ $(i = 1, \dots, l+1)$, then corollary 4.5 gives the linear independence over $\mathbb{Q}$ of at least two of the numbers

$$F(t), F(qt), \dots, F(q^{m-1} t) \tag{4.18}$$

for all $q = r/s, t \in \mathbb{Q}^*$ satisfying $s > |r|^m$. If $d = 2$, then any two of the numbers (4.18) being linearly independent over $\mathbb{K}$ have a linear independence measure $\omega = \mu - 1$, where

$$\mu = m \frac{\log(r/s)}{\log(r^m/s)}.$$

From Amou *et al.* [1] it follows that the numbers (4.18) are irrational for all $q = r/s$, $t \in \mathbb{Q}^*$, satisfying $s > |r|^{\Gamma_1}$, where $\Gamma_1 = \Gamma_1(m)$ (is a computable positive

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number) with an irrationality measure

$$\mu_{\Gamma_1} = \Gamma_1 \frac{\log(r/s)}{\log(r^{\Gamma_1}/s)}.$$

When $a_i \in \mathbb{Q}$ ($i = 1, \dots, l$), Stihl [19] gives the linear independence of all the numbers

$$1, F(t), F(qt), \dots, F(q^{m-1}t) \tag{4.19}$$

for all $q = r/s$, $t \in \mathbb{Q}^*$, satisfying $s > |r|^{\Gamma_2}$, where

$$\Gamma_2 = \Gamma_2(m, l) = \frac{m}{2(m-l)^2}((1+2m)(m-l) + l^2 - l \\ + \sqrt{(1+4m^2)(m-l^2)^2 + 4(m-l)(1+m)(l^2-l)}),$$

with a linear independence measure $\omega_{\Gamma_2} = \mu_{\Gamma_2} - 1$,

$$\mu_{\Gamma_2} = \Gamma_2 \frac{\log(r/s)}{\log(r^{\Gamma_2}/s)}.$$

Here we have chosen the best possible Γ_2, i.e. the case when all $a_i = q^{k_i}$ for some $k_i \in \mathbb{Z}^+$.

The numerical values

$$\Gamma_1(2) = 28.58, \quad \Gamma_1(3) = 39.54, \quad \Gamma_2(2,0) = 4.562, \quad \Gamma_2(3,0) = 6.542 \tag{4.20}$$

(and of course the general form of Γ_2) show that our results are valid in considerably larger set of variable q than Amou *et al*. [1] and Stihl [19] have and also our measures are rather sharp compared to measures given in [1] and [19]. However, we note that Stihl has the full linear independence with measures.

5. Iterations of the functional equation

Let the operator J be defined by

$$JF(t) = F(qt)$$

for any function, vector or matrix $F(t)$. It is readily seen that

$$J(FG) = JFJG, \tag{5.1}$$

whenever the scalar or matrix product of F and G is defined.

Let $F(t)$ satisfy the linear homogeneous q-functional equation

$$NF(q^m t) = P_{0,1}F(q^{m-1}t) + P_{0,2}F(q^{m-2}t) + \dots + P_{0,m}F(t) \tag{5.2}$$

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of lowest order $m \geqslant 2$, where $N = N(q,t)$, $P_{0,1} = P_{0,1}(q,t), \dots, P_{0,m} = P_{0,m}(q,t) \in \mathbb{K}[q,t]$. We shall write equation (5.2) in the matrix form

$$NJ\begin{pmatrix} J^{m-1}F \\ J^{m-2}F \\ \cdot \\ \cdot \\ \cdot \\ J^0F \end{pmatrix} = \begin{pmatrix} P_{0,1} & P_{0,2} & \cdots & \cdot & P_{0,m} \\ N & 0 & \cdots & \cdot & 0 \\ 0 & N & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & N & 0 \end{pmatrix}\begin{pmatrix} J^{m-1}F \\ J^{m-2}F \\ \cdot \\ \cdot \\ \cdot \\ J^0F \end{pmatrix}$$

or

$$NJ\varLambda = \mathcal{P}_0\varLambda, \tag{5.3}$$

where J operates to the m-vector $\varLambda$ and $\mathcal{P}_0$ is the $m \times m$ matrix.

Let us use the notation

$$[F]_n = \prod_{i=0}^{n-1} J^iF, \quad [F]_{-n} = 1 \quad \forall n \in \mathbb{N}$$

and

$$N_{n,k} = \frac{[N]_{n+1}}{[N]_{n-k+2}} \quad \forall k = 1, \dots, n+2.$$

We shall also write

$$[N]_{n+1}J^{n+m}F = P_{n,1}J^{m-1}F + P_{n,2}J^{m-2}F + \cdots + P_{n,m}F \quad \forall n \geqslant -m, \tag{5.4}$$

where $P_{-j,j} = 1$ and $P_{-j,i} = 0$ for all $i, j = 1, \dots, m$, $i \neq j$. Then

$$[N]_{n+1}J^{n+1}\varLambda = \mathcal{P}_n\varLambda \quad \forall n \in \mathbb{N}, \tag{5.5}$$

where

$$\mathcal{P}_n = \begin{pmatrix} N_{n,1}P_{n,1} & N_{n,1}P_{n,2} & \cdots & N_{n,1}P_{n,m} \\ N_{n,2}P_{n-1,1} & N_{n,2}P_{n-1,2} & \cdots & N_{n,2}P_{n-1,m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ N_{n,m}P_{n-m+1,1} & N_{n,m}P_{n-m+1,2} & \cdots & N_{n,m}P_{n-m+1,m} \end{pmatrix}.$$

The equation (5.5) operated by J gives

$$\left(\prod_{l=1}^{n+1} J^lN\right)J^{n+2}\varLambda = J\mathcal{P}_nJ\varLambda. \tag{5.6}$$

Multiplying by N and using (5.3) we get

$$[N]_{n+2}J^{n+2}\varLambda = J\mathcal{P}_n\mathcal{P}_0\varLambda. \tag{5.7}$$

Using the fact that equation (5.2) is of the lowest order implies

$$\mathcal{P}_{n+1} = J\mathcal{P}_n\mathcal{P}_0 \quad \forall n \in \mathbb{N}. \tag{5.8}$$

On the other hand, equation (5.3) operated by J^{n+1} implies

$$J^{n+1}NJ^{n+2}\Lambda = J^{n+1}\mathcal{P}_0 J^{n+1}\Lambda. \tag{5.9}$$

Multiplication of (5.9) by $\prod_{l=0}^{n} J^l N$ and the use of (5.5) give

$$[N]_{n+2}J^{n+2}\Lambda = J^{n+1}\mathcal{P}_0\mathcal{P}_n\Lambda. \tag{5.10}$$

Hence by (5.5) and (5.10) we get

$$\mathcal{P}_{n+1} = J^{n+1}\mathcal{P}_0\mathcal{P}_n \quad \forall n \in \mathbb{N}. \tag{5.11}$$

Equations (5.8) and (5.11) may be considered *the fundamental recurrence forms* for the functional equation (5.2).

THEOREM 5.1. *The polynomials $P_{n,1}(q,t), P_{n,2}(q,t), \ldots, P_{n,m}(q,t)$ satisfy the linear recurrences*

$$\left.\begin{aligned} P_{n,1}(q,t) &= N_{n-1,1}J^nP_{0,1}P_{n-1,1}(q,t) + \cdots + N_{n-1,m}J^nP_{0,m}P_{n-m,1}(q,t),\\ P_{n,2}(q,t) &= N_{n-1,1}J^nP_{0,1}P_{n-1,2}(q,t) + \cdots + N_{n-1,m}J^nP_{0,m}P_{n-m,2}(q,t),\\ &\cdots\\ P_{n,m}(q,t) &= N_{n-1,1}J^nP_{0,1}P_{n-1,m}(q,t) + \cdots + N_{n-1,m}J^nP_{0,m}P_{n-m,m}(q,t), \end{aligned}\right\} \tag{5.12}$$

for all $n \geqslant 0$.

Proof. From the formula (5.11) we get

$$\begin{pmatrix} N_{n,1}P_{n,1} & N_{n,1}P_{n,2} & \cdots & N_{n,1}P_{n,m}\\ N_{n,2}P_{n-1,1} & N_{n,2}P_{n-1,2} & \cdots & N_{n,2}P_{n-1,m}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ N_{n,m}P_{n-m+1,1} & N_{n,m}P_{n-m+1,2} & \cdots & N_{n,m}P_{n-m+1,m} \end{pmatrix}$$
$$= \begin{pmatrix} J^nP_{0,1} & J^nP_{0,2} & \cdots & \cdot & J^nP_{0,m}\\ J^nN & 0 & \cdots & \cdot & 0\\ 0 & J^nN & \cdots & \cdot & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ 0 & 0 & \cdots & J^nN & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} N_{n-1,1}P_{n-1,1} & N_{n-1,1}P_{n-1,2} & \cdots & N_{n-1,1}P_{n-1,m}\\ N_{n-1,2}P_{n-2,1} & N_{n-1,2}P_{n-2,2} & \cdots & N_{n-1,2}P_{n-2,m}\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ N_{n-1,m}P_{n-m,1} & N_{n-1,m}P_{n-m,2} & \cdots & N_{n-1,m}P_{n-m,m} \end{pmatrix},$$

which directly implies the recurrences (5.12). □

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THEOREM 5.2.

$$\det \mathcal{P}_n = (-1)^{(m+1)(n+1)}[P_{0,m}]_{n+1}[N]_{n+1}^{m-1}. \tag{5.13}$$

Proof. From the formula (5.11) we get

$$\mathcal{P}_n = (J^n\mathcal{P}_0)(J^{n-1}\mathcal{P}_0)\dots(J\mathcal{P}_0)\mathcal{P}_0. \tag{5.14}$$

For each term in the product (5.14) the determinant

$$\det J^k\mathcal{P}_0 = \det\begin{pmatrix} J^kP_{0,1} & J^kP_{0,2} & \cdots & \cdot & J^kP_{0,m} \\ J^kN & 0 & \cdots & \cdot & 0 \\ 0 & J^kN & \cdots & \cdot & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & J^kN & 0 \end{pmatrix}$$
$$= (-1)^{m+1}J^kP_{0,m}(J^kN)^{m-1}$$

follows easily from the definition of $\mathcal{P}_0$. Thus the formula (5.13) follows. □

6. Applications

Let $q, t \in \mathbb{K}^*$ and $|q|_v < 1$. In order to apply theorems 3.1 and 3.3 and corollaries 3.4–3.7 to the solutions of the functional equation (4.1) we shall denote $\alpha_i = F(q^{m-i}t)$ for all $i = 1, \dots, m$ and use the notation

$$R_n = [N]_{n+1}J^{n+m}F = P_{n,1}\alpha_1 + P_{n,2}\alpha_2 + \dots + P_{n,m}\alpha_m \quad \forall n \geqslant -m, \tag{6.1}$$

for *the remainder term* R_n.

Proof of theorem 4.1. First we have to estimate the upper bounds $P_w(n)$ for *the approximation polynomials* $P_{n,1}, \dots, P_{n,m}$. Let

$$P(z) = \sum_{k=0}^{n} p_kz^k, \qquad Q(z) = \sum_{k=0}^{m} q_kz^k \in \mathbb{K}[z],$$

if $\|p_k\|_w \leqslant \|q_k\|_w$ for all $k \in \mathbb{N}$, then we shall use the notation

$$P(z) \underset{w}{\leqslant} Q(z)$$

in any valuation w. The recurrences in (5.12) are of the form

$$P_n = a_1P_{n-1} + \dots + a_mP_{n-m}, \tag{6.2}$$

where

$$a_j = N_{n-1,j}J^nP_{0,j}$$

and the polynomials $N = N(q,t)$, $P_{0,i} = P_{0,i}(q,t) \in \mathbb{K}[q,t]$ are of degree

$$r_0 = r + s = \deg_t N, \qquad r_i = \deg_t P_{0,i}$$

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with respect to t. Because the degrees r_j and $[\mathbb{K}:\mathbb{Q}]$ are finite there exists an $h \in \mathbb{Q}$ such that

$$N(t) \underset{w}{\leqslant} \|h\|_w(1+\cdots+t^{r_0}) \tag{6.3}$$

and

$$P_{0,j}(t) \underset{w}{\leqslant} \|h\|_w(1+\cdots+t^{r_j}) \quad \forall j=1,\ldots,m \tag{6.4}$$

for all places w.

From (4.2) it follows that $(j-1)r + r_j < s$ and especially $r, r_j < s \leqslant r_0 < 2s$ for all $j = 1,\ldots,m$. Hence

$$\begin{aligned} a_j &= J^{n-1}N\ldots J^{n+1-j}NJ^nP_{0,j} \\ &\underset{w}{\leqslant} \|q\|_w^{*\,d_j}\|h\|_w^j(1+t+\cdots+t^{r_0})^{j-1}(1+t+\cdots+t^{r_j}) \\ &\underset{w}{\leqslant} \|q\|_w^{*\,d_j}\|h\|_w^j(r_0+1)^{j\delta_w}(1+t+\cdots+t^{jr_0}) \quad \forall j=1,\ldots,m, \end{aligned} \tag{6.5}$$

where $\delta_w = 0$, if $w \not| \infty$, $\delta_w = 1$, if $w|\infty$ and

$$d_j = (j-1)\frac{2n-j}{2}r_0 + nr_j. \tag{6.6}$$

Taking into account the estimate (6.5) with (6.6) and the recurrence (6.2) we get

$$\begin{aligned} P_n(t) &\underset{w}{\leqslant} m^{\delta_w}\Big\{\max_{j=1,\ldots,m}\|q\|_w^{*\,d_j}\|h\|_w^j(r_0+1)^{j\delta_w}(1+t+\cdots+t^{jr_0})P_{n-j}\Big\} \\ &\underset{w}{\leqslant} m^{\delta_w}\|q\|_w^{*\,An^2+smn}\|h\|_w^{mn}(r_0+1)^{mn\delta_w}(1+t+\cdots+t^{mr_0})^n \\ &\underset{w}{\leqslant} \|q\|_w^{*\,An^2+smn}\|h\|_w^{mn}(mr_0+1)^{mn\delta_w}(1+t+\cdots+t^{mr_0n}) \end{aligned} \tag{6.7}$$

because

$$d_j + A(n-j)^2 + sm(n-j) \leqslant An^2 + smn, \quad \forall j=1,\ldots,m. \tag{6.8}$$

Thus we may take

$$P_w(n) = (\|h\|_w^m(mr_0+1)^{m\delta_w}\|t\|_w^{*\,mr_0}\|q\|_w^{*\,sm})^n\|q\|_w^{*\,An^2}, \tag{6.9}$$

where

$$c_w = \|h\|_w^m(mr_0+1)^{m\delta_w}\|t\|_w^{*\,mr_0}\|q\|_w^{*\,sm}$$

satisfies the condition that the product

$$\begin{aligned} \prod_w c_w &= \prod_w \|h\|_w^m \prod_{w|\infty}(mr_0+1)^m \prod_w \|t\|_w^{*\,mr_0}\prod_w \|q\|_w^{*\,sm} \\ &= (mr_0+1)^{m\kappa}H(t)^{mr_0}H(q)^{sm} \end{aligned} \tag{6.10}$$

is finite.

Secondly, we shall study the upper bound $R_v(n)$ for the remainder term R_n. Let

$$N(q,t) = t^sM(q,t) = t^s(m_0+m_1t+\cdots+m_rt^r),$$

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then

$$|R_n|_v = |t^{s(n+1)}q^{s\binom{n+1}{2}}|_v|F(tq^{n+m})|_v \prod_{k=0}^{n} |m_0 + m_1tq^k + \cdots + m_rt^rq^{rk}|_v. \quad (6.11)$$

So

$$|R_n|_v \leqslant |t|_v^{s(n+1)}|q|_v^{s\binom{n+1}{2}} c_{12}c_{11}^n|m_0|_v^n \quad (6.12)$$

for some $c_{12} > 0$ because the product

$$\prod_{k=0}^{n}\left|1 + \frac{m_1tq^k + \cdots + m_rt^rq^{rk}}{m_0}\right|_v$$

converges for every $|q|_v < 1$ and $|F(tq^{n+m})|_v \leqslant c_{11}^{n+m}$ by the assumption (4.3). Hence we may take

$$R_v(n) = c_{13}^n\|t\|_v^{sn}\|q\|_v^{sn^2/2}, \quad c_{13} > 0. \quad (6.13)$$

In order to study the determinant condition (3.4) we start from the definitions (3.4) and (5.5) to get

$$\det \mathcal{P}_n = N_{n,1}N_{n,2}\ldots N_{n,m}\Delta(n), \quad (6.14)$$

which together with theorem 5.2 imply

$$\Delta(n) = (-1)^{(m+1)(n+1)}[P_{0,m}]_{n+1}[N]_n\ldots[N]_{n-m+2}. \quad (6.15)$$

Hence for all q, t satisfying

$$P_{0,m}(q, q^kt) \neq 0, \quad N(q, q^kt) \neq 0 \quad \forall k \in \mathbb{N}$$

the determinant condition (3.4) is valid because $\Delta(n) \neq 0$ by (6.15).

So, the assumptions (3.2)–(3.4) are valid with

$$P_w(n) = c_w^n\|q\|_w^{*\,An^2}, \qquad R_v(n) = c_1^n\|q\|_v^{Bn^2},$$

where

$$A = \max_{j=2,\ldots,m}\left\{r_1, \frac{(j-1)(s+r)+r_j}{2j}\right\}, \quad B = \frac{s}{2}.$$

The assumption $B + \lambda A > 0$ gives the dimension estimate

$$d \geqslant \frac{l}{-\lambda} = \frac{B}{-\lambda A} > 1 \quad (6.16)$$

by theorem 3.1 and, if $d = 2$, then theorem 3.3 gives the measure

$$\mu = \frac{B}{B+\lambda A}.$$

□

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Proof of example 4.3. Equation (4.5) is irreducible and thus solutions of (4.5) do not satisfy any lower degree linear q-functional equation over $\mathbb{K}[q,t]$. Now

$$A = \max\{0, \tfrac{3}{4}, \tfrac{2}{3}\} = \tfrac{3}{4} \quad \text{and} \quad B = \tfrac{2}{2} = 1$$

giving $\lambda > -\frac{4}{3}$ and a measure $\mu = 4/(4+3\lambda)$. The solution (4.7) is entire because

$$f_n = \frac{q^{\binom{n}{2}}}{\prod_{k=1}^{n+1}(1-q^{2k})} g_n, \tag{6.17}$$

where

$$|g_n|_v \leqslant c_{14}^n$$

for some $c_{14} > 0$ for every q satisfying $|q|_v < 1$. □

Proof of corollary 4.5. Now

$$A = \max\left\{\frac{(j-1)s}{2j}\right\} = \left(1 - \frac{1}{m}\right)\frac{s}{2}, \qquad B = \frac{s}{2}. \tag{6.18}$$

Thus $\lambda > -m/(m-1)$ and $\mu = 1/(1+\lambda(1-1/m))$. □

Proof of example 4.7. By using the definition for $\lambda = \lambda_q = \log H(q)/\log\|q\|_v$ the irrationality measure exponent $\mu = \mu(\lambda)$ can be written as follows

$$\begin{aligned}\mu &= \frac{B}{B + (d-1)(\log H(q)/\log\|q\|_v)A} \\ &= \frac{B}{B-(d-1)A} \frac{\log\|q\|_v}{\log\|q\|_v^{B/(B-(d-1)A)} H^{(d-1)A/(B-(d-1)A)}} \\ &= \mu(-1)\frac{\log\|q\|_v}{\log\|q\|_v{}^{\mu(-1)} H^{\mu(-1)-1}}\end{aligned} \tag{6.19}$$

for all $|q|_v < 1$ satisfying the condition (3.9), which reads $\|q\|_v^B H(q)^{(d-1)A} < 1$. Thus, if $q = r/s$ and $d = 2$, then

$$\mu = \mu(-1)\frac{\log|r|/s}{\log|r|^{\mu(-1)}/s} \tag{6.20}$$

under the condition $|r|^{\mu(-1)} < s$. By corollary 4.5 we know that $\mu(-1) = m$. □

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