Student Problems

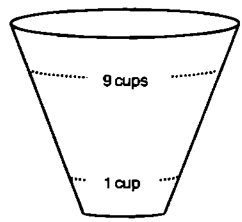
Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by January 20th 2012. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. The names of all successful solvers will be published in the March 2012 edition.

Problem 2011.5

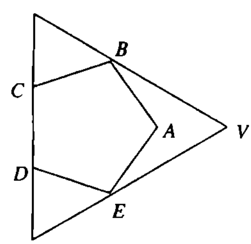
A conical measuring jug has marks for '1 cup' and '9 cups' at 4 cm and 12 cm, respectively, from the base.

How far from the base should be the mark for '5 cups'?



Problem 2011.6

The diagram shows a regular pentagon *ABCDE* inscribed in an equilateral triangle with *V* as one vertex. Find angle *CVD*.



Solutions to 2011.3 and 2011.4

No correct solutions were received for Problem 2011.3. Problem 2011.4 was solved by James Nevin (Berguliet High School, Cape Town).

Problem 2011.3

ABCD is a quadrilateral with *ABC* an equilateral triangle and $AD = BD + CD$. Prove that $AB^2 = AD^2 - BD \times CD$.

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Solution

By the converse of Ptolemy's theorem, $ABDC$ is a cyclic quadrilateral with the vertices in this order around the circle. Then $\angle BDC = 120°$.

By the cosine rule in triangle BDC,

$$BC^2 = BD^2 + CD^2 - 2 \times BD \times CD \cos 120°$$

$$\Rightarrow BC^2 = BD^2 + CD^2 + BD \times CD.$$

Also

$$AD^2 = BD^2 + CD^2 + 2 \times BD \times CD$$

$$\Rightarrow BC^2 = AD^2 - BD \times CD$$

$$\Rightarrow AB^2 = AD^2 - BD \times CD.$$

Problem 2011.4

What happens to the value of the product $\prod_{i=m}^{2m-1} \left(1 + \frac{1}{2i}\right)$ as the positive integer m becomes large?

Solution

Denote the product by X_m. Then

$$\frac{X_m}{X_{m+1}} = \frac{1 + \frac{1}{2m}}{\left(1 + \frac{1}{4m}\right)\left(1 + \frac{1}{4m+2}\right)} = \frac{1 + \frac{1}{2m}}{1 + \frac{1}{2m} - \frac{1}{4m(4m+2)}}.$$

As noted by James, this is greater than 1 and so the value of the product steadily decreases as m increases.

The limiting value for X_m can be determined as follows.

$$\left(1 + \frac{1}{2i}\right)^2 = \left(\frac{4i^2 + 4i + 1}{4i^2 + 4i}\right)\left(\frac{i+1}{i}\right)$$

and therefore $e^{1/(4i^2)}\left(\frac{i+1}{i}\right) > \left(1 + \frac{1}{4i^2}\right)\left(\frac{i+1}{i}\right) > \left(1 + \frac{1}{2i}\right)^2 > \frac{i+1}{i}$.

Now $\prod_{i=m}^{i=2m-1} \left(\frac{i+1}{i}\right) = 2$ and $e^{1/(4m)} > \prod_{i=m}^{i=2m-1} e^{1/(4i^2)}$.

Therefore $2e^{1/(4m)} > X_m^2 > 2$ and so $\lim_{m \to \infty} X_m = \sqrt{2}$.

A prize of £20 is awarded to James Nevin.

STAN DOLAN

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