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Globally Asymptotic Stability of a Delayed Integro-Differential Equation With Nonlocal Diffusion

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Abstract. We study a population model with nonlocal diffusion, which is a delayed integro-differential equation with double nonlinearity and two integrable kernels. By comparison method and analytical technique, we obtain globally asymptotic stability of the zero solution and the positive equilibrium. The results obtained reveal that the globally asymptotic stability only depends on the property of nonlinearity. As an application, we discuss an example for a population model with age structure.

1 Introduction

Consider an integro-differential equation

$$\frac{\partial w}{\partial t} = D_m \mathcal{A} w(t,x) - d_m w + g\Big(\int_{\mathbb{R}} k(x-y) b(w(t-r,y))\,dy\Big) \tag{1.1}$$

for $x \in \mathbb{R}$, $t > 0$, where $\mathcal{A}w(t,x) := \int_{\mathbb{R}} J(x-y)[w(t,y) - w(t,x)]\,dy$. We introduce the following assumptions on kernel functions J and k.

(J) J is a nonnegative Lebesgue measurable function defined on $\mathbb{R}$ such that $J(x) = J(-x)$ for $x \in \mathbb{R}$, $J \in L_1(\mathbb{R})$ and $\int_{\mathbb{R}} J(y)\,dy = 1$.

(K) k is a nonnegative Lebesgue measurable function defined on $\mathbb{R}$ such that $k(x) = k(-x)$ for $x \in \mathbb{R}$, $k \in L_1(\mathbb{R})$, and $\int_{-\infty}^{\infty} k(s)\,ds = 1$.

Equation (1.1) is in regard to a nonlocal diffusion system, since its diffusion process is modelled by a nonlocal operator $\mathcal{A}w(t,x)$ that uses a probability density function J and a convolution function to describe the diffusion of density w at a position x from the value w at all positions $y \in \mathbb{R}$. Furthermore, (1.1) also involves a delayed nonlocal reaction term, which includes a kernel function k and two nonlinear functions g and b. In fact, (1.1) takes some well-known equations as its special cases. We now give some examples in the following.

- Let $J(x) = \delta(x)$ (Dirac-delta function). Then (1.1) yields

$$\frac{\partial w}{\partial t} = -d_m w + g\Big(\int_{\mathbb{R}} k(x-y) b(w(t-r,y))\,dy\Big), \quad x \in \mathbb{R}, t > 0. \tag{1.2}$$

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- Let $J(x) = \delta(x) + \delta^{(2)}(x)$, $k(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$, and $g(v) = \varepsilon v$. Then (1.1) yields

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x-y)^2}{4\alpha}} b(w(t-r,y))\,dy, \quad x \in \mathbb{R}, t > 0. \tag{1.3}$$

- Let $J(x) = \delta(x) \pm \delta^{(1)}(x) + \delta^{(2)}(x)$, $k(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{(x\mp\alpha)^2}{4\alpha}}$, and $g(v) = \varepsilon v$. Then (1.1) yields

$$\begin{aligned}\frac{\partial w}{\partial t} = D_m \Big[\frac{\partial^2 w}{\partial x^2}(t,x) \mp \frac{\partial w}{\partial x}(t,x) \Big] - d_m w \\ + \varepsilon \int_{\mathbb{R}} \frac{1}{\sqrt{4\alpha\pi}} e^{-\frac{(x-y\mp\alpha)^2}{4\alpha}} b(w(t-r,y))\,dy\end{aligned} \tag{1.4}$$

  for $t > 0, x \in \mathbb{R}$.
- Let $k(x) = \delta(x)$ and $b(w) = w$. Then (1.1) yields

$$\frac{\partial w}{\partial t} = D_m \mathcal{A} w(t,x) - d_m w + g(w(t-r,x)), \quad x \in \mathbb{R}, t > 0. \tag{1.5}$$

Equations similar to those in (1.2)–(1.5) with $r \geq 0$ were studied in the existing literature [1–5, 11–13], where most of the results are about the existence, uniqueness, and wave tail behaviors of travelling wave solutions.

Another equation, more general than (1.1), is

$$\frac{\partial w}{\partial t} = D\mathcal{A}_\rho w(t,x) + f(w(t,x), \int_0^\infty \int_{\mathbb{R}} G(s,x-y) b(w(t-s,y))\,dy ds) \tag{1.6}$$

for $x \in \mathbb{R}, t > 0$, where $\mathcal{A}_\rho w(t,x) := \int_{\mathbb{R}} \frac{1}{\rho} J(\frac{x-y}{\rho})[w(t,y) - w(t,x)]\,dy$. Wu and Liu [16], Zhang [20], and Xu and Weng [17] investigated, respectively, the existence of travelling wave solutions for

$$D = 1,\ \rho = 1,\ f(w,v) = -l(w) + f(v),\ b(w) = w, G(t,x) = 0 \ \text{ for } t \geq \tau \geq 0,$$

$$\rho = 1,\ f(w,v) = -g(w) + l(w)v,\ G(t,x) = \delta(t-\tau)h(x),$$

$$\rho = 1,\ f(w,v) = -h(w) + v,\ G(t,x) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

As for the general form (1.6), the existence, uniqueness, and wave tail behaviors were discussed in [19] for $D = 1$, and a special case as $b(w) = w$ was discussed in [18].

It is obvious that (1.1) is taken from (1.6) while

$$f(w,v) = -d_m w + g(v) \quad \text{and} \quad G(t,x) = \delta(t-r)k(x).$$

There are two reasons for us to only consider the situation of $f(w,v) = -d_m w + g(v)$. One is that the population model with age structure has the character that the growth function depends on a fixed maturation period r, but the mortality function does not (see [8], where d_m is the mortality rate). Furthermore, the rate of population change for single species is generally of the form $\frac{dw}{dt}$ = births − deaths + migration [10]. The other reason is the consideration of technical simplicity on analysis.

The main concern of this article is the globally asymptotic stability of the zero solution and the positive equilibrium. To the best of our knowledge, there is little in the literature concerning this problem, and the results obtained in this article are new.

The organization of this paper is as follows. In Section 2, we give some basic theory, as well as two comparison lemmas. In Section 3, we obtain results for globally asymptotic stability, global attractivity, as well as persistence. The last section is devoted to the application and discussion.

2 Basic Theory and Comparison Lemmas

We first state some basic spaces and denotations as follows.

$$\widehat{J}(\omega) = \int_{\mathbb{R}} e^{i\omega x} J(x)\,dx, \quad G(x,\alpha) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{\alpha(\widehat{J}(\omega)-1)} e^{-i\omega x}\,d\omega,$$

$$X := \{\phi: \mathbb{R} \to \mathbb{R} \mid \phi = \{\phi(x)\}_{x\in\mathbb{R}} \text{ is bounded and uniformly continuous}\},$$

$$X^+ := \{\phi \in X \mid \phi(x) \geq 0 \text{ for } x \in \mathbb{R}\},$$

$$\mathcal{A}\phi(x) := \int_{\mathbb{R}} J(x-y)[\phi(y)-\phi(x)]\,dy, \quad \forall \phi \in X,$$

$$T(t)\phi(x) := e^{-d_m t}\int_{-\infty}^{+\infty} G(x-y, D_m t)\phi(y)dy = e^{-(d_m+D_m)t}\sum_{k=0}^{\infty}\frac{(D_m t)^k}{k!} a_k(\phi)(x), \forall \phi \in X, t > 0,$$

where $a_0(\phi) = \phi$, $a_k(\phi) = J * a_{k-1}(\phi)$ for $k \geq 1$ (see [8] for the properties of $G(x,\alpha)$). Clearly, X^+ is a closed cone of X under the partial ordering induced by X^+. We shall express the supremum norm of X by $\|\phi\| := \sup_{x\in\mathbb{R}} |\phi(x)|$. Then X is a Banach lattice, and $T(t): X \to X$ is a linear operator with $T(t)X^+ \subseteq X^+$ for $t > 0$.

Let $\mathcal{C} = C([-r,0], X)$ be the Banach space of continuous functions mapping from $[-r,0]$ into X with the supremum norm $\|\varphi\| = \max_{\theta\in[-r,,0]} \|\varphi(\theta)\|$ (where for every θ, $\|\varphi(\theta)\|$ is the norm in X). We define $\mathcal{C}^+ := \{\varphi \in \mathcal{C} \mid \varphi(s) \in X^+, s \in [-r,0]\}$. Clearly, $\mathcal{C}^+$ is a closed (positive) cone of $\mathcal{C}$. As usual, we identify an element $\varphi \in \mathcal{C}$ as a function from $[-r,0]\times\mathbb{R}$ into $\mathbb{R}$ defined by $\varphi(\theta,x) = \varphi(\theta)(x)$. For any continuous function $w: [-r,a) \to X$, $a > 0$, we define $w_t \in \mathcal{C}$, $t \in [0,a)$ by $w_t(\theta) = w(t+\theta)$, $\theta \in [-r,0]$. Then $t \to w_t$ is a continuous function from $[0,a)$ to $\mathcal{C}$.

Furthermore, we need some assumptions.

(P_1) $g, b \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b(0) = 0$, $g(0) = 0$ and $g(b(w)) > 0$ for $w > 0$; b (g) is globally Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constant L_b (L_g).

(P_2) $b(\cdot)$ and $g(\cdot)$ are nondecreasing on $[0,+\infty)$.

(P_2') There exists a number $w_{\max} > 0$ such that $g(b(w))$ is nondecreasing for $0 < w < w_{\max}$ and decreasing for $w > w_{\max}$.

(P_3) There exists a constant $K > 0$ such that $g(b(K)) = d_m K$, and $g(b(w)) > d_m w$ for $w \in (0,K)$, $g(b(w)) < d_m w$ for $w > K$.

(P_3') $d_m w > g(b(w))$ for $w > 0$.

Together with (1.1), we introduce an initial value condition

$$\begin{cases} w(s,x) = \varphi(s,x) \text{ for } (s,x) \in [-r,0]\times\mathbb{R}, & \varphi \in \mathcal{C}, \\ \varphi(s,x) \geq 0 \text{ for } (s,x) \in [-r,0]\times\mathbb{R}. \end{cases}$$

For getting a nontrivial solution, we need $\varphi(0,\cdot) > 0$, where $\varphi(0,\cdot) > 0$ implies

$$\varphi(0,x) \geq 0, \varphi(0,x) \not\equiv 0 \quad x \in \mathbb{R}.$$

Define $F: \mathcal{C} \to X$ by $F(\varphi)(x) := g\big(\int_{\mathbb{R}} k(x-y)b(\varphi(-r,y))\,dy\big)$. Then the initial value problem of (1.1) can be rewritten as

$$\begin{cases} w'(t) = D_m \mathcal{A} w(t) - d_m w(t) + F(w_t),\ t > 0, \\ w_0 = \varphi \in \mathcal{C}^+. \end{cases} \tag{2.1}$$

We have known [9] that (2.1) is equivalent to

$$\begin{cases} w(t,\cdot) = T(t)\varphi(0,\cdot) + \int_0^t T(t-s)F(w_s)(\cdot)\,ds, \quad t > 0, \\ w(\theta,\cdot) = \varphi(\theta,\cdot), \quad \theta \in [-r,0]. \end{cases} \tag{2.2}$$

The following two lemmas were originally shown in [8] for system (1.1) with $g(z) = \varepsilon z$. For general system (1.1), by using the Lipschitz condition, we have

$$\begin{aligned} \Big|g\Big(\int_{\mathbb{R}} k(x-y)b(w_1(t-r,y))dy\Big) - g\Big(\int_{\mathbb{R}} k(x-y)b(w_2(t-r,y))\,dy\Big)\Big| \\ \leq L_g \Big|\int_{\mathbb{R}} k(x-y)[b(w_1(t-r,y)) - b(w_2(t-r,y))]\,dy\Big| \\ \leq L_g L_b \int_{\mathbb{R}} k(x-y)|w_1(t-r,y)) - w_2(t-r,y)|\,dy, \end{aligned}$$

and thus we can derive the conclusions by similar arguments as in [8].

Lemma 2.1 *Assume* (J), (K) *and* (P_1) *hold. Then for any* $\varphi \in \mathcal{C}^+$, (2.1) *has a unique nonnegative solution* $w(t,x;\varphi)$ *for* $t > 0$. *Furthermore, if* $\varphi(0) \in \operatorname{Int} X^+$, *then* $w(t) \in \operatorname{Int} X^+$ *for* $t \geq 0$; *if* $\varphi(0) > 0$, *then* $w(t) \in \operatorname{Int} X^+$ *for* $t > 0$, *and* $w_t \in \operatorname{Int} \mathcal{C}^+$ *for* $t > r$.

In view of Lemma 2.1, we know $T(t)$ is strongly positive if $D_m t > 0$, and $T(t)$ is a C_0 semigroup on X.

Remark 2.2 Assume that $\varphi(0,\cdot) \geq \delta > 0$. We have from (2.2) and the conclusion of Lemma 2.1 that $w(t,x) \geq T(t)\varphi(0,x) \geq e^{-d_m t} \inf_{x\in\mathbb{R}} \varphi(0,x) \geq e^{-d_m t}\delta$ for $(t,x) \in [0,\infty) \times \mathbb{R}$.

Lemma 2.3 *Assume that* (J), (K) *and* (P_1) *hold. Then for any* $\varphi \in \mathcal{C}^+$, *the solution* $w(t,x;\varphi)$ *of* (1.1) *satisfies* $0 \leq w(t,x;\varphi) \leq \overline{M}$ *for any* $(t,x) \in [0,+\infty) \times \mathbb{R}$, *where* $\Theta := \max_{\theta\in[-r,0]} \sup_{x\in\mathbb{R}} \varphi(\theta,x)$ *and*

$$\overline{M} := \begin{cases} \max\{g(b(w_{\max}))/d_m, \Theta\} & \text{if } (P_2') \text{ holds}, \\ \max\{K,\Theta\}, & \text{if } (P_2) \text{ and } (P_3) \text{ hold}, \\ \Theta & \text{if } (P_3') \text{ hold}. \end{cases}$$

We give two comparison lemmas.

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Lemma 2.4 *Assume that* (J), (K), (P_1)–(P_2) *and* (P_3') *are satisfied. Let* $\overline{w}(t,x)$ *and* $\underline{w}(t,x)$ *be such that* $M \geq \overline{w}(s,x) \geq \underline{w}(s,x)$ *for all* $(s,x) \in [-r,0] \times \mathbb{R}$ *and*

$$\frac{\partial \overline{w}}{\partial t} \geq D_m \mathcal{A}\overline{w}(t,x) - d_m \overline{w} + g\Big(\int_{\mathbb{R}} k(x-y) b(\overline{w}(t-r,y))\, dy \Big),$$
$$\frac{\partial \underline{w}}{\partial t} \leq D_m \mathcal{A}\underline{w}(t,x) - d_m \underline{w} + g\Big(\int_{\mathbb{R}} k(x-y) b(\underline{w}(t-r,y))\, dy \Big),$$

for $(t,x) \in [0,+\infty) \times \mathbb{R}$. *Then* $\overline{w}(t,x) \geq \underline{w}(t,x)$ *for* $(t,x) \in [0,\infty) \times \mathbb{R}$. *Moreover, if* $\overline{w}(\theta,x) \geq \underline{w}(\theta,x)$ *for* $\theta \in [-r,0]$ *with* $\overline{w}(0,x) \not\equiv \underline{w}(0,x)$, *then there holds* $\overline{w}(t,x) > \underline{w}(t,x)$ *for all* $(t,x) \in (0,\infty) \times \mathbb{R}$.

Proof Under (P_2), $F(\varphi)$ is a nondecreasing functional of φ. Note that the above inequalities lead to the following:

$$\begin{cases} \overline{w}(t,\cdot) \geq T(t)\overline{w}(0,\cdot) + \int_0^t T(t-s)F(\overline{w}_s)(\cdot)\, ds, & t > 0, \\ \underline{w}(t,\cdot) \leq T(t)\underline{w}(0,\cdot) + \int_0^t T(t-s)F(\underline{w}_s)(\cdot)\, ds, & t > 0, \\ \overline{w}(\theta,\cdot) \geq \underline{w}(\theta,\cdot), \quad \theta \in [-r,0]. \end{cases}$$

Rewrite $\overline{w}_0(x) = \overline{w}(\theta,x)$, $\underline{w}_0(\theta) = \underline{w}(\theta,x)$ for $(\theta,x) \in [-r,0] \times \mathbb{R}$. Then from [9, Corollary 5] we have that the solutions of (2.2) satisfy $0 \leq w(t,x;\underline{w}_0) \leq w(t,x;\overline{w}_0)$ for $(t,x) \in (0,\infty) \times \mathbb{R}$. We have from (P_3') that

$$\frac{\partial M}{\partial t} \geq D_m \mathcal{A} M - d_m M + g\Big(\int_{\mathbb{R}} k(x-y) b(M) dy \Big), \quad (t,x) \in [0,+\infty) \times \mathbb{R}.$$

Again applying [9, Corollary 5] with

$$[v^+(t,\cdot) = M, v^-(t,\cdot) = \underline{w}(t,\cdot)] \quad \text{and} \quad [v^+(t,\cdot) = \overline{w}(t,\cdot), v^-(t,\cdot) = 0],$$

respectively, we obtain that, for $t \geq 0$,

$$\underline{w}(t,\cdot) \leq w(t,\cdot;\underline{w}_0) \leq M, \quad \text{and} \quad 0 \leq w(t,\cdot;\overline{w}_0) \leq \overline{w}(t,\cdot).$$

Combining the above three inequalities, we have $\underline{w}(t,x) \leq \overline{w}(t,x)$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

Let $v = \overline{w} - \underline{w}$. Then we already know that $v(t,x) \geq 0$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$. We have from the monotonicity of F and the positive property of $G(x-y, D_m t)$ that for $t > 0$,

$$v(t,x) \geq T(t)v(0,\cdot) = e^{-d_m t} \int_{-\infty}^{+\infty} G(x-y, D_m t) v(0,y)\, dy. \tag{2.3}$$

Therefore, it follows that $v(t,x) > 0$ for $t > 0$ if $v(0,x) \not\equiv 0$ on $\mathbb{R}$. ∎

Lemma 2.5 *Assume that* (J), (K) *and* (P_1) *hold, and there exists* $M > 0$ *such that* $0 \leq w^-(t,x) \leq w^+(t,x) \leq M$ *for* $(t,x) \in [-r,+\infty) \times \mathbb{R}$ *such that for any function* ρ *with* $0 \leq w^-(t,x) \leq \rho(t,x) \leq w^+(t,x)$ *for* $(t,x) \in [-r,+\infty) \times \mathbb{R}$, *we have*

$$\frac{\partial w^+(t,x)}{\partial t} \geq D_m \mathcal{A} w^+(t,x) - d_m w^+(t,x) + g\Big(\int_{\mathbb{R}} k(x-y) b(\rho(t-r,y))\, dy \Big) \tag{2.4}$$

and

$$\frac{\partial w^-(t,x)}{\partial t} \leq D_m \mathcal{A} w^-(t,x) - d_m w^-(t,x) + g\Big(\int_{\mathbb{R}} k(x-y) b(\rho(t-r,y))\, dy \Big) \tag{2.5}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}$. *Then for any function* ρ *with*

$$0 \le w^-(t,x) \le \rho(t,x) \le w^+(t,x) \quad \text{for } (t,x) \in [-r,0]\times\mathbb{R},$$

we have

$$0 \le w^-(t,x) \le w(t,x;\rho) \le w^+(t,x) \quad \text{for } (t,x) \in [0,+\infty)\times\mathbb{R},$$

where $w(t,x;\rho)$ *is the solution of* (1.1) *with the initial value* $\rho \in \mathcal{C}^+$, *and* w^- *and* w^+ *are called a pair of sub- and super-solutions of* (1.1).

Proof For any ρ with $w^-(t,x) \le \rho(t,x) \le w^+(t,x)$ for $(t,x) \in [-r,0]\times\mathbb{R}$, let $v(t,x) := w^+(t,x) - w(t,x;\rho)$ for $(t,x) \in [-r,+\infty)\times\mathbb{R}$. In (2.4), let $\rho(t,x) = \rho(t,x)$ for $(t,x) \in [-r,0]\times\mathbb{R}$. Then for $(t,x) \in (0,r]\times\mathbb{R}$, we have

$$\frac{\partial v(t,x)}{\partial t} \ge D_m\mathcal{A}v(t,x) - d_m v(t,x) + \Big[g\Big(\int_{\mathbb{R}} k(x-y) b(\rho(t-r,y))\,dy\Big) - g\Big(\int_{\mathbb{R}} k(x-y) b(w(t-r,y;\rho))\,dy\Big)\Big],$$

which leads to $\frac{\partial v(t,x)}{\partial t} \ge D_m\mathcal{A}v(t,x) - d_m v(t,x)$ for $(t,x) \in (0,r]\times\mathbb{R}$, and therefore,

$$v(t,x) \ge T(t)v(0,x) \ge 0 \quad \Longrightarrow \quad w^+(t,x) \ge w(t,x;\rho) \text{ for } (t,x) \in (0,r]\times\mathbb{R}.$$

Similarly, we obtain from (2.5) that $w^-(t,x) \le w(t,x;\rho)$ for $(t,x) \in (0,r]\times\mathbb{R}$.

In (2.4), let $\rho(t,x) = w(t,x;\rho)$ for $(t,x) \in [0,r]\times\mathbb{R}$. Then for $(t,x) \in (r,2r]\times\mathbb{R}$, we have

$$\frac{\partial v(t,x)}{\partial t} \ge D_m\mathcal{A}v(t,x) - d_m v(t,x) + \Big[g\Big(\int_{\mathbb{R}} k(x-y) b(w(t-r,y;\rho))\,dy\Big) - g\Big(\int_{\mathbb{R}} k(x-y) b(w(t-r,y;\rho))\,dy\Big)\Big],$$

which leads to $w^+(t,x) \ge w(t,x;\rho)$ for $(t,x) \in (r,2r]\times\mathbb{R}$. Similarly, we have $w(t,x;\rho) \ge w^-(t,x)$ for $(t,x) \in (r,2r]\times\mathbb{R}$. By mathematical induction, we can obtain $w^+(t,x) \ge w(t,x;\rho) \ge w^-(t,x)$ for $(t,x) \in (0,\infty)\times\mathbb{R}$. ∎

In what follows, we always assume that (J), (K) and (P_1) hold.

3 Global Stability

In this section, we shall discuss the global asymptotic stability of the zero solution and the positive equilibrium of (1.1). The main technique is the comparison between the solution of (1.1) and the solution of a delay differential equation without spacial variable x. The idea is motivated by the work of Wu, Weng, and Ruan [15].

The following proposition is from [6].

Proposition 3.1 *Necessary and sufficient conditions for every root of the equation* $(\lambda + A)e^{\lambda} + B = 0$ *to have negative real part are*

$$A > -1, \quad A + B > 0, \quad B < \eta\sin\eta - A\cos\eta,$$

where η *is the root of* $\eta = -A\tan\eta, 0 < \eta < \pi$ *if* $A \ne 0$, *and* $\eta = \frac{\pi}{2}$ *if* $A = 0$.

Consider a functional differential equation:

$$\frac{du(t)}{dt} = -d_m u(t) + g\big(b(u(t-r))\big), \quad t > 0. \tag{3.1}$$

Lemma 3.2 *The following conclusions hold.*

(i) *Assume that* (P_3') *is satisfied. Then the zero solution of* (3.1) *is asymptotically stable.*

(ii) *Assume that* (P_2)–(P_3) *are satisfied. Then the positive equilibrium* $u(t) \equiv K$ *of* (3.1) *is asymptotically stable.*

Proof Under (P_1), $w = 0$ is a solution of (3.1) and $\frac{d}{dw} g(b(w))|_{w=0} = g'(0)b'(0) \geq 0$. The eigen-equation at the zero solution is $(\lambda + d_m)e^{\lambda r} - g'(0)b'(0) = 0$.

Assuming (P_3'), we have $d_m > g'(0)b'(0)$. Let $A := d_m, B := -g'(0)b'(0)$. One can verify that $A > -1$, $A + B > 0$ are satisfied. Moreover, since the solution of $\eta = -d_m \tan(\eta)$ satisfies $\eta \in (\frac{\pi}{2}, \pi)$, we have $\cos\eta < 0$ and thus

$$\eta \sin\eta - A\cos\eta = -\frac{d_m}{\cos\eta}(\sin^2\eta + \cos^2\eta) = -\frac{d_m}{\cos\eta} > -g'(0)b'(0).$$

By Proposition 3.1, every root of $(\lambda + d_m)e^{\lambda r} - g'(0)b'(0) = 0$ has negative real part. Therefore, we have conclusion (i).

Assuming (P_2)–(P_3), let $\epsilon > 0$ be an arbitrary constant. Then we have

$$\begin{aligned} g'(b(K))b'(K) &= \frac{d}{du} g(b(u))|_{u=K} = \lim_{\epsilon\to 0} \frac{g(b(K+\epsilon)) - g(b(K))}{\epsilon} \\ &< \lim_{\epsilon\to 0} \frac{d_m(K+\epsilon) - d_m K}{\epsilon} = d_m. \end{aligned}$$

Note that K is the unique positive equilibrium of (3.1) and $g'(b(K))b'(K) \geq 0$. The eigen-equation at $u(t) \equiv K$ is $(\lambda + d_m)e^{\lambda r} - g'(b(K))b'(K) = 0$. Let $A := d_m, B := -g'(b(K))b'(K)$. Similar to the above argument, we know that conclusion (ii) holds. ∎

Let $\|\varphi\| := \max_{\theta\in[-r,0]} \sup_{x\in\mathbb{R}} \varphi(\theta, x)$ and

$$\mathcal{C}^+_{\delta,M} := \Big\{ \varphi \in \mathcal{C}^+ : \|\varphi\| \leq M \text{ and } \inf_{x\in\mathbb{R}} \varphi(0,x) \geq \delta \Big\}$$

for any given constants $\delta \geq 0$ and $M > 0$.

In this section, we shall assume that (J), (K), and (P_1) hold without any extra illustration.

3.1 Stability of Zero Solution

If there exists no positive equilibrium, the following theorem says that the trivial equilibrium is globally stable regardless of the monotonicity of $g(b(\cdot))$.

Theorem 3.3 *Assume that* (P_3') *holds. Then for any* $M > 0$,

$$\lim_{t\to\infty} \|w(t, \cdot; \varphi)\| = 0 \text{ uniformly for } \varphi \in \mathcal{C}^+_{0,M}.$$

Furthermore, the zero solution of (1.1) *is globally asymptotically stable.*

Proof First assume that $\delta > 0$ and $M > \delta$ are arbitrary and $\varphi \in \mathcal{C}^+_{\delta,M}$. From Lemma 2.3 and Remark 2.2, we have $\delta e^{-d_m t} < w(t,x;\varphi) \le M$ for any $(t,x) \in [0,\infty)\times\mathbb{R}$. In view of condition (P_1) and (P'_3), $g(b(\cdot))$ is either non-decreasing or non-monotone on $[0,\infty)$.

If $g(b(\cdot))$ is non-decreasing on $[0,\infty)$, by Lemma 2.4, we have

$$0 \le w(t,x;\varphi) \le u(t;M) \text{ for } (t,x) \in [0,\infty)\times\mathbb{R},$$

where $u(t;M)$ is the unique solution of the following initial value problem:

$$\begin{cases} \frac{du(t)}{dt} = -d_m u(t) + g(b(u(t-r))), \ t > 0, \\ u(s) = M, \ s \in [-r,0]. \end{cases} \tag{3.2}$$

Since $\lim_{t\to\infty} u(t;M) = 0$ (by using the assumption (P'_3) and the fluctuation lemma, see the argument in [15, p. 69]), the conclusion $\lim_{t\to\infty}\|w(t,\cdot;\varphi)\| = 0$ is true and uniform for $\varphi \in \mathcal{C}^+_{\delta,M}$.

Assume that $g(b(\cdot))$ is non-monotone on $[0,\infty)$. Let

$$B^+(u) := \max_{v\in[0,u]} g(b(v)), \ u \in [0,\infty). \tag{3.3}$$

It is easy to see that $B^+(0) = 0$, $B^+(u)$ is non-decreasing on $[0,\infty)$, $B^+(u) \ge g(b(u))$ for any $u \ge 0$, and $B^+(u) < d_m u$ for any $u > 0$. Replace u and $g(b(\cdot))$ in (3.2) with u^+ and $B^+(\cdot)$, respectively. Then we know that the corresponding solution $u^+(t;M)$ satisfies $\lim_{t\to\infty} u^+(t;M) = 0$. Let $v(t) := u^+(t;M) - w(t,x;\varphi)$ for $(t,x) \in [-r,\infty)\times\mathbb{R}$. Then $v(t) \ge 0$ for $(t,x) \in [-r,0]\times\mathbb{R}$, and we can obtain on $(t,x) \in [0,r]\times\mathbb{R}$ that

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t} &= D_m \mathcal{A} v(t,x) - d_m v(t,x) + B^+(u^+(t-r)) - g\big(b(u(t-r,x;\varphi))\big) \\ &\ge D_m \mathcal{A} v(t,x) - d_m v(t,x) + B^+(u^+(t-r)) - B^+(u(t-r,x;\varphi)) \\ &\ge D_m \mathcal{A} v(t,x) - d_m v(t,x),\end{aligned}$$

which implies (2.3) on $(t,x) \in [0,r]\times\mathbb{R}$. Inductively, we have (2.3) on $[0,\infty)\times\mathbb{R}$, which leads to $u^+(t;M) \ge w(t,x;\varphi)$ for $(t,x) \in [0,\infty)\times\mathbb{R}$. That is,

$$\lim_{t\to\infty}\|w(t,\cdot;\varphi)\| = 0$$

for $\varphi \in \mathcal{C}^+_{\delta,M}$. In fact, since $\lim_{t\to\infty} u^+(t;M) = 0$ is independent on $x \in \mathbb{R}$ and $\varphi \in \mathcal{C}^+_{\delta,M}$, this convergence of $w(t,x;\varphi)$ is uniform for $x \in \mathbb{R}$ and $\varphi \in \mathcal{C}^+_{\delta,M}$.

Note that $\delta > 0$ is arbitrary. Letting $\delta \to 0$, we, in fact, obtain that

$$\lim_{t\to\infty}\|w(t,\cdot;\varphi)\| = 0 \text{ uniformly for } \varphi \in \mathcal{C}^+_{0,M}.$$

In view of Lemma 3.2 (i), we obtain the asymptotic stability of the zero solution of (1.1). Combining with the global attractivity, we have the global asymptotic stability of the zero solution for (1.1). ■

3.2 Stability of Positive Equilibrium

If there exists a positive equilibrium K, we shall discuss its stability in the monotone case (Theorem 3.4) and nonmonotone cases (Theorem 3.5), respectively.

Theorem 3.4 *Assume that* (P_2)–(P_3) *hold. Then for any* $\delta > 0$ *and* $M > 0$,

$$\lim_{t\to\infty} \|w(t,\cdot;\varphi) - K\| = 0 \text{ uniformly for } \varphi \in \mathcal{C}^+_{\delta,M}.$$

Furthermore, the positive equilibrium $w \equiv K$ *of* (1.1) *is globally asymptotically stable.*

Proof If (P_3) holds, then from Lemma 2.3 and Remark 2.2, we have

$$\delta e^{-d_m t} \le w(t,x;\varphi) \le \overline{K} = \max\{K, M\}$$

for any $(t,x) \in [0,+\infty) \times \mathbb{R}$ and $\varphi \in \mathcal{C}^+_{\delta,M}$. Let $\overline{w}(t)$ and $\underline{w}(t)$ solve the following problems:

$$\begin{cases} \frac{d\overline{w}(t)}{dt} = -d_m \overline{w}(t) + g(b(\overline{w}(t-r))), \ t > r, \\ \overline{w}(s) = \overline{K}, \ s \in [0,r], \end{cases}$$

and

$$\begin{cases} \frac{d\underline{w}(t)}{dt} = -d_m \underline{w}(t) + g(b(\underline{w}(t-r))), \ t > r, \\ \underline{w}(s) = \delta e^{-d_m r}, \ s \in [0,r], \end{cases}$$

respectively. It then follows from Lemma 2.4 that $\underline{w}(t) \le w(t,x;\varphi) \le \overline{w}(t)$ for $(t,x) \in [r,\infty) \times \mathbb{R}$. Moreover, using Kuang [7, Theorem 9.1], we have $\lim_{t\to\infty} \underline{w}(t) = \lim_{t\to\infty} \overline{w}(t) = K$, and hence the first assertion follows.

In view of Lemma 3.2 (ii), we obtain the asymptotic stability of the positive equilibrium of (1.1). Combining with the global attractivity, we have the global asymptotic stability of the positive equilibrium for (1.1). ■

Theorem 3.5 *Let* $\overline{\theta} := \frac{1}{d_m} g(b(w_{\max}))$. *Assume that* (P_2') *and* (P_3) *are satisfied. Furthermore, assume that one of the following conditions holds:*

(i) $K \le w_{\max}$;

(ii) $K > w_{\max}$, $\frac{1}{d_m} g(b(\overline{\theta})) > w_{\max}$ *with one of the following:*

(a) $w g(b(w))$ *is strictly increasing on* $(0, \overline{\theta}]$,

(b)

$$\frac{1}{d_m} g(b(w)) \begin{cases} < 2K - w, & \text{if } w \in [w_{\max}, K), \\ \ge 2K - w, & \text{if } w \in [K, 2K]. \end{cases}$$

Then for any $\delta > 0$ *and* $M > 0$, *we have* $\lim_{t\to\infty} \|w(t,\cdot;\varphi) - K\| = 0$ *uniformly for* $\varphi \in \mathcal{C}^+_{\delta,M}$.

Proof Let $\overline{K} := \max\{M, \overline{\theta}\}$. From Lemma 2.3 and Remark 2.2, we have $\delta e^{-d_m t} \le w(t,x;\varphi) \le \overline{K}$ for any $(t,x) \in [0,+\infty) \times \mathbb{R}$ and $\varphi \in \mathcal{C}^+_{\delta,M}$. Let $W_1(t)$ be the solution of

$$\begin{cases} \frac{dW_1(t)}{dt} = -d_m W_1(t) + g(b(w_{\max})), & t > r, \\ W_1(s) = \overline{K}, & s \in [0,r]. \end{cases} \tag{3.4}$$

in view of Lemma 2.5, for any $\varphi \in \mathcal{C}^+_{\delta,M}$, the solution $w(t,x;\varphi)$ of (1.1) satisfies $0 \le w(t,x;\varphi) \le W_1(t)$ for $(t,x) \in [0,\infty) \times \mathbb{R}$. This leads to

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} w(t,x;\varphi) \le \lim_{t\to\infty} W_1(t) = \overline{\theta}.$$

(i) If $K < w_{\max}$, then by (P_3), we have $K = \frac{1}{d_m} g(b(K)) \le \overline{\theta} < w_{\max}$. Therefore, there exists $T > 0$ such that $w(t,x;\varphi) \le W_1(t) < w_{\max}$ for $t \ge T$. Noting that $g(b(\cdot))$ is non-decreasing on $[0, w_{\max}]$, similar to Theorem 3.4, one can show $\lim_{t\to\infty} | w(t,\cdot;\varphi) - K\| = 0$ uniformly for $\varphi \in \mathcal{C}^+_{\delta,M}$.

If $K = w_{\max}$, we define $B^+(w)$ as in (3.3). Let $u^+(t;M)$ be the solution of

$$\begin{cases} \frac{du(t)}{dt} = -d_m u(t) + B^+(u(t-r)), & t > 0,\\ u(s) = M, & s \in [-r, 0]. \end{cases}$$

Then in view of [7, Theorem 9.1], we have $\lim_{t\to\infty} u^+(t;M) = K$. Furthermore, for any given $\epsilon \in (0,1)$, there exists $T_1 > 0$ such that $w(t,x;\varphi) \le u^+(t;M) \le K + \epsilon$ for $t \ge T_1$. Define $B^-_\epsilon(w) := \min\{g(b(w)), g(b(K+\epsilon))\}$ for $w \in [0, K+\epsilon]$. Note that $B^-_\epsilon(w)$ is non-decreasing on $[0, K+\epsilon]$ and $B^-_\epsilon(w) = d_m w$ admits a unique solution K_ϵ with $0 < K - K_\epsilon \to 0$ as $\epsilon \to 0$. Let $u^-(t)$ be the solution of

$$\begin{cases} \frac{du(t)}{dt} = -d_m u(t) + B^-_\epsilon(u(t-r)), & t > T_1 + r,\\ u(s) = \delta e^{-d_m(T_1+r)}, & s \in [T_1, T_1 + r]. \end{cases}$$

Then similarly, we have $w(t,x;\varphi) \ge u^-(t)$ for $t \ge T_1$ and $\lim_{t\to\infty} u^-(t) = K_\epsilon$. Summarizing the above argument, we obtain $u^-(t) \le w(t,x;\varphi) \le u^+(t;M)$ for $t \ge T_1$. Therefore, the conclusion follows.

(ii) Assume $K > w_{\max}$ and $\frac{1}{d_m} g(b(\overline{\theta})) > w_{\max}$. Let

$$\underline{B}(w) := \min\{g(b(w)), g(b(\overline{\theta}))\}, \quad \theta \text{ satisfies } \theta = \frac{1}{d_m}\underline{B}(\theta).$$

Then $\underline{B}(w)$ is non-decreasing on $[0, \overline{\theta}]$ and decreasing on $[\overline{\theta}, \infty)$. Furthermore,

$$w_{\max} < \theta = \frac{1}{d_m} g(b(\overline{\theta})) < \frac{1}{d_m} g(b(K)) = K < \overline{\theta}. \tag{3.5}$$

Comparing $W_1(t)$ (see (3.4)), $w(t,x;\varphi)$ (see (1.1)) with solution $V_1(t)$ of the following problem

$$\begin{cases} \frac{dV_1(t)}{dt} = -d_m V_1(t) + \underline{B}(V_1(t-r)), & t > r,\\ V_1(s) = \delta e^{-d_m r}, & s \in [0, r], \end{cases} \tag{3.6}$$

we obtain $0 < V_1(t) \le w(t,x;\varphi) \le W_1(t)$ for $(t,x) \in (r,\infty)\times\mathbb{R}$, $\varphi \in \mathcal{C}^+_{\delta,M}$. Moreover, in view of the property of $\underline{B}(w)$ and the conclusion of (i), we have $\lim_{t\to\infty} V_1(t) = \theta$. Note that $\lim_{t\to\infty} W_1(t) = \overline{\theta}$. Therefore, this together with (3.5) leads to the existence of $T_2 > 0$ such that

$$u_{\max} < \underline{\theta} := \frac{1}{2}[u_{\max} + \theta] < V_1(t) < K, \quad K < W_1(t) < \frac{3}{2}\overline{\theta} \quad \text{for } t \ge T_2.$$

We now construct a sequence of pairs of sub- and super-solutions of (1.1):

$$\begin{cases} \frac{dV_n(t)}{dt} = -d_m V_n(t) + g(b(W_{n-1}(t-r))), & t > T_2 + r,\\ \frac{dW_n(t)}{dt} = -d_m W_n(t) + g(b(V_{n-1}(t-r))), & t > T_2 + r,\\ V_n(s) = \underline{\theta},\ W_n(s) = \overline{K}, & s \in [T_2, T_2 + r], \end{cases}$$

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where $\overline{K} = \max\{M, \overline{\theta}\}$. Since $g(b(\cdot))$ is decreasing on $(u_{\max}, \infty)$, one can show (see also [14]) that

$$V_1(t) \le \cdots \le V_{n-1}(t) \le V_n(t) \le K,$$
$$w(t, x; \varphi) \le W_n(t) \le W_{n-1}(t) \le \cdots \le W_1(t)$$

for $t > T_2 + r$.

By mathematical induction, we know that $\lim_{t\to\infty} W_n(t)$ and $\lim_{t\to\infty} V_n(t)$ exist. Set $W_n^* := \lim_{t\to\infty} W_n(t)$, $V_n^* := \lim_{t\to\infty} V_n(t)$. Then $d_m V_n^* = g((W_{n-1}^*))$, $d_m W_n^* = g(b(V_{n-1}^*))$. Let $V^* := \lim_{n\to\infty} V_n^*$ and $W^* := \lim_{n\to\infty} W_n^*$. Then we have $d_m V^* = g(b(W^*))$, $d_m W^* = g(b(V^*))$, and $V^* \le K \le W^* \le \overline{\theta}$.

We could show that either (a) or (b) can lead to $W^* = K = V^*$, but we omit the details [15, p. 79]. ■

In Theorem 3.5, for the case $K > w_{\max}$, without other assumptions in (ii), we can obtain the uniform persistence of system (1.1). Let $K^* := \overline{\theta} + 1$ and $K_* \in (0, K)$ with $B_-(K_*) = d_m K_*$ and

$$B_-(w) > d_m w \text{ for } w \in (0, K_*), \quad B_-(w) < d_m w \text{ for } w \in (K_*, \infty),$$

where $B_-(w)$ is defined by

$$B_-(w) := \begin{cases} \min_{u\in[w,K^*]} g(b(u)), & w \in [0, K^*], \\ g(b(w)), & w \in (K^*, \infty). \end{cases}$$

It is obvious that $B_-(w)$ is non-deceasing on $[0, K^*]$, and $g(b(w)) \ge B_-(w)$ on $[0, \infty)$.

Theorem 3.6 *Assume that* (P_2'), (P_3) *and* $K > w_{\max}$ *are satisfied. Then for any* $\varphi \in \mathcal{C}^+$ *with* $\inf_{x\in\mathbb{R}} \varphi(0, x) > 0$, *we have*

$$K_* \le \liminf_{t\to\infty} \inf_{x\in\mathbb{R}} w(t, x; \varphi) \le \limsup_{t\to\infty} \sup_{x\in\mathbb{R}} w(t, x; \varphi) \le \overline{\theta}.$$

Proof For any given $\varphi \in \mathcal{C}^+$, let $\delta := \inf_{x\in\mathbb{R}} \varphi(0, x)$, $M := \sup_{(s,x)\in[-r,0]\times\mathbb{R}} \varphi(s, x)$, and $\overline{K} := \max\{M, \overline{\theta}\}$. From Lemma 2.3 and Remark 2.2, we have

$$\delta e^{-d_m t} \le w(t, x; \varphi) \le \overline{K} \quad \text{for any } (t, x) \in [0, +\infty) \times \mathbb{R}.$$

Let $W_1(t)$ be the solution of (3.4) and $V_1(t)$ be the solution of (3.6) with $V_1(s) = \min\{K_*, \delta e^{-d_m r}\}$ on $s \in [0, r]$. Then the solution $w(t, x; \varphi)$ of (1.1) satisfies $V_1(t) \le w(t, x; \varphi) \le W_1(t)$ for $(t, x) \in [r, \infty) \times \mathbb{R}$. This leads to

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}} w(t, x; \varphi) \le \lim_{t\to\infty} W_1(t) = \overline{\theta}.$$

On the other hand, there exists $T_3 > 0$ such that $0 < V_1(t) \le \overline{\theta} + 1$ for $t \ge T_3$. Consider the solution $v(t)$ of

$$\begin{cases} \frac{dv(t)}{dt} = -d_m v(t) + B_-(v(t - r)), & t > T_3 + r, \\ v(s) = \min\{K_*, \min_{s\in[T_3,T_3+r]} V_1(s)\}, & s \in [T_3, T_3 + r]. \end{cases}$$

In view of the property of $B_-(w)$, we know $\lim_{t\to\infty} v(t) = K_*$. Furthermore, by comparison, we obtain $v(t) \le V_1(t) \le w(t,x;\varphi)$ for $t \ge T_3 + r$. Therefore, $K_* \le \liminf_{t\to\infty} \inf_{x\in\mathbb{R}} w(t,x;\varphi)$. ∎

4 Application and Discussions

Let $u(t,a,x)$ be the density of individuals with age a at a point x and time t. Let $r \ge 0$ be the length of the juvenile period. Let $\varepsilon = \exp\{-\int_0^r d_j(a)da\}$, $\alpha = \int_0^r D_j(a)da$, and the density of mature (or adult) individuals at point x and time t be denoted by $w(t,x) = \int_r^\infty u(t,a,x)da$. Based on the von Foerster type equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} = D_j(a) \displaystyle\int_{\mathbb{R}} J(x-y)[u(t,a,y) - u(t,a,x)]dy - d_j(a)u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } a \in [0,r], t > 0, x \in \mathbb{R}, \\ u(t,0,x,y) = b(w(t,x,y)) \quad \text{for } t \ge -r, x \in \mathbb{R}. \end{cases}$$

Then $w(t,x)$ satisfies

$$\frac{\partial w}{\partial t} = D_m \mathcal{A} w(t,x) - d_m w + \varepsilon \int_{\mathbb{R}} G(x-y,\alpha) b(w(t-r,y))dy, \quad t > 0, x \in \mathbb{R}. \tag{4.1}$$

Here $b(w)$ and $d_m w$ are the birth and mortality rates of mature individuals, respectively, $d_j(a) = d(a)(a \in [0,r])$ denotes the per capita mortality rate of juveniles at age a, $D_j(a)(a \in [0,r])$ and constant D_m $(a > r)$ are the diffusion coefficients of juveniles and maturities, respectively (see [8] for model derivation).

Assume that $J(x)$ satisfies (J). Then in view of [8, Lemma 3.1], $G(x,\alpha)$ satisfies (K). It is obviously that we obtain (4.1) from (1.1) by taking $g(z) = \varepsilon z$ and $k(x) = G(x,\alpha)$, and (1.2)–(1.4) are three special cases of (4.1) satisfying (J). By using εz to replace $g(z)$ in all assumptions (P_1)–(P_3) and (P_2')–(P_3'), we can immediately obtain corollaries for (4.1) from Theorems 3.3–3.6. We omit the details of the statements.

We want to mention a remark on model (1.4) with convection influence, where $k(x)$ in (1.4) is a parallel translation of kernel function

$$\frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}.$$

Our conclusion can be explained as follows: the convection has no effect on the stability of the system. Furthermore, from our arguments and results in this article, we also conclude that the delay r and the nonlocal diffusion have nothing to do with the stability of (1.1).

The other dynamical properties for equation (1.1) will be investigated in further research.

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