But

$$p(x+1) - p(x) = \sum_{j=0}^{k} a_j \left\{(x+1)^j - x^j\right\},$$

and one readily observes that $p(x+1) - p(x) = \pm 1$ holds only if $a_1 = \pm 1$ and $a_j = 0$ for $2 \leqslant j \leqslant k$. Therefore $p(x) = x + a$ or $p(x) = -x + a$, for some real constant a and it can be checked that these polynomials indeed satisfy the given functional equation.

We leave the reader with the following question to settle.

Determine all polynomials p with real coefficients that satisfy

$$p\left(x^2\right) = 1 + p(x-1)\,p(x+1)$$

for all real x.

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PRITHWIJIT DE
Homi Bhabha Centre for Science Education,
Tata Institute of Fundamental Research, Mumbai, India
e-mail: *de.prithwijit@gmail.com; prithwijit@hbcse.tifr.res.in*
SUTANAY BHATTACHARYA
B.S. Student, Indian Institute of Science,
Bengaluru, India
e-mail: *sutanay01@gmail.com*

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106.18 Impossibility of solving the quintic using Cardano's solution

In [1], the author formulated a modified version of Cardano's solution of the cubic using the binomial theorem of the third power and presented an alternative solution of the quartic by the modified method using the trinomial theorem of the fourth power. In the present paper, it is shown that solving the quintic and higher degree polynomial equations by the modified method using the quadrinomial theorem of the fifth power and the $(n-1)$th multinomial theorem of the nth power for $n > 5$, respectively, is impossible.

Cardano's solution of the cubic $x^3 + p_1x + p_2 = 0$ was found by del Ferro and Tartaglia, and published by Cardano in 1545 in his book *Ars Magna*. We show the solution in the following (e.g. [2]). In the cubic $x^3 + p_1x + p_2 = 0$, let $x = u_1 + u_2$. Then we have

$$u_1^3 + u_2^3 + \left(3u_1u_2 + p_1\right)\left(u_1 + u_2\right) + p_2 = 0.$$

Finding u_1 and u_2 such that $3u_1u_2 + p_1 = 0$ and $u_1^3 + u_2^3 + p_2 = 0$, we obtain Cardano's formulas:

$$x = \omega^k \sqrt[3]{-\frac{p_2}{2} + \sqrt{\frac{p_2^2}{4} + \frac{p_1^3}{27}}} + \omega^{3-k} \sqrt[3]{-\frac{p_2}{2} - \sqrt{\frac{p_2^2}{4} + \frac{p_1^3}{27}}}$$

with $k = 0, 1, 2$ where ω is a primitive cube root of unity and the product of the two cube roots is equal to $-\frac{p_1}{3}$.

We modify the above Cardano's method as follows. From the binomial theorem of the third power

$$(u_1 + u_2)^3 = \sum_{i=0}^{3} \binom{3}{i} u_1^{3-i} u_2^i,$$

we get the reduced cubic

$$x^3 + p_1 x + p_2 = 0 \tag{1}$$

with

$$p_1 = -3u_1u_2, \qquad p_2 = -u_1^3 - u_2^3 \tag{2}$$

where one of its solutions is $x = u_1 + u_2$. Then, from (2), we obtain the Cardano's formulas of (1) by his method.

Next, we show how to solve the quartic by the modified method using the trinomial theorem of the fourth power instead of the binomial theorem of the third power. From the trinomial theorem of the fourth power

$$(u_1 + u_2 + u_3)^4 = \sum_{i_1 + i_2 + i_3 = 4} \binom{4}{i_1,\, i_2,\, i_3} u_1^{i_1} u_2^{i_2} u_3^{i_3},$$

we get the reduced quartic

$$x^4 + p_1 x^2 + p_2 x + p_3 = 0 \tag{3}$$

with

$$u_1^2 + u_2^2 + u_3^2 = -\frac{p_1}{2}, \quad u_1^2u_2^2 + u_2^2u_3^2 + u_3^2u_1^2 = \frac{p_1^2 - 4p_3}{16}, \quad u_1^2u_2^2u_3^2 = \frac{p_2^2}{64} \tag{4}$$

where one of its solutions is $x = u_1 + u_2 + u_3$. Then, from (4), we have the resolvent cubic

$$X^3 + \frac{p_1}{2}X^2 + \frac{p_1^2 - 4p_3}{16}X - \frac{p_2^2}{64} = 0$$

which is the same as that of [3, p. 173]. Solving this cubic and finding u_1, u_2, u_3 with $u_1u_2u_3 = -\frac{1}{8}p_2$, we obtain the solutions of (3).

Let us consider solving the reduced quintic by the modified version of Cardano's solution of the cubic using the quadrinomial theorem of the fifth power. We transform the quadrinomial theorem of the fifth power

$$(u_1 + u_2 + u_3 + u_4)^5 = \sum_{i_1 + i_2 + i_3 + i_4 = 5} \binom{5}{i_1,\, i_2,\, i_3,\, i_4} u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4} \tag{5}$$

into the identity

$$\begin{aligned}(u_1 + u_2 + u_3 + u_4)^5 = A_{(1,l)}&(u_1 + u_2 + u_3 + u_4)^3 \\ &+ A_{(2,l)}(u_1 + u_2 + u_3 + u_4)^2 \\ &+ A_{(3,l)}(u_1 + u_2 + u_3 + u_4) + A_{(4,l)}\end{aligned} \tag{6}$$

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where $A_{(k,l)}$ $(k = 1, 2, 3, 4$ and $l = 2, 3, 4, 5)$ are certain polynomials consisting of

$$r_{(1,l)} = u_1^l + u_2^l + u_3^l + u_4^l,$$
$$r_{(2,l)} = u_1^l u_2^l + u_1^l u_3^l + u_1^l u_4^l + u_2^l u_3^l + u_2^l u_4^l + u_3^l u_4^l,$$
$$r_{(3,l)} = u_1^l u_2^l u_3^l + u_1^l u_2^l u_4^l + u_1^l u_3^l u_4^l + u_2^l u_3^l u_4^l,$$
$$r_{(4,1)} = u_1 u_2 u_3 u_4.$$

The degree of the quadrinomial $r_{(3,l)}$ is $3l$. For $l = 2, 3, 4, 5$, the degrees are 6, 9, 12, 15, respectively, which are all more than 5. For $l = 1$ in (6), we do not have the resolvent quartic because $x = u_1 + u_2 + u_3 + u_4$ is unknown. These facts show that we cannot transform (5) into (6). Setting

$$x = u_1 + u_2 + u_3 + u_4$$

and

$$p_k = -A_{(k,l)} \qquad (k = 1, 2, 3, 4)$$

in (6), we get the reduced quintic

$$x^5 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = 0$$

where one of its solutions is $x = u_1 + u_2 + u_3 + u_4$. Thus this quintic cannot be solved by the modified method using the quadrinomial theorem of the fifth power.

We close this paper with showing the impossibility of solving the general monic reduced equation of degree n by the modified method using the $(n - 1)$th multinomial theorem of the nth power for $n > 4$. We transform the $(n - 1)$th multinomial theorem of the nth power

$$(u_1 + u_2 + \cdots + u_{n-1})^n = \sum_{i_1 + i_2 + \cdots + i_{n-1} = n} \binom{n}{i_1, i_2, \cdots, i_{n-1}} u_1^{i_1} u_2^{i_2} \cdots u_{n-1}^{i_{n-1}} \qquad (7)$$

into the identity

$$(u_1 + u_2 + \cdots + u_{n-1})^n = B_{(1,l)}(u_1 + u_2 + \cdots + u_{n-1})^{n-2} + B_{(2,l)}(u_1 + u_2 + \cdots + u_{n-1})^{n-3} + \cdots + B_{(n-1,l)} \qquad (8)$$

where $B_{(k,l)}$ $(k = 1, 2, \cdots, n - 1$ and $l = 2, \cdots, n)$ are certain polynomials consisting of

$$s_{(1,l)} = u_1^l + u_2^l + \cdots + u_{n-1}^l,$$
$$s_{(2,l)} = u_1^l u_2^l + u_1^l u_3^l + \cdots + u_{n-2}^l u_{n-1}^l,$$
$$\cdots\cdots$$
$$s_{(n-2,l)} = u_1^l u_2^l \cdots u_{n-2}^l + u_1^l u_2^l \cdots u_{n-3}^l u_{n-1}^l + \cdots + u_2^l u_3^l \cdots u_{n-1}^l,$$
$$s_{(n-1,1)} = u_1 u_2 \cdots u_{n-1}.$$

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The degree of the polynomial $s_{(n-2,l)}$ is $(n - 2)l$. We consider the case of $l = 2$. If $2n - 4 > n$ (that is $n > 4$), then we cannot transform (7) into (8). In general, if $(n - 2)l > n$ (that is $n > \frac{2l}{l-1}$), then we cannot transform (7) into (8). Setting

$$x = u_1 + u_2 + \cdots + u_{n-1}$$

and

$$p_k = -B_{(k,l)} \qquad (k = 1, 2, \cdots, n - 1)$$

in (8), we get the reduced equation of degree n

$$x^n + p_1x^{n-2} + p_2x^{n-3} + \cdots + p_{n-2}x + p_{n-1} = 0$$

where one of its solutions is $x = u_1 + u_2 + \cdots + u_{n-1}$. Thus the equation of degree n cannot be solved by our modified method using the $(n - 1)$th multinomial theorem of the nth power for $n > 4$.

In the above, we have solved (1) by the modified method using (8) when $n = 3$ and $l = 3$. Also, we have obtained the solutions of (3) using (8) when $n = 4$ and $l = 2$ similarly. On the other hand, we have shown that the quintic cannot be solved using (8) when $n = 5$ and $l = 2, 3, 4, 5$.

As a result of formulating the modified Cardano's solution of the cubic, we can show suitably and simply to beginners and experts in other areas of mathematics the essential difference between the cubic and quartic equations and the quintic and higher degree polynomial equations.

It is noted that we can solve the reduced quadratic by the modified method using (8) when $n = 2$ and $l = 1$. However the solution by the modified method is more complicated than the usual one.

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HIROSHI OHYAMA
Graduate School of Science and Technology, Niigata University,
8050 Ikarashi 2-no-cho, Nishi-ku, Niigata, Japan
e-mail: *piron_0213@yahoo.co.jp*

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