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NOTES

OPTIMAL ECONOMIC GROWTH:

Test of Income/Wealth Conservation Laws in OECD Countries

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This paper attempts to derive several economic conservation laws and to test the validity of the optimal growth models using the income/wealth ratios for the United States, Japan, and other OECD countries. Conservation laws vary with the type of the objective function depicted, such as the maximization of the aggregate consumption or the maximization of per-capita consumption. The operational concept of "wealth-like quantity" is identified, although the Goldsmith-Kendrick standard definition of "net national wealth" should not always be used. The last section of the paper takes up an empirical analysis to determine how different economies have achieved long-term (optimal) growth. The U.S. economy has been operating rather efficiently, whereas the Japanese economy, after the oil shocks of the 1970's, has behaved differently, leading into the bubble period of the early 1990's.

Keywords: Dynamic Optimization, Intertemporal Choice and Growth, Macro Analysis of Economic Development

1. INTRODUCTION

For more than half a century, the theory of optimal economic growth has occupied a major position in modern economic analysis. Starting with the pioneering works of Ramsey (1928), Von Neumann (1945–1946), Samuelson and Solow (1956), Solow (1956), Cass (1965), and Sato and Davis (1971), to name a few, the theory of optimal economic growth has grown to include the more recent theory of "endogenous growth" (optimal or not optimal) of Paul Romer (1986), Robert Lucas (1988), and others. It is an old topic, but one that still presents new challenges for economists.

In this traditional approach, one major concern arises from determining the nature of appropriate choices in investment and how these decisions will influence

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the economy's path of long-run development. This method of analysis involves a typical application of "optimal" growth theory.

Another application of the optimal growth theory occurs when economists and statisticians attempt to justify and estimate the single index of "national income" from a theoretical point of view. This problem has also attracted the attention of many economists, including Weitzman (1976), Sato (1981), Sato and Maeda (1990), and Kemp and Long (1992), among others. It has been shown that the best justification for national income (NNP or GNP) is the welfare measurement of the present discounted value of future consumption in an "optimally growing economy" [see Weitzman (1976)]. Although these two applications of the optimal growth theory appear to be unrelated, the problems of optimal capital accumulation and of calculating the correct and justifiable index of a nation's income can be seen as two sides of the same coin, that is, "optimality."

Attempts have been made to empirically test the validity of optimal growth models by comparing the simulated optimal time paths for the capital/labor ratio and the saving ratio with the actual time paths of these ratios [e.g., Sakakibara (1970) and Lenard (1972)]. Sakakibara and Lenard attempted to determine whether optimal growth models have any value as empirical tools for explaining economic development.

However, little empirical work has been done to test the validity of the second aspect of the growth theory, namely the testing of an appropriate measure of national income. The correct measure of NNP and GNP is considered "impractical" since wealthlike measures of the present discounted value of future consumption may not have an operational counterpart.

The purpose of this paper is to unify both aspects of the optimal growth theory with a general theory of "economic conservation laws." The theory of conservation law involves the identification and discovery of hidden invariant quantities in a dynamic system. In this dynamic economic system, as in a dynamic physical system, it is suspected that a certain variable remains unchanged during its process of evolution, as long as the system follows an optimal trajectory. In a growing economy, the variable that is invariant is called the conservation law [see Sato (1981)]. By uncovering the existence of conservation laws and by formulating the operational concepts associated with them, one can test the validity of both the optimal growth models as well as the measurement of national income. It has been shown that in an optimally controlled economy, the ratio of income to wealth remains invariant. This is the income/wealth conservation law.

1.1. Warning

The income/wealth conservation law states that the ratio of income to wealth should be constant over the entire period $0 \leq t \leq \infty$, if the economy is on an optimal path. This should not be confused with the standard concept of the income/capital ratio.

In the present model, like any other models of optimal growth, the income/capital ratio is not constant unless the economy is in the steady state. This conservation law guarantees that the income/wealth ratio is constant regardless of whether the optimally controlled economy is inside or outside the steady state.

1.2. Brief Summary

In the next section the basic model is presented. Conservation laws vary with the type of the objective function depicted, such as the maximization of the aggregate consumption or the maximization of per-capita consumption. It also depends upon the existence of exogenous factors, such as the exogenously growing labor force and/or technical change. In addition, the operational concept of "wealth-like quantity" is identified, although one should not always use the standard definition of "net national wealth" of Goldsmith and Kendrick (1976). The last section of the paper takes up an empirical analysis to determine how different economies have achieved long-term (optimal) growth. It indirectly verifies the correct and justifiable index of national income. We employ data from major OECD countries including the United States, Japan, Great Britain, Canada, and Germany.

Applications of various models illustrate that the U.S. economy has been operating rather efficiently along its optimal paths, despite the problems of business cycles and other maladjustments. The income/wealth ratio has remained almost constant for more than a century, with a distinct shift upward after World War II. On the other hand, the Japanese economy, after the oil shocks of the 1970's, has behaved rather differently. The income/wealth ratio has consistently declined, reaching the minimal value during the bubble period of the early 1990's. The economies of most of the other OECD countries, except those of Italy and Greece, have behaved similarly to that of the United States.

2. MODEL

Let consumption of the economy depend on output, which in turn depends on a vector of capital goods, a vector of investment and labor input, so that

$$C = C[Y(K; \dot{K}; L; \dot{L})], \tag{1}$$

where $Y =$ output, which depends on $K = (K_1, \ldots, K_n) = n$ capital goods, $\dot{K} = (\dot{K}_1, \ldots, \dot{K}_n) = dK/dt =$ investment and L labor input, exogenously given by $\dot{L} = \lambda L$. [See Liviatian and Samuelson (1969).]

Let Y be a homogeneous function of the first degree with respect to its arguments $K, \dot{K}, L$, and $\dot{L}$, together with $\partial Y/\partial K_i > 0$, $i = 1, \ldots, n$, $\partial Y/\partial L > 0$, $\partial Y/\partial \dot{K}_i$, ≤ 0, $i = 1, \ldots, n$, $\partial Y/\partial \dot{L} \leq 0$. Also assume that $dC/dY > 0$ and Y is concave with respect to its argument. A special function of (1) is a separable case with $dC/dY = 1$.

$$C = Y(K, L) - G(\dot{K}, \dot{L}; K, L) \tag{2}$$

or simply

$$C = Y(K, L) - G(\dot{K}, \dot{L}) \tag{3}$$

The simplest case of (3) is the standard model of saving = investment; that is,

$$C = Y(K, L) - \dot{K} \qquad (i = 1). \tag{3a}$$

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Another example of (3) may be a well-known form [(Caton and Shell (1971), Sato (1981), and Samuelson (1990)]

$$C = \prod_{i=1}^{n} K_i^{\alpha_i} L^{1-\Sigma\alpha_i} \left(\frac{1}{2} \sum_{i=1}^{n} \dot{K}_i^2 + \frac{1}{2} \dot{L}^2 \right)^{\frac{1}{2}} \tag{4}$$

with $1 > \alpha_i > 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i < 1$.

Without loss of generality, we can assume that $(dC)/(dY) = 1$ and (1) can be written as

$$C = F(K;\, \dot{K};\, L;\, \dot{L}). \tag{1$'$}$$

One of the distinct features of this model is that $\partial^2 F/\partial \dot{K}_i \partial \dot{K}_j \neq 0$. That is, the "investment function" is nonlinear, as shown by the example [equation (4)].

Consumption per capita is given by

$$\begin{aligned} c = \frac{C}{L} &= F\left(\frac{K_1}{L}, \ldots, \frac{K_n}{L}; \frac{\dot{K}_1}{L}, \ldots, \frac{\dot{K}_n}{L}; 1; \frac{\dot{L}}{L} \right) \\ &= F(k_1, \ldots, k_n; \dot{k}_1 + \lambda k_1, \ldots, \dot{k}_n + \lambda k_n; 1; \lambda), \end{aligned}$$

where

$$k_i = \frac{K_i}{L} \quad \text{and} \quad \dot{k}_i = \left(\frac{\dot{K}_i}{K_i} - \lambda \right) k_i \qquad i = 1, \ldots, n.$$

Thus we have

$$c = f(k_1, \ldots, k_n; \dot{k}_1 + \lambda k_1, \ldots, \dot{k}_n + \lambda k_n; \lambda) = f(k; \dot{k} + \lambda k; \lambda). \tag{5}$$

The society's objective is to maximize the discounted future value of consumption per capita, $c(t) = f(t)$, as

$$J = \int_0^\infty e^{-\rho t} f(k; \dot{k} + \lambda k; \lambda)\, dt \to \text{Max}. \tag{6}$$

The necessary condition for the optimal solution is that the Euler–Lagrange equations vanish:

$$\begin{aligned} E_i &= \frac{\partial}{\partial k_i}(e^{-\rho t} f) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{k}_i}(e^{-\rho t} f) \right) \\ &= e^{-\rho t}\left[\frac{\partial f}{\partial k_i} + \rho \frac{\partial f}{\partial \dot{k}_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{k}_i} \right) \right] = 0, \qquad i = 1, \ldots, n. \end{aligned} \tag{7}$$

If we define the supply price of the ith capital per labor input as

$$P_i = -\frac{\partial f}{\partial \dot{k}_i}, \tag{8}$$

then the time derivative of the supply price of capital is

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$$\dot{P}_i = -\frac{\partial f}{\partial k_i} + \rho P_i. \tag{8a}$$

It is very convenient to use the so-called Noether theorem [see Noether (1918), Nôno (1968), and Nôno and Mimura (1975, 1976, 1977, 1978)] to uncover both hidden and unhidden quantities working along the trajectory of the optimally controlled economy—known as the conservation law [Sato (1981, 1985), Sato, Nôno, and Mimura (1983), and Sato and Maeda (1990)]. Using the Noether theorem and its invariance principle, the general expression for the conservation law is

$$\Omega = \left(e^{-\rho t} f - e^{-\rho t} \sum_{i=1}^{n} \frac{\partial f}{\partial \dot{k}_i} \dot{k}_i\right) \tau + e^{-\rho t} \sum_{i=1}^{n} \frac{\partial f}{\partial \dot{k}_i} \xi^i - \Phi = \text{constant}, \tag{9}$$

where τ and ξ^i are the so-called infinitesimal transformations of t and k_i, respectively [see Appendix A, equation (A.7)].

In a special case of (9) in which $\tau = 1$, $\xi^i = 0$, $i = 1, \ldots, n$, $d\Phi/dt = -\rho e^{-\rho t} f$, we obtain the well-known conservation law of the income/wealth ratio as

$$f(k) - \sum_{i=1}^{n} \frac{\partial f}{\partial \dot{k}_i} \dot{k}_i = \rho \int_t^{\infty} e^{-\rho(s-t)} f(s)\, ds$$

or

$$c(t) + \sum_{i=1}^{n} P_i \dot{k}_i = \rho \int_t^{\infty} e^{-\rho(s-t)} f(s)\, ds \tag{10}$$

or

Consumption + Value of Investment per capita
$= \rho \times$ (Wealth measured in terms of per-capita future Consumption)

or

$$\text{Income per capita} = \rho \times \text{Wealth per capita}$$

or

$$\rho = \frac{\text{Income per capita}}{\text{Wealth per capita}}. \tag{11}$$

We call the preceding the Per Capita Income/Wealth Conservation Law.

Note that equation (11) is *not* equal to (income per capita)/(capital per capita). Equation (11) is always constant for all, $0 \le t \le \infty$, while the income (per capita)/capital (per capita) ratio varies for $0 \le t < \infty$ and is constant only at $t = \infty$, or in the steady state in the long run. [See Samuelson (1970) and Sato (1981).]

Because we are interested in time path of the aggregate economy, the above conservation law must be converted to the Aggregate Income/Aggregate Wealth

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Law. This is not simply derived by multiplying both sides of (10) by L, because aggregate income must be defined by $C + \sum_{i=1}^{n} P_i \dot{K}_i$, which is not equal to $L(c + \sum_{i=1}^{n} P_i \dot{k}_i)$. In fact, $L(c + \sum_{i=1}^{n} P_i \dot{k})$ can be expanded as

$$L \cdot c + \sum_{i=1}^{n} P_i L\dot{k} = C + \sum_{i=1}^{n} P_i(\dot{K}_i - \lambda K_i), \tag{12}$$

where

$$P_i = -\frac{\partial C}{\partial \dot{K}} = -\frac{\partial c}{\partial \dot{k}}$$

and

$$L\dot{k}_i = L\frac{d}{dt}\left(\frac{\dot{K}_i}{L}\right) = L\left[\frac{(\dot{K}_i L - \dot{L} K_i)}{L^2}\right] = \dot{K}_i - \lambda K_i.$$

However, because $\dot{K}_i - \lambda K_i$ is the adjusted net capital formation ($\dot{K}_i + \delta K_i$ being gross investment with $\delta =$ depreciation rate > 0, we call this an adjustment), we get the aggregate (adjusted) income/wealth conservation law as

$$C + \sum_{i=1}^{n} P_i(\dot{K}_i - \lambda K_i) = \rho e^{\lambda t} \int_t^{\infty} e^{-\rho(s-t)} f(s)\, ds \tag{13}$$

or

$$\text{(Adjusted) National Income} = \rho \times \text{National Wealth}.$$

If we assume that "Labor" is also endogenous $K_{n+1} = L$, and the society's goal is to maximize the "aggregate" consumption, or if $\lambda = 0$, then (13) will reduce to the well-known result by Weitzman (1976) and Sato (1981, 1985).

From an empirical point of view, it is very important to formulate a model in such a way that the conservation laws can be tested against some observable data.

We have seen that when a factor such as labor is exogenously given to the system, maximization of consumption per labor input gives different conservation laws compared with the case of aggregate consumption maximization. Much of the literature in theoretical studies up to this point has basically ignored models of optimization with exogenous factors. As far as I know, there are only two exceptions: one where discount rate is changing over time and the other where there exists a kind of technical change (or taste change) in the system [see Sato (1985)]. These cases, in general, require a sophisticated data set to verify the underlying optimality conditions. We only take up a special case, the Harrod neutral type of technical change.

2.1. Harrod-Neutral Technical Change

If the aggregate consumption function is expressed as

$$C = F[K; \dot{K}; BL; (\dot{BL})], \tag{14}$$

where $BL =$ effective labor input with $B = B_0\, e^{\beta t}$, $B_0 = 1$ and $\beta \geq 0$, or Harrod neutral technical change, then per-capita consumption is written as

$$c = \frac{C}{L} = BF\left[\frac{K}{BL}; \frac{\dot{K}}{BL}; 1; \frac{(\dot{BL})}{BL}\right]$$

$$c = e^{\beta t} f[\bar{k}; \dot{\bar{k}} + (\lambda + \beta)\,\bar{k}; (\lambda + \beta)], \quad \textbf{(15)}$$

where

$$\bar{k} = (\bar{k}_1, \ldots, \bar{k}_n,), \dot{\bar{k}} = (\dot{\bar{k}}_1, \ldots, \dot{\bar{k}}_n) \quad \text{and} \quad k_i = \frac{K_i}{BL}, \dot{\bar{k}}_i = \frac{d\bar{k}_i}{dt}, \quad i = 1, \ldots, n.$$

The society's objective is to maximize

$$J = \int_0^\infty \mathrm{e}^{-\rho \mathrm{t}} c\, dt = \int_0^\infty e^{-(\rho-\beta)\mathrm{t}} f[\bar{k}; \bar{k} + (\lambda + \beta)\dot{\bar{k}}; (\lambda + \beta)]\, dt, \quad \textbf{(16)}$$

where $\rho > \beta \geq 0$.

This is identical to (6) with ρ replaced by $(\rho - \beta)$, λ by $(\lambda + \beta)$, and k by $\bar{k}$, etc. Since the supply price of the ith capital

$$P_i = \frac{\partial F}{\partial \dot{K}_i} = \frac{\partial f}{\partial \dot{\bar{k}}}, \qquad (i = 1, \ldots, n),$$

we obtain

$$c + \frac{1}{L}\sum_{i=1}^{n} P_i[\dot{K}_i - (\beta + \lambda)K_i] = (\rho - \beta)\int_\mathrm{t}^\infty e^{-\rho(s-t)+\beta s} f(s)\, ds. \quad \textbf{(17)}$$

Again, by multiplying both sides of (17) by L, we get an aggregate expression for the conservation law:

$$C + \sum_{i=1}^{n} P_i[\dot{K}_i - (\beta + \lambda)K_i] = (\rho - \beta)e^{\lambda t}\int_\mathrm{t}^\infty e^{-\rho(s-t)+\beta s} f(s)\, ds \quad \textbf{(18)}$$

(Adjusted) National Income = $(\rho - \beta) \times$ National Wealth.

This is the modified conservation law; ρ is replaced by $(\rho - \beta)$ and the term expressing national wealth now contains $e^{\beta s} = B(s)$, as labor becomes more efficient by Harrod neutral technical progress. However, the conservation laws expressed by (13) and (18) are basically the same, which will be used for empirical applications.

3. HOW TO MEASURE WEALTHLIKE PRESENT VALUE OF THE STREAM OF FUTURE CONSUMPTION?

We next want to ask if there is any operational method and empirical counterpart to measure the wealthlike quantity $\int_\mathrm{t}^\infty e^{-\rho(s-t)} C(s)\, ds$. We begin with the simplest case of $\lambda = 0$ and $\beta = 0$, or the endogenous labor force $K_n = L$. Then, the conservation law is

$$\text{Aggregate Income} = \rho \times \text{“Wealthlike Quantity.”}$$

Since the conservation law can be also written in (time) derivative form as [see Appendix A, equation (A.9)]

$$\frac{d}{dt}\left(C - \sum_{i=1}^{n} \dot{K}_i \frac{\partial F}{\partial \dot{K}_i}\right) = -\rho \sum_{i=1}^{n} \dot{K}_i \frac{\partial F}{\partial \dot{K}_i}, \tag{19}$$

where $C = F(K_1, \ldots, K_n; \dot{K}_1, \ldots, \dot{K}_n)$. Let $P_i = -\partial F/\partial \dot{K}_i$ and $Y = C + \sum_{i=1}^{n} P_i \dot{K}_i$, and then

$$\frac{dY}{dt} = \rho \sum_{i=1}^{n} P_i \dot{K}_i.$$

Then, by integrating both sides and utilizing (8a),

$$\dot{P}_i = -\frac{\partial F}{\partial K_i} + \rho P_i,$$

we have

$$\int_0^{\mathrm{t}} dY(t) = \rho \int_0^t \sum_{i=1}^{n} P_i \dot{K}_i \, ds.$$

$$\begin{aligned}
Y(t) - Y(0) &= \rho \left\{ \left(\sum_{i=1}^{n} P_i K_i \right)_0^{\mathrm{t}} - \int_0^{\mathrm{t}} \sum_{i=1}^{n} \frac{d}{ds}[P_i(s)] K_i(s) \, ds \right\} \\
&= \rho \left[V(t) - V(0) + \int_0^{\mathrm{t}} \sum_{i=1}^{n} \frac{d}{ds}\left(F_{\dot{K}_i}\right) K_i \, ds \right] \\
&= \rho \left[V(t) - V(0) + \int_0^{\mathrm{t}} \sum_{i=1}^{n} \frac{\partial F}{\partial K_i} K_i \, ds - \rho \int_0^{\mathrm{t}} \sum_{i=1}^{n} P_i K_i \, ds \right] \\
&= \rho \left[V(t) - V(0) + \int_0^t \prod(s) \, ds - \rho \int_0^t V(s) \, ds \right],
\end{aligned} \tag{20}$$

where $V(t) = \sum_{i=1}^{n} P_i K_i(t) =$ value of capital or standard definition of wealth, $\prod(t) = \sum_{i=1}^{n} r_i K_i =$ total profit ($r_i =$ return space to K_i), and $(d/dt)(F_{\dot{K}_i}) = F_{K_i} + \rho F_{\dot{K}_i} V(t)$ is usually known as wealth in the national wealth literature. Our conservation law tells us that $Y \neq \rho V(t)$. Alternatively, we can present an operational expression from (19) as

$$\frac{dY}{dt} = \rho(Y - C) = \rho I. \tag{21}$$

By integrating, we immediately derive

$$Y(t) - Y(0) = \rho \int_0^{\mathrm{t}} I(s)\, ds = \rho[W(t) - W(0)]. \tag{22}$$

The wealthlike quantity $\int_t^\infty e^{-\rho(s-t)} C(s)\, ds$ is exactly equal to $W(t)$. Hence the operational measure of wealthlike value of future consumption is nothing but the accumulated sum of the value of investments. In summary, we can use either (20), (21), or (22), because (20) requires ρ to be known a priori.

The next simplest case of maximization of utility of consumption per capita when $\lambda = 0$ requires several modifications. To apply aggregate data such as GNP or national wealth, we must use (13):

$$C + \sum_{i=1}^{n} P_i(\dot{K}_i - \lambda K_i) = \rho e^{\lambda t} \int_{\mathrm{t}}^{\infty} e^{-\rho(s-t)} f(s)\, ds, \tag{13}$$

which can be expressed as

$$C(t) + P(\dot{K} - \lambda K) - L(t)y(0) = \rho L(t) \int_0^t \frac{I(s) - \lambda V(s)}{L(s)}\, ds. \tag{23}$$

The right-hand side represents the adjustments to GNP by $\lambda PK = \lambda V(t)$ and the initial condition $L(t)y(0)$, which are all identifiable from empirical data.

The above relationship can be applied to the case of Harrod neutral technical change by simply replacing λ with $\lambda + \beta$ and ρ with $\rho - \beta$ as equation (18) suggests. However, for the purpose of empirical estimation, both λ and β must be known before the above equation is tested for optimality.

In summary, we have presented two conservation laws. For each theoretical equation, there are at least three empirical counterparts. For the simplest of endogenous labor (or $\lambda = 0$), the three equations are (20), (21), and (22). In the empirical analysis presented in the next section, we used (20) and (21) to supplement and verify the value of ρ.

4. TEST OF OPTIMALITY AND CONSERVATION LAWS

We begin with the application of the simplest optimal growth model (of per capita consumption maximization) for 12 OECD countries, including the United States, Canada, Japan, and Great Britain. However, as the first-order approximation we make *no* adjustments: that is, (a) no subtraction of $\lambda PK = \lambda V(t)$ on the left-hand side of (13) from GNP and (b) no adjustment in the exact calculation of "wealth" data [the right-hand side of (20)]. Instead, we simply use GNP data from the national income account and "net national wealth" data or value of capital goods in each country; that is,

$$Y = \text{GNP} \approx \rho \times \text{net national wealth} \approx \rho \times \sum_{i=1}^{n} P_i K_i = \rho V(t)$$

or

$$\frac{Y(t)}{V(t)} = \rho.$$

Obviously, when $\dot{P}_i$ is relatively close to zero, we have $V(t) \approx W(t)$. The results are remarkably consistent in general, and show that most economies can be viewed as operating along the optimal trajectories determined by the model. For instance, the U.S. economy for the most part showed a remarkable consistency in maintaining a relatively stable value of ρ around the value of 0.25 before the war (Figure 1) and around 0.3 in the post-World War II period (Figure 2). On the other hand, the Japanese economy behaved very differently from the U.S. economy in that ρ is consistently declining and approaches its lowest value during the bubble period

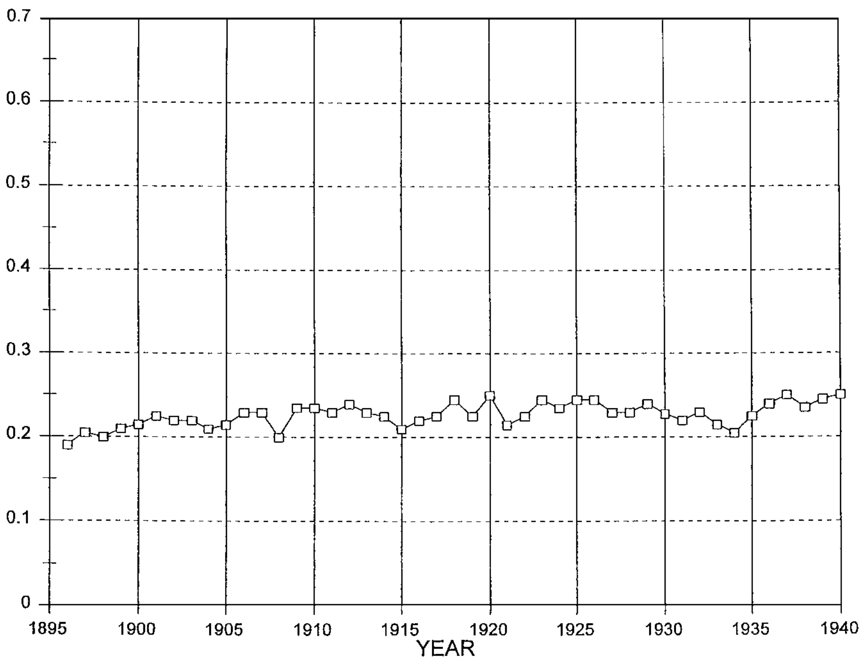

Figure 1. U.S. Income/wealth ratio (prewar).

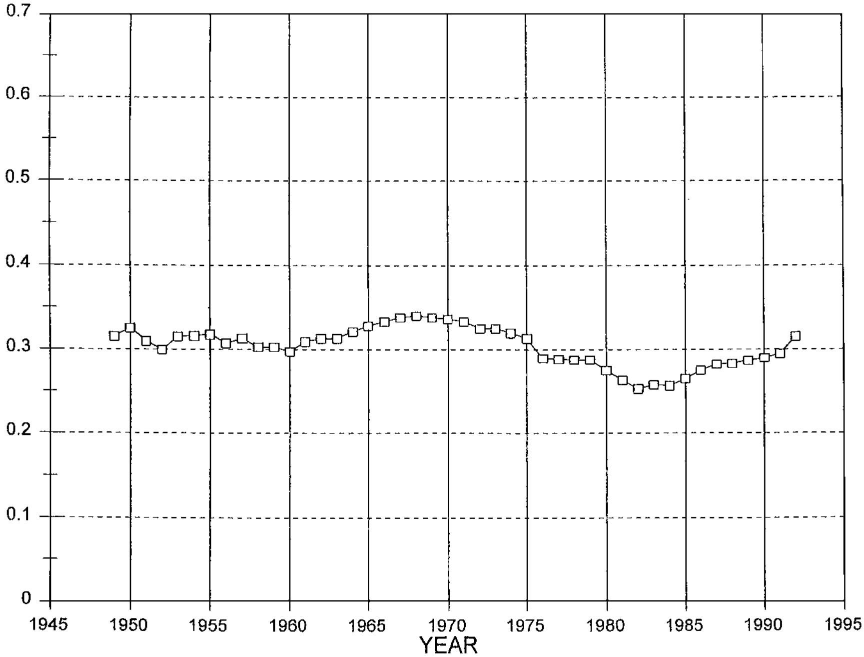

Figure 2. U.S. Income/wealth ratio (postwar).

of the early 1990's. This suggests that either the discount rate of the Japanese economy may be *variable* or that the Japanese economy may *not* be operating along the trajectories prescribed in the simple model. We take up each country separately.

5. INCOME/WEALTH RATIO FOR THE UNITED STATES

The ratio of income to wealth for the United States has been historically stable, indicating that the income/wealth law may in fact be operating. No observable trend upward or downward occurs for the entire period 1896–1992 (see Figure 3). Note that, for a short period surrounding World War II (1941–1946), the ratios exhibit a pattern of sudden upward shift which is not quite consistent with the rest of the historical period under study and is not exhibited in the data point of the graphs. One important observation, however, is that this war period seems to indicate a structural change for the U.S. economy. Between the two periods before and after the war, there is a hysteretical break in the U.S. ratios, with the post-war ratios fluctuating approximately about a mean permanently higher than its prewar level.

From the prewar period of 1896 and 1940, the U.S. ratio displayed stable fluctuations around its constant mean, although, during the Great Depression era of the

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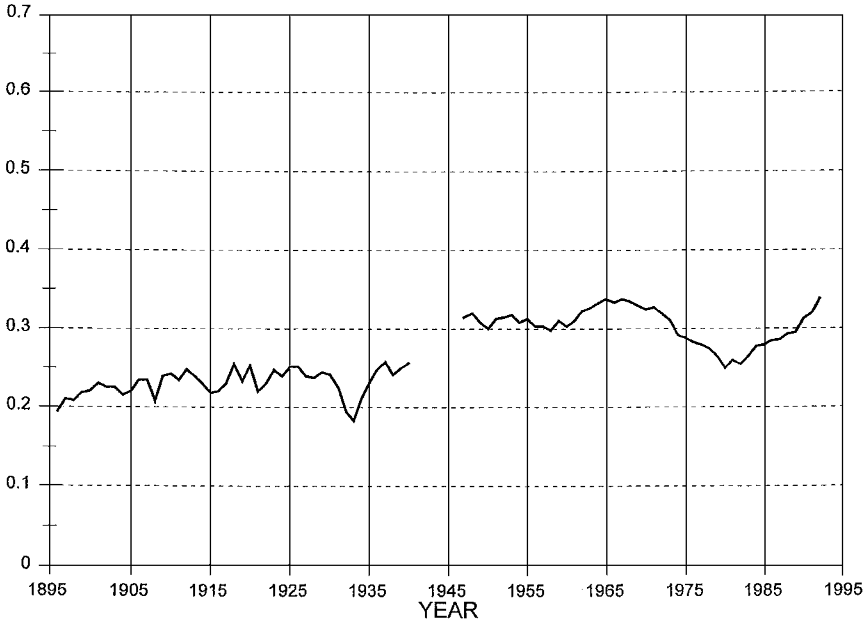

Figure 3. U.S. ratio (GNP/national private wealth).

early 1930's, income declined so that the ratio declined somewhat. The mean for this entire prewar period is $\rho = 0.23$, with variance 0.0003. In contrast, the mean for the postwar period 1947–1992 is higher at $\rho = 0.30$. Fluctuations about the postwar mean, however, are again stable with variance 0.0006.

Note that the postwar ratios display some business-cycle tendencies within their range. For the period immediately after the war and throughout the 1950's, the U.S. ratios are highly constant. This is followed in turn by a period in the 1960's that is largely of high ratios, due to a rise in income from the Vietnam War boom. The Ford-Carter era of the 1970's took the full shock of the oil crises, and, coupled with high asset price inflation, the ratio declined throughout the decade to a level below 0.3. It was not until Reagan's stimulative policy of the 1980's that the U.S. ratio turned upward again, and during the resultant Reagan boom of the decade, the ratio steadily climbed back. Meanwhile, the 1990's has seen a characteristic continuation in the rise of this ratio combined with a sharp decline in asset prices as the United States runs into a recession.

For the U.S. ratio, we have also studied cases of different levels of lag in wealth. One-year, two-year, and five-year lags were considered. Since none of these lags dramatically changes our results obtained without lags except that the ratios with lags tend to be lower, we will not include the results here. Those interested, however, may inquire about our results.

6. STATISTICAL TESTS FOR THE UNITED STATES

We have performed statistical tests to see if the conservations law holds for the United States and if there is autocorrelation in the disturbance. Our tests are based on earlier data derived from the President's Report (1992) covering the postwar period 1957–1990; see Appendix B.

To check for the conservation law, we first ran the following regression of GNP on net wealth, using 44 observations. The standard errors for the estimates are given inside the parentheses:

$$Y = a + bW + \varepsilon,$$
$$a = -13.5096\,(84.6638),$$
$$b = 0.3331\,(0.0027).$$

For the conservation law to hold, we require that the coefficient b equal the discount rate. This in turn requires that the constant term a be zero. Thus, we constructed a hypothesis,

$$H_0 : a = 0$$
$$H_\mathrm{A} : a \neq 0$$

Since the t-statistics in this case are equal to the ratio of the estimate for the constant term over its standard error, with the critical value for 5% significance level with 42 degrees of freedom being about 2.00, we can conclude that the null hypothesis is not rejected.

We next ran a regression on a specification without the constant term, and obtained the following result

$$Y = bW + \varepsilon$$
$$b = 0.3316\,(0.0019)$$

Thus, the estimated discount rate according to our conservation law is 0.3316 for the economy.

As for the autocorrelation test, our hypothesis was

$$H_0 : r = 0$$
$$H_A : r \neq 0 \quad (r\text{: correlation parameter})$$

For the sample size of 44 and 2 variables, the lower and the upper bounds of the Durbin–Watson distribution are 1.475 and 1.566, respectively. The obtained statistics equaled 1.61, which exceeded the upper bound. Therefore, we can conclude that the null hypothesis is not rejected and that there is no obvious autocorrelation in the disturbance term.

7. INCOME/WEALTH RATIO FOR JAPAN

For the Japanese case, we used GDP data as income. For both GDP and net wealth, data series were taken directly from the original sources: Economic Planning Agency, Government of Japan, *Report on National Accounts from 1955 to 1989* for both GDP and net wealth data for the period 1955–1984; and the same agency's *Annual Report on National Account from 1993* for their continuing series for 1985–1991.

For the observed period of 1955 through 1991, the Japanese ratios are comparably lower than the U.S. ratios, with the mean equal to 0.21 and variance 0.0026 (Figure 4). The Japanese variance is higher than that of the United States because the Japanese ratios tend to exhibit a downward trend over time. The Japanese discount rate is considerably lower than that of the United States. This may be due to the fact that the Japanese are more long-sighted.

One similarity with the United States is a business-cycle-like movement of the Japanese ratio for the 1960's and 1970's, whereby high ratios are first observed for the 1960's and reflect a decade of high economic growth in Japan, which was then followed by a decade of decline in the ratios for the 1970's. Their rise and fall during these two decades seem to be contemporaneous with the U.S. case.

The real divergence in patterns comes in the 1980's, during which the Japanese ratio continues to decline while the U.S. ratio steadily rises. This persistent decline

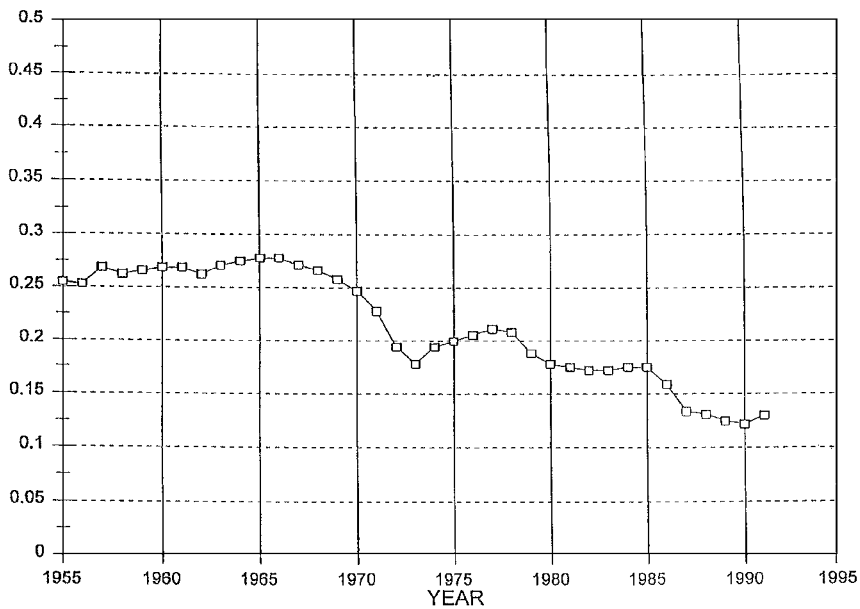

FIGURE 4. Ratio of GDP/net wealth in Japan.

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in the Japanese ratio coincides with the period of bubbles, during which asset prices skyrocketed two- or three-fold in Japan (see Figure 4).

Has the Japanese economy not grown along the optimal trajectories? Not necessarily. First, the standard model of a constant discount rate, $\rho(t) = \text{constant}$, may not be relevant to the Japanese case. Second, because the capital price P_i may bediverging from its equilibrium price, i.e., $\dot{P}_{\dot{K}} \neq 0$, much faster than in the United States, the true measure of the wealthlike quantity $W(t)$ and the value of capital $V(t)$ may be very different; most likely, $V(t)$ may be much greater than $W(t)$, especially during the bubble period. Further research will be undertaken in the near future.

8. INCOME/WEALTH RATIO FOR OTHER OECD COUNTRIES

Among the 24 OECD countries, we could collect data for 13 countries including the U.S. and Japan. The remaining 11 countries are: Australia, Canada, Finland, France, Germany, Greece, Iceland, Italy, Norway, Sweden, and the United Kingdom. The major data sources for these 11 countries is OECD, Department of Economics and Statistics, *Flows and Stocks of Fixed Capital: 1964–1989* (OECD: 1991).

Although we maintained Kendrick's concept of net wealth for OECD countries, because of a data availability problem, for some countries we were forced to use data that did not fully contain all components of net wealth as described in the section on U.S. wealth data. Also because of other data problems (i.e., observational sizes, data treatment procedures, etc., all slightly different across countries), our results are not for direct comparison between the levels of ratios across countries. In the following, therefore, we simply focus on one aspect: whether or not the constancy property of the ratio has been met. The 11 countries mentioned above can be roughly grouped into three categories: countries having constant ratios, those with declining ratios, and those with neither characteristic.

The first category consists of Australia, Canada, Norway, and Sweden (see Figures 5 through 8). The mean values of the ratios for their respective covered periods (see figures) are 0.35 for Australia, 0.31 for Canada, 0.27 for Norway, and 0.31 for Sweden. For these countries, the ratios seem to be highly constant over time. Thus, the income/wealth conservation law is present there.

On the other hand, Finland, Germany, Greece, and the United Kingdom make up the second group and have declining ratios (see Figures 9 through 12), a pattern similar to the Japanese case. Their respective means are 0.34 for Finland, 0.39 for Germany, 0.33 for Greece, and 0.38 for the United Kingdom.

Finally, Iceland and Italy follow neither of the above patterns (see Figure 13 and 14). For Iceland, the ratio is unstable, although there does not seem to be a trend, whereas for Italy the ratio goes up in the beginning of the series and comes down again. The means for those countries are 0.32 and 0.22, respectively.

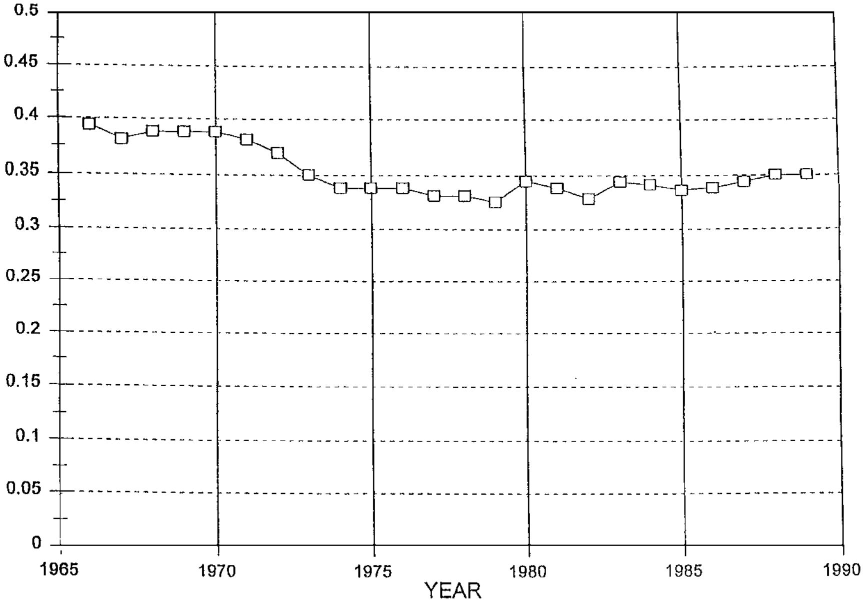

Figure 5. Australia.

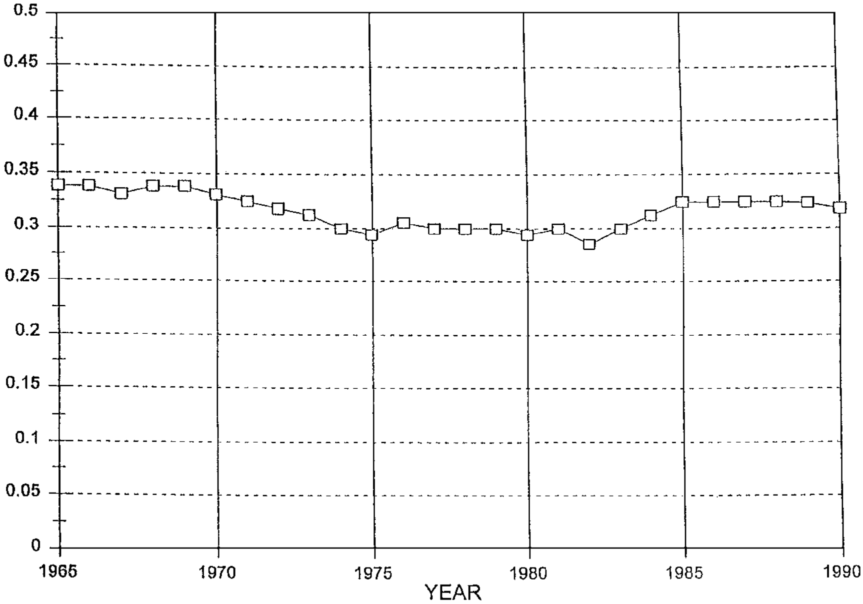

Figure 6. Canada.

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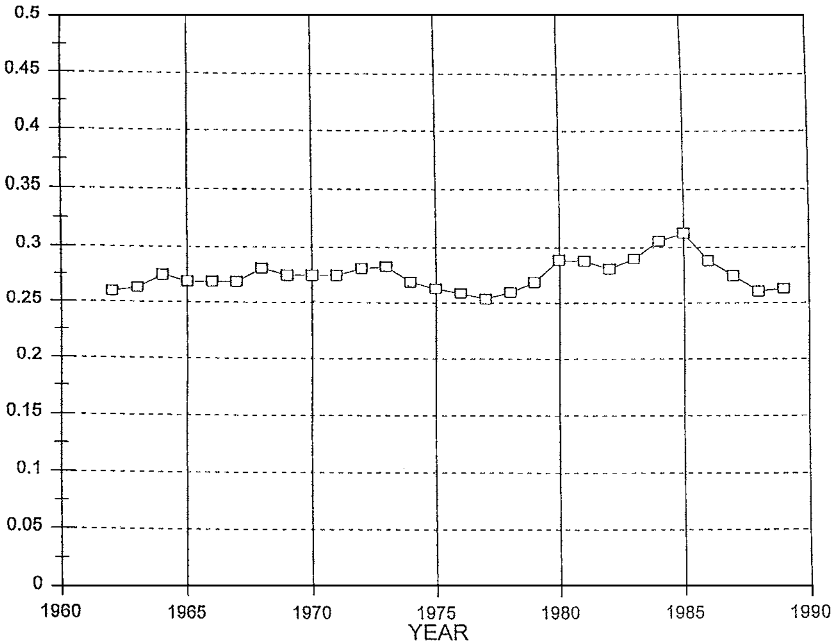

FIGURE 7. Norway.

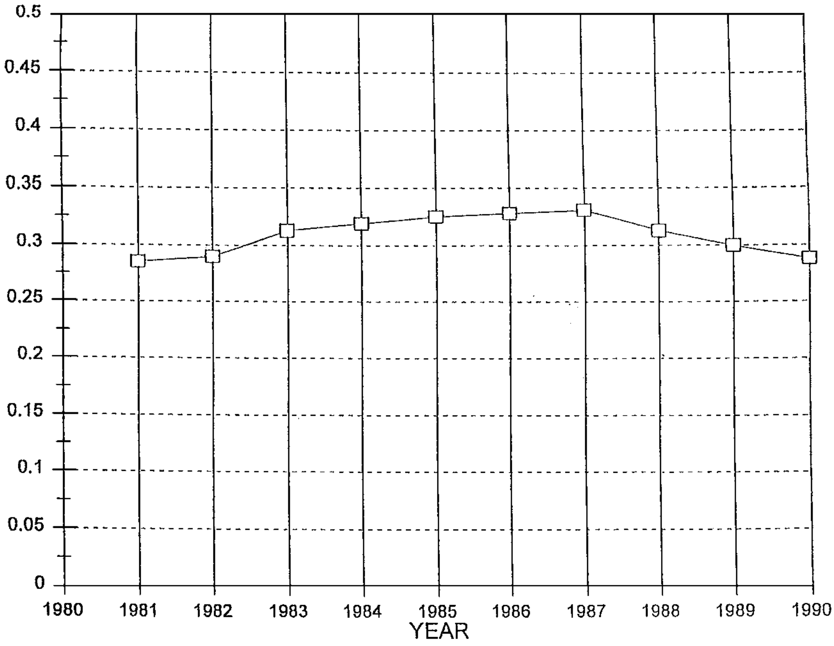

FIGURE 8. Sweden.

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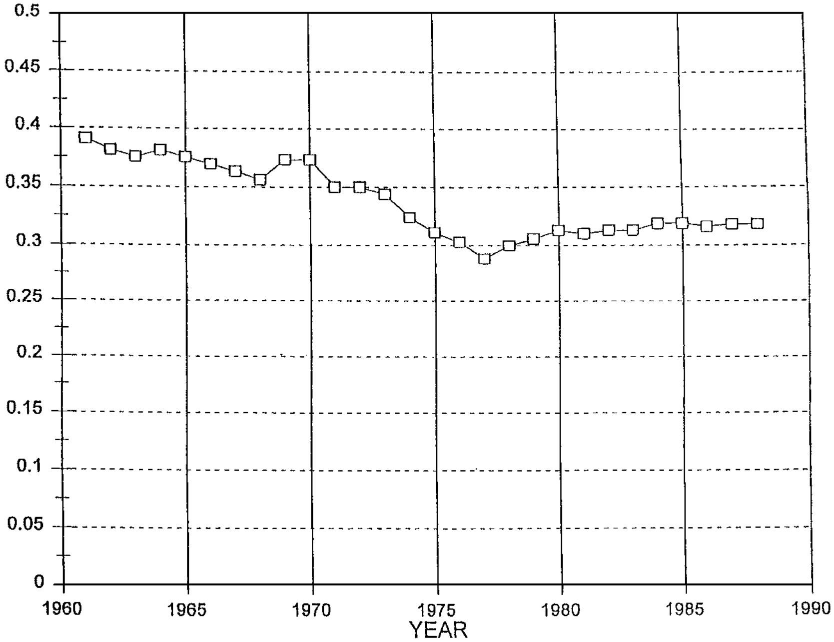

Figure 9. Finland.

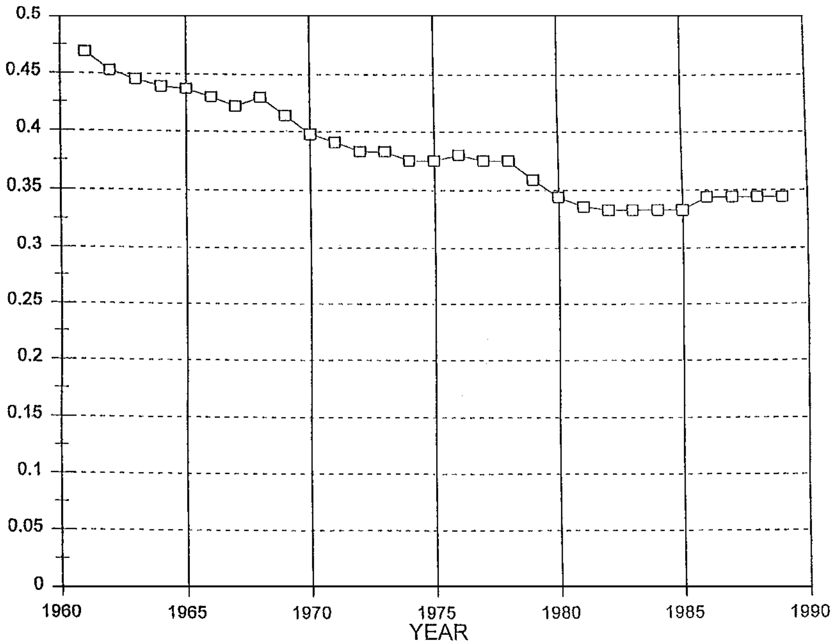

Figure 10. Germany.

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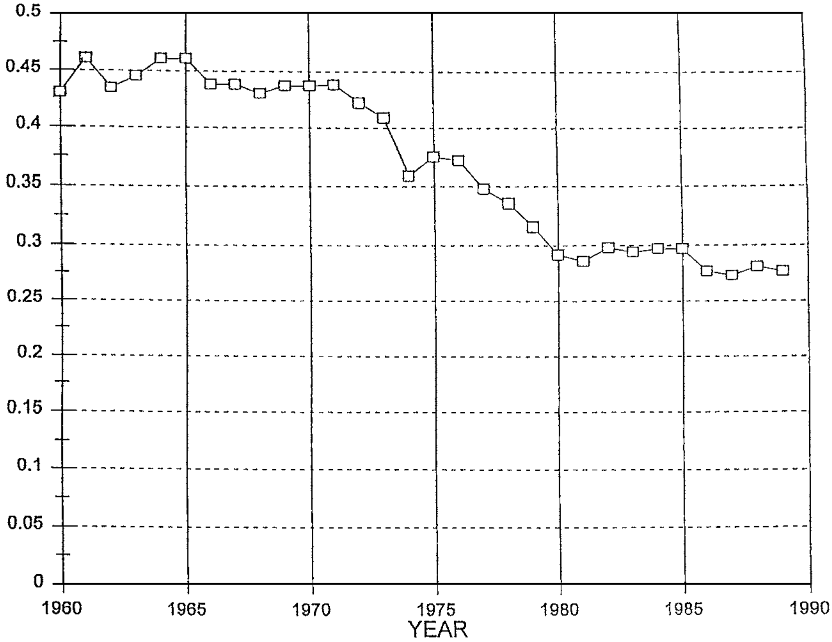

FIGURE 11. Greece.

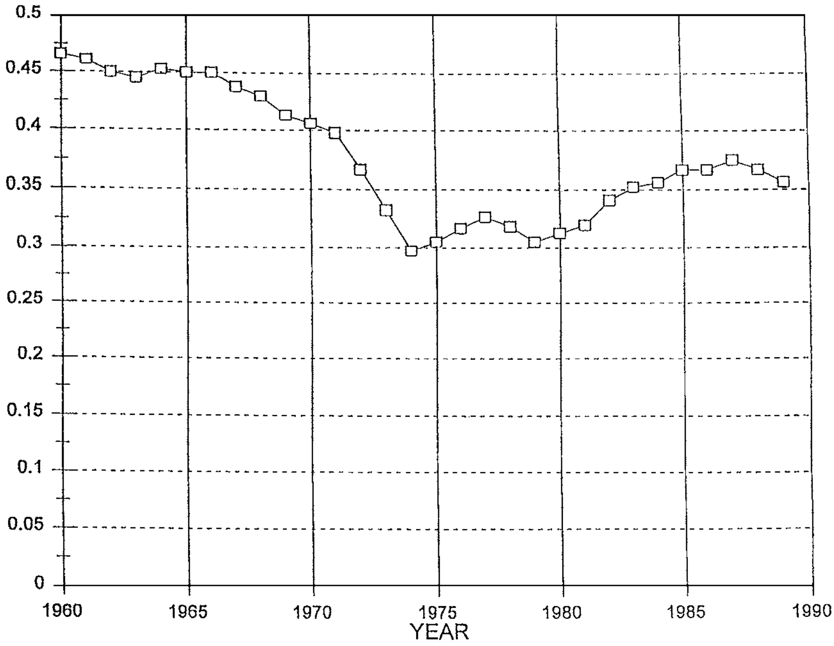

FIGURE 12. United Kingdom.

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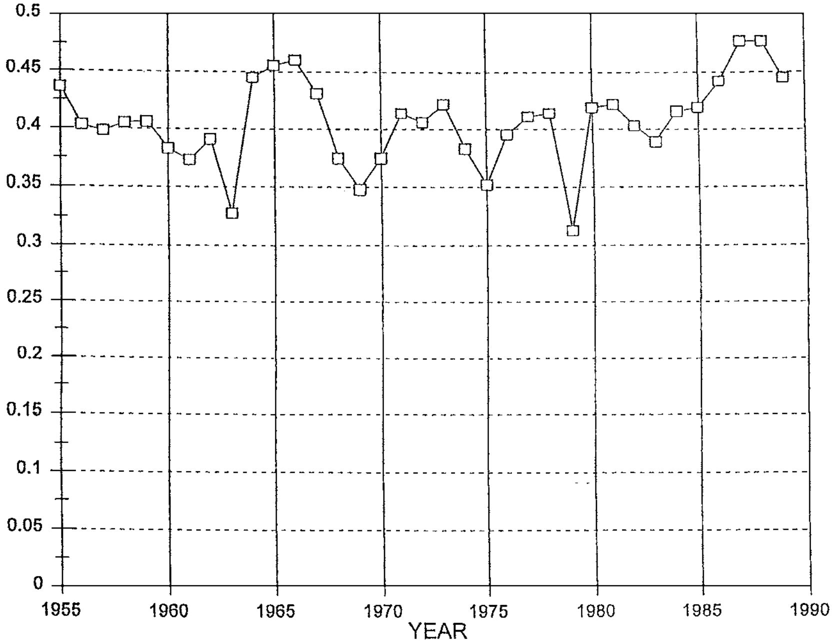

FIGURE 13. Iceland.

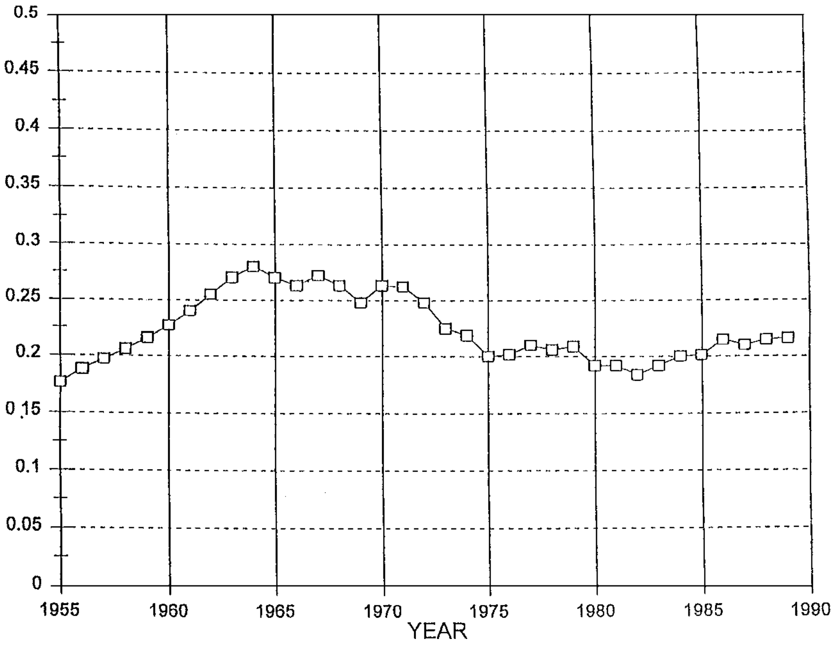

FIGURE 14. Italy.

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APPENDIX A

Consider the utility maximization problem, [See Sagan (1969), Gelfland and Fomin (1963), and Rund (1966)

$$\int_0^\infty U[t, k(t), \dot{k}(t)]\, dt \to \text{Max}, \qquad \textbf{(A.1)}$$

where $k(t) = [k_1(t), \ldots, k_n(t)]$, $\dot{k}(t) = [\dot{k}_1(t), \ldots, \dot{k}_n(t)]$, and t in U represents exogenous factors such as labor force $L(t) = L_0 e^{\lambda t}$, technical change and/or taste change.

Also consider r-parameter transformation [see Lie (1891) and Sato (1981)],

$$T : \bar{\mathrm{t}} = \phi(t, k; \varepsilon), \varepsilon = (\varepsilon^1, \ldots, \varepsilon^{\mathrm{r}}), \qquad \bar{\kappa}_i = \psi^i(t, k; \varepsilon), \qquad i = 1, \ldots, n, \qquad \textbf{(A.2)}$$

where

$$\phi(t, k; 0) = t, \qquad \psi^i(t, k; 0) = k_i, \qquad i = 1, \ldots, n. \qquad \textbf{(A.3)}$$

The infinitesimal transformations are given by

$$\tau_s(t, k) = \frac{\partial \phi}{\partial \varepsilon^s}(t, k; 0), \qquad \xi_s^i(t, k) = \frac{\partial \psi^i}{\partial \varepsilon^s}(t, k; 0), \qquad s = 1, \ldots, n,$$

and

$$X_s = \tau_s(t, k)\frac{\partial}{\partial t} + \xi_s^i(t, k)\frac{\partial}{\partial k_i} + \left(\frac{d\xi_s^i}{dt} - \dot{k}_i \frac{d\tau_s}{dt}\right)\frac{\partial}{\partial \dot{k}_i}, \qquad \textbf{(A.4)}$$

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where $\xi_s^i(t, k)(\partial/\partial k)$ is the Einstein notation for $\sum_{i=1}^n \xi_s^i(t, k)(\partial/\partial k_i)$, etc. Let (A.1) be given by

$$\int_0^\infty e^{-\rho t} U[k(t), \dot{k}(t)]\, dt \to \text{Max.} \tag{A.5}$$

Then, Noether's dynamic invariance condition with nullity Φ gives us [Noether (1918), Logan (1977), Sato (1981, p. 244, eq. 17)],

$$e^{-\rho t}\left[\rho U \tau + \frac{\partial U}{\partial k_i}\xi^i + \frac{\partial U}{\partial \dot{k}_i}\left(\frac{d\xi^i}{dt} - \dot{k}_i \frac{d\tau}{dt}\right) + U\frac{d\tau}{dt}\right] = \frac{d\Phi}{dt}$$

or

$$\left(U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\frac{d\tau}{dt} = \rho U \tau - \frac{\partial U}{\partial k_i}\xi^i - \frac{\partial U}{\partial \dot{k}_i}\frac{d\xi^i}{dt} + e^{\rho t}\frac{d\Phi}{dt} \tag{A.6}$$

(since $s = 1$, $\tau_1 = \tau$, and $\xi_1^i = \xi^i$). The above will yield the conservation law

$$\Omega = \left(e^{-\rho t} U - e^{-\rho t}\dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\tau + e^{-\rho t}\frac{\partial U}{\partial \dot{k}_i}\xi^i - \Phi = \text{const.} \tag{A.7}$$

or

$$\begin{aligned}\frac{d\Omega}{dt} &= -\rho e^{-\rho t}\left(U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\tau + e^{-\rho t}\frac{d}{dt}\left[\left(U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\tau\right] + e^{-\rho t}\left(U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\frac{d\tau}{dt} \\ &\quad - \rho e^{-\rho t}\frac{\partial U}{\partial \dot{k}_i}\xi^i + e^{-\rho t}\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{k}_i}\xi^i\right) - \frac{d\Phi}{dt} = 0\end{aligned} \tag{A.8}$$

By eliminating $d\Phi/dt$ between (A.6) and (A.8) and setting $\xi = 0$, $\tau = 1$, we obtain our first "income/wealth" conservation law as

$$\frac{d}{dt}\left[U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right] = -\rho \dot{k}_i \frac{\partial U}{\partial \dot{k}_i} \tag{A.9}$$

or

$$\frac{d}{dt}(\text{income at } t) = \rho \times (\text{Utility Value of Investment at } t),$$

where $-\partial U/\partial \dot{k}_i = P_i =$ price of capital. Also, from (A.6) and (A.8), we have

$$\frac{d}{dt}\left[e^{-\rho t}\left(U - \dot{k}_i \frac{\partial U}{\partial \dot{k}_i}\right)\right] = -\rho e^{-\rho t} U. \tag{A.10}$$

Integrating the above,

$$\Phi|_t^\infty = e^{-\rho s}\left\{\left[U(s) - \dot{k}_i(s)\frac{\partial U(s)}{\partial \dot{k}_i(s)}\right]\right\}_t^\infty = -\rho \int_t^\infty e^{-\rho s} U(s)\, ds$$

and using the transversality condition $\Phi(\infty) = 0$, we have

$$e^{-\rho s}\left[U(t) - \dot{k}_i(t)\frac{\partial U(t)}{\partial \dot{k}_i(t)}\right] = \rho \int_t^\infty e^{-\rho s} U(s)\, ds,$$

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which will reduce to Weitzman's original result [Weitzman (1976)],

$$\text{Income} = U(t) - \dot{k}_i(t)\frac{\partial U(t)}{\partial \dot{k}_i(t)} = \rho \int_t^{\infty} e^{-\rho(s-t)} U(s)\, ds; \tag{A.11}$$

that is,

$$\text{Income} = \rho \times \text{“Wealthlike Quantity”} \tag{A.12}$$

[see Sato (1985), Sato and Maeda (1990)].

APPENDIX B

B.1. NOTES ON U.S. WEALTH (NET WORTH) DATA

For the purpose of our study, we followed the John Kendrick's (1976), definition of wealth in which wealth is taken as the stock of productive capacity resulting from past investments. Furthermore, Kendrick confines wealth to the conventional tangible assets not including human capital.

Domestic financial assets do not come into play for our purposes because consolidation of all domestic assets results in cancellation of domestic financial claims and assets held by all sectors of the economy—households, corporations, and the government—because every domestic claim is a liability at the same time within the national boundaries. The domestic net worth therefore reduces to the total amount of tangible assets held in the domestic economy. By adding to this the net surplus in foreign assets held by the domestic sectors, one will obtain national wealth. Following this set of definitions thus leads to wealth and net worth being synonymous at the national level.

The U.S. net worth data used in our study is the private sector's national wealth derived from two major data sources. Our data for the period 1896 through 1945 is based on Goldsmith's Table W1 estimates. We derived our private wealth series by subtracting the government and the public wealth from the national wealth in the original Table W1. This procedure was used to make this series comparable with another series that we had obtained for the subsequent 1946–1992 period providing only the private tangible assets. For this subsequent period, we used Table B11 of the Board of Governors of the Federal Reserve System, *Balance Sheets for the U.S. Economy 1945–1992.* Since this table gives a series for domestic private wealth, we added to this a series for U.S. net foreign assets from the same source to derive our national private wealth series for this period.

In the original tables of both Goldsmith et al. and the Federal Reserve Board, the asset figures are given at current cost, with the latter being the net of straight-line depreciation. The national wealth in these sources is in line with Kendrick's definition of wealth mentioned earlier, and consists of reproducible and nonreproducible tangible assets and net foreign assets. The reproducible assets include residential and nonresidential structures, consumer and producer durables, inventories, and monetary gold, silver, and SDRs, and the nonreproducible assets consist mainly of land.

B.2. DATA SPLICING FOR A CONTINUOUS HISTORICAL SERIES

To construct a continuous historical series of the U.S. GNP/wealth ratios, we spliced two series of GNP, one prewar and the other postwar, into one continuous series and did the same for wealth. For the splicing of wealth series, noting that the two separate wealth series do have an overlapping period of from 1945 to 1949, we looked at the gap between the two during this period and inflated the prewar series (by Goldsmith et al.) by 6% to give an upward lift. The fitting went reasonably well. As for the GNP series, we repeated the same steps and inflated the prewar series by 0.69% to match with the later series. Inflating at this percentage gave a perfect fit for 1929–1945 in which the two series overlap.

For 1896–1945, our GNP series was based on U.S. Bureau of the Census (1975, Series F1), while for 1946–1992, we used the original data from the Bureau of Economic Analysis (1992, 1993).