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Amenable dynamical systems over locally compact groups

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Abstract. We establish several new characterizations of amenable W^*- and C^*-dynamical systems over arbitrary locally compact groups. In the W^*-setting we show that amenability is equivalent to (1) a Reiter property and (2) the existence of a certain net of completely positive Herz–Schur multipliers of (M, G, α) converging point weak* to the identity of $G\bar{\ltimes}M$. In the C^*-setting, we prove that amenability of (A, G, α) is equivalent to an analogous Herz–Schur multiplier approximation of the identity of the reduced crossed product $G \ltimes A$, as well as a particular case of the positive weak approximation property of Bédos and Conti [On discrete twisted C^*-dynamical systems, Hilbert C^*-modules and regularity. *Münster J. Math.* **5** (2012), 183–208] (generalized to the locally compact setting). When $Z(A^{**}) = Z(A)^{**}$, it follows that amenability is equivalent to the 1-positive approximation property of Exel and Ng [Approximation property of C^*-algebraic bundles. *Math. Proc. Cambridge Philos. Soc.* **132**(3) (2002), 509–522]. In particular, when $A = C_0(X)$ is commutative, amenability of $(C_0(X), G, \alpha)$ coincides with topological amenability of the G-space (G, X).

Key words: dynamical systems, crossed products, locally compact groups, amenable actions
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1. *Introduction*

Amenability and its various manifestations have played an important role in the study of dynamical systems and their associated operator algebras. Zimmer introduced a dynamical version of amenability [46] of an action of a locally compact group on a standard measure space through a generalization of Day's fixed-point criterion, which has proven very useful in ergodic theory and von Neumann algebras.

Motivated by the structure of crossed products, Anantharaman-Delaroche generalized Zimmer's notion of amenability to the level of W^*-dynamical systems (M, G, α) [2]. In [4, Théorème 3.3] she characterized amenability of (M, G, α) with G discrete through a Reiter type property involving asymptotically G-invariant functions in $C_c(G, M)$, generalizing Reiter's condition for amenable groups. She also introduced a notion of amenability for discrete C^*-dynamical systems (A, G, α), and showed, among other things, that a commutative discrete C^*-dynamical system $(C_0(X), G, \alpha)$ is amenable precisely when the transformation groupoid $G \ltimes X$ is topologically amenable in the sense of Renault [37].

Various approaches to amenability for non-discrete C^*-dynamical systems have been studied, including amenable transformation groups (e.g., [5]) and the approximation property of Exel and Ng [19]. Recently, a notion of amenability for arbitrary C^*-dynamical systems was introduced by Buss, Echterhoff and Willett [15], who performed an in-depth study of this notion in relation to amenability of the universal W^*-dynamical system [26], measurewise amenability, and the weak containment problem (among other things).

In this work we establish several new characterizations of amenable W^*- and C^*-dynamical systems over arbitrary locally compact groups. For W^*-systems we generalize [4, Théorème 3.3] to the locally compact setting, giving a Reiter property for arbitrary amenable (M, G, α) (see Theorem 3.6). Our approach relies on a continuous version of [4, Lemme 3.1], whose validity was required by Anantharaman-Delaroche in that paper. We therefore answer this question in the affirmative. We also characterize amenability of arbitrary (M, G, α) through a 'fundamental unitary' W_α associated to the action, and through Herz–Schur multipliers on the crossed product $G \bar{\ltimes} M$ [9, 31, 30]. Our results in this context can be summarized as follows.

THEOREM 1.1. *Let (M, G, α) be a W^*-dynamical system with $M \subseteq \mathcal{B}(H)$. The following conditions are equivalent.*

(1) (M, G, α) *is amenable.*

(2) *There exists a net (ξ_i) in $C_c(G, Z(M)_c)$ such that:*
   - (a) $\langle \xi_i, \xi_i \rangle \leq 1$ *for all i;*
   - (b) $\langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i \rangle \to 1$ *weak*, uniformly on compact subsets of G.*

(3) *There exists a net (ξ_i) in $C_c(G, Z(M)_c)$ such that:*
   - (a) $\langle \xi_i, \xi_i \rangle \leq 1$ *for all i;*
   - (b) $\langle \xi_i, \xi_i \rangle \to 1$ *weak*;*
   - (c) $\| W_\alpha(\xi_i \otimes_\alpha \eta) - \xi_i \cdot \eta \|_{L^2(G \times G, H)} \to 0$, $\eta \in L^2(G, H)$.

(4) *there exists a net (ξ_i) in $C_c(G, M_c)$ such that:*
   - (a) $\langle \xi_i, \xi_i \rangle \leq 1$ *for all i;*
   - (b) $\Theta(h_{\xi_i}) \to \mathrm{id}_{G \bar{\ltimes} M}$ *point weak*,*

   where $h_{\xi_i}(s)(a) = \langle \xi_i, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi_i \rangle$, are the associated completely positive Herz–Schur multipliers in the sense of [31]*, and $\Theta(h_{\xi_i})$ are the induced mappings on $G \bar{\ltimes} M$.*

The equivalence between (1) and (4) in Theorem 3.12 may be viewed as a dynamical systems analogue of [23, Theorem 1.13], which characterizes amenability of a locally compact group G through a net (u_i) of normalized positive definite functions on G whose multipliers converge to the identity of $VN(G)$ in the point weak* topology.

For C^*-dynamical systems, we complement the recent work of Buss, Echterhoff and Willett [**15**] by showing the equivalence between their notion of amenability, amenability of the universal enveloping W^*-system, and a particular case of the 1-positive weak approximation property of Bédos and Conti [**8**] (suitably generalized to the locally compact setting). We also obtain an analogous Herz–Schur multiplier characterization at the level of the reduced crossed product. Our results in this context are summarized as follows.

THEOREM 1.2. *Let (A, G, α) be a C^*-dynamical system. The following conditions are equivalent.*

(1) *(A, G, α) is amenable in the sense of* [**15**].

(2) *There exists a net (ξ_i) in $C_c(G, \ell^2(A))$ such that:*

  (a) *$\langle \xi_i, \xi_i\rangle \le 1$ for all i;*

  (b) *$h_{\xi_i}(e) \to \mathrm{id}_A$ in the point norm topology;*

  (c) *$\Theta(h_{\xi_i}) \to \mathrm{id}_{G\ltimes A}$ in the point norm topology,*

  where $h_{\xi_i}(s)(a) = \langle \xi_i, (1\otimes 1\otimes a)(\lambda_s \otimes 1\otimes \alpha_s)\xi_i\rangle$ are the associated completely positive Herz–Schur multipliers in the sense of [**31**], *and $\Theta(h_{\xi_i})$ are the induced mappings on $G \ltimes A$.*

(3) *There exists a net (ξ_i) in $C_c(G, \ell^2(A))$ such that $\langle \xi_i, \xi_i\rangle \le 1$ for all i, and*

$$\|h_{\xi_i}(f(s)) - f(s)\| \to 0, \quad f \in C_c(G, A),$$

  uniformly for s in compact subsets of G.

(4) *The universal W^*-dynamical system $(A''_\alpha, G, \overline{\alpha})$ (from* [**26**]*) is amenable.*

Moreover, when $Z(A^{**}) = Z(A)^{**}$, the net (ξ_i) can be chosen in $C_c(G, Z(A))$, in which case $h_{\xi_i}(s)(a) = a\langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i\rangle$, $s \in G$, $a \in A$.

The equivalence (1) $\Leftrightarrow$ (4) generalizes the corresponding result for exact locally compact groups [**15**, Proposition 3.10].

As a corollary to Theorem 1.2, when $Z(A^{**}) = Z(A)^{**}$, amenability of (A, G, α) is equivalent to the 1-positive approximation property of Exel and Ng [**19**]†. It follows that a commutative C^*-dynamical system $(C_0(X), G, \alpha)$ is amenable in the sense of [**15**] if and only if the action $G \curvearrowright X$ is topologically amenable (see Corollary 4.14). This generalizes [**4**, Théorème 4.9] from discrete groups to arbitrary locally compact groups, and shows that amenability of $(C_0(X), G, \alpha)$ coincides with topological amenability of the transformation groupoid $G \ltimes X$. Combining Corollary 4.14 with the recent result [**15**, Theorem 5.16] of Buss, Echterhoff and Willett, we obtain a positive answer to the long-standing open question whether topological amenability and measurewise amenability coincide for actions $G \curvearrowright X$ when G and X are second countable.

The paper is organized as follows. We begin in §2 with preliminaries on dynamical systems and vector-valued integration. Section 3 contains our results on amenable

† After this paper appeared in preprint, Ozawa and Suzuki showed (using Theorem 1.1) that amenability and the positive approximation property coincide for arbitrary C^*-dynamical systems [**32**].

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W^*-dynamical systems as well as results of independent interest which build on the recent theory of Herz–Schur multipliers for crossed products [9, 10, 30, 31]. Section 4 contains our results on amenable C^*-dynamical systems.

2. *Preliminaries*

2.1. *Vector-valued integration.* Throughout this subsection S will be a locally compact Hausdorff space with positive Radon measure μ.

For a Banach space B, we let $L^1(S, B)$ denote the space of (locally almost everywhere (a.e.) equivalence classes of) Bochner integrable functions $f : S \to B$ with the norm $\|f\| = \int_S \|f\| \, d\mu(s)$. By the Pettis measurability theorem and Bochner's theorem (see [38, Section 2.3]), for $f : S \to B$ supported on a σ-finite set, $f \in L^1(S, B)$ if and only if f is weakly measurable, essentially separably valued, and satisfies $\int_S \|f(s)\| \, d\mu(s) < \infty$. In particular, there is a canonical map $C_c(S, B) \to L^1(S, B)$, where $C_c(S, B)$ denotes the continuous B-valued functions of compact support. It is well known that $L^1(S, B) \cong L^1(S, \mu) \otimes^\pi B$ isometrically, where $\otimes^\pi$ is the Banach space projective tensor product (see, for example, [42, Proposition IV.7.14]).

If M is a von Neumann algebra we have the following canonical identifications:

$$(L^\infty(S, \mu)\overline{\otimes}M)_* \cong L^1(S, M_*) \cong L^1(S, \mu) \otimes^\pi M_*,$$

where $\overline{\otimes}$ denotes the von Neumann tensor product (see [42, Proposition IV.7.14 and Theorem IV.7.17].) We remark that $L^\infty(S, \mu)\overline{\otimes}M$ does not necessarily coincide with the space $L^\infty(S, M)$ of essentially bounded w^*-locally measurable functions from S to M since we do not assume that M_* is separable (see [39, 41]). However, by [42, Theorem IV.7.17], for each $F \in L^\infty(S)\overline{\otimes}M$, there exists a weak*-measurable function $\tilde{F} : S \to M$ such that, for every $g \in L^1(S, M_*)$, the function $s \mapsto \langle \tilde{F}(s), g(s)\rangle$ is a measurable function on S, and

$$\langle F, g\rangle = \int_S \langle \tilde{F}(s), g(s)\rangle \, d\mu(s), \quad g \in L^1(S, M_*).$$

In this case, we will say that $\tilde{F}$ *represents* F, and usually abuse notation by omitting the tilde in the latter centered equation. There are some pitfalls that one must take care to avoid though; for example, if $S = [0, 1]$ with Lebesgue measure, and $M = \ell^\infty[0, 1]$ is the space of all bounded functions on $[0, 1]$, then the function $f : S \to M$, $f(t) = \chi_{\{t\}}$, is non-zero everywhere, but f represents $0 \in L^\infty(S)\overline{\otimes}M$.

LEMMA 2.1. *If M is a von Neumann algebra and $\omega \in M_*$, there is a map $\tilde{\omega} : L^1(S, M) \to L^1(S, M_*)$ determined by the formula*

$$\langle \tilde{\omega}(g)(s), x\rangle = \langle \omega, g(s)x\rangle$$

for $g \in L^1(S, M)$, $s \in S$, and $x \in M$. Moreover, $\|\tilde{\omega}\| \leq \|\omega\|$.

Proof. Using the canonical identifications, the map $\tilde{\omega}$ is just $\mathrm{id} \otimes \omega_0 : L^1(S) \otimes^\pi M \to L^1(S) \otimes^\pi M_*$, where $\omega_0 : M \to M_*$ is the operator satisfying $\langle \omega_0(y), x\rangle = \langle \omega, yx\rangle$ for $x, y \in M$. The norm inequality is obvious. □

If A is a C^*-algebra, we let $L^2(S, A)$ denote the Hilbert module completion of $C_c(S, A)$ under the A-valued inner product

$$\langle \xi, \eta \rangle = \int_S \xi(s)^* \eta(s)\, d\mu(s), \quad \xi, \eta \in C_c(S, A).$$

2.2. *Dynamical systems.* A W^*-dynamical system (M, G, α) consists of a von Neumann algebra M endowed with a homomorphism $\alpha : G \to \mathrm{Aut}(M)$ of a locally compact group G such that, for each $x \in M$, the map $G \ni s \to \alpha_s(x) \in M$ is weak* continuous. In this case, the canonical action $G \curvearrowright M_*$ is norm continuous (see [43, Proposition 1.2]). We let M_c denote the unital C^*-subalgebra consisting of those $x \in M$ for which $s \mapsto \alpha_s(x)$ is norm continuous. By [34, Lemma 7.5.1], M_c is weak* dense in M.

The action α induces a normal injective unital $*$-homomorphism

$$\alpha : M \ni x \to (s \mapsto \alpha_{s^{-1}}(x)) \in L^\infty(G)\overline{\otimes} M$$

defined by

$$\langle \alpha(x), F \rangle = \int_G \langle \alpha_{s^{-1}}(x), F(s) \rangle\, ds \quad \text{for } F \in L^1(G, M_*).$$

A normal covariant representation (π, u) of (M, G, α) consists of a normal representation $\pi : M \to \mathcal{B}(H)$ and a unitary representation $u : G \to \mathcal{B}(H)$ such that $\pi(\alpha_s(x)) = u_s \pi(x) u_{s^{-1}}$ for all $x \in M$, $s \in G$. When (π, u) is a normal covariant representation of (M, G, α) (this includes the case when $M \subseteq \mathcal{B}(H)$ is standardly represented, since in this case there exists a unique strongly continuous unitary representation $u : G \to \mathcal{B}(H)$ such that $\alpha_s(x) = u_s x u_{s^{-1}}$ by [22, Corollary 3.6]), there is corresponding generator $U \in L^\infty(G)\overline{\otimes}\mathcal{B}(H)$, defined by

$$\langle U, F \rangle = \int_G \langle u_s, F(s) \rangle \quad \text{for } F \in L^1(G, M_*),$$

and we have $\alpha(x) = U^*(1 \otimes x)U$, $x \in M$. Moreover, for any $\xi \in L^2(G, H)$,

$$\begin{aligned} U(\lambda_s \otimes 1)\xi(t) &= u_t((\lambda_s \otimes 1)\xi(t)) = u_t(\xi(s^{-1}t)) \\ &= u_s u_{s^{-1}t}(\xi(s^{-1}t)) \\ &= u_s(U\xi(s^{-1}t)) \\ &= (\lambda_s \otimes u_s)U\xi(t). \end{aligned}$$

Hence, $U(\lambda_s \otimes 1) = (\lambda_s \otimes u_s)U$ for any $s \in G$.

A C^*-dynamical system (A, G, α) consists of a C^*-algebra endowed with a homomorphism $\alpha : G \to \mathrm{Aut}(A)$ of a locally compact group G such that for each $a \in A$, the map $G \ni s \mapsto \alpha_s(a) \in A$ is norm continuous.

A covariant representation (π, σ) of (A, G, α) consists of a representation $\pi : A \to \mathcal{B}(H)$ and a unitary representation $\sigma : G \to \mathcal{B}(H)$ such that $\pi(\alpha_s(a)) = \sigma_s \pi(a) \sigma_{s^{-1}}$ for all $a \in A$, $s \in G$. Given a covariant representation (π, σ), we let

$$(\pi \times \sigma)(f) = \int_G \pi(f(t))\sigma_t \, dt, \quad f \in C_c(G, A).$$

The full crossed product $G \ltimes_f A$ is the completion of $C_c(G, A)$ in the norm

$$\|f\| = \sup_{(\pi,\sigma)} \|(\pi \times \sigma)(f)\|$$

where the sup is taken over all covariant representations (π, σ) of (A, G, α).

Let $A \subseteq \mathcal{B}(H)$ be a faithful non-degenerate representation of A. Then $(\alpha, \lambda \otimes 1)$ is a covariant representation on $L^2(G, H)$, where

$$\alpha(a)\xi(t) = \alpha_{t^{-1}}(a)\xi(t), \quad (\lambda \otimes 1)(s)\xi(t) = \xi(s^{-1}t), \quad \xi \in L^2(G, H).$$

The reduced crossed product $G \ltimes A$ is defined to be the norm closure of $(\alpha \times (\lambda \otimes 1))(C_c(G, A))$. This definition is independent of the faithful non-degenerate representation $A \subseteq \mathcal{B}(H)$. We often abbreviate $\alpha \times (\lambda \otimes 1)$ as $\alpha \times \lambda$. Recall that $C_c(G, A)$ is a $*$-algebra under the operations

$$f \star g(s) = \int_G f(t)\alpha_t(g(t^{-1}s)) \, dt, \quad f^*(s) = \Delta(s^{-1})\alpha_s(f(s^{-1})^*), \quad f, g \in C_c(G, A),$$

and that $\alpha \times \lambda$ is a $*$-homomorphism.

Analogously to the group setting, dual spaces of crossed products can be identified with certain A^*-valued functions on G. We review aspects of this theory below and refer the reader to [**34**, Chs. 7.6, 7.7] for details.

For each C^*-dynamical system (A, G, α) there is a universal covariant representation (π, σ) such that

$$G \ltimes_f A \subseteq C^*(\pi(A) \cup \sigma(G)) \subseteq M(G \ltimes_f A).$$

Each functional $\varphi \in (G \ltimes_f A)^*$ then defines a function $\Phi : G \to A^*$ by

$$\langle \Phi(s), a\rangle = \varphi(\pi(a)\sigma_s), \quad a \in A, \ s \in G. \tag{1}$$

Let $B(G \ltimes_f A)$ denote the resulting space of A^*-valued functions on G. An element $\Phi \in B(G \ltimes_f A)$ is *positive definite* if it arises from a positive linear functional φ as above. We let $A(G \ltimes_f A)$ denote the subspace of $B(G \ltimes_f A)$ whose associated functionals φ are of the form

$$\varphi(x) = \sum_{n=1}^{\infty} \langle \xi_n, \alpha \times \lambda(x)\eta_n\rangle, \quad x \in G \ltimes_f A,$$

for sequences (ξ_n) and (η_n) in $L^2(G, H)$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$. Then $A(G \ltimes_f A)$ is a norm closed subspace of $(G \ltimes_f A)^*$ which can be identified with $((G \ltimes A)'')_*$. Explicitly, the duality is given as follows. Suppose $x \in (G \ltimes A)'' \subseteq \mathcal{B}(L^2(G, H))$ and $\Phi \in A(G \ltimes_f A)$. Take (ξ_n), (η_n) in $L^2(G, H)$ such that $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$, $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$, and $\langle \Phi(s), a\rangle = \sum_{n=1}^{\infty} \langle \xi_n, \alpha \times \lambda(\pi(a)\sigma_s)\eta_n\rangle$ for all $s \in G$, $a \in A$. Then

$$\langle \Phi, x\rangle = \sum_{n=1}^{\infty} \langle \xi_n, x\eta_n\rangle.$$

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A function $h : G \to A$ is of positive type (with respect to α) if for every $n \in \mathbb{N}$, and $s_1, \ldots, s_n \in G$, we have

$$[\alpha_{s_i}(h(s_i^{-1}s_j)] \in M_n(A)^+.$$

We let $P_1(A, G, \alpha)$ denote the convex set of positive type functions with $\|h_i(e)\| \leq 1$.

Every C^*-dynamical system (A, G, α) admits a unique universal W^*-dynamical system $(A''_\alpha, G, \overline{\alpha})$ [26]. We review this construction, taking an $L^1(G)$-module perspective. In [15], the authors study $(A''_\alpha, G, \overline{\alpha})$ from a different, equivalent perspective.

First, A becomes a right operator $L^1(G)$-module in the canonical fashion by slicing the corresponding non-degenerate representation

$$\alpha : A \ni a \mapsto (s \mapsto \alpha_{s^{-1}}(a)) \in C_b(G, A) \subseteq L^\infty(G)\overline{\otimes}A^{**}.$$

Explicitly, this action is given by

$$a * f = \int_G f(s)\alpha_{s^{-1}}(a)\, ds$$

for $a \in A$, $f \in L^1(G)$. By duality we obtain a left operator $L^1(G)$-module structure on A^* via

$$\alpha^*|_{L^1(G)\widehat{\otimes}A^*} : L^1(G)\widehat{\otimes}A^* \to A^*,$$

where $\widehat{\otimes}$ denotes the operator space projective tensor product (see, for example, [12, 1.5.11]). Then G acts in a norm-continuous fashion on the essential submodule

$$A^*_c := \langle L^1(G) * A^*\rangle,$$

where $\langle\cdot\rangle$ denotes closed linear span. The same argument in [34, Lemma 7.5.1] shows that A^*_c coincides with the norm-continuous part of A^* (that is, the set of $\varphi \in A^*$ such that each map $G \to A^*$, $s \mapsto \varphi \circ \alpha_s$ is norm continuous), hence the notation. This fact was also noted by Hamana in [24, Proposition 3.4(i)]. We therefore obtain a point-weak* continuous action of G on the dual space $(A^*_c)^*$ by surjective complete isometries. Clearly

$$(A^*_c)^* \cong A^{**}/(A^*_c)^\perp \tag{2}$$

completely isometrically and weak*-weak* homeomorphically as right $L^1(G)$-modules, where the canonical $L^1(G)$-module structure on A^{**} is obtained by slicing the normal cover of α, which is the normal $*$-homomorphism

$$\widetilde{\alpha} = (\alpha^*|_{L^1(G)\widehat{\otimes}A^*})^* : A^{**} \to L^\infty(G)\overline{\otimes}A^{**}.$$

Note that $\widetilde{\alpha}|_{M(A))}$ is the unique strict extension of α, and is therefore injective [29, Proposition 2.1]. However, on A^{**}, $\widetilde{\alpha}$ can have a large kernel. On the one hand, its kernel is of the form $(1 - z)A^{**}$ for some projection $z \in Z(A^{**})$. On the other hand, by definition of the $L^1(G)$-action on A^{**}, $\mathrm{Ker}(\widetilde{\alpha}) = (A^*_c)^\perp$. It follows that $(A^*_c)^*$ is completely isometrically weak*-weak* order isomorphic to zA^{**}, where we equip $(A^*_c)^*$ with the quotient operator system structure from A^{**}. We can therefore transport the point-weak* continuous G-action on $(A^*_c)^*$ to $A''_\alpha := zA^{**}$, yielding a W^*-dynamical system $(A''_\alpha, G, \overline{\alpha})$, where $\overline{\alpha} : G \to \mathrm{Aut}(A''_\alpha)$ is given by

$$\overline{\alpha}_t(zx) = z((\alpha_t)^{**}(x)), \quad x \in A^{**},\ t \in G.$$

The associated normal injective $*$-homomorphism

$$\overline{\alpha} : A''_{\alpha} \to L^{\infty}(G)\overline{\otimes}A''_{\alpha}$$

is $(\mathrm{id} \otimes \mathrm{Ad}(z)) \circ \widetilde{\alpha}|_{A''_{\alpha}}$. Hence, the $L^1(G)$-action on A''_{α} satisfies

$$(zx) * f = (f \otimes \mathrm{id})\overline{\alpha}(x) = \mathrm{Ad}(z)((f \otimes \mathrm{id})\widetilde{\alpha}(x)) = z(x * f),$$

for $f \in L^1(G)$, and $x \in A^{**}$. We emphasize that with this structure A''_{α} is not necessarily an $L^1(G)$-submodule of A^{**}, rather $\mathrm{Ad}(z) : A^{**} \to A''_{\alpha}$ is an $L^1(G)$-complete quotient map.

Finally, as $\widetilde{\alpha}|_{M(A)}$ is an injective $*$-homomorphism, for all $x \in M(A)$ we have

$$\|x\| = \|\widetilde{\alpha}(x)\| = \|\widetilde{\alpha}(zx)\| = \|zx\|.$$

It follows that $\mathrm{Ad}(z) : M(A) \hookrightarrow A''_{\alpha}$ is a G-equivariant isometry.

2.3. *Operator space theory.* In §4 below we will freely use several gadgets from operator space theory. We give a quick review of some of the topics we shall need. See [16] or [12, Ch. 1] for more details.

For a Hilbert space H, the *column Hilbert space* H_c is the operator space attained by equipping H with matrix norms arising from fixing any norm-one $\eta \in H$ and considering the injection $H \to \mathcal{B}(H)$, $\zeta \mapsto \zeta \otimes \eta$, where $\zeta \otimes \eta$ is the rank-one operator $\xi \mapsto \langle \xi, \eta \rangle \zeta$.

Given two operator spaces X, Y, the *Haagerup tensor product* $X \otimes^h Y$ is the unique operator space with following universal property: for every completely bounded bilinear map $u : X \times Y \to W$ into an operator space W, there exists a completely bounded linear map $\tilde{u} : X \otimes^h Y \to W$ such that $\|\tilde{u}\|_{cb} = \|u\|_{cb}$ and $\tilde{u}(x \otimes y) = u(x, y)$ for all $x \in X$, $y \in Y$. (See [12, 1.5.4] for the definition of a completely bounded bilinear map.)

We will implicitly use the fact that for a Hilbert space H and any operator space X, $H_c \otimes^h X = H_c \otimes_{\min} X$, where the latter is the operator space minimal, or spatial tensor product.

Given an operator space X, the *operator space dual* X^* is the usual dual Banach space X^* equipped with matrix norms from the canonical identification $M_n(X^*) = \mathcal{CB}(X, M_n)$. For two dual operator spaces X^*, Y^*, the *weak* Haagerup tensor product* $X^* \otimes^{w^*h} Y^*$ is defined to be the operator space dual $(X \otimes^h Y)^*$. It turns out that $X^* \otimes^{w^*h} Y^*$ contains $X^* \otimes^h Y^*$ completely isometrically [12, 1.5.9].

3. *Amenable W^*-dynamical systems*

A W^*-dynamical system (M, G, α) is *amenable* [2] if there exists a projection of norm one $P : L^{\infty}(G)\overline{\otimes}M \to M \cong 1 \otimes M$ such that $P \circ (\lambda_s \otimes \alpha_s) = \alpha_s \circ P$, $s \in G$, where λ denotes the left translation action on $L^{\infty}(G)$. For example, $(L^{\infty}(G), G, \lambda)$ is always amenable, and G is amenable if and only if the trivial action $G \curvearrowright \{x_0\}$ is amenable, in which case P becomes a left invariant mean on $L^{\infty}(G)$. In this section we first establish a Reiter property for amenability, and then apply this result to obtain the Herz–Schur multiplier characterization from Theorem 1.1.

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3.1. *A Reiter property.* In this subsection we establish a Reiter property for amenable W^*-dynamical systems, generalizing [4, Théorème 3.3] from discrete groups to arbitrary locally compact groups. We require several preparations. The first is a continuous version of [4, Lemme 3.1].

Given a locally compact Hausdorff space S with positive Radon measure μ, and a von Neumann algebra M, we let

$$K_1^+(S, Z(M)_c) = \left\{ g \in C_c(S, Z(M)_c^+) \mid \int_S g(s)\, d\mu(s) \le 1 \right\},$$

where $C_c(S, Z(M)_c^+)$ is the space of norm-continuous $Z(M)_c^+$-valued functions on S with compact support. Let $\mathcal{B}_M(L^\infty(S)\overline{\otimes}M, M)$ denote the Banach space of bounded M-bimodule maps from $L^\infty(S)\overline{\otimes}M$ to M, and let $\mathcal{P}$ denote the convex subset of $\mathcal{B}_M(L^\infty(S)\overline{\otimes}M, M)$ given by the positive contractive M-bimodule maps. Every map $P \in \mathcal{P}$ is automatically completely positive, so that $\|P\| = \|P(1)\|$.

Each $g \in K_1^+(S, Z(M)_c)$ gives rise to an element $P_g \in \mathcal{P}$ by means of the formula

$$\langle P_g(F), \omega\rangle = \int_S \langle F(s)g(s), \omega\rangle\, d\mu(s), \quad F \in L^\infty(S)\overline{\otimes}M,\ \omega \in M_*.$$

The latter expression makes sense irrespective of the choice of representative of F since it is equal to $\langle \tilde{\omega}(g), F\rangle$, viewing $g \in L^1(S, M)$. We will usually shorten the previously displayed formula by writing $P_g(F) = \int_S F(s)g(s)\, d\mu(s)$ for $F \in L^\infty(S)\overline{\otimes}M$.

Let $\mathcal{P}_K := \{P_g \mid g \in K_1^+(S, Z(M)_c)\} \subseteq \mathcal{P}$.

LEMMA 3.1. *Let S be a locally compact Hausdorff space with positive Radon measure μ and let M be a commutative von Neumann algebra. Then $\mathcal{P}_K$ is dense in $\mathcal{P}$ in the point-weak* topology of $\mathcal{B}(L^\infty(S)\overline{\otimes}M, M)$.*

Proof. The majority of the proof follows that of [4, Lemme 3.1], but we include some details for the convenience of the reader. First,

$$\mathcal{B}_M(L^\infty(S)\overline{\otimes}M, M) = (L^\infty(S)\overline{\otimes}M \otimes_M^\pi M_*)^*,$$

where $\otimes_M^\pi$ is the M-bimodule Banach space projective tensor product. By definition of the projective tensor norm together with the Radon–Nikodym theorem, every element in $(L^\infty(S)\overline{\otimes}M) \otimes_M^\pi M_*$ is the equivalence class of an element of the form $F \otimes \varphi$ with $F \in L^\infty(S)\overline{\otimes}M$ and $\varphi \in M_*^+$, as shown in [4, Lemme 3.1]. By convexity it suffices to show that $\mathcal{P}$ is contained in the bipolar of $\mathcal{P}_K$. Let $F_0 \in L^\infty(S)\overline{\otimes}M$ and $\varphi \in M_*^+$ be such that

$$\mathrm{Re}\langle P_g, F_0 \otimes \varphi\rangle = \mathrm{Re}\, \varphi\left(\int_S F_0(s)g(s)\, d\mu(s) \right) \le 1, \quad g \in K_1^+(S, M_c).$$

If $H_0 = \mathrm{Re}(F_0)$, then

$$\varphi\left(\int_S H_0(s)g(s)\, d\mu(s) \right) \le 1, \quad g \in K_1^+(S, M_c).$$

Let C denote the weak* closure of $\{\int_S H_0(s)g(s)\, d\mu(s) \mid g \in K_1^+(S, M_c)\}$ in M. Given $x_1, x_2 \in C$ and a projection $e \in M$, we have $x_1 e + x_2(1-e) \in C$. Indeed, pick nets

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(g_i), (f_j) in $K_1^+(S, M_c)$ such that

$$x_1 = w^* \lim_i \int_S H_0(s) g_i(s)\, d\mu(s), \quad x_2 = w^* \lim_j \int_S H_0(s) f_j(s)\, d\mu(s).$$

Without loss of generality, we can assume the nets (g_i) and (f_j) have the same index set. Since M_c is weakly dense in M, by Kaplansky's density theorem, pick a net (p_k) of positive operators in the unit ball of M_c such that $p_k \to e$ strongly (and hence weak*, by boundedness). Then

$$x_1 e + x_2(1-e) = w^* \lim_k\ w^* \lim_i \int_S H_0(s)(g_i(s) p_k + f_i(s)(1-p_k))\, d\mu(s).$$

Since $g_i(1 \otimes p_k) + f_i(1 \otimes (1-p_k)) \in K_1^+(S, M_c)$, combining the iterated limit into a single net, we see that $x_1 e + x_2(1-e) \in C$. Then C is closed under finite suprema using the Stonian structure of the spectrum of M, as in [4, Lemme 3.1].

Now, fix a $*$-monomorphism $\rho : L^\infty(S, \mu) \to \ell^\infty(S, \mu)$, satisfying $q \circ \rho = \mathrm{id}_{L^\infty(S,\mu)}$, where $\ell^\infty(S, \mu)$ is the C^*-algebra of bounded μ-measurable functions on S, and $q : \ell^\infty(S, \mu) \to L^\infty(S, \mu)$ is the canonical quotient map. Such a lifting exists by [27, Corollary 2]. For $s \in S$, $e_s := \mathrm{ev}_s \circ \rho \in L^\infty(S, \mu)^*$ is a state on $L^\infty(S, \mu)$.

Given $F \in L^\infty(S) \overline{\otimes} M$, define $F_\rho : S \to M$ by

$$\langle F_\rho(s), \omega \rangle = \langle e_s, (\mathrm{id} \otimes \omega) F \rangle, \quad s \in G,\ \omega \in M_*,$$

We claim that F_ρ represents F. First, by definition, $\langle F_\rho(s), \omega \rangle = \rho((\mathrm{id} \otimes \omega)F)(s)$ for every $\omega \in M_*$, hence the function $s \mapsto F_\rho(s)$ is weak* measurable as $\rho((\mathrm{id} \otimes \omega)F) \in \ell^\infty(S, \mu)$. It follows that, for any simple tensor $g \in L^1(S, M_*) = L^1(S, \mu) \otimes^\pi M_*$, the function $\varphi_g : s \mapsto \langle F_\rho(s), g(s) \rangle$ is measurable as φ_g is a product of measurable functions. Since a pointwise a.e. limit of measurable functions is measurable, it follows that φ_g is measurable for any $g \in L^1(S, M_*)$. Next, if $g \in L^1(S, \mu)$ and $\omega \in M_*$,

$$\begin{aligned}
\int_S g(s) \langle F_\rho(s), \omega \rangle d\mu(s) &= \int_S g(s) \rho((\mathrm{id} \otimes \omega)F)(s)\, d\mu(s) \\
&= \int_S g(s) q(\rho((\mathrm{id} \otimes \omega)F))(s)\, d\mu(s) \\
&= \langle g, (\mathrm{id} \otimes \omega) F \rangle \\
&= \langle g \otimes \omega, F \rangle,
\end{aligned}$$

where the second equality uses the fact that integration only depends on a.e. equivalence classes. The formula $\langle F, g \rangle = \int_S \langle F_\rho(s), g(s) \rangle\, d\mu(s)$ is proved for general $g \in L^1(S, M_*)$ using the observation that F_ρ is bounded.

Let (g_i^s) be a net of states in $L^1(S, \mu)$ approximating e_s weak*. By a further approximation using norm density of $C_c(S)$ in $L^1(S, \mu)$, we may take each $g_i^s \in C_c(S)^+$ with $\int_S g_i^s(t)\, d\mu(t) \leq 1$. Viewing $g_i^s \in K_1^+(S, M_c)$ in the canonical way (M_c is unital), for every $F \in L^\infty(S) \overline{\otimes} M$, it follows that

$$F_\rho(s) = w^* \lim_i \int_S g_i^s(t) F(t)\, d\mu(t) = w^* \lim_i P_{g_i^s}(F).$$

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Since $(H_0)_\rho(s) \in C$, we have $(H_0)_\rho(s)^+ = (H_0)_\rho(s) \vee 0 \in C$. Define $m = \sup_{s\in S}(H_0)_\rho(s)^+ \in M$. Then, by normality of φ, we have $\varphi(m) \leq 1$. Since $(H_0)_\rho(s) \leq m$ in M for all s, it follows that $H_0 \leq 1 \otimes m$ in $L^\infty(S)\overline{\otimes}M$. Indeed, if $g \in L^1(S, M_*)^+$ is a positive normal functional on $L^\infty(S)\overline{\otimes}M$, then

$$\langle H_0, g\rangle = \int_S \langle (H_0)_\rho(s), g(s)\rangle \, d\mu(s) \leq \int_S \langle m, g(s)\rangle \, d\mu(s) = \langle 1 \otimes m, g\rangle.$$

Thus, for every $P \in \mathcal{P}$ we have

$$P(H_0) \leq P(1 \otimes m) = mP(1) \leq m,$$

so that

$$\mathrm{Re}\langle P, F_0 \otimes \varphi\rangle = \varphi(P(H_0)) \leq \varphi(m) \leq 1.$$

Hence, P belongs to the bipolar of $\mathcal{P}_K$. □

Remark 3.2. In the special case where $M = L^\infty(X, \nu)$ and (X, ν) and (S, μ) are both σ-finite, the conclusion of Lemma 3.1 follows from [7, Lemma 1.2.6].

Similarly to [4], we consider the following two locally convex topologies on the Bochner space $L^1(S, M)$, where S and M are as in Lemma 3.1. The first, denoted by τ_n, is generated by the family of semi-norms $\{p_\omega \mid \omega \in M_*^+\}$, where

$$p_\omega(g) = \left\langle \omega, \int_S |g(s)| \, d\mu(s)\right\rangle = \int_S \langle |g(s)|, \omega\rangle \, d\mu(s).$$

This is indeed well defined since $s \mapsto |g(s)|$ is Bochner integrable whenever g is. The second, denoted by τ_F, is generated by the family of semi-norms

$$\{p_{F,\omega} \mid F \in L^\infty(S)\overline{\otimes}M, \ \omega \in M_*^+\} \quad \text{where } p_{F,\omega}(g) = \left| \int_S \langle g(s)F(s), \omega\rangle \, d\mu(s)\right|.$$

To see that this is well defined, define $\tilde{\omega}(g) : S \to M_*$ by $\langle x, \tilde{\omega}(g)(s)\rangle = \langle g(s)x, \omega\rangle$ for $x \in M$. Then, by Lemma 2.1, $\tilde{\omega}(g) \in L^1(S, M_*)$. A routine argument then shows that $s \mapsto \langle F(s), \tilde{\omega}(g)(s)\rangle = \langle g(s)F(s), \omega\rangle$ is measurable, and integrability of this function is easy to check.

Since $p_{F,\omega}(g) \leq \|F\| p_\omega(g)$, it follows that τ_n is stronger than τ_F.

LEMMA 3.3. *Let V be a convex subset of $L^1(S, M)$ such that every function in V is supported on a σ-finite subset. Then $\overline{V}^{\tau_F} = \overline{V}^{\tau_n}$.*

Proof. Since τ_n is stronger than τ_F, it suffices to show that $\overline{V}^{\tau_F} \subseteq \overline{V}^{\tau_n}$. Let (g_i) be a net in V converging to zero with respect to τ_F. Then, by definition of τ_F, $\tilde{\omega}(g_i) \to 0$ weakly in $L^1(S, M_*)$ for all $\omega \in M_*^+$. By Mazur's theorem, there exists a net $(g_{K,\varepsilon})$ in V indexed by finite subsets K of M_*^+ and $\varepsilon > 0$ such that

$$\|\tilde{\omega}(g_{K,\varepsilon})\|_{L^1(S,M_*)} < \varepsilon, \quad \omega \in K.$$

For (an a.e. representative of) $g \in L^1(S, M)$ and $s \in S$, let $g(s) = u_s|g(s)|$ be the polar decomposition in M. Then, since

$$\langle \omega, |g(s)|\rangle = |\langle \tilde{\omega}(g)(s), u_s^*\rangle| \leq \sup\{|\langle \tilde{\omega}(g)(s), x\rangle| : x \in M_{\|\cdot\|\leq 1}\} = \|\tilde{\omega}(g)(s)\|_{M_*}$$

for all $\omega \in M_*^+$ and $s \in S$, we have

$$p_\omega(g) = \int_S \langle \omega, u_s^* g(s)\rangle \, d\mu(s) \leq \int_S \|\tilde{\omega}(g)(s)\|_{M_*} \, d\mu(s) = \|\tilde{\omega}(g)\|_{L^1(S,M_*)}$$

for all $g \in L^1(S, M)$. It follows that $g_{K,\varepsilon} \to 0$ with respect to τ_n. □

The next lemma will be used to upgrade pointwise asymptotic G-invariance in Reiter's property to uniform asymptotic G-invariance on compacta. This is a generalization of the equivalence of the classical finite and compact Reiter's properties. Our proof generally follows that of [35, Proposition 6.10].

First, we record a useful, simple lemma, the proof of which is omitted.

LEMMA 3.4. *Suppose X is a Banach space and (φ_t) is a bounded net in X^*. Then $\varphi_t \to \varphi$ weak* in X^* if and only if $\varphi_t(x) \to \varphi(x)$ uniformly on compact subsets of X.*

LEMMA 3.5. *Let (M, G, α) be a commutative W^*-dynamical system. The following conditions are equivalent.*

(1) *There exists a net (g_i) in $K_1^+(G, M_c)$ satisfying $w^* \lim_i \int_G g_i(s)\, ds = 1$ such that*

$$w^* \lim_i \int_G |g_i(s) - (\lambda_t \otimes \alpha_t)(g_i)(s)| \, ds = 0 \quad \textit{for all } t \in G.$$

(2) *There exists a net (g_i) in $K_1^+(G, M_c)$ satisfying $w^* \lim_i \int_G g_i(s)\, ds = 1$ such that*

$$w^* \lim_i \int_G |g_i(s) - (\lambda_t \otimes \alpha_t)(g_i)(s)| \, ds = 0 \quad \textit{uniformly on compact subsets of } G.$$

Proof. Since (2) clearly implies (1), we only need to show (1) implies (2). In preparation, note that $C_c(G, M_c)$ is a left module over the algebra $\mathcal{M}_c(G)$ of compactly supported Radon measures on G via the action

$$\mu \star g(s) = \int_G (\lambda_t \otimes \alpha_t)(g)(s) \, d\mu(t) = \int_G \alpha_t(g(t^{-1}s)) \, d\mu(t),$$

for $\mu \in \mathcal{M}_c(G)$, $g \in C_c(G, M_c)$, and $s \in G$. (In fact, this action extends to give the injective Banach space tensor product $L^1(G) \otimes^\epsilon M_c$ a left Banach $\mathcal{M}(G)$-module action, but we will not need this.) Note also that for $g \in C_c(G, M_c)$, $\mu \in \mathcal{M}_c(G)$, and $\omega \in M_*^+$,

$$\begin{aligned}\left\langle \omega, \int_G |\mu \star g(s)| \, ds \right\rangle &= \int_G \left\langle \omega, \left| \int_G \alpha_t(g(t^{-1}s)) \, d\mu(t) \right| \right\rangle ds \\ &\leq \int_G \left\langle \omega, \int_G |\alpha_t(g(t^{-1}s))| \, d|\mu|(t) \right\rangle ds \\ &= \int_G \int_G \langle \omega, |\alpha_t(g(t^{-1}s))|\rangle \, ds \, d|\mu|(t)\end{aligned}$$

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$$
\begin{aligned}
&= \int_G \int_G \langle \omega, \alpha_t(|g(s)|)\rangle \, ds \, d|\mu|(t) \\
&= \int_G \int_G \langle (\alpha_t)_*(\omega), |g(s)|\rangle \, ds \, d|\mu|(t) \\
&= \int_G \left\langle (\alpha_t)_*(\omega), \int_G |g(s)| \, ds \right\rangle d|\mu|(t) \qquad (3) \\
&\leq \|\omega\| \|\mu\| \left\| \int_G |g(s)| \, ds \right\|. \qquad (4)
\end{aligned}
$$

(Note that commutativity of M_c is used in the first inequality.)

Let (g_i) be a net as in (1), and fix $f \in C_c(G)^+$ with $\int_G f(s)\, ds = 1$. We will show that $(f \star g_i)$ satisfies (2). It is immediate that $f \star g_i \in C_c(G, M_c^+)$, and the inequality $\int_G f \star g_i(s)\, ds \leq 1$ is straightforward to check. Now let $\omega \in M_*$. Since $(\int_G g_i(s)\, ds)$ is a bounded net in M converging weak* to 1, and $\{(\alpha_t)_*(\omega) \mid t \in \mathrm{supp}(f)\}$ is norm compact in M_* by norm continuity of the action $G \curvearrowright M_*$, we have, by Lemma 3.4,

$$
\begin{aligned}
\int_G \langle \omega, f \star g_i(s)\rangle \, ds &= \int_G f(t) \int_G \langle (\alpha_t)_*(\omega), g_i(t^{-1}s)\rangle \, ds \, dt \\
&= \int_G f(t) \int_G \langle (\alpha_t)_*(\omega), g_i(s)\rangle \, ds \, dt \\
&\to \int_G f(t) \langle (\alpha_t)_*(\omega), 1\rangle \, dt \\
&= \int_G f(t) \langle \omega, 1\rangle \, dt \\
&= \langle \omega, 1\rangle,
\end{aligned}
$$

so that $\int_G f \star g_i(s)\, ds \to 1$ weak* in M.

The remainder of the proof closely follows that of [35, Proposition 6.10]. Let $C \subseteq G$ be compact, $\omega \in (M_*)^+_{\|\cdot\|=1}$, and $\varepsilon > 0$. Put $C_1 = C \cup \{e\}$ and $\delta = \varepsilon/6$. There exists a neighborhood U of e such that $\|\lambda_y f - f\|_{L^1(G)} < \delta$, whenever $y \in U$ (by [25, 20.4]). Since C_1 is compact, there exists a compact neighborhood V of e such that $t^{-1}Vt \subseteq U$ for every $t \in C_1$ (by [25, 4.9]). Hence, for every $r \in V$ and $t \in C_1$, $\|\lambda_{t^{-1}rt} f - f\|_{L^1(G)} < \delta$. Let $h = |V|^{-1}\chi_V$. Then, for $t \in C_1$, we have

$$
\begin{aligned}
\|h * (\lambda_t f) - \lambda_t f\|_{L^1(G)} &= \int_G \left| \int_G h(r) f(t^{-1}r^{-1}s)\, dr - \int_G h(r) f(t^{-1}s)\, dr \right| ds \\
&\leq \int_G h(r) \left(\int_G |f(t^{-1}r^{-1}ts) - f(s)|\, ds \right) dr \\
&= \int_V h(r) \|\lambda_{t^{-1}rt} f - f\|_{L^1(G)}\, dr < \delta. \qquad (5)
\end{aligned}
$$

Let $C' = C_1 \,\mathrm{supp}(f)$. Then C' is compact, and for every $t \in C_1$,

$$
\int_{G \setminus C'} f(t^{-1}s)\, ds = 0. \qquad (6)
$$

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Pick a compact neighborhood W of e such that $\|h * \delta_t - h\|_{L^1(G)} = \|\lambda_{t^{-1}} h^* - h^*\|_{L^1(G)} < \delta$ for every $t \in W$ (by [25, 20.4] again). Then there is an open neighborhood W' of e for which $W'W'^{-1} \subseteq W$. As C' is compact, there are $c_1, \ldots, c_m \in C'$ such that $C' \subseteq \cup_{i=1}^m W'c_i$. Since each $W'c_i$ satisfies $W'c_i(W'c_i)^{-1} \subseteq W$, there exists a finite partition $\{B_j \mid j = 1, \ldots, n\}$ for C' consisting of non-empty Borel sets such that $B_j B_j^{-1} \subseteq W$ for all j. For every $j = 1, \ldots, n$, choose $b_j \in B_j$. Then, for all $t \in B_j$,

$$\|h * \delta_t - h * \delta_{b_j}\|_{L^1(G)} = \|h * \delta_{tb_j^{-1}} - h\|_{L^1(G)} < \delta. \tag{7}$$

Now, using norm compactness of $\{(\alpha_t)_*(\omega) \mid t \in V\}$ and Lemma 3.4 again, condition (1) implies that, for some index value i_0,

$$\langle (\alpha_t)_*(\omega), \int_G |g_i(s) - (\delta_{b_j} \star g_i)(s)|\, ds \rangle < \delta,$$

for every $i \geq i_0$, $j = 1, \ldots, n$ and $t \in V$. To simplify notation for the following calculations, fix $g = g_i$ for $i \geq i_0$. Then, for $j = 1, \ldots, n$, it follows by inequality (3) above that

$$\begin{aligned}\left\langle \omega, \int_G |h \star \delta_{b_j} \star g(s) - h \star g(s)|\, ds \right\rangle &\leq \int_G h(t) \left\langle (\alpha_t)_*(\omega), \int_G |(\delta_{b_j} \star g - g)(s)|\, ds \right\rangle dt \\ &< \int_G h(t)\delta\, dt \\ &= \delta. \end{aligned} \tag{8}$$

Now, for $t \in C_1$, we have, by inequalities (4) and (5),

$$\begin{aligned}\int_G \langle \omega, |(\lambda_t f) \star g(s) - h \star (\lambda_t f) \star g(s)| \rangle\, ds &= \int_G \langle \omega, |(\lambda_t f - h * (\lambda_t f)) \star g(s)| \rangle\, ds \\ &\leq \|\omega\| \|\lambda_t f - h * (\lambda_t f)\| \left\| \int_G |g(s)|\, ds \right\| \\ &< \delta. \end{aligned} \tag{9}$$

Also, if $f' = \lambda_t f$ for $t \in C_1$, then f' is a state and, *applying* (6), (7), (8), and (4), we see that

$$\begin{aligned}&\int_G \langle \omega, |(h \star f' \star g - h \star g)(s)| \rangle\, ds \\ &\quad\leq \int_G \int_G f'(r) \langle \omega, |(h \star \delta_r \star g - h \star g)(s)| \rangle\, dr\, ds \\ &\quad= \int_G \int_{C'} f'(r) \langle \omega, |(h \star \delta_r \star g - h \star g)(s)| \rangle\, dr\, ds \\ &\quad\leq \int_G \left(\sum_{j=1}^n \int_{B_j} f'(r) \langle \omega, |((h * \delta_r - h * \delta_{b_j}) \star g)(s)| \rangle\, dr \right.\\ &\qquad\left. + \sum_{j=1}^n \int_{B_j} f'(r) \langle \omega, |(h \star \delta_{b_j} \star g - h \star g)(s)| \rangle\, dr \right) ds\end{aligned}$$

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$$= \sum_{j=1}^{n} \int_{B_j} f'(r) \int_G \langle \omega, |((h * \delta_r - h * \delta_{b_j}) \star g)(s)| \rangle \, ds \, dr$$
$$+ \sum_{j=1}^{n} \int_{B_j} f'(r) \int_G \langle \omega, |(h \star \delta_{b_j} \star g - h \star g)(s)| \rangle \, ds \, dr$$
$$< 2\delta. \tag{10}$$

Finally, let $t \in C$. Since $t \in C_1$ and $e \in C_1$, by (9) and (10) we have

$$\begin{aligned}
&\int_G \langle \omega, |(\delta_t \star (f \star g) - f \star g)(s)| \rangle \, ds \\
&\leq \int_G \langle \omega, |(f' \star g - h \star (f' \star g))(s)| \rangle \, ds + \int_G \langle \omega, |(h \star (f' \star g) - h \star g)(s)| \rangle \, ds \\
&\quad + \int_G \langle \omega, |(h \star g - h \star f \star g)(s)| \rangle \, ds + \int_G \langle \omega, |(h \star f \star g - f \star g)(s)| \rangle \, ds \\
&< \delta + 2\delta + 2\delta + \delta = \varepsilon.
\end{aligned}$$

It follows that the net $(f \star g_i)$ satisfies

$$w^* \lim_i \int_G |f \star g_i(s) - (\lambda_t \otimes \alpha_t)(f \star g_i)(s)| \, ds = 0,$$

uniformly for t in compact subsets of G. □

We are now in a position to generalize [4, Théorème 3.3] to locally compact groups. The equivalences in the next theorem were independently obtained for exact locally compact groups using different techniques by Buss, Echterhoff and Willett in the recent work [15].

THEOREM 3.6. *Let (M, G, α) be a W^*-dynamical system. The following conditions are equivalent.*

(1) *There exists a net (h_i) of positive type functions in $C_c(G, Z(M)_c)$ such that:*
   (a) $h_i(e) \leq 1$ *for all* i;
   (b) $\lim_i h_i(t) = 1$ *weak*, uniformly on compact subsets.*
(2) *There exists a net (ξ_i) in $C_c(G, Z(M)_c)$ such that:*
   (a) $\langle \xi_i, \xi_i \rangle \leq 1$ *for all* i;
   (b) $\langle \xi_i, (\lambda_t \otimes \alpha_t)\xi_i \rangle \to 1$ *weak*, uniformly on compact subsets.*
(3) *There exists a net (g_i) in $K_1^+(G, Z(M)_c)$ such that:*
   (a) $\int_G g_i(s) \, ds \to 1$ *weak*;*
   (b) $\int_G |(\lambda_t \otimes \alpha_t)g_i(s) - g_i(s)| \, ds \to 0$ *weak*, uniformly on compact subsets.*
(4) *There exists a G-equivariant projection of norm one from $L^\infty(G)\overline{\otimes}M$ onto M.*
(5) *There exists a G-equivariant projection of norm one from $L^\infty(G)\overline{\otimes}Z(M)$ onto $Z(M)$.*

Proof. (2) ⇒ (1) is obvious by taking $h_i(t) = \langle \xi_i, (\lambda_t \otimes \alpha_t)\xi_i \rangle$ (noting that the compact support of ξ_i implies that the range of h_i indeed lies in the norm-closed subalgebra $Z(M)_c$).

(1) $\Rightarrow$ (2): By [4, Proposition 2.5] there exists a net (ξ_i) in $L^2(G, Z(M)_c)$ satisfying properties (2)(a) and (2)(b). By norm density of $C_c(G, Z(M)_c)$ in $L^2(G, Z(M)_c)$, a further approximation yields the desired net in $C_c(G, Z(M)_c)$.

(2) $\Leftrightarrow$ (3) follows more or less immediately from [4, Lemme 3.2] applied to the commutative C^*-dynamical system $(Z(M)_c, G, \alpha)$.

(3) $\Rightarrow$ (4): Suppose there exists a net (g_i) in $K_1^+(G, Z(M)_c)$ satisfying condition (3) above. By the properties of (g_i), each P_{g_i} is a positive contractive M-bimodule map. Passing to a subnet, we may assume that (P_{g_i}) converges weak* to some P in $\mathcal{B}(L^\infty(G)\overline{\otimes}M, M)$, which is necessarily a projection of norm one from property (3)(a).

Fix $t \in G$, $F \in (L^\infty(G)\overline{\otimes}M)^+$, and $\omega \in M_*^+$. Choose a representation for F with values in M_+. Then

$$\begin{aligned}\langle P(\lambda_t \otimes \alpha_t(F)), \omega\rangle &= \lim_i \int_G \langle g_i(s)(\lambda_t \otimes \alpha_t)(F)(s), \omega\rangle\, ds\\ &= \lim_i \int_G \langle \alpha_t(\alpha_{t^{-1}}(g_i(s))F(t^{-1}s)), \omega\rangle\, ds\\ &= \lim_i \int_G \langle \alpha_{t^{-1}}(g_i(s))F(t^{-1}s), (\alpha_t)_*(\omega)\rangle\, ds\\ &= \lim_i \int_G \langle \alpha_{t^{-1}}(g_i(ts))F(s), (\alpha_t)_*(\omega)\rangle\, ds\\ &= \lim_i \int_G \langle ((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i)(s)F(s), (\alpha_t)_*(\omega)\rangle\, ds.\end{aligned}$$

Since $(\lambda_t \otimes \alpha_t)g_i(s) - g_i(s) \in Z(M)_c$ is self-adjoint for each $s \in G$, we have

$$\begin{aligned}&\langle ((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i - g_i)(s)F(s), (\alpha_t)_*(\omega)\rangle\\ &\quad= \langle \sqrt{F(s)}((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i - g_i)(s)\sqrt{F(s)}, (\alpha_t)_*(\omega)\rangle\\ &\quad\le \langle \sqrt{F(s)}|((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i - g_i)(s)|\sqrt{F(s)}, (\alpha_t)_*(\omega)\rangle\\ &\quad= \langle |((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i - g_i)(s)|F(s), (\alpha_t)_*(\omega)\rangle\\ &\quad\le \|F\|\langle |((\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})g_i - g_i)(s)|, (\alpha_t)_*(\omega)\rangle,\end{aligned}$$

for every $s, t \in G$. Property (3) of (g_i) then implies that

$$\langle P(\lambda_t \otimes \alpha_t(F)), \omega\rangle = \lim_i \int_G \langle g_i(s)F(s), (\alpha_t)_*(\omega)\rangle = \langle \alpha_t(P(F)), \omega\rangle,$$

which yields (4).

(4) $\Rightarrow$ (5) is obvious by restriction, using the $1 \otimes M - M$-bimodule property of projections of norm one $L^\infty(G)\overline{\otimes}M \to M$.

(5) $\Rightarrow$ (3) is all that remains. Let $P : L^\infty(G)\overline{\otimes}Z(M) \to Z(M)$ be a G-equivariant projection of norm one. By Lemma 3.1 applied to the commutative von Neumann algebra $Z(M)$, P lies in the point-weak* closure of $\mathcal{P}_K$. Hence, there is a net (g_i) of functions in $K_1^+(G, Z(M)_c)$ satisfying

$$P(F) = w^* \lim_i P_{g_i}(F) = w^* \lim_i \int_G g_i(s)F(s)\, ds, \quad F \in L^\infty(G)\overline{\otimes}Z(M).$$

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In particular, $1 = w^* \lim_i \int_G g_i(s)\, ds$. The G-equivariance of P implies that

$$\lim_i \int_G \langle g_i(s)(\lambda_t \otimes \alpha_t)(F)(s), \omega\rangle ds = \lim_i \int_G \langle g_i(s)F(s), (\alpha_t)_*(\omega)\rangle\, ds$$

for all $F \in L^\infty(G)\overline{\otimes}Z(M)$, $\omega \in Z(M)_*$ and $t \in G$. But

$$\int_G \langle g_i(s)(\lambda_t \otimes \alpha_t)(F)(s), \omega\rangle\, ds = \int_G \langle (\lambda_{t^{-1}} \otimes \alpha_{t^{-1}})(g_i)(s)F(s), (\alpha_t)_*(\omega)\rangle\, ds,$$

as shown above, so it follows that $((\lambda_t \otimes \alpha_t)(g_i) - g_i) \to 0$ with respect to τ_F (on $L^1(G, Z(M))$) for all $t \in G$. Just as in [4, pp. 307], one can use Lemma 3.3 applied to $V = K_1^+(G, Z(M)_c)$ and an argument involving direct sums of $Z(M)$ with copies of $L^1(G, Z(M))$ to show the existence of a net (g_j) in $K_1^+(G, Z(M)_c)$ such that $((\lambda_t \otimes \alpha_t)(g_j) - g_j) \to 0$ with respect to τ_n for all $t \in G$, which implies a pointwise version of (3)(b), and $\int_G g_j(s)\, ds \xrightarrow{w^*} 1$ in $Z(M)$, which is property (3)(a). Property (3)(b) then follows from Lemma 3.5. □

As a corollary to Theorem 3.6 (and its proof), we obtain a different proof of the fact that a W^*-dynamical system (M, G, α) over an arbitrary locally compact group G is amenable if and only if the restricted action $(Z(M), G, \alpha)$ is amenable [3, Corollaire 3.6].

For actions of second countable locally compact groups G on standard Borel spaces (X, μ) with a quasi-invariant measure μ, amenability of $(L^\infty(X, \mu), G, \alpha)$ implies that π_X is weakly contained in λ [6, Corollary 3.2.2], where π_X is the associated unitary representation of G on $L^2(X, \mu)$. As a corollary to Theorem 3.6, we obtain a generalization of this fact to arbitrary (M, G, α).

COROLLARY 3.7. *Let (π, u) be a normal covariant representation of a W^*-dynamical system (M, G, α). If (M, G, α) is amenable then u is weakly contained in λ.*

Proof. Let (ξ_i) be as in Theorem 3.6 (2). Fix $v \in H$, and define $\eta_i : G \to H$ by $\eta_i(t) = u(t^{-1})\xi_i(t)v$. Then $\eta_i \in L^2(G, H)$, and a calculation similar to one in the proof of [5, Theorem 5.3] gives

$$\langle \eta_i, \lambda_s \eta_i\rangle = \langle v, \langle \xi_i, (\alpha_s \otimes \lambda_s)\xi_i\rangle u_s v\rangle \to \langle v, u_s v\rangle$$

uniformly on compact subsets of G. □

3.2. *Herz–Schur multipliers.* The theory of Herz–Schur multipliers has recently been generalized to the setting of dynamical systems [9, 10, 30, 31]. In this subsection we build on this work by providing an explicit representation of Herz–Schur multipliers arising from compactly supported positive type functions for arbitrary (M, G, α), along with a multiplier characterization of amenability. We begin with preliminaries on Hilbert C^*-modules associated to dynamical systems.

Let (A, G, α) be a C^*-dynamical system. We let $L^2(G, A)$ be the right Hilbert A-module given by the completion of $C_c(G, A)$ under $\|\xi\| = \|\langle \xi, \xi\rangle\|_A^{1/2}$, where

$$\langle \xi, \zeta\rangle = \int_G \xi(s)^* \zeta(s)\, ds, \quad \xi \cdot a(s) = \xi(s)a, \quad \xi, \zeta \in C_c(G, A),\ a \in A.$$

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To simplify notation we let $\widetilde{\alpha}_t \in \mathcal{B}(L^2(G, A))$ denote the isometry

$$\widetilde{\alpha}_t\xi(s) := (\lambda_t \otimes \alpha_t)\xi(s) = \alpha_t(\xi(t^{-1}s)), \quad \xi \in C_c(G, A).$$

By left invariance of the Haar measure and continuity of the action, it follows that

$$\langle \widetilde{\alpha}_t\xi, \widetilde{\alpha}_t\zeta\rangle = \alpha_t(\langle \xi, \zeta\rangle), \quad \xi, \zeta \in L^2(G, A),\ t \in G.$$

We assume throughout that $A \subseteq \mathcal{B}(H)$ non-degenerately. Then $\alpha : A \to C_b(G, A) \subseteq \mathcal{B}(L^2(G, H))$ is a strict $*$-homomorphism, and, viewing $L^2(G, H)$ as a right Hilbert C^*-module over $\mathbb{C}$, we may form the interior tensor product $L^2(G, A) \otimes_\alpha L^2(G, H)$ [29, Proposition 4.5]. This becomes a Hilbert space with inner product given on simple tensors by

$$\langle \xi_1 \otimes_\alpha \eta_1, \xi_2 \otimes_\alpha \eta_2\rangle = \langle \eta_1, \alpha(\langle \xi_1, \xi_2\rangle)\eta_2\rangle.$$

Letting $\pi : A \ni a \mapsto 1 \otimes a \in \mathcal{B}(L^2(G, H))$, we also implicitly use the interior tensor product $L^2(G, A) \otimes_\pi L^2(G, H)$, which is a Hilbert space under the inner product

$$\langle \xi_1 \otimes_\pi \eta_1, \xi_2 \otimes_\pi \eta_2\rangle = \langle \eta_1, (1 \otimes \langle \xi_1, \xi_2\rangle)\eta_2\rangle.$$

The map

$$L^2(G, A) \otimes_\pi L^2(G, H) \ni \xi \otimes_\pi \eta \mapsto \xi \cdot \eta \in L^2(G \times G, H)$$

extends to a unitary operator, where

$$\xi \cdot \eta(s, t) = \xi(s)\eta(t), \quad s, t \in G.$$

Indeed, for any $\xi_1, \ldots, \xi_n \in C_c(G, A)$ and $\eta_1, \ldots, \eta_n \in C_c(G, H)$,

$$\begin{aligned}\bigg\| \sum_{i=1}^n \xi_i \cdot \eta_i \bigg\|^2_{L^2(G\times G,H)} &= \iint \sum_{i,j=1}^n \langle \xi_i(s)\eta_i(t), \xi_j(s)\eta_j(t)\rangle_H \, ds\, dt \\ &= \sum_{i,j=1}^n \iint \langle \eta_i(t), \xi_i(s)^*\xi_j(s)\eta_j(t)\rangle_H \, ds\, dt \\ &= \sum_{i,j=1}^n \int \langle \eta_i(t), \langle \xi_i, \xi_j\rangle\eta_j(t)\rangle_H \, dt \\ &= \sum_{i,j=1}^n \langle \eta_i, (1 \otimes \langle \xi_i, \xi_j\rangle)\eta_j\rangle_{L^2(G,H)} \\ &= \sum_{i,j=1}^n \langle \xi_i \otimes_\pi \eta_i, \xi_j \otimes_\pi \eta_j\rangle \\ &= \bigg\| \sum_{i=1}^n \xi_i \otimes_\pi \eta_i \bigg\|^2. \end{aligned} \tag{11}$$

The map is therefore an isometry. That it also has dense range follows from non-degeneracy of $A \subseteq \mathcal{B}(H)$ using a bounded approximate identity (bai) for A.

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We let $W_\alpha : L^2(G, A) \otimes_\alpha L^2(G, H) \to L^2(G \times G, H)$ be the map determined by

$$W_\alpha(\xi \otimes \eta)(s, t) = (\widetilde{\alpha}_{t^{-1}}\xi)(s)\eta(t) = \alpha_{t^{-1}}(\xi(ts))\eta(t), \quad \xi \in C_c(G, A),\ \eta \in C_c(G, H).$$

Since

$$\begin{aligned} W_\alpha(\xi \cdot a \otimes \eta)(s, t) &= \alpha_{t^{-1}}(\xi(ts)a)\eta(t) = \alpha_{t^{-1}}(\xi(ts))\alpha_{t^{-1}}(a)\eta(t) \\ &= \alpha_{t^{-1}}(\xi(ts))(\alpha(a)\eta)(t) \\ &= W_\alpha(\xi \otimes \alpha(a)\eta)(s, t), \end{aligned}$$

it follows that W_α induces a unitary $W_\alpha : L^2(G, A) \otimes_\alpha L^2(G, H) \to L^2(G \times G, H)$, since

$$\begin{aligned} \left\| W_\alpha\left(\sum_{i=1}^n \xi_i \otimes_\alpha \eta_i \right) \right\|^2 &= \iint \left\| \sum_{i=1}^n W_\alpha(\xi_i \otimes_\alpha \eta_i)(s, t) \right\|_H^2 ds\, dt \\ &= \iint \left\| \sum_{i=1}^n \widetilde{\alpha}_{t^{-1}}\xi_i(s)\eta_i(t) \right\|_H^2 ds\, dt \\ &= \iint \sum_{i,j=1}^n \langle \widetilde{\alpha}_{t^{-1}}\xi_i(s)\eta_i(t), \widetilde{\alpha}_{t^{-1}}\xi_j(s)\eta_j(t)\rangle_H\, ds\, dt \\ &= \iint \sum_{i,j=1}^n \langle \eta_i(t), \widetilde{\alpha}_{t^{-1}}\xi_i(s)^*\widetilde{\alpha}_{t^{-1}}\xi_j(s)\eta_j(t)\rangle_H\, ds\, dt \\ &= \int \sum_{i,j=1}^n \langle \eta_i(t), \langle \widetilde{\alpha}_{t^{-1}}\xi_i, \widetilde{\alpha}_{t^{-1}}\xi_j\rangle \eta_j(t)\rangle_H\, dt \\ &= \int \sum_{i,j=1}^n \langle \eta_i(t), \alpha_{t^{-1}}(\langle \xi_i, \xi_j\rangle)\eta_j(t)\rangle_H\, dt \\ &= \sum_{i,j=1}^n \langle \eta_i, \alpha(\langle \xi_i, \xi_j\rangle)\eta_j\rangle_{L^2(G,H)} \\ &= \sum_{i,j=1}^n \langle \xi_i \otimes_\alpha \eta_i, \xi_j \otimes_\alpha \eta_j\rangle \\ &= \left\| \sum_{i=1}^n \xi_i \otimes_\alpha \eta_i \right\|^2. \end{aligned}$$

This fact was observed for discrete dynamical systems in [9, Lemma 4.9]. By covariance of $(\alpha, \lambda \otimes 1)$, one easily sees that $\widetilde{\alpha}_t \otimes (\lambda_t \otimes 1)$ induces an invertible map on $L^2(G, A) \otimes_\alpha L^2(G, H)$, and the standard argument shows that

$$W_\alpha^*(1 \otimes (\lambda_t \otimes 1))W_\alpha = \widetilde{\alpha}_t \otimes (\lambda_t \otimes 1), \quad t \in G. \tag{12}$$

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Also, whenever $a \in A$ commutes with the range of $\xi \in L^2(G, A)$, in particular when $\xi \in L^2(G, Z(A))$, we have

$$\begin{aligned} W_\alpha(\xi \otimes_\alpha \alpha(a)\eta)(s,t) &= W_\alpha(\xi \cdot a \otimes \eta)(s,t) \\ &= \alpha_{t^{-1}}(\xi \cdot a(ts))\eta(t) \\ &= \alpha_{t^{-1}}(a\xi(ts))\eta(t) \\ &= \alpha_{t^{-1}}(a)\alpha_{t^{-1}}(\xi(ts))\eta(t) \\ &= (1 \otimes \alpha(a))(W_\alpha(\xi \otimes_\alpha \eta))(s,t). \end{aligned}$$

Thus,

$$W_\alpha(\xi \otimes_\alpha \alpha(a)\eta) = (1 \otimes \alpha(a))W_\alpha(\xi \otimes_\alpha \eta). \tag{13}$$

When $(A, G, \alpha) = (\mathbb{C}, G, \text{trivial})$, W_α is simply the fundamental unitary of the quantum group $VN(G)$.

The following are special cases of [31, Definitions 3.1,3.3] when F is assumed bounded and continuous.

Definition 3.8. [31, Definitions 3.1,3.3] Let (A, G, α) be a C^*-dynamical system, and let $\mathcal{CB}(A)$ denote the space of completely bounded maps $\varphi : A \to A$ with the completely bounded norm $\|\varphi\|_{cb} = \sup_n \|\varphi_n\|$, where $\varphi_n : M_n(A) \to M_n(A)$ is the map $[a_{ij}] \mapsto [\varphi(a_{ij})]$, and $M_n(A)$ has its canonical C^*-algebra norm.

A bounded continuous function $F : G \to \mathcal{CB}(A)$ is:

(1) a (completely positive) *Herz–Schur* (A, G, α)*-multiplier* if the map

$$\Theta(F)(\alpha \times \lambda)(f) = (\alpha \times \lambda)(F \cdot f), \quad f \in C_c(G, A),$$

extends to a completely (positive) bounded map on $G \ltimes A$, where $F \cdot f(s) = F(s)(f(s))$, $s \in G$.

(2) a (completely positive) *Herz–Schur multiplier* if the map

$$\Theta(F)(\alpha(a)(\lambda_s \otimes 1)) = \alpha(F(s)(a))(\lambda_s \otimes 1), \quad a \in A,\ s \in G,$$

extends to a normal completely (positive) bounded map on $(G \ltimes A)''$ (the weak* closure of $G \ltimes A$ in $\mathcal{B}(L^2(G, H))$).

By [31, Remark 3.4], when A is separable, a Herz–Schur multiplier is automatically a Herz–Schur (A, G, α)-multiplier. Their argument (for continuous F and $f \in C_c(G, A)$) extends verbatim to arbitrary (A, G, α). As mentioned on [31, pp. 403], when $A = \mathbb{C}$, both conditions are equivalent to F defining a completely bounded multiplier of the Fourier algebra $A(G)$, as in that case, the associated maps on $C^*_\lambda(G)$ admit canonical weak* continuous extensions to $VN(G)$. Such an extension is not ensured to exist in general, hence the two definitions.

We now show that any element $\xi \in C_c(G, A)$ defines a completely positive Herz–Schur multiplier via $h_\xi(s)(a) = \langle \xi, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi\rangle$. For discrete dynamical systems, this latter fact follows from [30, Theorem 2.8] and/or [9, Theorem 4.8].

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PROPOSITION 3.9. *Let (A, G, α) be a C^*-dynamical system. For each $\xi \in C_c(G, A)$, the function $h : G \to \mathcal{CB}(A)$ given by*

$$h(s)(a) = \langle \xi, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi \rangle, \quad s \in G, \ a \in A,$$

defines a normal completely positive map $\Theta(h)$ on $(G \ltimes A)''$ satisfying $\|\Theta(h)\|_{cb} = \|h(e)\|$,

$$\Theta(h)(\alpha(a)(\lambda_s \otimes 1)) = \alpha(h(s)(a))(\lambda_s \otimes 1), \quad a \in A, \ s \in G, \tag{14}$$

and

$$\Theta(h)(\alpha \times \lambda(f)) = \alpha \times \lambda(h \cdot f), \quad f \in C_c(G, A). \tag{15}$$

When $(A, G, \alpha) = (M_c, G, \alpha)$ for a W^*-dynamical system (M, G, α), we have that

$$\Theta(h)(\alpha(x)(\lambda_s \otimes 1)) = \alpha(h(s)(x))(\lambda_s \otimes 1), \quad x \in M, \ s \in G. \tag{16}$$

Proof. We first consider the map at the level of $B(G \ltimes_f A)$. Let $\Phi \in B(G \ltimes_f A)^+$, and let φ, σ be as in equation (1) in §2.2. We claim that $h^* \cdot \Phi \in B(G \ltimes_f A)^+$, where

$$h^* \cdot \Phi \ni G \ni s \mapsto h(s)^*(\Phi(s)) \in A^*.$$

By [34, Proposition 7.6.8], it suffices to show

$$\sum_{j,k=1}^{n} \langle (h^* \cdot \Phi)(t_j^{-1} t_k), \alpha_{t_j^{-1}}(a_j^* a_k) \rangle \geq 0$$

for any $t_1, \ldots, t_n \in G$ and $a_1, \ldots, a_n \in A$. We compute

$$\begin{aligned}
\sum_{j,k=1}^{n} \langle (h^* \cdot \Phi)(t_j^{-1} t_k), \alpha_{t_j^{-1}}(a_j^* a_k) \rangle &= \sum_{j,k=1}^{n} \langle \Phi(t_j^{-1} t_k), h(t_j^{-1} t_k)(\alpha_{t_j^{-1}}(a_j^* a_k)) \rangle \\
&= \sum_{j,k=1}^{n} \langle \Phi(t_j^{-1} t_k), \langle \xi, (1 \otimes \alpha_{t_j^{-1}}(a_j^* a_k))(\lambda_{t_j^{-1} t_k} \otimes \alpha_{t_j^{-1} t_k})\xi \rangle \rangle \\
&= \sum_{j,k=1}^{n} \langle \Phi(t_j^{-1} t_k), \langle (1 \otimes \alpha_{t_j^{-1}}(a_j))(\lambda_{t_j} \otimes 1)\xi, (1 \otimes \alpha_{t_j^{-1}}(a_k))(\lambda_{t_k} \otimes \alpha_{t_j^{-1} t_k})\xi \rangle \rangle \\
&= \sum_{j,k=1}^{n} \langle \Phi(t_j^{-1} t_k), \alpha_{t_j^{-1}}(\langle (1 \otimes a_j)(\lambda_{t_j} \otimes \alpha_{t_j})\xi, (1 \otimes a_k)\lambda_{t_k} \otimes \alpha_{t_k}\xi \rangle) \rangle \\
&= \int_G \sum_{j,k=1}^{n} \langle \Phi(t_j^{-1} t_k), \alpha_{t_j^{-1}}(\alpha_{t_j}(\xi(t_j^{-1}s))^* a_j^* a_k \alpha_{t_k}(\xi(t_k^{-1}s))) \rangle \, ds \\
&= \int_G \sum_{j,k=1}^{n} \langle \varphi, \pi(\alpha_{t_j^{-1}}(\alpha_{t_j}(\xi(t_j^{-1}s))^* a_j^* a_k \alpha_{t_k}(\xi(t_k^{-1}s))))\sigma(t_j^{-1} t_k) \rangle \, ds \\
&= \int_G \sum_{j,k=1}^{n} \langle \varphi, \sigma(t_j^{-1})\pi(\alpha_{t_j}(\xi(t_j^{-1}s))^* a_j^* a_k \alpha_{t_k}(\xi(t_k^{-1}s)))\sigma(t_k) \rangle \, ds
\end{aligned}$$

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$$= \int_G \varphi\bigg(\bigg(\sum_{j=1}^n a_j\alpha_{t_j}(\xi(t_j^{-1}s))\sigma(t_j)\bigg)^*\bigg(\sum_{k=1}^n a_k\alpha_{t_k}(\xi(t_k^{-1}s))\sigma(t_k)\bigg)\bigg)\bigg\rangle\, ds$$
$$\geq 0.$$

We therefore obtain a well-defined linear map on $B(G \ltimes_f A) = \text{span} B(G \ltimes_f A)^+$ by the Jordan decomposition.

Since $(M_n(\mathbb{C}) \otimes A, G, \text{id}_{M_n} \otimes \alpha)$ is a C^*-dynamical system satisfying $M_n(\mathbb{C}) \otimes (G \ltimes_f A) \cong G \ltimes_f (M_n(\mathbb{C}) \otimes A)$ canonically (by [44, Lemma 2.75]), and since [34, Proposition 7.6.8] applies to any C^*-dynamical system, the matricial analogue of the above argument together with the previous identification shows that the linear map

$$h^* : B(G \ltimes_f A) \ni \Phi \mapsto h^* \cdot \Phi \in B(G \ltimes_f A)$$

is completely positive. Moreover, since h is compactly supported and compactly supported elements of $B(G \ltimes_f A)^+$ lie in $A(G \ltimes_f A)^+$ [34, Lemma 7.7.6], it follows that

$$h^* : B(G \ltimes_f A) \ni \Phi \mapsto h^* \cdot \Phi \in A(G \ltimes_f A)$$

Since $A(G \ltimes_f A) \subseteq B(G \ltimes_f A)$ and $A(G \ltimes_f A) = (G \ltimes A)''_* \subseteq (G \ltimes A)^*$, by restriction, h^* induces a completely positive map on $A(G \ltimes_f A)$, whose adjoint $\Theta(h)$ is normal and completely positive on $(G \ltimes A)''$. Moreover, for each $a \in A$, $s \in G$, and $v \in A(G \ltimes_f A)$,

$$\begin{aligned}\langle \Theta(h)(\alpha(a)(\lambda_s \otimes 1)), v\rangle &= \langle \alpha(a)(\lambda_s \otimes 1), h^*(v)\rangle \\ &= \langle a, h(s)^*(v(s))\rangle \\ &= \langle h(s)(a), v(s)\rangle \\ &= \langle \alpha(h(s)(a))(\lambda_s \otimes 1), v\rangle.\end{aligned}$$

Hence, $\Theta(h)$ satisfies equation (14). A similar argument shows that

$$\Theta(h)(\alpha \times \lambda(f)) = \alpha \times \lambda(h \cdot f), \quad f \in C_c(G, A),$$

where $h \cdot f(s) = h(s)(f(s))$, $s \in G$. Taking a bai (a_i) for A which converges strictly (and hence weak*) to the identity of the non-degenerate representation space H of A, we have

$$\Theta(h)(1_{(G\ltimes A)''}) = w^* \lim_i \Theta(h)(\alpha(a_i)) = w^* \lim_i \alpha(h(e)(a_i)) = \alpha(\langle \xi, \xi\rangle).$$

By complete positivity,

$$\|\Theta(h)\|_{cb} = \|\Theta(h)(1)\| = \|\langle \xi, \xi\rangle\| = \|h(e)\|.$$

When $(A, G, \alpha) = (M_c, G, \alpha)$ for a W^*-dynamical system (M, G, α), equation (16) follows from (14), weak* density of M_c in M and normality of $\Theta(h)$ and α. Note that in this case we view

$$h(s)(x) = \langle \xi, (1 \otimes x)(\lambda_s \otimes \alpha_s)\xi\rangle \in M$$

in the obvious way as $\xi \in L^2(G, M_c) \subseteq L^2(G, M)$. □

Remark 3.10. For $\xi \in C_c(G, \ell^2(A))$, the function $h(s)(a) = \langle \xi, (1 \otimes 1 \otimes a)(\lambda_s \otimes 1 \otimes \alpha_s)\xi\rangle$ also satisfies the conclusions of Proposition 3.9. This may be seen by applying

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Proposition 3.9 to the functions h_k associated to $\xi_k = P_k \circ \xi$, where $P_k : \ell^2(A) \to A$ is the canonical kth coordinate projection. Then $h(s) = \sum_{k=1}^{\infty} h_k(s)$ and $\Theta(h) = \sum_{k=1}^{\infty} \Theta(h_k)$.

If the range of ξ in Lemma 3.9 lies in $Z(A)$, then $\Theta(h)$ admits an explicit representation in terms of the fundamental unitary W_α, which we now show. It is not clear whether this particular representation is valid for all $\xi \in C_c(G, A)$, although related representations are known to exist at the level of equivariant representations of discrete dynamical systems (see the proof of [**9**, Theorem 4.8]).

In the following, $\omega_\xi \otimes_\alpha \mathrm{id}$ denotes the map $\mathcal{B}(L^2(G, A) \otimes_\alpha L^2(G, H)) \to \mathcal{B}(L^2(G, H))$, defined so that, for $T \in \mathcal{B}(L^2(G, A) \otimes_\alpha L^2(G, H))$, $(\omega_\xi \otimes_\alpha \mathrm{id})(T)$ is the operator in $\mathcal{B}(L^2(G, H))$ determined by the sesquilinear form $(\eta_1, \eta_2) \mapsto \langle \xi \otimes_\alpha \eta_1, T(\xi \otimes_\alpha \eta_2)\rangle$.

PROPOSITION 3.11. *Let (A, G, α) be a C^*-dynamical system. Let $\xi \in C_c(G, Z(A))$ and*

$$h(s) = \langle \xi, (\lambda_s \otimes \alpha_s)\xi \rangle, \quad s \in G,$$

be the associated positive type function. Viewing $h : G \to \mathcal{CB}(A)$ via multiplication, $h(s)(a) = h(s)a$, the Herz–Schur multiplier $\Theta(h)$ satisfies

$$\Theta(h)(x) = (\omega_\xi \otimes_\alpha \mathrm{id})(W_\alpha^*(1 \otimes x)W_\alpha), \quad x \in (G \ltimes A)''. \tag{17}$$

Proof. By equation (15),

$$\Theta(h)(\alpha \times \lambda(f)) = \int_G \alpha(h(s)f(s))(\lambda_s \otimes 1)\, ds, \quad f \in C_c(G, A).$$

Represent $A \subseteq \mathcal{B}(H)$ non-degenerately and view $G \ltimes A \subseteq \mathcal{B}(L^2(G) \otimes H)$. Fix $\eta \in C_c(G, H)$. Then, for any $f \in C_c(G, A)$, the commutation relations (12) and (13) imply that

$$\begin{aligned}
\langle \eta, \Theta(h)((\alpha \times \lambda)(f))\eta\rangle &= \int_G \langle \eta, \alpha(h(s)f(s))(\lambda_s \otimes 1)\eta\rangle\, ds \\
&= \int_G \langle \eta, \alpha(\langle \xi, \widetilde{\alpha}_s\xi\rangle)\alpha(f(s))(\lambda_s \otimes 1)\eta\rangle\, ds \\
&= \int_G \langle \xi \otimes_\alpha \eta, (\widetilde{\alpha}_s\xi) \otimes_\alpha (\alpha(f(s))(\lambda_s \otimes 1)\eta)\rangle\, ds \\
&= \int_G \langle \xi \otimes_\alpha \eta, (\widetilde{\alpha}_s\xi) \otimes_\alpha (\lambda_s \otimes 1)(\alpha(\alpha_{s^{-1}}(f(s)))\eta)\rangle\, ds \\
&= \int_G \langle \xi \otimes_\alpha \eta, (\widetilde{\alpha}_s \otimes_\alpha (\lambda_s \otimes 1))(\xi \otimes_\alpha (\alpha(\alpha_{s^{-1}}(f(s)))\eta)\rangle\, ds \\
&= \int_G \langle \xi \otimes_\alpha \eta, W_\alpha^*(1 \otimes (\lambda_s \otimes 1))W_\alpha(\xi \otimes_\alpha \alpha(\alpha_{s^{-1}}(f(s)))\eta)\rangle\, ds \\
&= \int_G \langle W_\alpha(\xi \otimes_\alpha \eta), (1 \otimes (\lambda_s \otimes 1)\alpha(\alpha_{s^{-1}}(f(s))))W_\alpha(\xi \otimes_\alpha \eta)\rangle\, ds \\
&= \int_G \langle W_\alpha(\xi \otimes_\alpha \eta), (1 \otimes \alpha(f(s))(\lambda_s \otimes 1))W_\alpha(\xi \otimes_\alpha \eta)\rangle\, ds \\
&= \langle W_\alpha(\xi \otimes_\alpha \eta), (1 \otimes (\alpha \times \lambda)(f))W_\alpha(\xi \otimes_\alpha \eta)\rangle
\end{aligned}$$

$$=\langle \xi \otimes_\alpha \eta, W_\alpha^*(1 \otimes (\alpha \times \lambda)(f)) W_\alpha(\xi \otimes_\alpha \eta)\rangle$$
$$=\langle \eta, (\omega_\xi \otimes_\alpha \mathrm{id})(W_\alpha^*(1 \otimes (\alpha \times \lambda)(f)) W_\alpha)\eta\rangle.$$

It follows that

$$\Theta(h)(x) = (\omega_\xi \otimes_\alpha \mathrm{id})(W_\alpha^*(1 \otimes x) W_\alpha), \quad x \in G \ltimes A.$$

By normality, the above representation extends to all $x \in (G \ltimes A)''$. □

Using the 'fundamental unitary' W_α associated to the C^*-dynamical system (M_c, G, α), we now rephrase the convergence in Theorem 3.6 (2) at a Hilbert space level. This characterization is a dynamical systems analogue of the fundamental unitary characterization of (co)amenability of locally compact (quantum) groups, and it leads to an approximation of the identity of $G \bar{\ltimes} M$ by completely positive Herz–Schur multipliers.

For the following theorem, fix a non-degenerate normal representation $M \subseteq \mathcal{B}(H)$.

THEOREM 3.12. *Let (M, G, α) be a W^*-dynamical system. The following conditions are equivalent.*

(1) *(M, G, α) is amenable.*

(2) *There exists a net (ξ_i) in $C_c(G, Z(M)_c)$ such that:*
   (a) *$\langle \xi_i, \xi_i\rangle \leq 1$ for all i;*
   (b) *$\langle \xi_i, \xi_i\rangle \to 1$ weak*;*
   (c) *$\|W_\alpha(\xi_i \otimes_\alpha \eta) - \xi_i \cdot \eta\|_{L^2(G\times G,H)} \to 0$, $\eta \in L^2(G, H)$.*

(3) *There exists a net (ξ_i) in $C_c(G, M_c)$ such that:*
   (a) *$\langle \xi_i, \xi_i\rangle \leq 1$ for all i;*
   (b) *$\Theta(h_{\xi_i}) \to \mathrm{id}_{G\bar{\ltimes} M}$ point weak*.*

Proof. (1) ⇒ (2): If (M, G, α) is amenable, by Theorem 3.6 there exists a net (ξ_i) in $C_c(G, Z(M)_c)$ such that $\langle \xi_i, \xi_i\rangle \leq 1$ for all i and $\langle \xi_i, \widetilde{\alpha}_t \xi_i\rangle \to 1$ weak*, uniformly on compact subsets. Let $\eta = \eta_1 \otimes \eta_2$ with $\eta_1 \in C_c(G)$ and $\eta_2 \in H$. Let ω_{η_2} be the associated vector functional on $\mathcal{B}(H)$. By norm continuity of the action $G \curvearrowright M_*$ and Lemma 3.4, it follows that

$$\omega_{\eta_2}(\langle \widetilde{\alpha}_{t^{-1}}\xi_i - \xi_i, \widetilde{\alpha}_{t^{-1}}\xi_i - \xi_i\rangle) = \omega_{\eta_2}(\alpha_{t^{-1}}(\langle \xi_i, \xi_i\rangle) - 2\mathrm{Re}\langle \xi_i, \widetilde{\alpha}_{t^{-1}}\xi_i\rangle + \langle \xi_i, \xi_i\rangle) \to 0$$

uniformly on compact subsets of G. Hence,

$$\begin{aligned}
\|W_\alpha(\xi_i \otimes_\alpha \eta) - \xi_i \cdot \eta\|^2_{L^2(G\times G,H)} &= \iint \|(\widetilde{\alpha}_{t^{-1}}\xi_i(s) - \xi_i(s))\eta(t)\|^2_H \, ds \, dt \\
&= \iint |\eta_1(t)|^2 \|(\widetilde{\alpha}_{t^{-1}}\xi_i(s) - \xi_i(s))\eta_2\|^2_H \, ds \, dt \\
&= \iint |\eta_1(t)|^2 \langle \eta_2, (\widetilde{\alpha}_{t^{-1}}\xi_i(s) - \xi_i(s))^*(\widetilde{\alpha}_{t^{-1}}\xi_i(s) - \xi_i(s))\eta_2\rangle_H \, ds \, dt \\
&= \int |\eta_1(t)|^2 \langle \eta_2, \langle \widetilde{\alpha}_{t^{-1}}\xi_i - \xi_i, \widetilde{\alpha}_{t^{-1}}\xi_i - \xi_i\rangle \eta_2\rangle_H \, dt \\
&\to 0.
\end{aligned}$$

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Since linear combinations of simple tensors $\eta_1 \otimes \eta_2$ with $\eta_1 \in C_c(G)$ and $\eta_2 \in H$ are dense in $L^2(G, H)$, boundedness of W_α and (ξ_i), together with the inequality $\|\xi \cdot \eta\| \leq \|\xi\|\|\eta\|$ (which follows from (11)), shows that

$$\|W_\alpha(\xi \otimes_\alpha \eta) - \xi \cdot \eta\|_{L^2(G\times G,H)} \to 0,$$

for all $\eta \in L^2(G, H)$.

(2) $\Rightarrow$ (3): Pick a net (ξ_i) in $C_c(G, Z(M)_c)$ satisfying (2). If (h_i) denotes the corresponding positive type functions in $C_c(G, Z(M)_c)$, then Propositions 3.9 and 3.11 applied to the C^*-dynamical system (M_c, G, α) imply that

$$\Theta(h_i)(x) = (\omega_{\xi_i} \otimes_\alpha \mathrm{id})(W_\alpha^*(1 \otimes x)W_\alpha), \quad x \in G \bar{\ltimes} M,$$

and $\|\Theta(h_i)\|_{cb} = \|\langle \xi_i, \xi_i\rangle\| \leq 1$.

By boundedness of $(\Theta(h_i))$, it suffices to show that, for any $x \in G\bar{\ltimes}M$ and $\eta \in C_c(G, H)$,

$$|\langle \eta, \Theta(h_i)(x)\eta\rangle - \langle \eta, x\eta\rangle| \to 0.$$

To show this, first note that by the representation (17),

$$\langle \eta, \Theta(h_i)(x)\eta\rangle = \langle W_\alpha(\xi_i \otimes_\alpha \eta), (1 \otimes x)W_\alpha(\xi_i \otimes_\alpha \eta)\rangle,$$

so condition (2)(c) implies that

$$|\langle \eta, \Theta(h_i)(x)\eta\rangle - \langle \xi_i \cdot \eta, (1 \otimes x)\xi_i \cdot \eta\rangle| \to 0. \tag{18}$$

Second, since the range of each ξ_i belongs to $Z(M)_c$, for any $y \in M_c$ and $r, s, t \in G$, we have

$$\begin{aligned}((1 \otimes \alpha(y)(\lambda_r \otimes 1))(\xi_i \cdot \eta))(s, t) &= \alpha_{t^{-1}}(y)((\lambda_r \otimes 1)(\xi_i \cdot \eta)(s, t)) \\ &= \alpha_{t^{-1}}(y)((\xi_i \cdot \eta)(s, r^{-1}t)) \\ &= \alpha_{t^{-1}}(y)(\xi_i(s)\eta(r^{-1}t)) \\ &= \xi_i(s)\alpha_{t^{-1}}(y)\eta(r^{-1}t) \\ &= \xi_i(s)\alpha(y)(\eta(r^{-1}t)) \\ &= \xi_i(s)(\alpha(y)(\lambda_r \otimes 1)\eta(t)) \\ &= \xi_i \cdot (\alpha(y)(\lambda_r \otimes 1)\eta)(s, t).\end{aligned}$$

Thus, $\xi_i \cdot (x_0\eta) = (1 \otimes x_0)\xi_i \cdot \eta$ for any $x_0 \in \mathrm{span}\{\alpha(y)(\lambda_r \otimes 1) \mid y \in M_c, r \in G\}$. Since this space is strong operator topology dense in $G\bar{\ltimes}M$, and the $\cdot$ operation is separately norm continuous (again from (11)), it follows that $\xi_i \cdot (x\eta) = (1 \otimes x)\xi_i \cdot \eta$. Hence, condition (2)(b) implies

$$\begin{aligned}\langle \xi_i \cdot \eta, (1 \otimes x)\xi_i \cdot \eta\rangle &= \langle \xi_i \cdot \eta, \xi_i \cdot (x\eta)\rangle \\ &= \langle \eta, (1 \otimes \langle \xi_i, \xi_i\rangle)x\eta\rangle \\ &\to \langle \eta, x\eta\rangle.\end{aligned}$$

Combining this limit with (18) yields the desired conclusion.

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(3) ⇒ (1): By property (3), there exists a net (ξ_i) in $C_c(G, M_c)$ with $\langle \xi_i, \xi_i \rangle \leq 1$ whose corresponding Herz–Schur multipliers $\Theta(h_{\xi_i})$ converge to $\mathrm{id}_{G \bar{\ltimes} M}$ point weak*. By Proposition 3.9 applied to (M_c, G, α), it follows that

$$\alpha(h_{\xi_i}(s)(x))(\lambda_s \otimes 1) = \Theta(h_{\xi_i})(\alpha(x)(\lambda_s \otimes 1)) \xrightarrow{w^*} \alpha(x)(\lambda_s \otimes 1),$$

and therefore $\alpha(h_{\xi_i}(s)(x)) \to \alpha(x)$ weak*, for each $x \in M$ and $s \in G$. Since $\alpha : M \to L^\infty(G)\overline{\otimes}M$ is a weak*-weak* homeomorphism onto its range, it follows that

$$\langle \xi_i, (1 \otimes x)(\lambda_s \otimes \alpha_s)\xi_i \rangle = h_{\xi_i}(s)(x) \xrightarrow{w^*} x, \quad x \in M, \; s \in G.$$

Hence, (ξ_i) satisfies condition (7) of [15, Proposition 3.10]. Since the implication (7) ⇒ (8) of [15, Proposition 3.10] is valid for arbitrary locally compact groups, it follows that (M, G, α) is amenable. □

4. *Amenable C^*-dynamical systems*

In their recent study of amenability and weak containment for C^*-dynamical systems [15], Buss, Echterhoff and Willett introduced the following definitions.

Definition 4.1. [15] Let (A, G, α) be a C^*-dynamical system. Then (A, G, α) is:

- *von Neumann amenable* if the universal W^*-dynamical system (A''_α, G, α) is amenable;
- *amenable* if there exists a net of norm-continuous, compactly supported, positive type functions $h_i : G \to Z(A''_\alpha)$ such that $\|h_i(e)\| \leq 1$ for all i, and $h_i(s) \to 1$ weak* in A''_α, uniformly for s in compact subsets of G;
- *strongly amenable* if there exists a net $(h_i) \in P_1(Z(M(A)), G, \alpha) \cap C_c(G, Z(M(A)))$ such that $h_i(s) \to 1$ strictly, uniformly on compact subsets of G.

It was shown in [15, Proposition 3.10] that amenability always implies von Neumann amenability and that the conditions are equivalent when G is exact. It follows from Theorem 3.6 that amenability and von Neumann amenability coincide for arbitrary C^*-dynamical systems.

Strong amenability always implies amenability [15, Remark 3.6]; however, results of Suzuki [40] imply that, for non-commutative A, amenability is, in general, strictly weaker than strong amenability. For commutative A and discrete G, strong amenability coincides with amenability by [4, Théorème 4.9]. We show in Corollary 4.14 that the two notions coincide for arbitrary commutative C^*-dynamical systems.

Another approach to amenability is through Exel's approximation property of Fell bundles over discrete groups [18]. This property was later generalized by Exel and Ng in [19] to Fell bundles over locally compact groups. Specializing to the case of crossed products of C^*-dynamical systems, they defined the *C-approximation property* of (A, G, α) to be the existence of nets (ξ_i) and (η_i) in $C_c(G, A)$ for which $\|\langle \xi_i, \xi_i \rangle\| \|\langle \eta_i, \eta_i \rangle\| \leq C$ and, for any $f \in C_c(G, A)$

$$\int_G \xi_i(t)^* f(s)\alpha_s(\eta_i(s^{-1}t))\, dt \to f(s)$$

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in norm, uniformly in $(s, f(s))$. If one can take $\eta_i = \xi_i$, then (A, G, α) has the *C-positive approximation property*. Exel and Ng showed that when A is nuclear and G is discrete, the approximation property implies amenability of (A, G, α), and conversely, the two notions are equivalent whenever G is discrete and A is commutative or finite-dimensional (see [19, §4]).

In [8], Bédos and Conti generalized this notion by defining the *C-weak approximation property* as the existence of an equivariant representation (ρ, v) of (A, G, α) on a Hilbert A-module E (see, for example, [9, pp. 40]), and nets (ξ_i) and (η_i) in $C_c(G, E)$ for which $\|\langle \xi_i, \xi_i\rangle\| \|\langle \eta_i, \eta_i\rangle\| \leq C$ and

$$\langle \xi_i, \rho(a)v(s)\eta_i\rangle \to a, \quad a \in A,$$

uniformly for s in compact subsets of G. (This property was defined for discrete dynamical systems in [8], the definition above being the natural generalization.) Again, if one can take $\xi_i = \eta_i$, then (A, G, α) has the *C-positive weak approximation property*. For discrete dynamical systems with A unital, Bédos and Conti showed that the weak approximation property implies that the full and reduced crossed products coincide [10, Theorem 4.32].

By [15, Theorem 3.25], it follows that the C-positive approximation property implies amenability. Below we establish a partial converse, showing the equivalence of amenability and a particular case of the 1-positive weak approximation property of Bédos and Conti, when $E = \ell^2(A)$. When A is commutative, or, more generally, when $Z(A^{**}) = Z(A)^{**}$, we can take $E = A$, in which case amenability is equivalent to the 1-positive approximation property. This is a consequence of the following theorem, our main result of this section.

To explain some of the notation below, for a C^*-dynamical system (A, G, α) and Hilbert A-module E, there is a canonical A-valued inner product on $C_c(G, E)$ given by

$$\langle \xi, \eta\rangle = \int_G \langle \xi(s), \eta(s)\rangle\, ds, \quad \xi, \eta \in C_c(G, E).$$

We use this notation below in the case when $E = \ell^2(A)$ with the A-valued inner product $\langle (a_n), (b_n)\rangle = \sum_n a_n^* b_n$.

THEOREM 4.2. *Let (A, G, α) be a C^*-dynamical system. The following conditions are equivalent.*

(1) *(A, G, α) is amenable.*

(2) *There exists a net (ξ_i) in $C_c(G, \ell^2(A))$ such that:*
   (a) *$\langle \xi_i, \xi_i\rangle \leq 1$ for all i;*
   (b) *$h_{\xi_i}(e) \to \mathrm{id}_A$ in the point norm topology;*
   (c) *$\Theta(h_{\xi_i}) \to \mathrm{id}_{G \ltimes A}$ in the point norm topology,*

   where $h_{\xi_i}(s)(a) = \langle \xi_i, (1 \otimes 1 \otimes a)(\lambda_s \otimes 1 \otimes \alpha_s)\xi_i\rangle$ are the associated completely positive Herz–Schur multipliers.

(3) *There exists a net (ξ_i) in $C_c(G, \ell^2(A))$ such that $\langle \xi_i, \xi_i\rangle \leq 1$ for all i and*

$$\|h_{\xi_i}(s)(f(s)) - f(s)\| \to 0, \quad f \in C_c(G, A),$$

   uniformly for s in compact subsets of G.

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(4) (A, G, α) *is von Neumann amenable.*

Moreover, when $Z(A^{**}) = Z(A)^{**}$*, the net* (ξ_i) *can be chosen in* $C_c(G, Z(A))_{\|\cdot\|_{L^2(G,Z(A))}\leq 1}$*, in which case* $h_i(s)(a) = a\langle\xi_i, (\lambda_s \otimes \alpha_s)\xi_i\rangle$*,* $s \in G$*,* $a \in A$*.*

The outline of the proof is as follows. We first use the Kaplansky density theorem for Hilbert modules to obtain a C^*-Reiter type property from amenability, which is then used to deduce the Herz–Schur multiplier convergence (Proposition 4.6). The equivalence of (2) and (3) follows from a more general equivalence at the level of compactly supported completely positive multipliers (Theorem 4.10). The final step uses the techniques from [1, Lemma 6.5] to deduce von Neumann amenability from the weak approximation property in (3), at which point amenability follows from Theorem 3.6.

We begin with the following estimate, which will be used several times in the sequel.

LEMMA 4.3. *Let* A *be a* C^**-algebra and* E *be an inner product* A*-module. Then, for any state* $\mu \in A^*$ *and* $\xi, \xi', \eta, \eta' \in E$*,*

$$|\mu(\langle\xi, \xi'\rangle - \langle\eta, \eta'\rangle)| \leq \|\langle\xi, \xi\rangle\|^{1/2}\mu(\langle\xi' - \eta', \xi' - \eta'\rangle)^{1/2} + \|\langle\eta', \eta'\rangle\|^{1/2}\mu(\langle\xi - \eta, \xi - \eta\rangle)^{1/2}.$$

Proof. By the Schwarz inequality for completely positive maps, for any $a \in A$ we have $|\mu(a)|^2 \leq \mu(a^*a), \mu(aa^*)$. Combining this with the Cauchy–Schwarz inequality [29, Proposition 1.1] for E, we have

$$\begin{aligned}|\mu(\langle\xi, \xi'\rangle - \langle\eta, \eta'\rangle)| &= |\mu(\langle\xi, \xi' - \eta'\rangle + \langle\xi - \eta, \eta'\rangle)|\\ &\leq \mu(\langle\xi' - \eta', \xi\rangle\langle\xi, \xi' - \eta'\rangle)^{1/2} + \mu(\langle\xi - \eta, \eta'\rangle\langle\eta', \xi - \eta\rangle)^{1/2}\\ &\leq \|\langle\xi, \xi\rangle\|^{1/2}\mu(\langle\xi' - \eta', \xi' - \eta'\rangle)^{1/2}\\ &\quad + \|\langle\eta', \eta'\rangle\|^{1/2}\mu(\langle\xi - \eta, \xi - \eta\rangle)^{1/2}.\end{aligned}$$

□

The next lemma is known; we include a proof for completeness.

LEMMA 4.4. *Let* G *be a locally compact group and* A *be a* C^**-algebra. Then* $L^2(G, A) \cong L^2(G)_c \otimes^h A$ *completely isometrically.*

Proof. Let $(e_i)_{i\in I}$ be an orthonormal basis of $L^2(G)$. Given $\xi_1, \ldots, \xi_n \in L^2(G)$ and $a_1, \ldots, a_n \in A$, for each i, let $b_i = \sum_{k=1}^n \langle e_i, \xi_k\rangle a_k$. Then, on the one hand,

$$\left\|\sum_{k=1}^n \xi_k \otimes a_k\right\|_h = \left\|\sum_{i\in I}\sum_{k=1}^n \langle e_i, \xi_k\rangle e_i \otimes a_k\right\|_h = \left\|\sum_{i\in I} e_i \otimes b_i\right\|_h = \left\|\sum_{i\in I} b_i^* b_i\right\|^{1/2}.$$

On the other hand,

$$\begin{aligned}\left\|\sum_{k=1}^n \xi_k \otimes a_k\right\|_{L^2(G,A)} &= \left\|\sum_{k,l=1}^n \langle\xi_k, \xi_l\rangle a_k^* a_l\right\|^{1/2} = \left\|\sum_{i\in I}\sum_{k,l=1}^n \langle\xi_k, e_i\rangle\langle e_i, \xi_l\rangle a_k^* a_l\right\|^{1/2}\\ &= \left\|\sum_{i\in I} b_i^* b_i\right\|^{1/2}.\end{aligned}$$

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Thus there is an isometric isomorphism $\theta : L^2(G)_c \otimes^h A \to L^2(G, A)$ acting as the identity on simple tensors. Equipping the space $L^2(G)_c \otimes^h A$ with the canonical C^*-A-module structure (see [12, Theorem 8.2.11]), standard calculations show that θ is an A-module map satisfying $\theta(x\langle y, z\rangle) = \theta(x)\langle\theta(y), \theta(z)\rangle$ for all $x, y, z \in L^2(G)_c \otimes^h A$. Thus, if we equip $L^2(G, A)$ with its canonical operator space structure (see [12, Section 8.2]), it follows by [12, Lemma 8.3.2] that θ is completely isometric. □

Let A be a C^*-algebra. The self-dual completion of a Hilbert A-module E is the space $E'' := B_A(E, A^{**})$ of bounded A-module maps from E into A^{**}. By [33, Corollary 4.3] (see also [45, Proposition 2.2]) there is a Hilbert A^{**}-module structure on E'', whose norm coincides with the operator norm induced from $B_A(E, A^{**})$.

LEMMA 4.5. *Let G be a locally compact group and A be a C^*-algebra. The map*

$$j : L^2(G, A^{**}) \ni \xi \mapsto \left(\eta \mapsto \langle \xi, \eta\rangle_{A^{**}} = \int_G \xi(s)^*\eta(s)\, ds\right) \in L^2(G, A)''$$

is an isometric A^{**}-module map.

Proof. Fix $\xi \in C_c(G, A^{**})$. Then

$$\begin{aligned} \|j(\xi)\|_{L^2(G,A)''} &= \sup\{\|\langle \xi, \eta\rangle_{A^{**}}\| \mid \eta \in L^2(G, A),\ \|\eta\| \le 1\} \\ &\le \sup\{\|\langle \xi, \eta\rangle_{A^{**}}\| \mid \eta \in L^2(G, A^{**}),\ \|\eta\| \le 1\} \\ &= \|\xi\|_{L^2(G,A^{**})}. \end{aligned}$$

For the reverse inequality, first note that by self-duality of the Haagerup tensor product [13, Corollary 3.4], the canonical inclusion

$$L^2(G)_c \otimes^h A^{**} = L^2(G)_c^{**} \otimes^h A^{**} \hookrightarrow L^2(G)_c^{**} \otimes^{w^*h} A^{**}$$

is a complete isometry. Further, by [17, Theorem 5.7], the canonical injection

$$L^2(G)_c^{**} \otimes^{w^*h} A^{**} \hookrightarrow (L^2(G)_c^* \otimes^{w^*h} A^*)^* = (L^2(G)_c \otimes^h A)^{**}$$

is a complete isometry. Hence, $L^2(G)_c \otimes^h A^{**} \subseteq (L^2(G)_c \otimes^h A)^{**}$, canonically. Let (ξ_i) be a net in $(L^2(G)_c \otimes^h A)_{\|\cdot\| \le \|\xi\|}$ which converges to ξ in the weak* topology of $(L^2(G)_c^* \otimes^{w^*h} A^*)^*$. Then, for every $\chi \in L^2(G)$ and $\mu \in A^*$, we have

$$\langle \xi, \chi \otimes \mu\rangle = \lim_i \int_G \langle \xi_i(s), \mu\rangle \overline{\chi(s)}\, ds,$$

uniformly for μ in compact subsets of A^* (by Lemma 3.4). Let $\chi = \chi_{\mathrm{supp}(\xi)} \in L^2(G)$. Then for every $\mu \in A^*$, the set $\{\mu \cdot \xi(s)^* \mid s \in G\}$ is norm compact in A^*, so that

$$\begin{aligned} \mu(\langle \xi, \xi_i\rangle_{A^{**}}) &= \int_G \langle \xi(s)^*\xi_i(s), \mu\rangle \overline{\chi(s)}\, ds \\ &= \int_G \langle \xi_i(s), \mu \cdot \xi(s)^*\rangle \overline{\chi(s)}\, ds \end{aligned}$$

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$$\to \int_G \langle \xi(s), \mu \cdot \xi(s)^* \rangle \overline{\chi(s)}\, ds$$
$$= \mu(\langle \xi, \xi \rangle_{A^{**}}).$$

Hence, $\langle \xi, \xi_i \rangle_{A^{**}} \to \langle \xi, \xi \rangle_{A^{**}}$ weak* in A^{**}, and so

$$\|\langle \xi, \xi \rangle_{A^{**}}\| \le \limsup_i \|\langle \xi, \xi_i \rangle_{A^{**}}\| \le \limsup_i \|j(\xi)\|_{L^2(G,A)''} \|\xi_i\|_{L^2(G,A)}$$
$$\le \|j(\xi)\|_{L^2(G,A)''} \|\xi\|_{L^2(G,A^{**})},$$

which implies that $\|\xi\|_{L^2(G,A^{**})} \le \|j(\xi)\|_{L^2(G,A)''}$. □

PROPOSITION 4.6. *Let (A, G, α) be an amenable C^*-dynamical system. Then there exists a net (h_i) of continuous compactly supported completely positive Herz–Schur multipliers satisfying:*

(1) $\|h_i(e)\|_{cb} \le 1$ *for all i;*
(2) $h_i(e) \to \mathrm{id}_A$ *in the point norm topology;*
(3) $\Theta(h_i) \to \mathrm{id}_{G \ltimes A}$ *in the point norm topology;*
(4) $h_i(s)(a) = \langle \xi_i, (1 \otimes 1 \otimes a)(\lambda_s \otimes 1 \otimes \alpha_s)\xi_i \rangle_A$, *for a contractive net (ξ_i) in $C_c(G, \ell^2(A))$.*

When $Z(A^{**}) = Z(A)^{**}$, the net (ξ_i) can be chosen in $C_c(G, Z(A))$, in which case $h_i(s)(a) = a\langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i \rangle$, $s \in G$, $a \in A$.

Proof. By Theorem 3.6, amenability of (A, G, α) implies the existence a net (ξ_i) in $C_c(G, Z(A''_\alpha)_c)$ whose corresponding positive type functions $h_i(s) = \langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i \rangle$ satisfy $h_i(e) = \langle \xi_i, \xi_i \rangle \le 1$ for all i, $\lim_i h_i(s) = 1$ weak*, uniformly on compact subsets.

Pick $\eta \in C_c(G)_{\|\cdot\|_2=1}$ and let $\xi_i' = (1 \otimes z)\xi_i + \eta \otimes (1 - z) \in C_c(G, Z(A^{**}))$. Then

$$\langle \xi_i', \xi_i' \rangle = \int_G z\xi_i(s)^*\xi(s) + |\eta(s)|^2(1 - z)\, ds = z\langle \xi_i, \xi_i \rangle + \|\eta\|^2(1 - z) \le 1.$$

By Lemma 4.5, $(j(\xi_i'))$ is a net in the unit ball of $L^2(G, A)''$. By the Kaplansky density theorem for Hilbert C^*-modules [**45**, Corollary 2.7], for each i, there exists a net $(\xi_{i,j})$ in $C_c(G, A)_{\|\cdot\|_{L^2(G,A)} \le 1}$ such that

$$\mu(\langle j(\xi_i') - j(\xi_{i,j}), j(\xi_i') - j(\xi_{i,j}) \rangle_{L^2(G,A)''})^{1/2} = \mu(\langle \xi_i' - \xi_{i,j}, \xi_i' - \xi_{i,j} \rangle_{A^{**}})^{1/2}$$
$$\to 0, \quad \mu \in (A^*)^+,$$

where the equality uses that j is an isometric A^{**}-module map (Lemma 4.5). We now observe two consequences of this approximation which will be combined into a single convexity argument to yield the desired properties (2)–(4) (property (1) being automatic).

First, for any state $\mu \in A^*$, applying Lemma 4.3 to the inner product A^{**}-module $E = C_c(G, A^{**})$, we have

$$\mu(1 - \langle \xi_{i,j}, \xi_{i,j} \rangle) = \mu(\langle \xi_i', \xi_i' \rangle - \langle \xi_{i,j}, \xi_{i,j} \rangle)$$
$$\le \|\langle \xi_i', \xi_i' \rangle\| \mu(\langle \xi_i' - \xi_{i,j}, \xi_i' - \xi_{i,j} \rangle)^{1/2}$$
$$+ \|\langle \xi_{i,j}, \xi_{i,j} \rangle\| \mu(\langle \xi_i' - \xi_{i,j}, \xi_i' - \xi_{i,j} \rangle)^{1/2}$$

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$$\leq 2\mu(\langle \xi_i' - \xi_{i,j}, \xi_i' - \xi_{i,j}\rangle)^{1/2}$$
$$\xrightarrow{j} 0.$$

Thus, $\langle \xi_{i,j}, \xi_{i,j}\rangle \to 1$ weak* in A^{**}, where we are considering the doubly indexed net as in [28, pp. 69]. Then, for each i and any state $\mu \in A^*$,

$$\mu(\langle (1\otimes a)(\xi_{i,j} - \xi_i'), (1\otimes a)(\xi_{i,j} - \xi_i')\rangle)^{1/2} \leq \|a\|\mu(\langle \xi_{i,j} - \xi_i', \xi_{i,j} - \xi_i'\rangle)^{1/2} \xrightarrow{j} 0$$

and, similarly,

$$\mu(\langle (\xi_{i,j} - \xi_i')(1\otimes a), (\xi_{i,j} - \xi_i')(1\otimes a)\rangle)^{1/2} = (a\cdot\mu\cdot a^*)(\langle \xi_{i,j} - \xi_i', \xi_{i,j} - \xi_i'\rangle)^{1/2} \xrightarrow{j} 0.$$

In addition, as ξ_i' takes values in $Z(A^{**})$, we have $(1\otimes a)\xi_i' = \xi_i'(1\otimes a)$ for each i and each $a \in A$. Using this, and applying similar estimates from the proof of Lemma 4.3, for any state μ and $a \in A$,

$$\begin{aligned}
&\mu(\langle \xi_{i,j}, (1\otimes a^*)\xi_{i,j}\rangle - a^*\langle \xi_{i,j}, \xi_{i,j}\rangle)\\
&\quad = \mu(\langle (1\otimes a)\xi_{i,j}, \xi_{i,j}\rangle - \langle \xi_{i,j}(1\otimes a), \xi_{i,j}\rangle)\\
&\quad = \mu(\langle (1\otimes a)(\xi_{i,j} - \xi_i'), \xi_{i,j}\rangle + \langle (\xi_i' - \xi_{i,j})(1\otimes a), \xi_{i,j}\rangle)\\
&\quad \leq \|\langle \xi_{i,j}, \xi_{i,j}\rangle\|^{1/2}\mu(\langle (1\otimes a)(\xi_{i,j} - \xi_i'), (1\otimes a)(\xi_{i,j} - \xi_i')\rangle)^{1/2}\\
&\qquad + \|\langle \xi_{i,j}, \xi_{i,j}\rangle\|^{1/2}\mu(\langle (\xi_{i,j} - \xi_i')(1\otimes a), (\xi_{i,j} - \xi_i')(1\otimes a)\rangle)^{1/2}\\
&\quad \to 0,
\end{aligned}$$

Thus, $\langle \xi_{i,j}, (1\otimes a)\xi_{i,j}\rangle - a\langle \xi_{i,j}, \xi_{i,j}\rangle \to 0$ weak* in A^{**}, and it follows that $\langle \xi_{i,j}, (1\otimes a)\xi_{i,j}\rangle \to a$ weak* in A^{**} for each $a \in A$.

Second, since $(1\otimes z)\xi_i'$ is equal to the original $\xi_i \in C_c(G, Z(A_\alpha'')_c)$, for any $\mu \in (A_\alpha'')_*^+ = z(A^*)^+$, we have

$$\mu(\langle \xi_i - z\xi_{i,j}, \xi_i - z\xi_{i,j}\rangle_{A_\alpha''})^{1/2} = \mu(\langle \xi_i' - \xi_{i,j}, \xi_i' - \xi_{i,j}\rangle_{A^{**}})^{1/2} \to 0,$$

where $z\xi_{i,j}$ is shorthand for $(1\otimes z)\xi_{i,j}$. Fix a state $\mu \in (A_\alpha'')_*^+$, $a \in A$, and let $\eta_i = (1\otimes a)^*\xi_i$ and $\eta_{i,j} = (1\otimes a)^* z\xi_{i,j}$. Then, by Lemma 4.3 applied to the inner product A_α''-module $E = C_c(G, A_\alpha'')$,

$$\begin{aligned}
&|\mu(\langle \xi_i, (1\otimes a)(\lambda_t\otimes\overline{\alpha}_t)\xi_i\rangle) - \mu(\langle z\xi_{i,j}, (1\otimes a)(\lambda_t\otimes\overline{\alpha}_t)z\xi_{i,j}\rangle)|\\
&\quad = |\mu(\langle \eta_i, (\lambda_t\otimes\overline{\alpha}_t)\xi_i\rangle) - \mu(\langle \eta_{i,j}, (\lambda_t\otimes\overline{\alpha}_t)z\xi_{i,j}\rangle)|\\
&\quad \leq \|\langle \eta_i, \eta_i\rangle\|^{1/2}\mu\circ\overline{\alpha}_t(\langle z\xi_{i,j} - \xi_i, z\xi_{i,j} - \xi_i\rangle)^{1/2}\\
&\qquad + \|\langle z\xi_{i,j}, z\xi_{i,j}\rangle\|^{1/2}\mu(\langle \eta_{i,j} - \eta_i, \eta_{i,j} - \eta_i\rangle)^{1/2}\\
&\quad = \|\langle (1\otimes a)^*\xi_i, (1\otimes a)^*\xi_i\rangle\|^{1/2}\mu\circ\overline{\alpha}_t(\langle z\xi_{i,j} - \xi_i, z\xi_{i,j} - \xi_i\rangle)^{1/2}\\
&\qquad + \mu(\langle (1\otimes a)^*(z\xi_{i,j} - \xi_i), (1\otimes a)^*(z\xi_{i,j} - \xi_i)\rangle)^{1/2}\\
&\quad \leq \|a\|\mu\circ\overline{\alpha}_t(\langle z\xi_{i,j} - \xi_i, z\xi_{i,j} - \xi_i\rangle)^{1/2}\\
&\qquad + \|a\|\mu(\langle z\xi_{i,j} - \xi_i, z\xi_{i,j} - \xi_i\rangle)^{1/2}.
\end{aligned}$$

Once again using the norm continuity of the predual action $G \curvearrowright (A''_\alpha)_*$ and Lemma 3.4, the above estimates imply that

$$|\mu(\langle \xi_i, (1 \otimes a)(\lambda_t \otimes \overline{\alpha}_t)\xi_i\rangle) - \mu(\langle z\xi_{i,j}, (1 \otimes a)(\lambda_t \otimes \overline{\alpha}_t)z\xi_{i,j}\rangle)| \xrightarrow{j} 0, \quad \mu \in (A''_\alpha)_*,$$

uniformly for t in compact subsets of G, and a in bounded subsets of A. Putting $h_{i,j}(t)(a) = \langle \xi_{i,j}, (1 \otimes a)(\lambda_t \otimes \alpha_t)\xi_{i,j}\rangle$, we obtain a net $(h_{i,j})$ of compactly supported completely positive Herz–Schur multipliers satisfying $\|h_{i,j}(e)\|_{cb} \leq 1$ and (recalling that each ξ_i takes central values),

$$|\mu(zh_{i,j}(t)(za)) - \mu(za)| \leq |\mu(zh_{i,j}(t)(za)) - \mu(h_i(t)za)| + |\mu(h_i(t)za) - \mu(za)| \xrightarrow{i,j} 0$$

for any $\mu \in (A''_\alpha)_*$, uniformly for (t, a) in compact subsets of $G \times A$ (the uniformity on compacta in A coming from the convergence of the second term above).

Fix $f \in C_c(G, zA)$. By [20, Lemme 3.2] there exists a linear combination $v \in A(G)$ of positive definite functions in $C_c(G)$ such that $v \equiv 1$ on $\mathrm{supp}(f)$. It follows that

$$v \cdot (\alpha \times \lambda)(f) = (\alpha \times \lambda)(v \cdot f) = (\alpha \times \lambda)(f),$$

where $\cdot$ is the canonical action of $A(G)$ on $G \ltimes zA$ via the dual coaction. Given $u \in (G \ltimes zA)^* \subseteq B(G \ltimes_f zA)$, by [34, Corollary 7.6.9], $v \cdot u$ is a linear combination of compactly supported positive definite functions in $B(G \ltimes_f zA)$. Hence, by [34, Lemma 7.7.6],

$$v \cdot u \in A(G \ltimes_f zA) = (G \ltimes zA)''_* \cong (G \bar{\ltimes} A''_\alpha)_*.$$

Then $\{v(s)u(s) \mid s \in G\}$ is a norm-compact subset of $(A''_\alpha)_*$, so boundedness of $\|h_{i,j}(s)\|$, the identification $A(G \ltimes_f zA) = (G \bar{\ltimes} A''_\alpha)_*$, and the weak* convergence $zh_{i,j}(s)(za) \to za$ imply that

$$\begin{aligned}
\langle u, \Theta(zh_{i,j})(\alpha \times \lambda(f))\rangle &= \langle v \cdot u, \Theta(zh_{i,j})(\alpha \times \lambda(f))\rangle \\
&= \int_G \langle v(s)u(s), zh_{i,j}(s)(f(s))\rangle \, ds \\
&\to \int_G \langle v(s)u(s), f(s)\rangle \, ds \\
&= \langle u, (\alpha \times \lambda)(f)\rangle.
\end{aligned}$$

By boundedness of $(\Theta(zh_{i,j}))$, it follows that $\Theta(zh_{i,j}) \to \mathrm{id}_{G \ltimes zA}$ in the point weak topology. Identifying A with $zA \subseteq A''_\alpha$, as well as the C^*-dynamical systems $(A, G, \alpha) \cong (zA, G, \overline{\alpha})$, it follows that $\Theta(h_{i,j}) \to \mathrm{id}_{G \ltimes A}$ in the point weak topology.

Now, for every $a_1, \ldots, a_n \in A$, $x_1, \ldots, x_m \in G \ltimes A$, consider the convex set

$$\begin{aligned}
C = \{&(h(e)(a_1) - a_1, \ldots, h(e)(a_n) - a_n, \\
&\Theta(h)(x_1) - x_1, \ldots, \Theta(h)(x_m) - x_m) \mid h \in \mathrm{conv}\{h_{i,j}\}\},
\end{aligned}$$

viewed inside the locally convex Hausdorff space

$$(A, w) \oplus \cdots \oplus (A, w) \oplus (G \ltimes A, w) \oplus \cdots \oplus (G \ltimes A, w),$$

where w denotes the weak topology. By the above analysis, 0 belongs to the closure of C. The standard convexity argument then shows that 0 belongs to the closure of C where

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all summands are equipped with the norm topology. It follows that there exists a net (h_i) of continuous compactly supported completely positive Herz–Schur multipliers $h_i : G \to \mathcal{CB}(A)$ satisfying properties (1)–(3), and each $h_i \in \mathrm{conv}\{h_{i,j}\}$. To see that (4) holds, use $h_i \in \mathrm{conv}\{h_{i,j}\}$ to write each h_i as

$$\begin{aligned} h_i(s)(a) &= \sum_{k=1}^{n_i} \lambda_k \langle \xi_{i_k,j_k}, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi_{i_k,j_k} \rangle \\ &= \langle \oplus_{k=1}^{n_i} \sqrt{\lambda_k}\xi_{i_k,j_k}, (1 \otimes 1 \otimes a)(\lambda_s \otimes 1 \otimes \alpha_s)(\oplus_{k=1}^{n_i} \sqrt{\lambda_k}\xi_{i_k,j_k})\rangle, \end{aligned}$$

where $\xi_i := \oplus_{k=1}^{n_i} \sqrt{\lambda_k}\xi_{i_k,j_k} \in \oplus_{k=1}^{n_i} C_c(G, A)$ lies in the unit ball of the Hilbert A-module $L^2(G, \ell^2(A))$.

Finally, when $Z(A^{**}) = Z(A)^{**}$, inspection of the proof shows that the $\xi_{i,j}$ from the Kaplansky density argument can be taken in $C_c(G, Z(A))$. In this case, $h_{i,j}(s)(a) = ak_{i,j}(s)$, where $k_{i,j}(s) = \langle \xi_{i,j}, (\lambda_s \otimes \alpha_s)\xi_{i,j}\rangle$ is a continuous compactly supported function $G \to Z(A)$ of positive type. It follows that the h_i from the final convexity argument satisfy $h_i(s)(a) = ak_i(s)$ for some continuous compactly supported function $k_i : G \to Z(A)$ of positive type, which, by [4, Proposition 2.5], is necessarily of the form $k_i(s) = \langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i\rangle$ for some contractive net $(\xi_i) \subseteq L^2(G, Z(A))$. The norm density of $C_c(G, Z(A))$ inside $L^2(G, Z(A))$ then yields the claim. □

Remark 4.7. Contrary to the well-known group case ($A = \mathbb{C}$), it is not clear whether every continuous completely positive Herz–Schur multiplier $h : G \to \mathcal{CB}(A)$ of compact support is necessarily of the form $h_i(s)(a) = \langle \xi, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi\rangle$ for some $\xi \in L^2(G, A)$. Indeed, this was already required for discrete dynamical systems in [10, Remark 4.29]. If this were true, then the net (ξ_i) in the conclusion of Proposition 4.6 can be taken in $C_c(G, A)$, and it would follow from the proof of Theorem 4.2 (see below) that amenability is equivalent to the 1-positive approximation property for arbitrary (A, G, α).

The following lemmas will be used to establish Theorem 4.10, which, as a corollary, entails the equivalence of conditions (2) and (3) of Theorem 4.2. The first is standard, and the second is surely known, but we include proofs for completeness.

LEMMA 4.8. *Let G be a locally compact group, and $f \in C_c(G)$. Then $\lambda(f) \geq 0$ if and only if $\Delta^{1/2} f$ is positive definite.*

Proof. This follows from the identity $\langle \Delta^{1/2} f, g^* * g\rangle = \langle \lambda(f)(\Delta^{1/2} g)^\vee, (\Delta^{1/2} g)^\vee\rangle$ for $f, g \in C_c(G)$, where the former pairing is the dual pairing $(B(G), C^*(G))$, the latter is the inner product on $L^2(G)$, and $(\Delta^{1/2} g)^\vee(t) = \sqrt{\Delta(t^{-1})}g(t^{-1})$ for $t \in G$. □

LEMMA 4.9. *Let (A, G, α) be a C^*-dynamical system. Then* $\mathrm{span}\{f^* \star f \mid f \in C_c(G, A)\}$ *is norm dense in $C_0(G, A)$.*

Proof. Let (f_i) be a bai for $L^1(G)$ consisting of states in $C_c(G)$ whose support goes to $\{e\}$. Let (a_j) be a bai for A, and let $f_{i,j} \in C_c(G, A)$ be $f_{i,j}(s) = f_i(s)\alpha_s(a_j)$. Then $(f_{i,j})$ is a bai for the convolution algebra $L^1(G, A)$ (see, for example, [36, Proposition 16.4.3]).

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By density of $C_c(G) \otimes A$ in $C_0(G, A)$ and a simple polarization argument, it suffices to show that $f_{i,j} \star (g \otimes a) \to (g \otimes a)$ uniformly in $C_0(G, A)$ for all $g \in C_c(G)$ and $a \in A$.

First, $g \in C_c(G)$ is uniformly continuous, so

$$f_i * g \to g \tag{19}$$

uniformly, where $*$ denotes convolution in $L^1(G)$. Second, by norm continuity of $\alpha_t(a)$ at the identity, the standard argument shows that

$$\int_G f_i(t)\|\alpha_t(a) - a\|\, dt \to 0. \tag{20}$$

Then (19) and (20), together with the fact that $a_j a \to a$, imply

$$\begin{aligned}
\|f_{i,j} \star (g \otimes a)(s) - g \otimes a(s)\| &= \left\| \int_G f_i(t)(g(t^{-1}s)\alpha_t(a_j a) - g(s)a)\, dt \right\| \\
&\leq \int_G |f_i(t)g(t^{-1}s)|\|\alpha_t(a_j a) - \alpha_t(a)\|\, dt \\
&\quad + \int_G |f_i(t)g(t^{-1}s)|\|\alpha_t(a) - a\|\, dt \\
&\quad + \left| \int_G f_i(t)(g(t^{-1}s) - g(s))\, dt \right| \|a\| \\
&\leq \|f_i\|_1 \|g\|_\infty \|a_j a - a\| \\
&\quad + \|g\|_\infty \int_G f_i(t)\|\alpha_t(a) - a\|\, dt \\
&\quad + \|a\| |f_i * g - g|(s) \\
&\xrightarrow{i,j} 0
\end{aligned}$$

uniformly in s. Thus, $f_{i,j} \star (g \otimes a) \to (g \otimes a)$ uniformly in $C_0(G, A)$, and the claim is verified. □

THEOREM 4.10. *Let (A, G, α) be a C^*-dynamical system and let (h_i) be a bounded net of continuous, compactly supported, completely positive Herz–Schur multipliers. The following conditions are equivalent.*

(1) $\|h_i(s)(f(s)) - f(s)\| \to 0$ *for every $f \in C_c(G, A)$, uniformly for s in compact subsets of G.*

(2) $h_i(e) \to \mathrm{id}_A$ *and* $\Theta(h_i) \to \mathrm{id}_{G \ltimes A}$ *in the respective point norm topologies.*

Proof. (1) ⇒ (2): First, pick $g \in C_c(G)$ with $g(e) = 1$. Given $a \in A$, applying condition (1) to $f = g \otimes a$ at $s = e$ implies that $\|h_i(e)(a) - a\| \to 0$.

Second, we have

$$\begin{aligned}
\|\alpha(h_i(s)(f(s)))(\lambda_s \otimes 1) - \alpha(f(s))(\lambda_s \otimes 1)\| &= \|\alpha(h_i(s)(f(s)) - f(s))\| \\
&= \|h_i(s)(f(s)) - f(s)\| \to 0
\end{aligned}$$

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for every $f \in C_c(G, A)$, uniformly for s in compact subsets of G. Hence, by definition of $\Theta(h_i)$, we have

$$\|\Theta(h_i)((\alpha \times \lambda(f))) - (\alpha \times \lambda(f))\| \leq \int_{\text{supp(f)}} \|\alpha(h_i(s)(f(s)))(\lambda_s \otimes 1) - \alpha(f(s))(\lambda_s \otimes 1)\| \, ds \to 0$$

for every $f \in C_c(G, A)$. By boundedness of (h_i), it follows that $\Theta(h_i) \to \mathrm{id}_{G \ltimes A}$ in the point norm topology.

(2) ⇒ (1): Identify A with $zA \subseteq A''_\alpha$, and identify the C^*-dynamical systems $(A, G, \alpha) \cong (zA, G, \overline{\alpha})$. We may also assume $A''_\alpha \subseteq \mathcal{B}(H)$ is standardly represented, so that $\alpha(x) = U^*(1 \otimes x)U$, for a unitary $U \in L^\infty(G)\overline{\otimes}\mathcal{B}(H)$.

Using the standard implementation U along with the commutation relation $U(\lambda_s \otimes 1) = (\lambda_s \otimes u_s)U$, for each $f \in C_c(G, A)$ we have

$$\begin{aligned}\int_G \lambda_s \otimes h_i(s)(f(s))u_s \, ds &= \int_G (1 \otimes h_i(s)(f(s)))(\lambda_s \otimes u_s) \, ds \\ &= \int_G U\alpha(h_i(s)(f(s)))U^*(\lambda_s \otimes u_s) \, ds \\ &= \int_G U\alpha(h_i(s)(f(s)))(\lambda_s \otimes 1)U^* \, ds \\ &= U\Theta(h_i)((\alpha \times \lambda)(f))U^* \\ &\to U(\alpha \times \lambda)(f)U^* \\ &= \int_G \lambda_s \otimes f(s)u_s \, ds,\end{aligned}$$

where the convergence is in the norm topology of $\mathcal{B}(L^2(G, H))$. Consequently, for any $\eta \in H$, with ω_η denoting the associated vector functional on $\mathcal{B}(H)$,

$$\begin{aligned}\int_G \langle \eta, h_i(s)(f(s))u_s\eta\rangle \lambda_s \, ds &= (\mathrm{id} \otimes \omega_\eta)\bigg(\int_G \lambda_s \otimes h_i(s)(f(s))u_s \, ds\bigg) \\ &\to (\mathrm{id} \otimes \omega_\eta)\bigg(\int_G \lambda_s \otimes f(s)u_s \, ds\bigg) \\ &= \int_G \langle \eta, f(s)u_s\eta\rangle \lambda_s \, ds, \qquad (21)\end{aligned}$$

where the convergence is in $(C^*_\lambda(G), \|\cdot\|)$ and is uniform for η in bounded subsets of H.

Let $f \in C_c(G, A)$ be positive in the sense that $f = f_0^* \star f_0$ in the convolution algebra $C_c(G, A)$. Then, by positivity of $\Theta(h_i)$,

$$\int_G \lambda_s \otimes h_i(s)(f(s))u_s \, ds = U\Theta(h_i)((\alpha \times \lambda)(f))U^* \geq 0$$

so that

$$\int_G \langle \eta, h_i(s)(f(s))u_s\eta\rangle \lambda_s \, ds = \lambda(v_{i,f,\eta}) \geq 0,$$

where $v_{i,f,\eta}(s) = \langle \eta, h_i(s)(f(s))u_s\eta\rangle$. Similarly,

$$\int_G \langle \eta, f(s)u_s\eta\rangle\lambda_s\, ds = \lambda(v_{f,\eta}) \geq 0,$$

where $v_{f,\eta}(s) = \langle \eta, f(s)u_s\eta\rangle$. By Lemma 4.8, $w_{i,f,\eta} := \Delta^{1/2}v_{i,f,\eta}$ and $w_{f,\eta} := \Delta^{1/2}v_{f,\eta}$ are positive definite functions on G. Applying the convergence (21) to $(\Delta^{1/2}g \otimes 1)f \in C_c(G, A)$, for $g \in C_c(G)$, it follows that

$$\|\lambda(w_{i,f,\eta}g) - \lambda(w_{f,\eta}g)\| \to 0, \quad g \in C_c(G),$$

uniformly for η in bounded subsets of H.

We now show that $w_{i,f,\eta} \to w_{f,\eta}$ weak* in $B(G)$, and that $\|w_{i,f,\eta}\|_{B(G)} \to \|w_{f,\eta}\|_{B(G)}$, both uniformly in η. First, observe that $(w_{i,f,\eta})$ is bounded in $B(G) = C^*(G)^*$ uniformly in $\|\eta\|$: since $w_{i,f,\eta}$ is positive definite, we have

$$\begin{aligned}\|w_{i,f,\eta}\|_{B(G)} &= w_{i,f,\eta}(e)\\ &= \langle \eta, h_i(e)(f(e))\eta\rangle\\ &\leq \|h_i(e)\|_{cb}\|f(e)\|\|\eta\|^2\end{aligned}$$

Given $g \in C_c(G)$, if we pick $v \in A(G)$ with $v \equiv 1$ on $\mathrm{supp}(g)$, then

$$\langle w_{i,f,\eta} - w_{f,\eta}, g\rangle = \langle w_{i,f,\eta} - w_{f,\eta}, vg\rangle = \langle \lambda(w_{i,f,\eta}g) - \lambda(w_{f,\eta}g), v\rangle \to 0.$$

Since the image of $C_c(G)$ under the universal representation of G is dense in $C^*(G)$ and $(w_{i,f,\eta})$ is bounded in $B(G) = C^*(G)^*$ (uniformly in $\|\eta\|$), we have $w_{i,f,\eta} \to w_{f,\eta}$ weak* in $B(G)$, uniformly for η in bounded subsets.

The convergence $\|w_{i,f,\eta}\|_{B(G)} \to \|w_{f,\eta}\|_{B(G)}$ and its uniformity in η follow from the point norm convergence $h_i(e) \to \mathrm{id}_A$:

$$\lim_i \|w_{i,f,\eta}\|_{B(G)} = \lim_i \langle \eta, h_i(e)(f(e))\eta\rangle = \langle \eta, f(e)\eta\rangle = \|w_{f,\eta}\|_{B(G)}.$$

Thus, in the notation of [21], $w_{i,f,\eta} \to w_{f,\eta}$ in $(B(G), \tau_{nw^*})$, uniformly for η in bounded subsets of H. By [21, Theorem A], it follows that $w_{i,f,\eta} \to w_{f,\eta}$ in the $A(G)$-multiplier topology, and therefore uniformly on compact sets, and the convergence is uniform for η in bounded subsets of H. Thus, given $K \subseteq G$ compact and $\varepsilon > 0$, pick i_ε such that

$$\sup_{s\in K} |w_{i,f,\eta}(s) - w_{f,\eta}(s)| < \frac{\varepsilon}{\sup_{s\in K} \Delta^{-1/2}(s)}, \quad i \geq i_\varepsilon.$$

Then, for all $i \geq i_\varepsilon$,

$$\sup_{s\in K} |v_{i,f,\eta}(s) - v_{f,\eta}(s)| = \sup_{s\in K} |\Delta^{-1/2}(s)||w_{i,f,\eta}(s) - w_{f,\eta}(s)| < \varepsilon,$$

and $v_{i,f,\eta} \to v_{f,\eta}$ uniformly on compact sets, uniformly for η in bounded subsets of H. In particular,

$$\sup_{\|\eta\|\leq 2} |\langle \eta, (h_i(s)(f(s)) - f(s))u_s\eta\rangle| = \sup_{\|\eta\|\leq 2} |\langle \eta, h_i(s)(f(s))u_s\eta\rangle - \langle \eta, f(s)u_s\eta\rangle| \to 0,$$

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uniformly for s in compact subsets of G. Hence, by polarization,

$$\begin{aligned}\|h_i(s)(f(s)) - f(s)\| &= \|(h_i(s)(f(s)) - f(s))u_s\| \\ &= \sup_{\|\eta_1\|,\|\eta_2\|\le 1} |\langle \eta_1, (h_i(s)(f(s)) - f(s))u_s\eta_2\rangle| \\ &\le \frac{1}{4}\sum_{k=0}^{3} \sup_{\|\eta_1\|,\|\eta_2\|\le 1} |\langle(\eta_1 + i^k\eta_2), (h_i(s)(f(s)) \\ &\quad - f(s))u_s(\eta_1 + i^k\eta_2)\rangle| \\ &\to 0\end{aligned}$$

for each $f \in C_c(G, A)$ of the form $f_0^* \star f_0$, uniformly for s in compact subsets of G. By boundedness of $h_i(s)$ in $\mathcal{CB}(A)$, Lemma 4.9 and a standard 3ε-argument, it follows that

$$\|h_i(s)(f(s)) - f(s)\| \to 0, \quad f \in C_c(G, A),$$

uniformly for s in compact subsets of G. □

Let (A, G, α) be a C^*-dynamical system. The space $L^2(G)_c \otimes^{w^*h} A''_\alpha$ is a Hilbert W^*-module over A''_α in the canonical fashion [11, Theorem 3.1]; see also [11, bottom of pp. 71], which explains that the tensor product referred to in [11, Theorem 3.1] coincides with the weak* Haagerup tensor product in this case; that is, $L^2(G)_c \otimes^{w^*h} A''_\alpha$ is the weak*-closure of the (interior or exterior) C^*-module tensor product $L^2(G) \otimes A''_\alpha$ in $\mathcal{B}(H, L^2(G, H))$, where $A''_\alpha \subseteq \mathcal{B}(H)$ is a faithful normal representation.

The next lemma is used to make sense of the 'diagonal' action of $L^\infty(G)\overline{\otimes}A''_\alpha$ from $L^2(G, A)$ into $L^2(G)_c \otimes^{w^*h} A''_\alpha$.

LEMMA 4.11. *Let G be a locally compact group and let M be a von Neumann algebra. There exists a contraction*

$$\pi : L^\infty(G)\overline{\otimes}M \to \mathcal{CB}(L^2(G)_c \otimes^{w^*h} M, L^2(G)_c \otimes^{w^*h} M)$$

such that, for every $F \in L^\infty(G)\overline{\otimes}M$, $\xi, \eta \in L^2(G)$, $a \in M$, and $\mu \in M_*$,

$$\langle \eta \otimes \mu, \pi(F)(\xi \otimes a)\rangle = \langle(\omega_{\eta,\xi} \otimes \mathrm{id})(F)a, \mu\rangle = \int_G \xi(s)\overline{\eta}(s)\langle \tilde{F}(s)a, \mu\rangle\, ds.$$

Proof. Fix a faithful normal representation $M \subseteq \mathcal{B}(H)$, and view $L^\infty(G)\overline{\otimes}M \subseteq \mathcal{B}(L^2(G, H))$ and $L^2(G)_c \otimes^{w^*h} M \subseteq \mathcal{B}(H, L^2(G, H))$ under the canonical embeddings. Define π to be the restriction of the map $\tilde{\pi} : \mathcal{B}(L^2(G, H)) \to \mathcal{CB}(\mathcal{B}(H, L^2(G, H)))$ defined by $\tilde{\pi}(T)(R) = TR$. By approximating with elements in $C_c(G, M)$, it is straightforward to check that $\pi(L^\infty(G)\overline{\otimes}M)(L^2(G)_c \otimes^{w^*h} M) \subseteq L^2(G)_c \otimes^{w^*h} M$, and that the desired formulae in the statement of the lemma hold. □

We now possess the ingredients to establish our main result of this section.

Proof of Theorem 4.2. (1) ⇒ (2) follows directly from Proposition 4.6.

(2) ⇔ (3) follows immediately from Theorem 4.10.

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(3) $\Rightarrow$ (4) follows from the techniques used in the proof of [1, Lemma 6.5], which, as shown in the proof of (7) $\Rightarrow$ (8) in [15, Proposition 3.10], extend to the locally compact case. We outline the construction, referring the reader to the proof of [15, Proposition 3.10] for details. Throughout the argument we identify A with $zA \subseteq A''_\alpha$.

Let $(\xi_i) \subset C_c(G, \ell^2(A))$ be a net from (3). Note that we may view (ξ_i) inside $\ell^2_c \otimes^h L^2(G, A)$, as

$$\begin{aligned}\ell^2_c \otimes^h L^2(G, A) &= \ell^2_c \otimes^h (L^2(G)_c \otimes^h A) \quad \text{(Lemma 4.4)}\\ &= (\ell^2_c \otimes^h L^2(G)_c) \otimes^h A\\ &= (\ell^2 \otimes L^2(G))_c \otimes^h A \quad [16, \text{Proposition 9.3.5}]\\ &= (L^2(G) \otimes \ell^2)_c \otimes^h A\\ &= L^2(G)_c \otimes^h (\ell^2_c \otimes^h A)\\ &= L^2(G, \ell^2(A)),\end{aligned}$$

where the Hilbert A-module structure on the latter space is

$$\langle \xi, \eta \rangle = \int_G \langle \xi(s), \eta(s) \rangle \, ds, \quad \xi, \eta \in L^2(G, \ell^2(A)).$$

Let $\Lambda = \{a \in A \mid 0 \leq a \leq 1\}$, which forms a bai for A under the natural ordering, and converges weak* to 1 inside A''_α. Define $P_{i,a} : L^\infty(G)\overline{\otimes}A''_\alpha \to A''_\alpha$ by

$$P_{i,a}(F) = \langle (1 \otimes 1 \otimes a^{1/2})\xi_i, (1 \otimes F)(1 \otimes 1 \otimes a^{1/2})\xi_i \rangle, \quad F \in L^\infty(G)\overline{\otimes}A''_\alpha,$$

where we write F for the map $\pi(F) : L^2(G, A) \to L^2(G)_c \otimes^{w^*h} A''_\alpha$ from Lemma 4.11. Then $P_{i,a}$ is a completely positive contraction.

Suppose $A''_\alpha \subseteq \mathcal{B}(H)$ and let $K = \oplus^2_{a \in \Lambda} H$. Then with $P_i := \oplus_a P_{i,a}$, we obtain a completely positive contraction from $L^\infty(G)\overline{\otimes}A''_\alpha$ into $\mathcal{B}(K)$. Passing to a subnet, we may assume that P_i converges to P in the weak* topology of $\mathcal{CB}(L^\infty(G)\overline{\otimes}A''_\alpha, \mathcal{B}(K))$. For each $a \in \Lambda$, let $P_a : L^\infty(G)\overline{\otimes}A''_\alpha \to A''_\alpha$ be the compression of P to the ath block, and let $Q_a : L^\infty(G)\overline{\otimes}Z(A''_\alpha) \to A''_\alpha$ be the restriction of P_a. The same monotonicity argument from [1, Lemma 6.5] shows that, for each positive $F \in L^\infty(G)\overline{\otimes}Z(A''_\alpha)$, $(Q_a(F))$ is increasing in a, and hence by boundedness it converges weak*. Let $Q : L^\infty(G)\overline{\otimes}Z(A''_\alpha) \to A''_\alpha$ be the resulting map. Using the fact that $a \mapsto 1 \otimes 1 \otimes a$ and $s \mapsto 1 \otimes \lambda_s \otimes \alpha_s$ is an equivariant representation of (A, G, α) on the direct sum $\oplus^\infty_{n=1} L^2(G, A) \cong \ell^2_c \otimes^h L^2(G, A)$, it follows more or less verbatim from the proof of [1, Lemma 6.5] (see also [15, Proposition 3.10]) that Q is a G-equivariant projection of norm one that takes values in $Z(A''_\alpha)$. Hence, (A, G, α) is von Neumann amenable.

(4) $\Rightarrow$ (1) follows immediately from Theorem 3.6.

Finally, when $Z(A^{**}) = Z(A)^{**}$, the particular conclusion from Proposition 4.6 yields the claim. □

Remark 4.12. In [15, Definition 3.24], Buss, Echterhoff and Willett defined a C^*-dynamical system (A, G, α) to have the (wAP) if there exists a bounded net

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$(\xi_i) \in C_c(G, A) \subseteq L^2(G, A)$ such that, for all $\mu \in A_c^*$, and $a \in A$,

$$\mu(\langle \xi_i, (1 \otimes a)(\lambda_s \otimes \alpha_s)\xi_i \rangle - a) \to 0,$$

uniformly on compact subsets of G. This notion is a weakening of Exel and Ng's positive approximation property, *a priori* unrelated to condition (3) of Theorem 4.2, which is a specific instance of the positive weak approximation property of Bédos and Conti. However, it was shown that the (wAP) coincides with amenability [**15**, Theorem 3.25]. Hence, it is equivalent to condition (3) of Theorem 4.2.

COROLLARY 4.13. *Let (A, G, α) be a C^*-dynamical system such that $Z(A^{**}) = Z(A)^{**}$. Then (A, G, α) is amenable if and only if it has the 1-positive approximation property.*

Proof. The forward direction follows immediately from the special case of Theorem 4.2. The reverse direction is always true, by [**15**, Theorem 3.25]. □

COROLLARY 4.14. *A commutative C^*-dynamical system $(C_0(X), G, \alpha)$ is amenable if and only if it is strongly amenable.*

Proof. Only one direction requires proof. If $(C_0(X), G, \alpha)$ is amenable, by the special case of Theorem 4.2 when $Z(A^{**}) = Z(A)^{**}$, there exists a net (ξ_i) in $C_c(G, C_0(X))$ whose corresponding positive type functions $h_i(s) = \langle \xi_i, (\lambda_s \otimes \alpha_s)\xi_i \rangle$ satisfy $\|h_i(e)\| \leq 1$ and

$$\|h_i(s)f - f\| \to 0, \quad f \in C_0(X),$$

uniformly for s in compact subsets of G. It follows that $h_i(s) \to 1$ strictly in $C_b(X)$, uniformly on compact subsets of G. Since the strict topology and the topology of uniform convergence on compacta agree on bounded subsets of $C_b(X)$ [**14**, Theorem 1], the associated functions $h_i : G \times X \to \mathbb{C}$ converge to 1 uniformly on compact subsets of $G \times X$. By norm density of $C_c(G) \otimes C_c(X)$ in $L^2(G, C_0(X))$, we may assume without loss of generality that $\xi_i \in C_c(G) \otimes C_c(X)$. Then the net (ξ_i) satisfies the conditions of [**5**, Proposition 2.5(2)], hence (G, X) is an amenable transformation group. □

Remark 4.15. After this paper appeared in preprint, Ozawa and Suzuki showed (using Theorem 1.1) that amenability and the positive approximation property coincide for arbitrary C^*-dynamical systems [**32**].

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