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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Clinton Conley (Managing Editor), Mark van Atten, Benno van den Berg, Thomas Colcombet, Samuel Coskey, Bradd Hart, Bernard Linsky, Antonio Montalbán, Rahim Moosa, Christian Retoré, and Nam Trang. Authors and publishers are requested to send, for review, copies of books to *ASL, Department of Mathematics, University of Connecticut, 341 Mansfield Road, U-1009, Storrs, CT 06269-1009, USA*.

J. H. SCHMERL, *Subsets coded in elementary end extensions*. ***Archive for Mathematical Logic***, vol. 53 (2014), no. 5–6, pp. 571–581.

J. H. SCHMERL, *Minimal elementary end extensions*. ***Archive for Mathematical Logic***, vol. 56 (2017), no. 5–6, pp. 541–553.

These papers study elementary end extensions of models of Peano Arithmetic (PA). They follow in the footsteps of the earliest results in the field, in particular the theorem of Robert MacDowell and Ernst Specker (1961), which asserts that every model of PA has an elementary end extension.

We write $\mathcal{M} \prec_{\text{end}} \mathcal{N}$ to denote that $\mathcal{N}$ is an *elementary end extension* of $\mathcal{M}$; that is, $\mathcal{M} \prec \mathcal{N}$ and for all $a \in M$ and $b \in N \setminus M$, $\mathcal{N} \models a < b$. Much of the study of models of arithmetic has resulted from modifying and refining the MacDowell–Specker Theorem. The focus on end extensions appears to be a bit of a historical accident: MacDowell and Specker themselves were interested in this result as a lemma in order to study additive groups of nonstandard models, rather than as part of a systematic study of nonstandard models of arithmetic.

Haim Gaifman (1970) improved the MacDowell–Specker Theorem by showing that every model of PA has a minimal elementary end extension. An elementary extension $\mathcal{M} \prec \mathcal{N}$ is called *minimal* if whenever $\mathcal{M} \preccurlyeq \mathcal{K} \preccurlyeq \mathcal{N}$, then either $\mathcal{M} = \mathcal{K}$ or $\mathcal{K} = \mathcal{N}$. By Gaifman's Splitting Theorem (1972), minimal elementary extensions can be either end extensions or cofinal extensions; in this discussion, we will restrict our attention only to end extensions.

Robert Philips (1974) noted that, in MacDowell–Specker and Gaifman's proofs, the resulting elementary extensions are *conservative*. An elementary extension $\mathcal{M} \prec \mathcal{N}$ is called *conservative*, denoted $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$, if for every set $X \subseteq N$ which is definable (with parameters) in $\mathcal{N}$, the set $X \cap M$ is definable (with parameters) in $\mathcal{M}$. It is easy to see that conservative extensions are necessarily end extensions. Furthermore, there are some models whose elementary end extensions are always conservative. More generally, given a first-order structure $\mathcal{M}$, we define $\text{Def}(\mathcal{M})$ to be the set of parametrically definable subsets of the universe M, and if $\mathcal{M} \prec \mathcal{N}$, we define $\text{Cod}(\mathcal{N}/\mathcal{M}) = \{X \cap M : X \in \text{Def}(\mathcal{N})\}$. A set $X \in \text{Cod}(\mathcal{N}/\mathcal{M})$ is said to be *coded* in $\mathcal{N}$. It is clear that $\text{Def}(\mathcal{M}) \subseteq \text{Cod}(\mathcal{N}/\mathcal{M})$; if $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$, then $\text{Def}(\mathcal{M}) = \text{Cod}(\mathcal{N}/\mathcal{M})$.

About nonconservative elementary end extensions, the strongest result known was given by Roman Kossak and Jeff Paris (1992). This result studied countable models $\mathcal{M} \models \text{PA}$ and characterized which $X \subseteq M$ can be coded in an elementary end extension. The first paper under review improves the result by Kossak–Paris in two important ways: it applies to uncountable models $\mathcal{M}$, and it considers collections of subsets of a model $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$. The main results in this paper characterize such collections in terms of their second-order theories.

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The theory RCA_0^* consists of Δ_1^0 comprehension (Δ_1^0-CA) and Σ_0^0 induction. WKL_0^* consists of RCA_0^* and WKL, Weak König's Lemma, asserting that every infinite binary tree has an infinite path. ACA_0 consists of Σ_1^0 induction and ACA, the arithmetical comprehension axiom. Given a model $\mathcal{M} \models \mathsf{PA}$ and a collection $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, we say $\mathfrak{X}_0$ generates $\mathfrak{X}$ if whenever $\mathfrak{X}_0 \subseteq \mathfrak{X}_1 \subseteq \mathfrak{X}$ and $(\mathcal{M}, \mathfrak{X}_1) \models \Delta_1^0$-$\mathsf{CA}$, then $\mathfrak{X}_1 = \mathfrak{X}$. $\mathfrak{X}$ is countably generated if there is a countable $\mathfrak{X}_0 \subseteq \mathfrak{X}$ which generates it.

Schmerl proves that if $\mathcal{M} \models \mathsf{PA}$ and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, then $\mathcal{M}$ has a finitely generated elementary end extension $\mathcal{N}$ such that $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$ if and only if $\mathrm{Def}(\mathcal{M}) \subseteq \mathfrak{X}$, $\mathfrak{X}$ is countably generated, and $(\mathcal{M}, \mathfrak{X}) \models \mathsf{WKL}_0^*$. This result generalizes the MacDowell–Specker Theorem: for any model $\mathcal{M} \models \mathsf{PA}$, $\mathrm{Def}(\mathcal{M})$ is countably generated and $(\mathcal{M}, \mathrm{Def}(\mathcal{M})) \models \mathsf{ACA}_0$, so $\mathcal{M}$ has a conservative elementary end extension. One direction of Schmerl's result is proved fairly routinely: if $\mathcal{M} \prec_{\mathrm{end}} \mathcal{N}$ is finitely (or even countably) generated, then $\mathrm{Def}(\mathcal{M}) \subseteq \mathrm{Cod}(\mathcal{N}/\mathcal{M})$, $\mathrm{Cod}(\mathcal{N}/\mathcal{M})$ is countably generated and $(\mathcal{M}, \mathrm{Cod}(\mathcal{N}/\mathcal{M})) \models \mathsf{WKL}_0^*$. The other direction involves constructing a sequence $\Phi_0(x) \subseteq \Phi_1(x) \subseteq \cdots$ whose union is a complete type $p(x)$. This sequence satisfies the following properties:

1. For any formula $\phi(u, x)$, there is $m < \omega$ and $X \in \mathfrak{X}$ such that
$$\{\phi(a, x) : a \in X\} \cup \{\neg\phi(a, x) : a \in M \setminus X\} \subseteq \Phi_m(x)\}.$$
2. If $X \in \mathfrak{X}$, there is a formula $\phi(u, x)$ and $m < \omega$ such that
$$\{\phi(a, x) : a \in X\} \cup \{\neg\phi(a, x) : a \in M \setminus X\} \subseteq \Phi_m(x)\}.$$

Suppose c realizes $p(x)$. Property (1) ensures that $\mathrm{Cod}(\mathcal{M}(c)/\mathcal{M}) \subseteq \mathfrak{X}$. Property (2) ensures that $\mathfrak{X} \subseteq \mathrm{Cod}(\mathcal{M}(c)/\mathcal{M})$.

Schmerl also studies this problem for countable models and shows that the same characterization suffices for minimal elementary end extensions. For uncountable models, more can be said about the sets coded in their minimal elementary end extensions, and Schmerl does so in the second paper under review. The main result states that if $\mathcal{M} \models \mathsf{PA}$ and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, then $\mathcal{M}$ has a minimal elementary end extension $\mathcal{N}$ such that $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$ if and only if $\mathrm{Def}(\mathcal{M}) \subseteq \mathfrak{X}$, $\mathfrak{X}$ is countably generated, $(\mathcal{M}, \mathfrak{X}) \models \mathsf{WKL}_0^*$, and every set that is Π_1^0-definable in $(\mathcal{M}, \mathfrak{X})$ is the union of countably many Σ_1^0-definable sets. This result in fact proves a weakening of the result for countable models from the first paper: if $\mathcal{M}$ and $\mathfrak{X}$ are countable, then every definable set in $(\mathcal{M}, \mathfrak{X})$ is a union of countably many Σ_1^0-definable sets, and so $\mathcal{M}$ has a minimal elementary end extension $\mathcal{N}$ such that $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. It also implies that if $\mathfrak{X}$ is countably generated and $(\mathcal{M}, \mathfrak{X}) \models \mathsf{ACA}_0$, then $\mathcal{M}$ has a minimal elementary end extension $\mathcal{N}$ such that $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. This result generalizes Gaifman's theorem: for any $\mathcal{M} \models \mathsf{PA}$, as before, $\mathrm{Def}(\mathcal{M})$ is countably generated and $(\mathcal{M}, \mathfrak{X}) \models \mathsf{ACA}_0$, so $\mathcal{M}$ has a conservative minimal elementary end extension.

The main result of the second paper seemingly has an extra condition; namely, that every Π_1^0 definable set is a union of countably many Σ_1^0 definable sets. That is, one might expect that if $\mathcal{M} \models \mathsf{PA}$ and $\mathfrak{X}$ is such that there is some finitely generated elementary end extension $\mathcal{N}$ such that $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$, then $\mathcal{M}$ would necessarily also have a minimal elementary end extension $\mathcal{N}$ with $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$. Schmerl shows that this is not the case: there are models $\mathcal{M} \models \mathsf{PA}$ and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ such that $\mathrm{Def}(\mathcal{M}) \subseteq \mathfrak{X}$, $\mathfrak{X}$ is countably generated, and $(\mathcal{M}, \mathfrak{X}) \models \mathsf{WKL}_0$, but there is a set X which is Π_1^0-definable in $(\mathcal{M}, \mathfrak{X})$ that is not the union of countably many Σ_1^0-definable sets. Therefore, there exists a finitely generated $\mathcal{N} \succ_{\mathrm{end}} \mathcal{M}$ with $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$, but $\mathcal{M}$ has no minimal elementary end extension $\mathcal{N}$ where $\mathrm{Cod}(\mathcal{N}/\mathcal{M}) = \mathfrak{X}$.

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