TRANSIENT ANALYSIS OF LINEAR BIRTH–DEATH PROCESSES WITH IMMIGRATION AND EMIGRATION

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Linear birth–death processes with immigration and emigration are major models in the study of population processes of biological and ecological systems, and their transient analysis is important in the understanding of the structural behavior of such systems. The spectral method has been widely used for solving these processes; see, for example, Karlin and McGregor [11]. In this article, we provide an alternative approach: the method of characteristics. This method yields a Volterra-type integral equation for the chance of extinction and an explicit formula for the z-transform of the transient distribution. These results allow us to obtain closed-form solutions for the transient behavior of several cases that have not been previously explicitly presented in the literature.

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1. INTRODUCTION

Birth–death processes with immigration and emigration are major models for the study of the population process of biological and ecological systems, and their transient analysis is important in the understanding of the structural behavior of such systems (see, e.g., Keyfitz [14]). However, analytic transient solutions for such systems have been derived only for some special cases (see, e.g., Kendall [13], Karlin and McGregor [11], Bailey [1], Bartlett [2], Iosifescu and Tautu [7], and Ismail, Letessier, and Valent [8], among others). In this article, we are concerned with the transient behavior of the general linear birth–death processes with both immigration and emigration.

Transient analysis of birth and death processes has also attracted much interest because of its connection with queuing models. Morse [20] studied the $M/M/1$ queue and obtained the time-dependent probabilities for the number of customers at any time t (see also Kleinrock [15]). Using the spectral measure method, Karlin and McGregor [10] analyzed the transient solutions of multiple-server queues. Saaty [26] considered the $M/M/s$ queue and derived the Laplace transform for the distribution of the number of customers in a system at any time t (see also Kelton and Law [12]). Rothkopf and Oren [25] studied some generalized $M/M/s$ queues with time-dependent arrival or service rates, and they obtained approximation results for transient solutions. The spectral function of the continuous time transient solution for $M/M/s$ queue was studied in van Doorn [27, Chap. 6], and further papers relevant to transient solution for $M/M/s$ queues include Whitt [29], Halfin and Whitt [6], and Pegden and Rosenshine [21]. All these queuing systems are modeled as and analyzed using simple birth–death processes.

The spectral method has been widely used in the literature in analyzing birth–death processes (see Karlin and McGregor [11]). In this article, we employ an alternative approach, the *method of characteristics*, to study birth and death processes. As will be seen in this article, this method not only provides a unified approach to the study of birth–death processes with immigration and emigration, but it also yields explicit solutions that have not been explicitly reported in the literature. The method can be considered as the dual method and it is complementary to the spectral method of Karlin and McGregor [11], further refined in Ismail et al. [8]. Karlin and McGregor's method is essentially a separation of variables which converts a time-dependent partial differential equation to a time-independent (stationary) problem; that is, the time variable is removed first. In contrast, our method converts a time-dependent partial differential equation to a set of ordinary differential equations with respect to time, in which the spatial variable x is a benign parameter. Put simply, our method is to remove x and Karlin and McGregor's is to remove t. Karlin and McGregor's eigenvalues and eigenfunctions approach is powerful. Nonetheless our method of characteristics gives a comprehensive integral representation of solutions for all the linear birth and death processes (see Theorem 1 in Section 2) and yields many explicit solutions (Section 4). Furthermore, the spectral method requires the compu-

tation of measures that depend heavily on Laguerre and Meixner polynomials, whereas our method is more elementary.

In this article, we present a transient analysis for the general linear birth–death processes with both immigration and emigration by the method of characteristics. Our analysis involves solving partial differential equations for the moment generating functions of the transient distribution. For the general case, we present a Volterra-type integral equation for the chance of extinction $P_0(t)$ (i.e., the probability that the system is empty at time t) and an explicit formula for the z-transform of the transient distribution in terms of $P_0(t)$. Furthermore, an explicit-form infinite-series solution can be found for $P_0(t)$. These results allow us to obtain explicit solutions for the transient solutions of several special cases that have not been previously presented in the literature.

This article is organized as follows. In the following section, we present the transient analysis for the general linear birth–death processes with immigration and emigration. In Section 3, we discuss the chance of extinction $P_0(t)$ and its asymptotic behavior. In Section 4, we present the closed-form solutions for several special cases. We conclude the article with a discussion in Section 5.

2. MODEL AND ANALYSIS

Consider a linear birth–death process with immigration and emigration. The state of the system is the population size. When the state of the system is n, $n = 0, 1, \ldots,$ the immigration rate is ν, the emigration rate is θ, the birth rate is $n\lambda$, and the death rate is $n\mu$.

Let $X(t)$ be the population size at time t. Then, $\{X(t); t \geq 0\}$ is a continuous-time Markov process with transition rates

$$q(n, n+1) = \nu + n\lambda, \qquad n \geq 0$$
$$q(n, n-1) = n\mu + \theta, \qquad n \geq 1.$$

We are concerned with the probability distribution of the population size at any time $t \geq 0$ [i.e., of $X(t)$]. Assume that the initial population distribution is

$$P(X(0) = n) = h_n, \qquad n = 0, 1, \ldots, \qquad (1)$$

and let

$$h(z) = \sum_{n=0}^{\infty} h_n z^n.$$

Note that if the system is initially empty, then $h(z) \equiv 1$.

Let

$$P_n(t) = P(X(t) = n), \qquad n = 0, 1, \ldots,$$

then the Kolmogorov forward differential equation for $P_n(t)$ is (see, e.g., Ross [24] and Bailey [1])

$$P'_n(t) = ((n-1)\lambda + \nu)P_{n-1}(t) + ((n+1)\mu + \theta)P_{n+1}(t)$$
$$- (n\lambda + n\mu + \nu + \theta)P_n(t), \qquad n = 1, 2, \dots,$$
$$P'_0(t) = (\mu + \theta)P_1(t) - \nu P_0(t). \tag{2}$$

The initial condition for these differential equations is $P_n(0) = h_n$, $n = 0, 1, \dots$.

Let

$$P(t, z) = \sum_{n=0}^{\infty} P_n(t) z^n$$

be the z-transform of $P_n(t), n \geq 0$, then it follows from (2) that $P(t, z)$ satisfies the partial differential equation (PDE)

$$\frac{\partial P(t, z)}{\partial t} = (\lambda z^2 - (\lambda + \mu)z + \mu) \frac{\partial P(t, z)}{\partial z}$$
$$+ \left(\nu z - \nu - \theta + \frac{\theta}{z}\right) P(t, z) + \theta \left(1 - \frac{1}{z}\right) P(t, 0), \tag{3}$$

with initial-boundary conditions

$$P(t, 1) = 1, \qquad P(0, z) = h(z). \tag{4}$$

Thus, the problem boils down to solving the PDE (3) with initial condition (4). We remark that the appearance of the boundary term $P(t, 0)$ in the equation makes the problem unusual from the view point of standard PDEs. Thus, a key step is to determine $P(t, 0) \equiv P_0(t)$, which is the probability that the population size at time t is zero.

To solve the differential equation, we first study its characteristics. The characteristic equation is (see, e.g., John [9])

$$\frac{dz}{dt} = -(\lambda z^2 - (\lambda + \mu)z + \mu), \qquad z(0) = z_0.$$

We note that this characteristic equation is the same as that studied in Chao and Zheng [3]. Solving this simple ordinary differential equation gives the characteristics of PDE (3):

$$z(t) = \begin{cases} \left[1 + e^{-(\lambda-\mu)t}\left(-\dfrac{1}{1-z_0} + \dfrac{\lambda}{\lambda-\mu}\left(1 - e^{-(\lambda-\mu)t}\right)\right)\right]^{-1} & \text{for } \lambda \neq \mu, \\[4mm] 1 + \left(-\dfrac{1}{1-z_0} + \lambda t\right)^{-1} & \text{for } \lambda = \mu. \end{cases} \tag{5}$$

See Chao and Zheng [3, Figs. 1–5] for the different scenarios of the characteristic curve. Some characteristic curves intersect the t-axis if and only if $\mu > 0$. We will assume $\mu > 0$ in this article.

Along any characteristic curve, the PDE becomes an ordinary differential equation (ODE):

$$\frac{dP(t, z(t))}{dt} = \left(\nu z(t) - \nu - \theta + \frac{\theta}{z(t)} \right) P(t, z(t)) + \theta \frac{z(t) - 1}{z(t)} P(t, 0).$$

For convenience, we let

$$a(t) = P(t, 0).$$

We now introduce the integration factor

$$\Lambda(\tau; z_0) = \exp\left(-\int_0^\tau \left(\nu z(\zeta) - \nu - \theta + \frac{\theta}{z(\zeta)} \right) d\zeta \right), \tag{6}$$

where $\tau > 0$ and $z_0 \in [0, 1]$ are two independent variables and the function $z(\zeta)$ depends on z_0. Then, the solution of the ODE can be expressed implicitly as

$$P(t, z(t)) = \frac{1}{\Lambda(t; z_0)} \left(h(z_0) + \theta \int_0^t \frac{z(\tau) - 1}{z(\tau)} a(\tau) \Lambda(\tau; z_0) \, d\tau \right). \tag{7}$$

where the parameter z_0 is an arbitrary number in $[0, 1]$ and $z(0) = z_0$.

To obtain $P(t, z)$ at a given point (t, z), we let $z(t) = z$ in (5) and solve for z_0 to obtain

$$z_0 = 1 + e^{(\lambda - \mu)t} \left(-\frac{1}{1 - z} + \frac{\lambda}{\lambda - \mu} \left(1 - e^{(\lambda - \mu)t} \right) \right)^{-1} \tag{8}$$

for $\lambda \neq \mu$ and

$$z_0 = 1 - \left(\lambda t + \frac{1}{1 - z} \right)^{-1} \tag{9}$$

for $\lambda = \mu$. Thus, the z-transform $P(t, z)$ is given by

$$P(t, z) = \frac{1}{\Lambda(t; z_0)} \left(h(z_0) + \theta \int_0^t \frac{z(\tau) - 1}{z(\tau)} a(\tau) \Lambda(\tau; z_0) \, d\tau \right), \tag{10}$$

where z_0 is given by (8) for $\lambda \neq \mu$ and by (9) for $\lambda = \mu$, and $z(\tau)$ is given by (5) (where t is replaced by τ), with z_0 as just defined. The remaining key issue is to obtain $a(t)$.

If we let $z_0 > 0$ be less than both μ/λ and 1, then the characteristic curve starting at z_0 must intersect the t axis at a finite time. See Figure 1. Denote this finite time by t^*. By setting $z(t) = 0$ in the characteristic solution (5), we find that
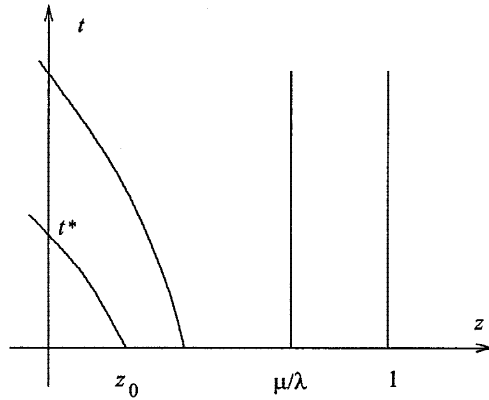
FIGURE 1. Characteristics connecting the two axes.

$$
t^* = \begin{cases} -\dfrac{1}{\lambda - \mu} \ln\left[\dfrac{1}{\mu}\left(\lambda - \dfrac{\lambda - \mu}{1 - z_0} \right) \right] & \text{if } \lambda \neq \mu, \\[3mm] \dfrac{z_0}{\lambda(1 - z_0)} & \text{if } \lambda = \mu \end{cases}
$$

and that

$$
z(t) = -\mu(t - t^*) + o(1)(t - t^*)
$$

as $t \to t^*$. Thus, the integration factor (6) satisfies

$$
\Lambda(t; z_0) = \exp\left(-\int_0^t \left(\nu z(\zeta) - \nu - \theta + \frac{\theta}{z(\zeta)} \right) d\zeta \right) \to 0
$$

as $t \to t^*$ (assuming $\theta > 0$). Since $P(t, z(t))$ is expected to approach $P(t^*, 0)$ as $t \to t^*$, we conclude that the second factor in (10) must approach zero for $z_0 \in [0, \min\{1, \mu/\lambda\})$; that is,

$$
h(z_0) + \theta \int_0^{t^*} \frac{z(\tau) - 1}{z(\tau)} a(\tau) \Lambda(\tau; z_0) \, d\tau = 0. \tag{11}
$$

This is the equation that determines $a(t)$, which is a key observation in our derivation.

The following is the main result of this section. The reader is referred to [5], [22], and [23] for the concept of Volterra integral equations (IEs).

THEOREM 1: *Suppose $\mu > 0$ and $\theta > 0$. Then, the following hold:*

(a) *The solution $P(t, z)$ of PDE (3) is given by (10) with $z(t)$ given by (5), $\Lambda(t; x)$ given by (6), z_0 given by (8) and (9), and $P(t, 0) \equiv a(t)$ given by integral equation (11).*

(b) The integral equation (11) can be simplified to a singular Volterra integral equation of the first kind

$$H(y) = \frac{\theta}{\mu} \int_0^y A(s)(1 + \mu y - \lambda s)^{-\nu/\lambda}(y - s)^{\theta/\mu - 1}\, ds \qquad (12)$$

for $0 \le y < 1/(\lambda - \mu)^+$, where $(\lambda - \mu)^+ = \max\{\lambda - \mu, 0\}$,

$$H(y) = h\left(\frac{\mu y}{1 + \mu y}\right) y^{\theta/\mu}(1 + \mu y)^{-\nu/\lambda} \qquad (13)$$

for $y \ge 0$ and $H(y) = 0$ for $y < 0$,

$$a(t) = A(s(t)),$$

and

$$s = s(t) = \frac{1 - e^{-(\lambda - \mu)t}}{\lambda - \mu} \quad \text{for } \lambda \neq \mu \text{ and } s(t) = t \text{ for } \lambda = \mu. \qquad (14)$$

(c) Once $A(s)$ is obtained from Volterra integral equation (12), the solution $P(t, z)$ of PDE (3) given in (10) can be simplified to

$$P(t, z) = \frac{\theta}{\mu} \int_0^1 A\left(s + \frac{zy(1 - s\lambda + s\mu)}{\mu(1 - z)}\right)$$

$$\times \left(1 - \frac{\lambda z y}{\mu}\right)^{-\nu/\lambda}(1 - y)^{(\theta/\mu) - 1}\, dy, \qquad (15)$$

where $s = s(t)$ is given in (14).

PROOF:

(a) Our earlier analysis has shown that any continuous solution $P(t, z)$ with a finite $P(t,0)$ yields a solution of the integral equation (11). Conversely, a finite solution of the integral equation means that the second factor of (10) vanishes as $t \to t^*$. Since the integral factor (6) goes to zero as $t \to t^*$, we can use L'Hôpital's rule on the ratio of the second factor over the integral factor, which yields that the right-hand side of (10) approaches the value $P(t^*, 0)$, which shows that $P(t, z)$ is continuous up to the boundary $z = 0$. The problem is trivial when $z > 0$. The proof of part (a) is thus completed.

(b) We first simplify (11). We use the substitution $s = s(t)$ defined in (14). The inverse formula is

$$t = -\frac{\ln(1 - (\lambda - \mu)s)}{\lambda - \mu} \quad \text{if } \lambda \neq \mu$$

and $t = s$ if $\lambda = \mu$. The differential relation is $ds = (1 - (\lambda - \mu)s)\, dt$. From the formula for t^*, we find the corresponding s^*:

$$s^* = \frac{z_0}{\mu(1 - z_0)}.$$

We can express the characteristic curves in the new variable s as

$$z(t(s)) = 1 + \frac{(1 - (\lambda - \mu)s)(1 - z_0)}{-1 + \lambda s(1 - z_0)},$$

$$\frac{z(t(s)) - 1}{z(t(s))} = \frac{1 - (\lambda - \mu)s}{\mu(s - s^*)}. \tag{16}$$

Thus, we find

$$\int_0^\tau \left(\nu(z(t) - 1) - \theta \frac{z(t) - 1}{z(t)} \right) dt$$

$$= \int_0^{s(\tau)} \left(\nu(z(t(s)) - 1) - \theta \frac{z(t(s)) - 1}{z(t(s))} \right) \frac{ds}{1 - (\lambda - \mu)s}$$

$$= \frac{\nu}{\lambda} \ln(1 - \lambda(1 - z_0)s(\tau)) - \frac{\theta}{\mu} \ln \left(1 - \frac{s(\tau)}{s^*} \right), \tag{17}$$

where, of course, $s(\tau)$ is as defined in (14). With the above simplification, the integral factor becomes

$$\Lambda(\tau; z_0) = (1 - \lambda(1 - z_0)s(\tau))^{-\nu/\lambda} \left(1 - \frac{s(\tau)}{s^*} \right)^{\theta/\mu} \tag{18}$$

and the integral equation (11) becomes

$$h(z_0) + \theta \int_0^{t^*} \frac{z(\tau) - 1}{z(\tau)} a(\tau)(1 - \lambda(1 - z_0)s(\tau))^{-\nu/\lambda} \left(1 - \frac{s(\tau)}{s^*} \right)^{\theta/\mu} d\tau$$

$$= 0.$$

Substituting τ into s of (14) we obtain

$$h(z_0)s^* = \frac{\theta}{\mu} \int_0^{s^*} a(\tau(s))(1 - \lambda(1 - z_0)s)^{-\nu/\lambda} \left(1 - \frac{s}{s^*} \right)^{\theta/\mu - 1} ds.$$

Finally, let us apply the change of variables

$$y = \frac{z_0}{\mu(1 - z_0)},$$

so that

$$z_0 = \frac{\mu y}{1 + \mu y}.$$

Notice that the range $0 \le z_0 < \min\{1, \mu/\lambda\}$ corresponds to $0 \le y < 1/(\lambda - \mu)$ for $\lambda > \mu$ or all $y \ge 0$ for $\lambda \le \mu$. For this range of y, the IE transforms to the pivotal equation (12).

(c) Assume that $A(s)$ has been obtained from the Volterra integral equation (12). For any given (t, z), we let $z(t) = z$ in (5), from which we solve for z_0 to obtain (8) and (9). Using this z_0, we find (10) in which $a(t) = A(s(t))$. Using (11) in (10) to eliminate $h(z_0)$, we find

$$P(t, z) = \frac{\theta}{\Lambda(t; z_0)} \int_t^{t^*} a(\tau) \frac{1 - z(\tau)}{z(\tau)} \Lambda(\tau; z_0) \, d\tau. \tag{19}$$

From the expression of the integral factor (18) and (16), we do a change of variable to find

$$P(t, z) = \frac{\theta}{\Lambda(t; z_0)} \int_{s(t)}^{s^*} a(\tau(w))(1 - \lambda(1 - z_0)w)^{-\nu/\lambda}$$

$$\times \left(1 - \frac{w}{s^*}\right)^{\theta/\mu} \frac{dw}{\mu(s^* - w)}. \tag{20}$$

It follows from (8) and (9) that

$$z_0 = \frac{J}{1 + J} \quad \text{where } J = s\mu + \frac{z(1 - s\lambda + s\mu)}{1 - z},$$

$$\frac{1 - z_0}{z_0} = \frac{1}{\mu s^*} = \frac{1 - z}{s\mu + z - s\lambda z},$$

$$1 - \frac{s}{s^*} = \frac{z(1 - s\lambda + s\mu)}{s\mu + z - s\lambda z},$$

$$s\mu + z - s\lambda z = \left(s\mu + \frac{z}{1 - z}(1 - s\lambda + s\mu)\right)(1 - z).$$

Thus,

$$\Lambda(t; z_0) = \left(1 - \frac{\lambda s(t)}{1 + J}\right)^{-\nu/\lambda} \left(\frac{z(1 - s\lambda + s\mu)}{(1 - z)J}\right)^{\theta/\mu}.$$

Let the integral in (20) be denoted by L,

$$L = \int_{s(t)}^{s^*} a(\tau(w))(1 - \lambda(1 - z_0)w)^{-\nu/\lambda} \left(1 - \frac{w}{s^*}\right)^{\theta/\mu} \frac{dw}{\mu(s^* - w)}.$$

Then, we have

$$L = \int_{s(t)}^{J/\mu} A(w) \left(1 - \frac{\lambda w}{1 + J}\right)^{-\nu/\lambda} (J - \mu w)^{(\theta/\mu)-1} J^{-\theta/\mu} \, dw.$$

Transforming the variable $w \to y$,

$$w = s(t) + \frac{z(1 - s\lambda + s\mu)}{\mu(1 - z)} y,$$

we further find

$$L = \int_0^1 A\left(s + \frac{z(1 - s\lambda + s\mu)}{\mu(1 - z)} y\right)$$

$$\times \left(1 - \frac{\lambda\mu s(1 - z) + \lambda z(1 - s\lambda + s\mu)y}{\mu(1 - z)(1 + J)}\right)^{-\nu/\lambda}$$

$$\times (1 - y)^{(\theta/\mu)-1} \left(\frac{z(1 - s\lambda + s\mu)}{(1 - z)J}\right)^{\theta/\mu} \frac{1}{\mu} \, dy.$$

Now, we use

$$1 + J - s\lambda = \frac{1 - s\lambda + s\mu}{1 - z},$$

thus

$$1 - \frac{\lambda\mu s(1 - z) + \lambda z(1 - s\lambda + s\mu)y}{\mu(1 - z)(1 + J)}$$

$$= \frac{1}{1 + J}\left(1 + J - s\lambda - \frac{\lambda z(1 - s\lambda + s\mu)y}{\mu(1 - z)}\right)$$

$$= \frac{1}{1 + J} \frac{1 - s\lambda + s\mu}{1 - z}\left(1 - \frac{\lambda zy}{\mu}\right).$$

With these, we find

$$P(t, z) = \frac{\theta}{\mu} \int_0^1 A\left(s + \frac{z(1 - s\lambda + s\mu)}{\mu(1 - z)} y\right)\left(1 - \frac{\lambda zy}{\mu}\right)^{-\nu/\lambda}$$

$$\times (1 - y)^{(\theta/\mu)-1} \, dy.$$

This completes the proof of Theorem 1.   ■

It follows from Theorem 1 that the solution $P(t, z)$ is obtained once we solve the singular Volterra integral equation (12). From the theory of integral equations, we know that the singular integral equation (12) can be reduced to a regular equation and this first-kind equation can be brought to the second kind, for which the iteration method can be used to construct an infinite-series solution. The details will not be given here and the reader is referred to [5], [22], or [23].

In the following sections, we consider the asymptotic behavior of the solution to this equation and we also consider several cases for which we are able to find closed-form solutions to this equation.

Remark 1: The probability $P_0(t) = a(t)$ is determined by (11). Once $P_0(t)$ is obtained, recursive formulas can be developed for $P_n(t)$, $n = 1, 2, \ldots$, using (2), which can then be used for calculating explicit numerical solutions. However, for many other purposes, we prefer to obtain a closed-form explicit formula for $P_n(t)$, which, among other things, shows more explicitly its dependencies on the systems parameters. Indeed, very few stochastic systems possess closed-form solutions for transient probabilities, and in Section 4, we focus on finding closed-form solutions for the process.

3. CHANCE OF EXTINCTION

The probability that the population size is zero at time t, i.e., $P_0(t)$, is referred to as the chance of extinction (Kendall [13]). In this section we apply Theorem 1 to analyze the limiting behavior of $P_0(t)$ as time tends to infinity.

THEOREM 2: *As time tends to infinity, we have the following:*

   (a) $P_0(t) \to 0$ *if* $\lambda \geq \mu$ *and* $\nu/\lambda \geq \theta/\mu$.
   (b) $P_0(t) \to (\theta - \nu)/\theta$ *for the case* $\lambda = \mu$ *and* $0 \leq \nu < \theta$.
   (c) $P_0(t) \to (\mu/\theta)[\int_0^1 (1 - \lambda s/\mu)^{-\nu/\lambda}(1 - s)^{(\theta/\mu)-1} ds]^{-1} \in (0,1)$ *for* $\lambda < \mu$ *and* $\nu > 0$.
   (d) $P_0(t) \to 1$ *for* $\lambda < \mu$ *and* $\nu = 0$.

PROOF: We notice that (12) can be written as

$$h\left(\frac{\mu y}{1 + \mu y}\right)(1 + \mu y)^{-\nu/\lambda} = \frac{\theta}{\mu} \int_0^1 A(sy)(1 + \mu y - \lambda sy)^{-\nu/\lambda}(1 - s)^{(\theta/\mu)-1} ds. \tag{21}$$

For the case $\lambda > \mu$, we let $y \to 1/(\lambda - \mu)$ in (21) to find

$$h\left(\frac{\mu}{\lambda}\right) = \frac{\theta}{\mu} \int_0^1 A\left(\frac{s}{\lambda - \mu}\right)(1 - s)^{-(\nu/\lambda)+(\theta/\mu)-1} ds. \tag{22}$$

If $\nu/\lambda \geq \theta/\mu$, the term $(1 - s)^{-\nu/\lambda+\theta/\mu-1}$ is not integrable on $[0,1]$. Thus, $A(s/(\lambda - \mu))$ must vanish as $s \to 1$; hence, $A(1/(\lambda - \mu)) = 0$. In this case, we have

$$a(t) = P_0(t) \to A(1/(\lambda - \mu)) = 0.$$

This proves the first half of part (a).

If $\lambda = \mu$, we let $y \to \infty$ in (21) and use $h(1) = 1$ to find

$$1 = \frac{\theta}{\mu} \int_0^1 A(\infty)(1 - s)^{-\nu/\lambda+\theta/\mu-1} ds. \tag{23}$$

This implies $A(\infty) = 0$ if $\nu \geq \theta$ and $A(\infty) = (\theta - \nu)/\theta$ if $\nu < \theta$. This proves part (b) and completes part (a).

If $\lambda < \mu$, we similarly let $y \to \infty$ in (21) and use $h(1) = 1$ to find

$$1 = \frac{\theta}{\mu} \int_0^1 A(\infty) \left(1 - \frac{\lambda s}{\mu}\right)^{-\nu/\lambda} (1-s)^{\theta/\mu - 1} \, ds. \tag{24}$$

Thus,

$$A(\infty) = \frac{\mu}{\theta} \left[\int_0^1 \left(1 - \frac{\lambda s}{\mu}\right)^{-\nu/\lambda} (1-s)^{\theta/\mu - 1} \, ds \right]^{-1}. \tag{25}$$

If $\nu = 0$, we find $A(\infty) = 1$. Otherwise, it is a number between zero and one since the factor $(1 - \lambda s/\mu)^{-\nu/\lambda}$ is greater than one. This proves parts (c) and (d). The proof of Theorem 2 is complete. ∎

Remark 2: The limit of $P_0(t)$ for the case $\lambda > \mu$ and $\nu/\lambda < \theta/\mu$ is generally dependent on the initial condition $h(z)$. For example, $a(t) \to 1$ if $h(z) \equiv 1$ for the case $\lambda > \mu$, $\nu = 0$, any $\theta > 0$, and $\mu > 0$. However, $a(t) \to 1 - ((\lambda - \mu)/\lambda)^2$ if $h(z) = z$ for the case $\lambda > \mu$, $\nu = 0$, and $\theta = \mu$. These results can be obtained from the explicit formulas of Corollaries 3 and 4 of the next section.

Remark 3: It follows from parts (b) and (d) of Theorem 2 that if $\lambda \leq \mu$ and $\nu = 0$, then $\lim_{t \to \infty} P_0(t) = 1$; that is, the population is "almost certain" to die out for linear birth–death process with emigration when $\lambda \leq \mu$. This result was proved in Kendall [13] for the special case $\nu = \theta = 0$.

4. CLOSED-FORM SOLUTIONS

Birth and death processes have been extensively studied in the literature and some cases have been solved explicitly a long time ago. For example, when $\lambda = \mu = 0$, we obtain an $M/M/1$ queue and its transient solution can be found, for example, in Kleinrock [15]. The case $\lambda = 0$ and $\theta = 0$ is equivalent to an $M/M/\infty$ queue and its transient solution can also be found in many texts on stochastic processes (e.g., Example 4.5 of Cox and Miller [4] and Massey and Whitt [19]). The case $\nu = 0$ and $\theta = 0$ is the typical linear birth–death process and the case $\theta = 0$ is a linear birth–death process with immigration, also referred to as the Kendall process (Kendall [13]). The transient solutions to all of these cases have been solved in closed forms (see, e.g., Bailey [1]).

Based on the analysis in Section 2, we present in this section the closed-form transient solutions for four cases. Our first case is covered by Ismail et al. [8], but we choose to include it here since our presentation is simple and more elementary. The transient solutions to the other cases appear to be new.

We first consider the case $\lambda = \mu$. In this case, the integral equation (12) becomes

$$H(y) = \frac{\theta}{\mu} \int_0^y a(s)(1 + \mu(y - s))^{-\nu/\lambda} (y - s)^{\theta/\mu - 1} \, ds.$$

We recognize that the right-hand side of this equation is a convolution. In what follows, we use $f * g$ to represent the convolution of f and g. Let

$$K_1(s) = \frac{\theta}{\mu} (1 + \mu s)^{-\nu/\lambda} s^{-1+(\theta/\mu)}, \qquad s > 0; \tag{26}$$

then,

$$H(y) = (a * K_1)(y) \equiv \int_0^y a(s) K_1(y - s) \, ds.$$

By using the Laplace transform $\mathcal{L}$, we have

$$a(s) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(H)}{\mathcal{L}(K_1)} \right), \qquad s \geq 0. \tag{27}$$

This analysis yields the following result.

COROLLARY 1: *The integral equation (12) has closed-form solutions if $\lambda = \mu$. In this case, $a(t) = A(t)$ is given by (27) via Laplace transform, and the solution is*

$$P(t, z) = \frac{\theta}{\mu} \int_0^1 a\left(t + \frac{zy}{\mu(1 - z)}\right)(1 - zy)^{-\nu/\mu}(1 - y)^{(\theta/\mu)-1} \, dy.$$

We next consider the case $\nu = 0$, resulting in a linear birth–death process with emigration. In this case, the integral equation (12) becomes

$$H(y) = \frac{\theta}{\mu} \int_0^y A(s)(y - s)^{\theta/\mu-1} \, ds, \tag{28}$$

where $0 \leq y < 1/(\lambda - \mu)^+$. We note that since $\nu = 0$, we can assume that (28) holds for all $y \geq 0$ (in this case, the term $(1 + \mu y - \lambda s)^{-\nu/\lambda}$ is not present).

Note that the right-hand side of the IE (28) is also a convolution. Let

$$K_2(s) = \frac{\theta}{\mu} s^{-1+(\theta/\mu)}, \qquad s > 0.$$

Then

$$H(y) = (A * K_2)(y).$$

Using the Laplace transform $\mathcal{L}$, we have

$$A(s) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(H)}{\mathcal{L}(K_2)} \right). \tag{29}$$

More specifically, we find

$$\mathcal{L}(K_2) = \frac{\theta}{\mu} \Gamma\left(\frac{\theta}{\mu}\right) s^{-\theta/\mu},$$

where Γ is the gamma function. Using $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$, we have

$$A(s) = \mathcal{L}^{-1}(s^{\theta/\mu}\mathcal{L}(H))/\Gamma\left(\frac{\theta}{\mu}+1\right).$$

If $\theta/\mu = m$ is a positive integer, then

$$A(s) = \frac{1}{m!} H^{(m)}(s), \qquad a(t) = A(s(t)) \tag{30}$$

since $H^{(n)}(0) = 0$ for all $n = 0,1,2,\ldots,m-1$. If $\theta/\mu = m + \alpha$, where m is a nonnegative integer and $\alpha \in (0,1)$, then

$$s^{m+\alpha}\mathcal{L}(H) = \frac{1}{s^{1-\alpha}} \mathcal{L}(H^{(m+1)}) = \frac{\mathcal{L}(s^{-\alpha})\mathcal{L}(H^{(m+1)})}{\Gamma(1-\alpha)}$$

since $H^{(n)}(0) = 0$ for all $n = 0,1,2,\ldots,m$. Using the convolution formula, we find

$$A(s) = \left[\Gamma\left(\frac{\theta}{\mu}+1\right)\Gamma(1-\alpha)\right]^{-1} \int_0^s \frac{1}{\tau^{\alpha}} H^{(m+1)}(s-\tau)\,d\tau. \tag{31}$$

We note that (31) is valid even for $\alpha = 0$; thus, (31) includes (30) as a special case.

We comment in passing that the equation for $A(s)$ for the case $\theta/\mu = \alpha \in (0,1)$ is an Abel's equation; see [5], [22], [23], or [28]. In particular, if $\theta/\mu = \frac{1}{2}$, then we have

$$A(s) = \frac{2}{\pi} \int_0^s \frac{H'(s-\tau)}{\sqrt{\tau}}\,d\tau.$$

COROLLARY 2: *If $\nu = 0$, $\mu = \lambda$, and $\theta = m\mu$ for some positive integer m, we have*

$$a(t) = \frac{1}{m!} G^{(m)}(\lambda t) = \frac{1}{m!} H^{(m)}(t),$$

where

$$G(y) = h\left(\frac{y}{1+y}\right) y^m \tag{32}$$

and

$$P(t,z) = \left(\frac{1-z}{z}\right)^m \left[G\left(\lambda t + \frac{z}{1-z}\right) - \sum_{k=0}^{m-1} \frac{1}{k!} G^{(k)}(\lambda t)\left(\frac{z}{1-z}\right)^k\right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+m)!} G^{(k+m)}(\lambda t)\left(\frac{z}{1-z}\right)^k, \tag{33}$$

with

$$P_n(t) = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!(k+m)!} G^{(k+m)}(\lambda t), \quad n \geq 1.$$

We remark that the function $G(y)$ is very similar to $H(y)$. We prefer to use $G(y)$ here. Also, the notation $G^{(k)}(\lambda t)$ denotes the kth derivative of $G(y)$ with respect to the generic variable y at λt. One can see that λ and t appear as a product in the formula, which is consistent with the equation.

PROOF: The function $a(t)$ follows from (30) and a simple change of variable. From (15), we obtain that

$$P(t, z) = m \int_0^1 a\left(t + \frac{zy/\mu}{1-z}\right)(1-y)^{m-1} \, dy. \tag{34}$$

Through Taylor expansion and the beta function, we find

$$P(t, z) = m \sum_{k=0}^{\infty} \frac{1}{k!} a^{(k)}(t) \left(\frac{z/\mu}{1-z}\right)^k \frac{k!(m-1)!}{(k+m)!}. \tag{35}$$

That simplifies to

$$P(t, z) = \sum_{k=0}^{\infty} \frac{1}{(k+m)!} G^{(k+m)}(\lambda t) \left(\frac{z}{1-z}\right)^k.$$

Using the expansion

$$(1-z)^{-k} = \sum_{l=0}^{\infty} \frac{1}{l!} k(k+1)\cdots(k+l-1)z^l,$$

we find for all $n \geq 1$ that

$$P_n(t) = \sum_{l+k=n} \frac{k(k+1)\cdots(k+l-1)}{l!(k+m)!} G^{(k+m)}(\lambda t),$$

where the sum is over all $k \geq 1$ such that $l + k = n$, and it simplifies to yield the result stated in Corollary 2. The expressions in (33) can be verified easily through either integration by parts on (34) or inspection of their Taylor expansion. ∎

COROLLARY 3: *For the case $\nu = 0$, $\mu \neq \lambda$, and $\theta = m\mu$ for some positive integer m, we have*

$$a(t) = \frac{1}{m!} G^{(m)}(\mu s(t)),$$

where $G(y) = h(y/(1 + y))y^m$ is the same as in (32), and

$$P(t, z) = \left(\frac{1 - z}{z(1 - \lambda s + \mu s)} \right)^m$$

$$\times \left[G\left(\mu s + \frac{z(1 - \lambda s + \mu s)}{1 - z} \right) - \sum_{k=0}^{m-1} \frac{1}{k!} G^{(k)}(\mu s) \left(\frac{z(1 - \lambda s + \mu s)}{1 - z} \right)^k \right]$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k + m)!} G^{(k+m)}(\mu s) \left(\frac{z(1 - \lambda s + \mu s)}{1 - z} \right)^k,$$

where $s = s(t)$ is defined in (14), with

$$P_n(t) = \sum_{k=1}^{n} \frac{(n - 1)!}{(n - k)!(k - 1)!(k + m)!} (1 - \lambda s + \mu s)^k G^{(k+m)}(\mu s), \qquad n \geq 1.$$

PROOF: Once we make use of the variable $s(t)$, the proof will be parallel to that of Corollary 2. We omit the details. ∎

COROLLARY 4: *Suppose $\nu = 0$, $\theta/\mu = m + \alpha > 0$ for some nonnegative integer m and $0 \leq \alpha < 1$, and $\mu > 0$, $\lambda > 0$, we have*

$$a(t) = A(s(t)),$$

$$A(w) = \left[\Gamma\left(\frac{\theta}{\mu} + 1 \right) \Gamma(1 - \alpha) \right]^{-1} \int_0^w \frac{1}{\tau^\alpha} H^{(m+1)}(w - \tau) \, d\tau,$$

and

$$P(t, z) = \frac{\theta}{\mu} \int_0^1 A\left(s + \frac{zy(1 - s\lambda + s\mu)}{\mu(1 - z)} \right)(1 - y)^{(\theta/\mu) - 1} \, dy, \tag{36}$$

where $s = s(t)$ is given in (14). The corresponding $P_n(t)$ for all $n \geq 1$ are

$$P_n(t) = \Gamma\left(\frac{\theta}{\mu} + 1 \right) \sum_{k=1}^{n} \frac{(n - 1)!}{(n - k)!(k - 1)! \Gamma\left(k + 1 + \frac{\theta}{\mu} \right)} A^{(k)}(s) \left(\frac{1 - s\lambda + s\mu}{\mu} \right)^k.$$

PROOF: It is a simple consequence of Theorem 1 and (31). ∎

We next present an alternative method. As used in (21) in the proof of Theorem 2, we notice that (12) can always be written as

$$F(y) = \frac{\theta}{\mu} \int_0^1 A(sy)(1 + \mu y - \lambda sy)^{-\nu/\lambda}(1 - s)^{\theta/\mu - 1} \, ds, \tag{37}$$

where we have defined

$$F(y) = H(y)y^{-\theta/\mu}. \tag{38}$$

When $\nu = 0$, we have

$$F(y) = \frac{\theta}{\mu} \int_0^1 A(sy)(1 - s)^{\theta/\mu - 1} \, ds. \tag{39}$$

It follows from (39) that the nth derivative of F at zero is

$$F^{(n)}(0) = A^{(n)}(0) \frac{\theta}{\mu} \int_0^1 s^n (1 - s)^{\theta/\mu - 1} \, ds$$

for all nonnegative integers n. Thus, for all n, we obtain $A^{(n)}(0)$ as

$$A^{(n)}(0) = \frac{(\mu/\theta) F^{(n)}(0)}{\beta(n + 1, \theta/\mu)},$$

where β is the beta function. Then, we find

$$A(s) = \sum_{n=0}^{\infty} \frac{1}{n!} A^{(n)}(0) s^n.$$

Since $a(\tau(s)) = A(s)$, we have

$$a(t) = A(s(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} A^{(n)}(0) \left(\frac{1 - e^{-(\lambda - \mu)t}}{\lambda - \mu} \right)^n. \tag{40}$$

This method can be used to deal with the general case. We can use Taylor expansion on $F(y)$, $A(sy)$, and $(1 + (\mu - \lambda s)y)^{-\nu/\lambda}$ at $y = 0$ in (37) to derive a recursive formula for $A^{(n)}(0)$:

$$\frac{F^{(n)}(0)}{n!} = \frac{\theta}{\mu} \frac{A^{(n)}(0)}{n!} \beta\left(n + 1, \frac{\theta}{\mu}\right)$$

$$+ \frac{\theta}{\mu} \sum_{k=1}^n \frac{A^{(n-k)}(0)}{(n - k)!} \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}$$

$$\times \int_0^1 s^{n-k} (\mu - \lambda s)^k (1 - s)^{(\theta/\mu) - 1} \, ds,$$

where $\alpha = -\nu/\lambda$. In this way we can find an expression for $a(t)$; however, since it is very complicated, we choose not to present its formula.

5. DISCUSSION

We presented the use of the method of characteristics for a transient analysis of the linear birth–death processes with immigration and emigration. For the general case, we presented a Volterra-type integral equation for extinction probability $P_0(t)$ and an explicit formula for $P(t, z)$ in terms of $P_0(t)$, which is ready for expansion to yield $P_n(t)$. For several special cases that have not been previously solved, we presented

closed-form solutions for $P(t, z)$ and $P_n(t)$ for all $n \geq 0$. We also give semiexplicit closed-form solutions $P(t, z)$ and $P_0(t)$ for the case $\lambda = \mu$.

Recently, there has been several studies that incorporate total catastrophes in the stochastic systems (see, e.g., Kumar and Arivudainambi[16], and Kyriakidis[17,18], Chao and Zheng[3], among others). The catastrophes arrive according to a Poisson process and its arriving effect is to reduce the total population size to zero. Let the catastrophes arrival rate be γ; then PDE (3) becomes

$$
\frac{\partial P(t, z)}{\partial t} = (\lambda z^2 - (\lambda + \mu)z + \mu) \frac{\partial P(t, z)}{\partial z}
$$

$$
+ \left(\nu z - \nu - \theta + \frac{\theta}{z} - \gamma \right) P(t, z) + \theta \left(1 - \frac{1}{z} \right) P(t, 0) + \gamma. \qquad \textbf{(41)}
$$

It is not hard to see from our analysis that the procedure used in solving PDE (3) equally applies to solving PDE (41). Thus, adding total catastrophes does not add difficulty in solving the problem. The details will not be given here.

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