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RESEARCH ARTICLE

Period realization of meromorphic differentials with prescribed invariants

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Abstract
We provide a complete description of realizable period representations for meromorphic differentials on Riemann surfaces with prescribed orders of zeros and poles, hyperelliptic structure and spin parity.

Contents

1. Introduction

Let $S_{g,n}$ be the connected and oriented surface of genus g and with n punctures, and let $\mathcal{M}_{g,n}$ be the moduli space of unmarked complex structures on $S_{g,n}$. For a complex structure $X \in \mathcal{M}_{g,n}$, let $\Omega(X)$ denote the space of holomorphic abelian differentials with at most finite-order poles at the punctures that we refer as *meromorphic differentials* on X. Let $\Omega\mathcal{M}_{g,n}$ denote the moduli space of pairs (X, ω), where X is a punctured Riemann surface and $\omega \in \Omega(X)$ is an abelian differential on X. Such a moduli space admits a natural stratification, given by the strata $\mathcal{H}_g(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$, enumerated by unordered partitions of $2g-2$ where we allow negative integers corresponding to the orders of poles.

A stratum $\mathcal{H}_g(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ consists of equivalent classes of pairs (X, ω), where ω has k zeros and n poles of orders $m_1, \ldots, m_k, -p_1, \ldots, -p_n$, respectively. We call $(m_1, \ldots, m_k, -p_1, \ldots, -p_n)$ the *signature* of the stratum. Sometimes, when necessary, we shall adopt an 'exponential' notation to denote multiple zeros or poles of the same order, for example $\mathcal{H}_1(2,2;\,-2,-2) = \mathcal{H}_1(2^2;\,-2^2)$.

Note that a stratum of differentials can be disconnected. For $g \geq 2$, a stratum can have up to three connected components according to the presence of a *hyperelliptic involution* or a topological invariant known as *spin structure*, while the number of connected components of a stratum in genus one depends on the greatest common divisor of the signature due to a structure called *rotation number*. For holomorphic differentials on compact Riemann surfaces, the connected components of the strata have been classified by Kontsevich–Zorich in [KZ03] and subsequently by Boissy in [Boi15] in the case of meromorphic differentials.

The *period character* of an abelian differential ω on a Riemann surface X is defined as the homomorphism

$$\chi\colon \mathrm{H}_1(X,\mathbb{Z}) \longrightarrow \mathbb{C} \;\text{ such that }\; \gamma \longmapsto \int_\gamma \omega \tag{1}$$

For holomorphic differentials on compact Riemann surfaces (i.e., $n = 0$), in [Hau20] Haupt provided necessary and sufficient conditions for a representation $\chi\colon \mathrm{H}_1(S_g,\mathbb{Z}) \longrightarrow \mathbb{C}$ to arise as the period character of some pair (X,ω). The same result has been subsequently rediscovered by Kapovich in [Kap20] by using Ratner's theory. It turns out that there are two obstructions for a representation χ to be realized as the period character of an abelian differential. In the following, we shall refer to these conditions as *Haupt's conditions*; see Section §2.1. Realizing a representation as a character in a prescribed stratum turns out to be a more subtle problem because, in the realizing process, the orders of zeros can no longer be ignored. In the same spirit of [Kap20], Le Fils has provided in [LF22] necessary and sufficient conditions for a representation χ to be a character in a given stratum. Around the same time, Bainbridge–Johnson–Judge–Park in [BJJP22] have provided, with an independent and alternative approach, necessary and sufficient conditions for a representation to be realized in a connected component of a prescribed stratum.

For meromorphic differentials on compact Riemann surfaces, equivalently holomorphic differentials on punctured complex curves (i.e., $n \geq 1$), in [CFG22, Theorem A] Chenakkod–Faraco–Gupta provided conditions for a representation $\chi\colon \mathrm{H}_1(S_{g,n},\,\mathbb{Z}) \longrightarrow \mathbb{C}$ to arise as a period character. In this case, no obstructions appear for realizing a representation χ as the period character of some meromorphic differential. Moreover, they have described necessary and sufficient conditions for realizing χ as the period of some (X,ω) in a prescribed stratum $\mathcal{H}_g(m_1,\ldots,m_k;-p_1,\ldots,-p_n)$; see [CFG22, Theorems B, C, D]. Around the same time, a slightly different version of [CFG22, Theorem A] has been proved in [FG24, Theorem C] in which the authors been provided necessary and sufficient condition for a representation χ to be the period character of some translation surface (X,ω) in which all zeros and poles are at the punctures. Finally, in [Far24] the author determined conditions under which a representation appears as the period character of some (X,ω) with nontrivial group of translations, namely automorphisms of the underlying complex structure that preserves the abelian differential ω. See [Far24] for further details.

The aim of the present paper is to extend the results in the [CFG22] study of the period map (7) to connected components of the strata of meromorphic differentials whenever they are not connected. Our first result states as follows.

Theorem A. *Let $\chi\colon \mathrm{H}_1(S_{g,n},\,\mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation arising as the period character of some meromorphic genus-g differential in a stratum $\mathcal{H}_g(m_1,\ldots,m_k;-p_1,\ldots,-p_n)$. Then χ can be realized in each of its connected components.*

The trivial representation $\chi\colon \mathrm{H}_1(S_{g,n},\,\mathbb{Z}) \longrightarrow \mathbb{C}$ is the homomorphism such that $\chi(\gamma) = 0$ for all $\gamma \in \mathrm{H}_1(S_{g,n},\,\mathbb{Z})$. Our second result handles this special representation which is exceptional as follows.

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Theorem B. *Suppose the trivial representation arises as the period character of some genus-g meromorphic differential in a stratum $\mathcal{H}_g(m_1,\ldots,m_k;-p_1,\ldots,-p_n)$. Then χ can be realized in each of its connected components with the only exceptions being the strata:*

1. $\mathcal{H}_1(3,3;\,-3,-3)$,
2. $\mathcal{H}_g(2^{g+2};\,-2^3)$, *for* $g\geq 1$,
3. $\mathcal{H}_g(2^{g+1};\,-4)$, *for* $g\geq 1$.

Moreover, in these exceptional strata, for $g=1$ these strata exhibit two connected components, one of which is primitive and the other is not. The trivial representation can only be realized in the nonprimitive component. For $g\geq 2$, the strata $\mathcal{H}_g(2^{g+2};\,-2^3)$ and $\mathcal{H}_g(2^{g+1};\,-4)$ have two connected components distinguished by the spin parity. The trivial representation can only be realized in the connected component with parity equal to $g \pmod 2$.

For the reader's convenience, in what follows we recall the main results of [CFG22] in §1.1 and then explain the new challenges as well as strategies to overcome them in §1.2.

1.1. Realizable representations in a given stratum

As already alluded above, the earlier work [CFG22] focuses on the study of which representations can be realized in a given stratum. As in the holomorphic case, to realize a given representation χ as the period character of some meromorphic differential in a given stratum is a subtle problem because the order of zeros and poles can be no longer ignored. Broadly speaking, we have three kinds of representations, namely χ can be

- the trivial representation if $\chi(\gamma)=0$ for all γ,
- integral representations if $\mathrm{Im}(\chi)\cong\mathbb{Z}$,
- generic if χ is not trivial nor integral.

We remark that being generic is a fairly general condition. In principle, the realization process consists of finding a pair (X,ω) having the given representation χ as the period character. According to [CFG22, Theorem C], a generic representation can be realized in a given stratum with signature $(m_1,\ldots,m_k;-p_1,\ldots,-p_n)$ if the residues of simple poles are nonzero. Aside this restriction, for a generic representation there are no other obstructions in the realization process. More precisely,

Theorem (Chenakkod - F. - Gupta, 2022). *Let χ be a generic representation. Then χ can be realized in a stratum with signature $(m_1,\ldots,m_k;-p_1,\ldots,-p_n)$ if $p_i\geq 2$ whenever $\chi(\gamma_i)=0$, where γ_i denotes a peripheral loop around the i^{th} puncture.*

Note that this formulation is slightly different from that in [CFG22]. On the other hand, the trivial representation as well as integral representations cannot be realized in every stratum.

Let us consider first integral representations. For an integral representation χ, obstructions appear only if we aim to realize χ in a stratum with *all* simple poles, that is, a stratum with signature $(m_1,\ldots,m_k;-1^n)$. This happens because every translation surface $(X,\omega)\in\mathcal{H}_g(m_1,\ldots,m_k;-1^n)$ with integral periods is always determined by a branched covering $f\colon X\longrightarrow\mathbb{C}\mathbf{P}^1$ and ω is defined as the pull-back of the meromorphic differential $\frac{1}{2\pi i}\frac{dz}{z}$ on $\mathbb{C}\mathbf{P}^1$ with two simple poles at $\{0,\infty\}$. Therefore, the realization of (X,ω) with integral periods in a stratum with signature $(m_1,\ldots,m_k;-1^n)$ is subject to the realization of a branched covering f with prescribed data. In particular, f is subject to the condition $m_i+1\leq\deg(f)$, coming from the Riemann–Hurwitz formula, and $\deg(f)$ is equal to the sum of the positive residues. As a consequence, the maximum zero order must be lower than $\deg(f)$. In [CFG22] this is precisely stated as

Theorem (Chenakkod - F. - Gupta, 2022). *Let $\chi\colon \mathrm{H}_1(S_{g,n},\mathbb{Z})\longrightarrow\mathbb{Z}$ be an integral representation. Let the residues around the positive punctures be given by the integer-tuple $\lambda\in\mathbb{Z}_+^r$ and the residues* *around the negative punctures be given by $-\mu$, where $\mu \in \mathbb{Z}_+^l$. Then χ can be realized in a stratum with signature $(m_1, \dots, m_k; -1^n)$, where $(m_1, \dots, m_k)$ satisfies the degree condition*

$$\sum_{i=1}^{k} m_i = 2g - 2 + n \tag{2}$$

if and only if

$$\sum_{i=1}^{r} \lambda_i = \sum_{j=1}^{l} \mu_j > \max\{m_1, \dots, m_k\}. \tag{3}$$

Finally, the other special case concerns the trivial representation, that is, $\operatorname{Im}(\chi) = \{0\}$. Every translation surface with poles and trivial periods, say (X, ω), is always determined by a rational map, namely a branched covering map, $f\colon X \longrightarrow \mathbb{C}\mathbf{P}^1$ and $\omega = f^*dz$ is an exact differential. Necessary conditions to realize the trivial representation in a stratum with signature $(m_1, \dots, m_k; -p_1, \dots, -p_n)$ are naturally imposed by this branched covering. [CFG22, Theorem B] states that these conditions are also sufficient, that is

Theorem (Chenakkod - F. - Gupta, 2022). *Let $g \geq 0$, and let $(m_1, \dots, m_k; -p_1, \dots, -p_n)$ be a signature satisfying*

$$\sum_{i=1}^{k} m_i - \sum_{j=1}^{n} p_j = 2g - 2. \tag{4}$$

There exists a meromorphic differential in the stratum $\mathcal{H}_g(m_1, \dots, m_k; -p_1, \dots, -p_n)$ with trivial periods if and only if

(i) $p_j \geq 2$ *for each* $1 \leq j \leq n$,

(ii) $m_i + 1 \leq \sum_{i=1}^{n} \left(p_j - 1 \right)$ *for each* $1 \leq i \leq k$, *and*

(iii) *if* $g > 0$, $k > 1$ *whenever* $n > 1$.

We remark that the third condition arises from the fact that a meromorphic differential (X, ω) with trivial periods has a single zero if and only if $(X, \omega) = (\mathbb{C}\mathbf{P}^1, z^m dz)$; see [CFG22, Proposition 8.1]. In particular, the trivial representation cannot be realized in strata of meromorphic differentials with simple poles.

Since every pair (X, ω) always determines a translation structure on $X \setminus \{\text{ poles of } \omega\}$, the realization process boils down to find a suitable collection of polygons such that, once glued together properly along slits, it provides a translation surface in a stratum with a single zero of maximal order and prescribed orders of poles. Then, by breaking the single zero into zeros of lower order, realizing a representation in a stratum with several zeros becomes a straightforward consequence of the former construction.

1.2. Realizing representations in connected components of strata

In the present paper, a series of new challenges that require distinct techniques appear for dealing with connected components of strata. In fact, for dealing with connected components, the gist of the idea is still the same, but now the way these polygons are glued together also does matter. This is particularly evident in Section §5 where, in order to realize a genus-one differential with prescribed rotation number, several copies of the standard differential $(\mathbb{C}, dz)$ need to be glued together by following a certain pattern. Different ways of gluing provide structures in the same stratum but lying in different connected components. This motivates the several constructions summarized in Appendix A. A similar phenomenon can be seen in Section §6 in which we aim to realize a hyperelliptic translation surface with prescribed period character in a given stratum. Although realizing genus-one differential requires a deeper and more detailed argument, realizing higher genus surfaces with prescribed parity simplifies considerably due to a double inductive foundation starting from minimal strata with fixed polar part, for example, the signature $(m; -2^n)$; see Section §7 for more details.

Let us now discuss the trivial representation. Theorem A states that whenever a nontrivial representation can be realized in a given stratum, then the same representation can be realized in every connected component of the stratum. On the other hand, the trivial representation is exceptional in the following sense: To realize an exact differential in a given stratum with prescribed rotation number (if $g = 1$), parity or hyperelliptic involution (if $g \geq 2$) turns out to be an arduous challenge and not always possible. In other words, the trivial representation may appear in a stratum without being realizable in each connected component of the same stratum; see Theorem B for the exceptional cases. The case of higher genus surfaces with prescribed parity relies on the realization of genus-one differentials with prescribed rotation number in a certain stratum. Even in this case we use an inductive foundation having genus-one exact differentials as the base case. Despite a geometric argument works for realizing an exact differential with prescribed invariants, it fails to show that the trivial representation cannot be realized in certain connected components of certain strata. In order to prove the second part of Theorem B we shall use an argument from the viewpoint of algebraic geometry. In fact, an exact differential (X, ω) yields a rational map $f\colon X \longrightarrow \mathbb{C}\mathbf{P}^1$ (*i.e.* a branched cover of the sphere) and we shall use a monodromy argument associated to the cover to determine whether the trivial representation can be realized in certain connected components of the exceptional strata listed above.

Remark. A few comments are in order. In the first place we point out that, in each section, we assume the representation satisfies the necessary conditions to be realized in a given stratum, for example, for a generic representation we shall assume the residue at simple poles to be nonzero. The crucial point is that all necessary conditions for a representation to appear in a certain stratum are also sufficient. In other words, one of the main results of [CFG22] is that a representation can be realized in a certain stratum if the necessary conditions hold. Despite relying on this result, most of our constructions are new and independent of those proposed in the earlier work [CFG22].

A second noteworthy comment is about the realization of integral representations in strata with all simple poles, that is, $\mathcal{H}_g(m_1, \ldots, m_k; -1^n)$. According to Boissy (see [Boi15]), if $g = 1$ or $g \geq 2$ and $n \geq 3$, these strata are all connected, and hence, Theorem A is nothing but [CFG22, Theorem D]. On the other hand, if $g \geq 2$ and $n = 2$ then strata with signature $(m_1, \ldots, m_k; -1^2)$ may have two connected components depending on the type of the signature; see §2.3. This leads us to consider a few special cases in separated subsections, that is, §6.4 and §7.2.3 of Sections §6 and §7 respectively. In fact, these are very specific to the realization of integral representations in these strata.

1.3. Period map and isoperiodic fibers

Besides strata, there is another type of subspaces in $\Omega\mathcal{M}_{g,n}$ which is worth studying, called *isoperiodic foliations*. These are defined as the (projections of) nonempty fibers of the so-called *period map* defined as the association that maps an abelian differential ω to its period character; see Equation (7). Unfortunately, for every $n \geq 0$, such a map is not well defined on $\Omega\mathcal{M}_{g,n}$; see the discussion in [Nag85, Section §4.1.1] for $n = 0$. In order to have a well-defined period map we need to consider a suitable covering space that we denote with $\mathcal{S}_{g,n}$.

It follows from the classical theory that the orbifold universal cover of $\mathcal{M}_{g,n}$ of the moduli space of unmarked complex structures on $S_{g,n}$ is biholomorphic to the Teichmüller space $\mathcal{T}_{g,n}$. The fiber over every point X identifies with the different possible identifications of the fundamental group of X with the fundamental group of a reference topological surface $S_{g,n}$. Under this perspective the covering group turns out to be isomorphic to the mapping class group $\mathrm{Mod}(S_{g,n})$. Since in what follows we only want to keep the information of the identification at the homology level of the complex curve we consider the

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quotient of the Teichmüller space $\mathcal{T}_{g,n}$ with the subgroup $\mathcal{I}(S_{g,n}) \subset \mathrm{Mod}(S_{g,n})$ formed by elements that act trivially in homology, that is, on $\mathrm{H}_1(S_{g,n}, \mathbb{Z})$. We then define the following space:

$$\mathcal{S}_{g,n} = \frac{\mathcal{T}_{g,n}}{\mathcal{I}(S_{g,n})}. \tag{5}$$

Remark. For $n = 0$, the group $\mathcal{I}(S_{g,n}) \subset \mathrm{Mod}(S_{g,n})$ is known as the Torelli group and the space $\mathcal{S}_{g,n}$ is thus named as the Torelli space.

Every point in $\mathcal{S}_{g,n}$ thus corresponds to an equivalent class of tuples (X, m), where X is an unmarked complex structure on $S_{g,n}$ and $m\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathrm{H}_1(X, \mathbb{Z})$ is an identification. Here two tuples (X, m_X) and (Y, m_Y) are equivalent if there exists a biholomorphism of marked Riemann surfaces, say $f\colon X \longrightarrow Y$, such that $m_Y = f_* \circ m_X$. The mapping class group $\mathrm{Mod}(S_{g,n})$ acts on $\mathcal{S}_{g,n}$ by precomposition on the marking and such an action yields a covering map

$$\mathcal{S}_{g,n} \longrightarrow \mathcal{M}_{g,n} \tag{6}$$

by construction. The moduli space of differentials $\Omega\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_{g,n}$ pulls back to $\mathcal{S}_{g,n}$, and hence, it defines $\Omega\mathcal{S}_{g,n} \longrightarrow \mathcal{S}_{g,n}$, where $\Omega\mathcal{S}_{g,n}$ denotes the moduli space of *homologically marked translation surfaces* or *homologically marked differentials*. The period map is then defined as the association

$$\mathrm{Per}\colon \Omega\mathcal{S}_{g,n} \longrightarrow \mathrm{Hom}\Big(\mathrm{H}_1(S_{g,n}, \mathbb{Z}), \mathbb{C}\Big) \tag{7}$$

that maps a marked translation surface (X, ω, m) to its period character. If $n \geq 1$, [CFG22, Theorem A] states that, unlike the holomorphic case corresponding to $n = 0$, such a map is surjective. Our Theorems A and B thus provide a better characterization of the period map.

An *isoperiodic fiber* is defined as the nonempty preimage of a representation χ via the period map (7) These are also known in literature as *absolute kernel foliations*. We call a fiber of the period map an *isoperiodic fiber*. These fibers clearly project to $\Omega\mathcal{M}_{g,n}$ and then define a foliation that, with a little abuse of notation, we may still call isoperiodic. Since these latter along with strata are both subspaces of $\Omega\mathcal{M}_{g,n}$, it sounds natural to inquire about their mutual intersections. As a corollary of our main Theorems A and B, we obtain the following result:

Corollary C. *For a nontrivial representation its isoperiodic fiber in $\Omega\mathcal{M}_{g,n}$ intersects each connected component of each stratum of meromorphic differentials. The same statement holds for the trivial representation except for the strata in Theorem B.*

We conclude with some additional comments. In the first place it is worth mentioning that $\Omega\mathcal{S}_{g,n}$ also admit a natural stratification given by strata, sometimes denoted by $\Omega\mathcal{S}_g(\mu)$, enumerated by unordered partitions $\mu = \big(m_1, \ldots, m_k; -p_1, \ldots, -p_n\big)$ of $2g - 2$ where, even in this case, negative integers are allowed and correspond to the orders of poles. Moreover, for every signature μ, the covering map (6) induces a natural covering map $\pi(\mu)\colon \Omega\mathcal{S}_g(\mu) \longrightarrow \mathcal{H}_g(\mu)$ between strata.

The period map restricted to strata of marked structures
The period map (7) just defined naturally restricts to a mapping

$$\mathrm{Per}(\mu)\colon \Omega\mathcal{S}_g(\mu) \longrightarrow \mathrm{Hom}\Big(\mathrm{H}_1(S_{g,n}, \mathbb{Z}), \mathbb{C}\Big) \tag{8}$$

for every signature μ and [CFG22, Theorems B,C, D] provide a full characterization of the image of $\mathrm{Per}(\mu)$. In fact, a representation χ can be realized in a stratum $\mathcal{H}_g(\mu)$ if and only if it can be realized in the marked stratum $\Omega\mathcal{S}_g(\mu)$. In one direction, if a triple (X, ω, m) has period character χ then (X, ω) has the same period character. Vice versa, let $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be any representation, fix a system of handle generators, say $\{\alpha_i, \beta_i\}_{1 \leq i \leq g}$, for $\mathrm{H}_1(S_{g,n}, \mathbb{Z})$ (see Definitions 4.1 and 4.2), and

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complete it with a collection $\{\delta_1, \ldots, \delta_n\}$ of n simple closed curves, where δ_i is a simple loop around the i^{th} puncture. Assume that χ appears as the period character of some pair $(X, \omega) \in \mathcal{H}_g(\mu)$. Then, in the realization process, we determine a system of handle generators $\{\alpha_i', \beta_i'\}_{1 \le i \le g}$ and n curves, say $\{\delta_1', \ldots, \delta_n'\}$ that generates $\mathrm{H}_1(X, \mathbb{Z})$ and such that

$$\int_{\alpha_i'} \omega = \chi(\alpha_i), \qquad \int_{\beta_i'} \omega = \chi(\beta_i) \quad \text{and} \quad \int_{\delta_i'} \omega = \chi(\delta_i). \tag{9}$$

The association $m(\alpha_i) = \alpha_i'$, $m(\beta_i) = \beta_i'$ and $m(\delta_i) = \delta_i'$ naturally extends to an isomorphism in homology, that is, a map $m \colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathrm{H}_1(X, \mathbb{Z})$ that provides the desired marking. Thus the triple (X, ω, m) has period character χ and belongs to $\Omega\mathcal{S}_g(\mu)$.

Notice that the stratum of marked differentials $\Omega\mathcal{S}_g(\mu)$ is generally disconnected; for example, if the underneat space $\mathcal{H}_g(\mu)$ itself is disconnected. In any case the isoperiodic fiber in $\Omega\mathcal{S}_g(\mu)$ intersects at least as many connected components as the isoperiodic fiber in $\mathcal{H}_g(\mu)$ does. In other words, Theorems A and B do not provide a full description of the mutual relations between strata in $\Omega\mathcal{S}_{g,n}$ and isoperiodic fibers. It naturally arises the following:

> *Question*: Let μ be a signature and let $\pi(\mu) \colon \Omega\mathcal{S}_g(\mu) \longrightarrow \mathcal{H}_g(\mu)$ be the associated covering projection. If a representation χ can be realized in a connected component, say C, of $\mathcal{H}_g(\mu)$, then can χ be realized in every connected component of $\pi(\mu)^{-1}(C) \subset \Omega\mathcal{S}_g(\mu)$?

Although a stratum $\mathcal{H}_g(\mu)$ has at most three connected components with the only exception being strata of genus-one meromorphic differentials, the corresponding stratum $\Omega\mathcal{S}_g(\mu)$ may have more connected components. This is the case of strata in $\Omega\mathcal{T}_g$ defined by pulling back the moduli space of differentials $\Omega\mathcal{M}_g \to \mathcal{M}_g$ to $\mathcal{T}_g$. For a signature μ made only of positive integers, the connected components of $\Omega\mathcal{T}(\mu)$ have been classified by Calderon in [Cal20] and Calderon-Salter in [CS21]. No classification is known for marked and homologically marked differentials on Riemann surfaces.

Geometry and topology of isoperiodic fibers

It is interesting to study under which conditions an isoperiodic fiber is connected. For holomorphic differentials it is known that the isoperiodic fibers in $\Omega\mathcal{S}_g$ are generically connected up to a few exceptions; see [CDF23, Theorems 1.2 and 1.3]. In [Win21], Winsor have studied the connectedness of the isoperiodic foliation for nonprincipal strata of holomorphic differentials in $\Omega\mathcal{M}_g$. For meromorphic differentials with two simple poles an analogous result has recently been obtained in [CD24]. It is worth mentioning that in [CDF23, CD24] the authors work with strata that have all simple zeros. It remains to study connected components of isoperiodic fibers for meromorphic differentials with general pole orders. The following statement is straightforward and follows by the fact that all strata of genus-zero differentials are connected.

Corollary D. *Let χ be the trivial representation. Then its isoperiodic fiber in $\Omega\mathcal{M}_{0,n}$ is disconnected with the connected components being strata $\mathcal{H}_0(\mu)$ where μ is an integer partition of -2.*

For nontrivial representation less is known. In [FTZ24], the second named author with Guillaume Tahar and Yongquan Zhang aims to study the isoperiodic foliations for $\mathcal{H}_1(1, 1; -2)$. Another direction is to study period realization of *k-differentials* for $k > 1$, for example, $k = 2$ is the case of quadratic differentials that is, half-translation surfaces. We plan to treat these questions in future works.

1.4. Isoresidual fibrations

Beyond the period map (7) defined above, we can define the so-called *residual map* that associates every translation surface (X, ω) its n−tuple of residues of ω at the punctures. Recall that according to our

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convention $X \in \mathcal{M}_{g,n}$ and ω is a holomorphic differential on X with at most finite-order poles at the punctures. More precisely, for a signature $(\mu;\kappa) = (m_1,\dots,m_k;-1^s,-p_1,\dots,-p_{n-s})$, where $p_j \geq 2$ for $j = 1,\dots,n-s$, we may consider the following space:

$$\mathcal{R}_g(\mu;\kappa) = \left\{ \left(r_1,\dots,r_s,r_{s+1},\dots,r_n\right) \in \left(\mathbb{C}^*\right)^s \times \mathbb{C}^{n-s} \;\middle|\; \sum_{i=1}^{n} r_i = 0 \right\}. \tag{10}$$

The residual map is thus defined as

$$\mathfrak{R}_g(\mu;\kappa)\colon \mathcal{H}_g(m_1,\dots,m_k;-1^s,-p_1,\dots,-p_{n-s}) \longrightarrow \mathcal{R}_g(\mu;\kappa) \tag{11}$$

that associates (X,ω) to $\left(\mathrm{Res}_{P_i}(\omega)\right)_{i=1,\dots,n}$ where the $\{P_i\}_{i=1,\dots,n}$ is the set of punctures of X. In [GT21], the authors showed that the restriction of the residual map $\mathfrak{R}_g(\mu;\kappa)$ to each connected component of $\mathcal{H}_g(\mu;\kappa)$ is surjective provided that $g \geq 1$; see [GT21, Theorem 1.1].

Remark. The residual map, however, generally fails to be surjective on strata of genus-zero differentials. [GT21, Theorem 1.2] provides necessary conditions for a tuple $\left(r_1,\dots,r_n\right)$ to lie in the image of $\mathfrak{R}_0(\mu;\kappa)$. The latter has been extended by the recent work [CFG22, Theorems B,C,D].

The main results of the work [CFG22] along with Theorem A provides another, independent, proof of [GT21, Theorem 1.1]; that is,

Corollary E. *Let $g \geq 1$. The residual map restricted to each connected component of $\mathcal{H}_g(\mu;\kappa)$ is surjective.*

Proof of Corollary E provided Theorem A holds. Let $\kappa = (-1^s,-p_1,\dots,-p_{n-s})$ be the polar part of some signature $(\mu;\kappa)$ and let $(r_1,\dots,r_n) \in \mathcal{R}_g(\mu;\kappa)$. Recall that $\pi_1\left(\mathbb{S}^2 \setminus \{P_1,\dots,P_n\}\right) \cong \mathbb{Z}^{n-1}$ is generated by a collection of peripheral loops $\gamma_1,\dots,\gamma_n$ such that $\gamma_1\cdots\gamma_{n-1}\gamma_n^{-1} = 1$. Define χ_n as the unique character such that $\chi_n(\gamma_i) = r_i$. Following [CFG22, Theorem B,C,D], χ_n can be realized as the period character of some translation surface in the stratum $\mathcal{H}_0(m_1,m_2;-1^s,-p_1,\dots,-p_{n-s})$. We next bubble g handles with positive volume (see Section §3.2) to realize a translation surface in $\mathcal{H}_g(m_1+m_2+2g\,;\,-1^s,-p_1,\dots,-p_{n-s})$. Finally break the zero (see Section §3.1), to get $(X,\omega) \in \mathcal{H}_g(\mu\,;\,-1^s,-p_1,\dots,-p_{n-s})$ as desired.

By construction (X,ω) belongs to the fibre of $(r_1,\dots,r_n)$ via the residual map $\mathfrak{R}_g(\mu;\kappa)$. Let χ be the period character of (X,ω) and notice that χ is not trivial. Since Theorem A ensures that it can be realized in every connected component of $\mathcal{H}_g(\mu\,;\,-1^s,-p_1,\dots,-p_{n-s})$, the result thus follows. □

1.5. Structure of the paper

This paper is organized as follows. In Section §2 we review basic concepts about differentials and translation surfaces, their strata with prescribed orders of zeros and poles, and the geometric invariants that can distinguish connected components of the strata. In Section §3 we introduce several surgery operations that can be used to construct translation surfaces of higher genera and study how the concerned invariants change under these operations. In Section §4 we discuss a system of handle generators for the period domain in order to carry out inductive constructions. In Sections §5, we first prove Theorem A for surfaces of genus one. More precisely, for a nontrivial representation χ we provide a direct construction to realize a meromorphic genus-one differential with period character χ and prescribed invariants. The proof involves a case-by-case discussion according to Table 1; see Appendix A. In Section §6, we consider higher genus surfaces; that is, we shall suppose $g \geq 2$. This section is entirely devoted to realize a representation as the period character of some hyperelliptic translation surface with poles in a give stratum. Next, in Section §7 we still consider higher genus surfaces and we aim to realize a given nontrivial representation as the period character of some translation structure with spin parity. The

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proof is based on an induction foundation having genus-one differentials as the base case. Therefore, this section is mainly devoted to show how to run the inductive process. We finally consider the trivial representation on its own right. In Section §8 we shall prove Theorem B for the cases of genus-one, hyperelliptic surfaces, surfaces with spin parity, and the trivial representation (which corresponds to exact differentials), respectively. Along the way we shall encounter and classify some exceptional strata in which the trivial representation cannot be realized for certain connected components. Finally in Appendix §A we provide a flowchart to illustrate the relations of the various cases in the course of the proof.

2. Translation surfaces with poles and meromorphic differentials

We begin by recalling the notion of translation structure on a topological surface by providing a geometric and a complex-analytic perspective.

A *translation structure* on a surface $S_{g,n}$ is a branched $(\mathbb{C}, \mathbb{C})$-structure; that is, the datum of a maximal atlas where local charts in $\mathbb{C}$ have the form $z \longmapsto z^k$, for $k \geq 1$, and transition maps given by translations on their overlappings. Any such an atlas defines an underlying complex structure X and the pullbacks of the 1-form dz on $\mathbb{C}$ via local charts globalize to a holomorphic differential ω on X. Vice versa, a holomorphic differential ω on a complex structure X defines a singular Euclidean metric with isolated singularities corresponding to the zeros of ω. In a neighborhood of a point P which is not a zero of ω, a local coordinate is defined as

$$z(Q) = \int_P^Q \omega \tag{12}$$

in which $\omega = dz$, and the coordinates of two overlapping neighborhoods differ by a translation $z \mapsto z + c$ for some $c \in \mathbb{C}$. Around a zero, say P of order $k \geq 1$, there exists a local coordinate z such that $\omega = z^k dz$. The point P is also called a *branch point* because any local chart around it is locally a branched $k+1$ covering over $\mathbb{C}$ which is totally ramified at P.

Definition 2.1 (Translation surfaces with poles). Let ω be a meromorphic differential on a compact Riemann surface $\overline{X} \in \mathcal{M}_g$. We define a *translation surface with poles* to be the structure induced by ω on the surface $X = \overline{X} \setminus \Sigma$, where Σ is the set of poles of ω.

Given a translation structure (X, ω) on a surface $S_{g,n}$, the local charts globalize to a holomorphic mapping $\mathrm{dev}\colon \widetilde{S}_{g,n} \longrightarrow \mathbb{C}$ called the *developing map*, where $\widetilde{S}_{g,n}$ is the universal cover of $S_{g,n}$. The translation structure on $S_{g,n}$ lifts to a translation structure $(\widetilde{X}, \widetilde{\omega})$ and the developing map turns out to be locally univalent away from the zeros of $\widetilde{\omega}$. The developing map, in particular, satisfies an equivariant property with respect to a representation $\chi\colon \mathrm{H}_1(X, \mathbb{Z}) \longrightarrow \mathbb{C}$ called *holonomy* of the translation structure. The following lemma establishes the relation between holonomy representations and period characters.

Lemma 2.2. *A representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is the period of some abelian differential $\omega \in \Omega(X)$ with respect to some complex structure X on $S_{g,n}$ if and only if it is the holonomy of the translation structure on $S_{g,n}$ determined by ω.*

This twofold nature of a representation χ permits us to tackle our problem by adopting a geometric approach. More precisely, in order to realize a representation χ on a prescribed connected component of some stratum of differentials, we shall realize it as the holonomy of some translation surface with poles (X, ω) with prescribed zeros and poles and, whenever they are defined, with prescribed spin structure or hyperelliptic involution. Some remarks are in order.

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Remark 2.3. Let ω be a meromorphic differential on a compact Riemann surface $\overline{X} \in \mathcal{M}_g$ which yields a translation surface (X, ω) with poles of finite orders $p_1, \dots, p_n$, $n \geq 1$. Let $m_1, \dots, m_k$ be the orders of zeros of ω. Then it is well known that the following equality holds:

$$\sum_{i=1}^{k} m_i - \sum_{j=1}^{n} p_j = 2g - 2. \tag{13}$$

Remark 2.4. Let (X, ω) be a translation surface, possibly with poles, and let us denote $X^* = X \setminus \{\text{zeros of } \omega\}$ and pick any point $x_0 \in X^*$. Since the structure is flat, the parallel transport induced by the flat connection yields a homomorphism from $\pi_1(X^*, x_0)$ to $\mathrm{SO}(2, \mathbb{R}) \cong \mathbb{S}^1$ which acts on the tangent space of x_0. Since $\mathbb{S}^1$ is abelian, such a homomorphism factors through the homology group, and hence, we have a well-defined homomorphism $\mathrm{PT}\colon \mathrm{H}_1(X^*, \mathbb{Z}) \to \mathrm{SO}(2, \mathbb{R}) \cong \mathbb{S}^1$. For translation surfaces this homomorphism turns out to be trivial in the sense that a parallel transport of a vector tangent to the Riemann surface X along any closed path avoiding the zeros of ω brings the vector back to itself.

Remark 2.5. Let $x_0 \in (X, \omega)$ be a regular point; that is, x_0 is not a zero for ω. A given tangent direction at x_0 can be extended to all other regular points by means of the parallel transport. This yields a nonsingular foliation which extends to a singular foliation with singularities at the branch points. Let $z = x + iy$ be a local coordinate at x_0. The *horizontal foliation* is the oriented foliation determined by the positive real direction in the coordinate z. Notice that this is well defined because different local coordinates differ by a translation. In the same fashion, the *vertical foliation* is the oriented foliation determined by the positive purely imaginary direction in the coordinate z.

2.1. Volume

We now discuss the notion of *volume* which plays an important role in the theory. For our purposes, we shall mostly interested in the algebraic volume which is a topological invariant associated to a representation χ.

Let us recall this notion in the holomorphic case. For a symplectic basis $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ of $\mathrm{H}_1(S_g, \mathbb{Z})$ we define the volume of a representation $\chi\colon \mathrm{H}_1(S_g, \mathbb{Z}) \longrightarrow \mathbb{C}$ as the quantity

$$\mathrm{vol}(\chi) = \sum_{i=1}^{g} \Im\big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\big), \tag{14}$$

where $\Im(\overline{z}w)$ is the usual symplectic form on $\mathbb{C}$. As a consequence, this algebraic definition of volume of a character χ is invariant under precomposition with any automorphisms in $\mathrm{Aut}\big(\mathrm{H}_1(S_g, \mathbb{Z})\big) \cong \mathrm{Sp}(2g, \mathbb{Z})$. The image of χ, provided it has rank $2g$, turns out to be a polarized module.

Haupt's conditions. As already alluded in the introduction, there are some obstructions for realizing χ as the period character of some holomorphic differential ω on a compact Riemann surface X and both of these concern the volume. We shall recall them here for the reader's convenience. The first requirement is that the volume of χ has to be positive with respect to some symplectic basis of $\mathrm{H}_1(S_g, \mathbb{Z})$. Indeed, one can show that this equals the area of the surface X endowed with the singular Euclidean metric induced by ω. There is a second obstruction that applies in the case $g \geq 2$ and only when the image of $\chi\colon \mathrm{H}_1(S_g, \mathbb{Z}) \to \mathbb{C}$ is a lattice, say Λ in $\mathbb{C}$. In fact, one can show that in this special case (X, ω) arises from a branched cover of the flat torus $\mathbb{C}/\Lambda$. In particular, the following inequality must hold: $\mathrm{vol}(\chi) \geq 2\,\mathrm{Area}(\mathbb{C}/\Lambda)$. Haupt's Theorem says that these are the only obstructions for realizing χ as the period of some holomorphic differential.

Remark 2.6. We provide here an alternative and more geometric definition of volume. Let X be a compact Riemann surface and let $\omega \in \Omega(X)$ be a (possibly meromorphic) differential with period character χ. Let E be the complex line bundle over X canonically associated to χ and let $f\colon X \longrightarrow E$ be a section. By lifting the map f to $\widetilde{f}\colon \widetilde{X} \longrightarrow \widetilde{X} \times \mathbb{C}$ and projecting onto the second coordinate, the natural volume form $\frac{i}{2} dz \wedge d\overline{z}$ on $\mathbb{C}$ pulls back to a volume form over X. The quantity given by the integration of this form over X, that is

$$\frac{i}{2}\int_X f^*\Big(dz \wedge d\overline{z}\Big) = \frac{i}{2}\int_X \omega \wedge \varpi, \quad \text{where } \omega = f^* dz \tag{15}$$

is the area of the singular Euclidean metric determined by ω on X. In particular, by means of Riemann's bilinear relations one can show that it agrees with the algebraic definition of volume as in Equation (14).

Definition 2.7. Let $\omega \in \Omega(X)$ be a holomorphic differential on a compact Riemann surface X with period character $\chi\colon \mathrm{H}_1(S_g, \mathbb{Z}) \longrightarrow \mathbb{C}$. We define the *algebraic volume* of ω as the quantity $\mathrm{vol}(\chi)$ defined in formula (14). Notice that the algebraic volume is well defined since the volume of χ does not depend on the choice of the symplectic basis for $\mathrm{H}_1(S_g, \mathbb{Z})$.

We now extend the notion of volume to meromorphic differentials. In this case, the notion of volume relies on a choice of a splitting of $S_{g,n}$ as the connected sum of the closed surface S_g and the n-punctured sphere. Let γ be a simple closed separating curve in $S_{g,n}$ that bounds a subsurface homeomorphic to $S_{g,1}$. There is a natural embedding $S_{g,1} \hookrightarrow S_{g,n}$ and thence an injection $\iota_g\colon \mathrm{H}_1(S_g, \mathbb{Z}) \longrightarrow \mathrm{H}_1(S_{g,n}, \mathbb{Z})$ because the curve γ is trivial in homology. In the same fashion, γ bounds a subsurface homeomorphic to $S_{0,n+1}$ and the natural embedding $S_{0,n+1} \hookrightarrow S_{g,n}$ yields an injection $\iota_n\colon \mathrm{H}_1(S_{0,n}, \mathbb{Z}) \to \mathrm{H}_1(S_{g,n}, \mathbb{Z})$. Therefore, given a splitting as above, any representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ gives rise to two representations $\chi_g = \chi \circ \iota_g$ and $\chi_n = \chi \circ \iota_n$ that determine χ completely.

Definition 2.8. A representation χ is said to be of *trivial-ends type* if χ_n is trivial, otherwise it is said to be of *nontrivial-ends type*. The representation χ_n is always well defined and in fact it does not depend on the choice of any splitting. In particular, if χ is the period character of some pair (X, ω), then χ_n records the residues of the differential at the n poles.

Remark 2.9. It is worth noticing that a representation χ_g generally depends on the embedding $S_{g,1} \hookrightarrow S_{g,n}$ and in fact on the curve γ along which we split $S_{g,n}$. However, it is not hard to see that χ_g is uniquely determined if and only if χ is of trivial-ends type.

We can now introduce the following:

Definition 2.10. Let $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \to \mathbb{C}$ be a representation of trivial-ends type. We define the volume of χ as the volume of χ_g as defined in Equation (14). Let $\omega \in \Omega(X)$ be a holomorphic differential on $X \in \mathcal{M}_{g,n}$ with finite-order poles at the punctures and such that its period character χ is of trivial-ends type. Then we can define the *algebraic volume* of ω as the volume of χ. Notice that the algebraic volume does not depend on the choice of the splitting.

Remark 2.11. In the case that a meromorphic differential admits poles with nonzero residues, that is, its period character has nontrivial χ_n part, then the algebraic volume as in Definition 2.10 does depend on the splitting. In fact if we alter a closed loop representing a homology class by making it go across a pole with nonzero residue, its period will change by the amount of the residue.

2.2. *Further invariants on translation surfaces*

In Section §2.1 we have introduced the volume as an algebraic invariant naturally attached to a representation χ. In the present section we are going to introduce geometric invariants which we shall use to distinguish the connected components of strata whenever they fail to be connected.

2.2.1. Rotation number

In this subsection $\overline{X}$ is a compact Riemann surface of genus one, that is, an elliptic curve $\mathbb{C}/\Lambda$ with Λ a lattice in $\mathbb{C}$. We will introduce an invariant very specific to genus-one translation surfaces with poles.

Let X be a complex structure on $S_{1,n}$, where $n \geq 1$, and let $\omega \in \Omega(X)$ be an abelian differential on X with finite order poles at the punctures. Recall that away from the zeros and poles of ω there is

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a well-defined horizontal direction and hence a nonsingular horizontal foliation on $X \setminus \{$ zeros of $\omega\}$. Such a foliation extends to a singular horizontal foliation over the zeros of ω. In fact, for a zero, say P, of order k there are exactly $k+1$ horizontal directions leaving from P. Let γ be a simple smooth curve parametrized by the arc length. Assume in addition that γ avoids all the zeros and poles of ω. We shall define the index of γ as a numerical invariant given by the comparison of the unit tangent field $\dot\gamma(t)$ and the unit vector field along γ given by unit vectors tangent to the horizontal direction. More precisely, let us denote by $u(t)$ the unit vector at $\gamma(t)$ tangent to the leaf of the horizontal foliation through $\gamma(t)$, then the assignment

$$t \longmapsto \theta(t) = \angle\big(\dot\gamma(t),\, u(t)\big) \tag{16}$$

defines a mapping $f_\gamma\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.

Definition 2.12. The index of γ is defined as $\deg(f_\gamma)$ and denoted by $\mathrm{Ind}(\gamma)$.

Convention. We agree that $\mathbb{S}^1$ is counterclockwise oriented. As a consequence we agree that the index of $\mathrm{Ind}(\gamma) = \deg(f_\gamma)$ is *positive* is γ is counterclockwise oriented.

The index $\mathrm{Ind}(\gamma)$ of a closed curve γ measures the number of times the unit tangent vector field spins with respect to the direction given by the horizontal foliation. Since for translation surfaces the parallel transport yields a trivial homomorphism; see Remark 2.4, the total change of the angle is $2\pi\,\mathrm{Ind}(\gamma)$.

We now allow perturbations of γ in its homotopy class. The index of any curve remains unchanged if, while deforming it, we do not cross a zero or pole of ω. On the other hand, whenever we cross a zero or pole of order k the index of γ changes by $\pm k$. As a consequence, the index of a curve is well defined for homotopy classes in $\pi_1(\overline{X} \setminus \{\text{zeros and poles of } \omega\})$. We can now define the rotation number.

Definition 2.13. Let $(X,\omega) \in \mathcal{H}_1(m_1,\dots,m_k;-p_1,\dots,-p_n)$ be a genus-one translation surface with poles. Let $\{\alpha,\beta\}$ be a symplectic basis of $\mathrm{H}_1(\overline{X},\mathbb{Z})$, the first homology group of $\overline{X}$. Let γ_α and γ_β be the representatives of α and β respectively. Assume they are both parametrized by the arc length and avoid the zeros and poles of ω. The *rotation number* of (X,ω) is defined as

$$\mathrm{rot}(X,\omega) = \gcd\big(\mathrm{Ind}(\gamma_\alpha),\, \mathrm{Ind}(\gamma_\beta),\, m_1,\dots,m_k, p_1,\dots,p_n\big). \tag{17}$$

In the next subsection we briefly introduce the rotation number in a more algebraic way that we shall use only in the final Section §8. Notice that the well-known properties of gcd make the rotation number $\mathrm{rot}(X,\omega)$ well defined for a given element $(X,\omega) \in \mathcal{H}_1(m_1,\dots,m_k;-p_1,\dots,-p_n)$. As we shall see in Section §2.3, the rotation number is used to distinguish the connected components of a generic stratum of genus-one differentials $\mathcal{H}_1(m_1,\dots,m_k;-p_1,\dots,-p_n)$.

2.2.2. Hyperelliptic translation surface

For a complex structure X on $S_{g,n}$ we shall denote by $\overline{X}$ the compact Riemann surface obtained by filling the punctures with complex charts. Let us now introduce the following:

Definition 2.14. A translation surface (X,ω) (possibly with poles) is said to be *hyperelliptic* if the Riemann surface $\overline{X}$ is hyperelliptic and ω is anti-invariant under the hyperelliptic involution τ, that is, $\tau^*\omega = -\omega$).

Remark 2.15. For a hyperelliptic translation surface (X,ω), the hyperelliptic involution τ realizes an isometry between the singular Euclidean structures determined by the differentials ω and $-\omega$ on $\overline{X}$. Therefore, not all strata can admit hyperelliptic translation surfaces. In fact, if ω has two zeros (or two poles) that are swapped by τ then they must have the same order, and if a zero (or a pole) is fixed by τ then it must have even order.

Note that there are no nonzero holomorphic one-forms on the sphere. As a result, every holomorphic differential ω on a hyperelliptic Riemann surface X is anti-invariant under the hyperelliptic involution,

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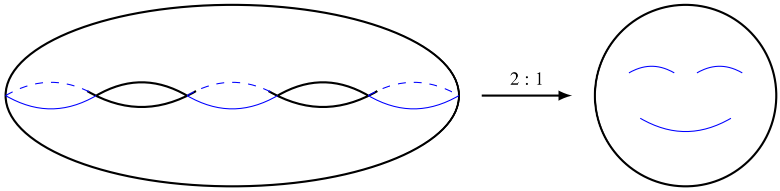

Figure 1. *A* 2*-fold branched covering* $\pi : X \longrightarrow \mathbb{C}\mathbf{P}^1$.

and hence, in this case (X, ω) is always a hyperelliptic translation surface. For meromorphic differentials such a scenario is no longer true as shown by the following example.

Example 2.16. Let us consider the translation structure with a single pole on the Riemann sphere $\mathbb{C}\mathbf{P}^1$ determined by the meromorphic differential dz. Let l_1, l_2, $l_3 \subset \mathbb{C}\mathbf{P}^1$ be three geodesic segments for the standard Euclidean structure and slit $\mathbb{C}\mathbf{P}^1$ along them. The resulting slit surface is homeomorphic to a sphere with three open disjoint disks removed. Geometrically, the resulting surface is a pair of pants equipped with a Euclidean structure and piece-wise geodesic boundary. In fact, each boundary component comprises two geodesic segments, say l_{ij} with $i = 1, 2, 3$ and $j = 1, 2$, and two corner points each one of angle 2π. We consider two copies of this latter surface and we glue them by identifying the boundary segments having the same label. The final surface turns out to be a compact Riemann surface X of genus two equipped with a meromorphic differential ω with six zeros of order 1 and two poles of order 2. The Riemann surface X, being compact of genus two, admits a hyperelliptic involution τ. We want to show with a direct computation that (X, ω) is not hyperelliptic.

Let $\pi\colon X \longrightarrow \mathbb{C}\mathbf{P}^1$ be the 2-fold branched covering that naturally arises from our construction. The differential ω satisfies the equation $\omega = \pi^*(dz)$. According to Definition 2.14 above, (X, ω) is hyperelliptic if and only if $\tau^*(\omega) = -\omega$. On the other hand, being of genus two, the hyperelliptic involution τ is an automorphism of X and commutes with the projection $\pi\colon X \longrightarrow \mathbb{C}\mathbf{P}^1$ in the sense that $\pi \circ \tau = \pi$. Therefore, $\omega = \pi^*(dz) = \tau^*(\omega)$ and hence

$$\tau^*(\omega) = -\omega \iff \omega = 0 \tag{18}$$

which provides the desired contradiction.

2.2.3. Spin structure

The last geometric invariant we shall introduce is the spin structure. Let X be a compact Riemann surface and let P be a principal $\mathbb{S}^1$-bundle over X. A *spin structure* on X is a choice of a linear functional $\xi\colon \mathrm{H}_1(P, \mathbb{Z}_2) \to \mathbb{Z}_2$ having nonzero value on the cycle representing the fiber $\mathbb{S}^1$ of P. This is equivalent to a choice of a double covering $Q \longrightarrow P$ whose restriction to each fiber of P is a 2-fold covering of $\mathbb{S}^1$.

Recall that an element of $\mathrm{H}_1(P, \mathbb{Z}_2)$ can be regarded as a *framed closed curve* in X, by which we mean a closed curve in X and a smooth unit vector field along it. For a simple closed curve γ in X there is a preferred frame given by its unit tangent vector field. We shall denote such a framed closed curve as $\vec{\gamma}$. This is well defined in the sense that two homologous closed curves $\gamma, \delta \in \mathrm{H}_1(X, \mathbb{Z}_2)$ yield homologous framed closed curves $\vec{\gamma}, \vec{\delta}$; see [Joh80, Section 3]. The *canonical lift* of a simple closed curve γ is defined as $\widetilde{\gamma} = \vec{\gamma} + \mathbf{z}$ where $\mathbf{z} \in \mathrm{H}_1(P, \mathbb{Z}_2)$ is the homology class represented by the fiber $\mathbb{S}^1$. The assignment $\gamma \longmapsto \widetilde{\gamma}$ defines a mapping of sets $\mathrm{H}_1(X, \mathbb{Z}_2) \to \mathrm{H}_1(P, \mathbb{Z}_2)$ which fails to be a homomorphism; see [Joh80]. Nevertheless, post-composition of such a map with a spin structure ξ yields a quadratic form $\Omega_\xi\colon \mathrm{H}_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ defined as

$$\Omega_\xi(\gamma) = \xi\big(\widetilde{\gamma}\big) = \langle \xi, \widetilde{\gamma} \rangle. \tag{19}$$

According to [Arf41] and [Joh80, Section 5] we shall introduce the following:

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Definition 2.17. For a symplectic basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ of $\mathrm{H}_1(X, \mathbb{Z}_2)$ the Arf-invariant of Ω_ξ is defined by the formula

$$\sum_{i=1}^{g} \Omega_\xi(\alpha_i)\Omega_\xi(\beta_i) \pmod 2. \tag{20}$$

The *parity* of a spin structure ξ is defined as the Arf-invariant of Ω_ξ. In particular, the parity of ξ does not depend on the choice of the symplectic basis.

Let X be a complex structure on $S_{g,n}$, with $g \geq 1$, and $\omega \in \Omega(X)$ be an abelian differential on X with poles of finite order at the punctures. Recall that we can regard ω as a meromorphic differential on a compact Riemann surface $\overline{X}$. Whenever $(X, \omega) \in \mathcal{H}_g(2m_1, \ldots, 2m_k; -2p_1, \ldots, -2p_n)$, then ω defines a natural spin structure on $\overline{X}$ as follows.

Let γ be a simple closed curve parametrized by the arc length, and let ϑ be a unit vector field along γ. The couple (γ, ϑ) defines a framed closed curve, that is, an element of $\mathrm{H}_1(P, \mathbb{Z}_2)$. Recall that ω defines a nonsingular horizontal foliation away from zeros and poles, and hence, the unit vector field ϑ can be compared to the unit vector field along γ tangent to the horizontal foliation. In fact, as in Section §2.2.1, this can be done by means of a mapping $f_{(\gamma,\vartheta)} : \mathbb{S}^1 \to \mathbb{S}^1$ defined as in Equation (16). The spin structure induced by the meromorphic differential ω on $\overline{X}$ is then defined as the $\mathbb{Z}_2$-value linear functional

$$\xi\colon \mathrm{H}_1(P, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2, \quad \text{defined as} \quad \xi(\gamma, \vartheta) = \deg\big(f_{(\gamma,\vartheta)}\big) \pmod 2. \tag{21}$$

A direct application of Equation (21) shows that, for the canonical lift $\widetilde{\gamma}$ of γ, the spin structure determined by ω satisfies the property

$$\xi(\widetilde{\gamma}) = \langle \xi, \widetilde{\gamma} \rangle = \langle \xi, \vec{\gamma} + \mathbf{z} \rangle = \mathrm{Ind}(\gamma) + 1 \pmod 2. \tag{22}$$

As a consequence, we can apply formula (20) to compute the parity $\varphi(\omega)$ of the spin structure determined by a meromorphic differential ω. Given a symplectic basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ of $\mathrm{H}_1(X, \mathbb{Z}_2)$, it follows that

$$\varphi(\omega) = \sum_{i=1}^{g} \big(\mathrm{Ind}(\alpha_i) + 1\big)\big(\mathrm{Ind}(\beta_i) + 1\big) \pmod 2. \tag{23}$$

It is straightforward to check that $\varphi(\omega)$ does not depend on the choice of the symplectic basis. Finally, it follows from works of Atiyah in [Ati71] and Mumford in [Mum71] that the spin parity is invariant under continuous deformations.

Remark 2.18. Let $(X, \omega) \in \mathcal{H}_g(2m_1, \ldots, 2m_k, -1, -1)$ be a genus-g meromorphic differential with two simple poles. The meromorphic differential ω does not define any spin structure as in Equation (21) because, for any curve γ, the index $\mathrm{Ind}(\gamma)$ is not longer well defined as an element of $\mathbb{Z}_2$; see formula (22). Nevertheless, we can still define an invariant that turns out to be the spin parity of some structure (Y, ξ) in the stratum $\mathcal{H}_{g+1}(2m_1, \ldots, 2m_k)$. Recall that a neighborhood of a simple pole is an infinite cylinder. Around a puncture of (X, ω), we can find a simple closed geodesic curve. In fact, there are infinitely many of such curves. Choose a waist geodesic curve on each cylinder and truncate (X, ω) along them. Since (X, ω) has only two simple poles and their residues are opposite (and nonzero), the curves above are isometric. We glue them and the resulting object is a translation structure on a closed surface of genus $g + 1$. We define this latter as (Y, ξ), and it lies in the stratum $\mathcal{H}_{g+1}(2m_1, \ldots, 2m_k)$ by construction. Therefore, it makes sense to consider a *parity of the spin structure* for differentials in $\mathcal{H}_g(2m_1, \ldots, 2m_k, -1, -1)$. See [Boi15, Section 5.3] for more details.

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2.3. *Moduli spaces and connected components*

As alluded in the introduction, the moduli space $\Omega\mathcal{M}_{g,n}$ admits a natural stratification given by unordered partitions $(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ of $2g-2$, where negative integers are allowed only in the case of meromorphic differentials. In the present subsection, we aim to recall for the reader's convenience the classifications of connected components of the strata both in the holomorphic and meromorphic cases.

Let us discuss first the case of holomorphic differentials which has been studied by Kontsevich–Zorich in [KZ03, Theorems 1 and 2]. For low-genus surfaces, that is, $g = 2, 3$, we have the following:

Theorem (Kontsevich–Zorich). *The moduli space of abelian differentials on a curve of genus $g = 2$ contains two strata: $\mathcal{H}_2(1,1)$ and $\mathcal{H}_2(2)$. Each of them is connected and coincides with its hyperelliptic component. Each of the strata $\mathcal{H}_3(2,2)$, $\mathcal{H}_3(4)$ of the moduli space of abelian differentials on a curve of genus $g = 3$ has two connected components: the hyperelliptic one and one having odd spin structure. The other strata are connected for genus $g = 3$.*

For surfaces of genus $g \geq 4$, the following classification holds.

Theorem (Kontsevich–Zorich). *All connected components of any stratum of abelian differentials on a curve of genus $g \geq 4$ are described by the following list:*

- *The stratum $\mathcal{H}_g(2g-2)$ has three connected components: the hyperelliptic one and two other components corresponding to even and odd spin structures;*
- *the stratum $\mathcal{H}_g(2l, 2l)$, $l \geq 2$ has three connected components: the hyperelliptic one and two other components distinguished by the spin parity;*
- *all the other strata of the form $\mathcal{H}_g(2l_1, \ldots, 2l_n)$, where all $l_i \geq 1$, have two connected components corresponding to even and odd spin structures;*
- *the strata $\mathcal{H}_g(2l-1, 2l-1)$, $l \geq 2$ have two connected components: One comprises hyperelliptic structures and the other does not; finally,*
- *all the other strata of abelian differentials on the curves of genus $g \geq 4$ are nonempty and connected.*

Let us discuss the case of meromorphic differentials and begin from the case of genus-one meromorphic differentials. Let $\mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ be a stratum in $\Omega\mathcal{M}_{1,n}$, and let d be the number of positive divisors of $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n)$. Recall that a stratum is nonempty as soon as the Gauss–Bonnet condition holds for some nonnegative integer g; see Remark 2.3, and $\sum p_i > 1$. In [Boi15, Theorem 1.1], Boissy showed the following.

Theorem (Boissy). *Let $\mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ be a nonempty stratum of genus-one meromorphic differentials. Denote by d the number of positive divisors of $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_k)$. Then the number of connected components of this stratum is:*

- *$d-1$ connected components if $k = n = 1$. In this case, the stratum is $\mathcal{H}_1(m, -m)$ and the connected components are parametrized by the positive divisors of m different from m itself.*
- *d otherwise. In this case, each connected component is parametrized by a positive divisor of $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_k)$.*

Common divisors of the zero and pole orders in the above result are called *rotation numbers*. It is worth mentioning that connected components of the strata in genus one can also be classified by another invariant from the algebraic viewpoint; see [CC14, Section §3.2]. For a stratum $\mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ of genus-one differentials, let l be any positive divisor of $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n)$ – with the only exception being $l = n$ for signatures of the form $(n; -n)$. We shall say that $(X, \omega) \in \mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ has *torsion number* l if l is the largest positive integer such that

$$\sum_{j=1}^{k} \left(\frac{m_j}{l}\right) Z_j - \sum_{i=1}^{n} \left(\frac{p_i}{l}\right) P_i \sim 0, \tag{24}$$

where $\{Z_i, \dots, Z_n\}$ and $\{P_1, \dots, P_n\}$ are, respectively, the set of zeros and poles of ω. Here, $\sim$ stands for linear equivalence which means there exists a meromorphic function on the underlying Riemann surface with zeros at $\{Z_1, \dots, Z_k\}$ and poles at $\{P_1, \dots, P_n\}$ with orders given by the coefficients in the relation. We shall use this characterisation in Section §8 to show a few lemmas concerning the exceptional cases listed in Theorem B.

Remark 2.19. The two notions of rotation numbers and torsion numbers coincide; see [Tah18, Section §3.4] and [CG22, Section §3.4 and Proposition 3.13].

For genus-g meromorphic differentials with $g \geq 2$, the classifications of connected components is similar to that of holomorphic differentials. Let $\mathcal{H}_g(m_1, \dots, m_k; -p_1, \dots, -p_n)$ be a stratum in $\Omega\mathcal{M}_{g,n}$. In [Boi15], Boissy has showed that a stratum admits a hyperelliptic component if and only if it is one of the following:

$$\mathcal{H}_g(2m, -2p),\ \mathcal{H}_g(m, m, -2p),\ \mathcal{H}_g(2m, -p, -p),\ \mathcal{H}_g(m, m, -p, -p), \tag{25}$$

for some $m \geq p \geq 1$, otherwise there is no hyperelliptic component. According to Boissy, we introduce the following.

Definition 2.20. For a stratum $\mathcal{H}_g(\kappa; -\nu)$ we shall say that the set of zeros and poles is

- of *hyperelliptic type* if $\kappa = \{2m\}$ or $\{m, m\}$ and $\nu = \{2p\}$ or $\{p, p\}$, where m and p are positive integers;
- of *even type* if $\kappa = (2m_1, \dots, 2m_k)$ and $\nu = (2p_1, \dots, 2p_n)$ or $\nu = (1, 1)$.

We shall say that a translation surface is of even type if it belongs to a stratum of even type, namely the set of zeros and poles is of even type.

Remark 2.21. A stratum of hyperelliptic type admits a connected component such that *every* (X, ω) parameterized by this component is a hyperelliptic translation surface as described in Definition 2.14. In contrast, certain strata that are not of hyperelliptic type can still contain lower-dimensional loci parameterizing hyperelliptic translation surfaces. For example, holomorphic differentials on genus-three hyperelliptic Riemann surfaces with a double zero fixed by the hyperelliptic involution τ and two simple zeros swapped by τ are hyperelliptic translation surfaces. However, their parameter space is a proper subset of the stratum $\mathcal{H}_3(2, 1, 1)$ which is not of hyperelliptic type.

We now state the following result due to Boissy concerning connected components of the strata of genus-g meromorphic differentials with $g \geq 2$.

Theorem (Boissy). *Let $\mathcal{H}_g(\kappa, -\nu) = \mathcal{H}_g(m_1, \dots, m_k; -p_1, \dots, -p_n)$ be a nonempty stratum of genus-g meromorphic differentials. We have the following:*

- *If $p_1 + \cdots + p_n$ is odd and greater than two, then the stratum is connected.*
- *If $\nu = \{2\}$ or $\nu = \{1, 1\}$ and $g = 2$, then we distinguish two cases:*
  - *if the set of poles and zeros is of hyperelliptic type, then there are two connected components, one hyperelliptic, the other not; in this case, these two components are also distinguished by the parity of the spin structure,*
  - *otherwise the stratum is connected*
- *If ν is not $\{2\}$, $\{1, 1\}$, that is, $p_1 + \cdots + p_n > 2$) or $g \geq 3$, then we distinguish two cases as follows*
  - *if the set of poles and zeros is of hyperelliptic type, there is exactly one hyperelliptic connected component and one or two nonhyperelliptic components that are described below. Otherwise, there is no hyperelliptic component.*
  - *if the set of poles and zeros is of even type, then the stratum contains exactly two nonhyperelliptic connected components that are distinguished by the parity of the spin structure. Otherwise, the stratum contains exactly one nonhyperelliptic component.*

This concludes the classification of the strata of meromorphic differentials.

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3. Surgeries and isoperiodic deformations

In this section, we recall a few surgeries we shall use in the sequel. These all basically consist of cutting a given translation surfaces along one, or possibly more, geodesic segment(s) and gluing them back along those segments in order to get new translation structures. Different gluings will provide different translation structures.

3.1. Breaking a zero

The surgery we are going to describe has been introduced by Eskin–Masur–Zorich in [EMZ03, Section 8.1] and it literally 'breaks up' a zero in two, or possibly more, zeros of lower orders. Complex-analytically this can be thought as the analogue to the classical Schiffer variations for Riemann surfaces; see [Nag85]. This surgery only modifies a translation surface on a contractible neighborhood of the initial zero. In particular, after the surgery the resulting surface has the same genus as the former one but the type of zero orders is changed. Moreover, the new translation surface we obtain after the surgery has the same period character as the original one. As a consequence, this operation produces small deformation of the original translation structure in the same isoperiodic fiber.

Remark 3.1. In the context of branched projective structures, such a surgery is also known as *movement of branched points* and it has been originally introduced by Tan in [Tan94, Chapter 6] for showing the existence of a complex one-dimensional continuous family of deformations of a given structure.

Let us now explain this surgery in more detail. Let (X, ω) be a translation surface possibly with poles. Breaking a zero is a procedure that takes place at the ε-neighbourhood of some zero of order m of the differential on which it looks like the pull-back of the form dz via a branched covering $z \mapsto z^{m+1}$. The differential is then modified by a surgery inside this ε-neighbourhood. Once this surgery is performed, we obtain a new translation structure with two zeros of order m_1 and m_2 such that $m_1 + m_2 = m$. Furthermore, the translation structure remains unchanged outside the ε-neighbourhood of such a zero of order m. The idea is to consider the ε-neighbourhood of a zero of order m as $m + 1$ copies of a disc D of radius ε whose diameters are identified in a specified way. We can see this family of discs as a collection of $m + 1$ upper half-discs and $m + 1$ lower half-discs. See figure 2.

We now break a zero of order m into two zeros of order m_1 and m_2. To break a zero consists of identifying the diameters of the starting $m + 1$ discs in a different way as follows. In order to do this, we modify the labelling on the upper half-disc indexed by 0, the lower half-disc indexed by m_1, and all upper and lower half-discs with index more than m_1 accordingly. The modified labelling is shown below in Figure 3 for the case of splitting a zero of order 2 into two zeros of order 1.

We now identify ul_i with ll_i and lr_i with ur_{i+1} as before with the added identification of um with lm. This identification gives two zeros A and B, where A is a zero of the differential of order k_1 and B is a zero of order k_2. We also get a geodesic line segment joining A and B. Given $c \in \mathbb{C} \setminus \{0\}$ with length less than 2ε, we can perform the surgery in such a way that the line segment joining A and B is c. It is also clear that such a deformation of the translation structure is only local. This procedure can be

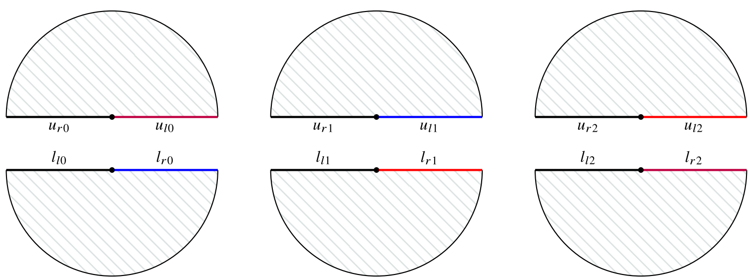

Figure 2. *An ε-neighbourhood of a zero of order* 2.

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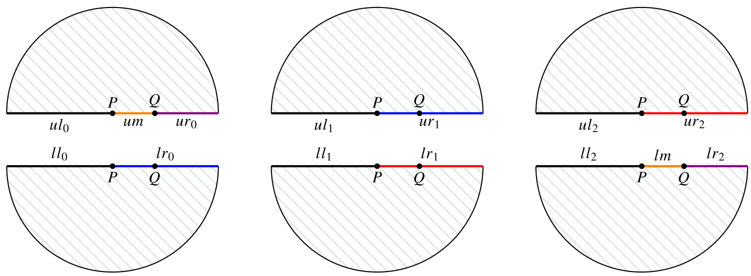

Figure 3. *New labelling for breaking up a zero of order* 2 *in two zeros of order* 1.

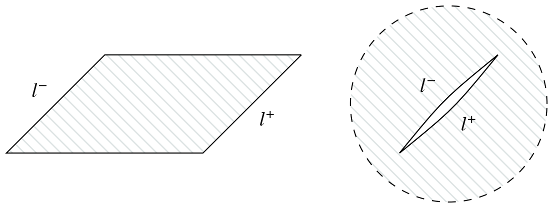

Figure 4. *Bubbling a handle with positive volume.*

repeated multiple times to obtain zeros of orders $m_1, \ldots, m_k$ from a single zero of order $m_1 + \cdots + m_k$. We shall frequently rely on this procedure of breaking a zero in the remaining part of this paper. The following lemmas are easy to establish.

Lemma 3.2. *Let (X, ω) be a genus-one differential with rotation number r. Let (Y, ξ) be a genus-one meromorphic differential in the stratum $\mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ which is obtained from (X, ω) by breaking a zero. If r divides* $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n)$*, then (Y, ξ) has rotation number r.*

Lemma 3.3. *Breaking a zero does not alter the spin structure of a genus-g differential for $g \geq 2$.*

Lemma 3.4. *Let $(X, \omega) \in \mathcal{H}_g(2m; -\nu)$ be a hyperelliptic translation surface, where $\nu = \{2m - 2g + 2\}$, or $\{m - g + 1,\ m - g + 1\}$, and let (Y, ξ) be the structure in $\mathcal{H}_g(m, m; -\nu)$ obtained from (X, ω) by breaking a zero. Then (Y, ξ) is hyperelliptic.*

3.2. Bubbling a handle with positive volume

We now describe a second surgery introduced by Kontsevich and Zorich in [KZ03]. In their paper, this surgery is defined for holomorphic differentials, but, as already observed by Boissy in [Boi15], the same surgery can be defined for meromorphic differentials because this is a local surgery. Topologically, *bubbling a handle with positive volume* consists of adding a handle, say Σ to a given surface. Let us see how this can be done metrically. Let $(X, \omega) \in \mathcal{H}_g(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ be a translation surface, possibly with finite order poles. Let $l \subset (X, \omega)$ be a geodesic segment with distinct end points.

We slit the translation surface (X, ω) along l, and we label the side that has the surface on its left as l^+ and the other side as l^-. We then paste the extremal points of the geodesic segment, and the resulting surface has two geodesic boundary components having (by construction) the same length. Let $\mathcal{P} \subset \mathbb{C}$ be a parallelogram such that the two opposite sides are both parallel to l (via the developing map) – the handle Σ is obtained by gluing the opposite sides of $\mathcal{P}$. We can paste such a parallelogram to the slit (X, ω), and the resulting topological surface is homeomorphic to $S_{g+1,n}$. Metrically, we have a new translation surface (Y, ξ). We can distinguish three mutually disjoint possibilities:

1. Both of the extremal points of $l \subset (X, \omega)$ are regular, then $(Y, \xi) \in \mathcal{H}_g(2, m_1, \ldots, m_k; -p_1, \ldots, -p_n)$,

2. One of the extremal points of $l \subset (X, \omega)$ is a zero of ω of order m_i, where $1 \leq i \leq k$, then (Y, ξ) belongs to the stratum $\mathcal{H}_g(m_1, \dots, m_i + 2, \dots, m_k; -p_1, \dots, -p_n)$,
3. Both of the extremal points are zeros of ω of orders m_i, m_j, respectively, then (Y, ξ) belongs to the stratum $\mathcal{H}_g(m_1, \dots, \widehat{m}_i, \dots, \widehat{m}_j, \dots, m_i + m_j + 2, \dots, m_k; -p_1, \dots, -p_n)$.

The following holds.

Lemma 3.5. *Let (X, ω) be a translation surface obtained from $(Y, \xi) \in \mathcal{H}_g(2m_1, \dots, 2m_k; -2p_1, \dots, -2p_n)$ by bubbling a handle along a slit. Let $2\pi(l+1)$ be the angle around one of the extremal points of the slit, where $l = 2m_i \geq 2$ for some $i = 1, \dots, k$ or $l = 0$. Then, the spin structures determined by ω and ξ are related as follows*

$$\varphi(\omega) - \varphi(\xi) = l \pmod 2. \tag{26}$$

In particular, bubbling a handle does not alter the spin parity.

Remark 3.6. It is worth noticing that this lemma differs from [KZ03, Lemma 11] because definition of bubbling is different in principle. More precisely, in the present paper a bubbling is performed along a geodesic segment that joins two points, possibly regular. In [KZ03], however, the bubbling of a handle is performed along a saddle connection that joins two zeros obtained after breaking a zero. This preliminary operation is the reason of a possible alteration of the spin parity because, by breaking a zero, the resulting structure may no longer be of even type. More precisely, if we break a zeros of ω on a structure of even type, the resulting zeros both have even order or odd order. In the former case, bubbling a handle does not alter the parity but in the latter case it does. [KZ03, Lemma 11] takes into account this possibility. Here, we do not break any zero for bubbling a handle, and hence, the spin parity remains unaltered.

Proof of Lemma 3.5. By keeping in mind Remark 3.6, the result follows as in [KZ03, Lemma 11]. □

3.3. Bubbling a handle with nonpositive volume

We have seen above how to glue a handle with positive volume on a given translation surface (X, ω). Here, we briefly describe a way to glue a handle with nonpositive volume and nontrivial periods; we shall describe a way to add handles with trivial periods in subsection §8.1. Topologically, this surgery deletes the interior of a parallelogram with distinct vertices on (X, ω) whose sides are given by the absolute periods of the handle we want to glue.

This includes the case in which the parallelogram is degenerate, that is, with empty interior, and the surgery reduces to slit along a segment. Since this construction is a local surgery, we explain how the gluing works in the complex plane because the same argument applies to any simply connected open neighborhood of (X, ω). Let $a, b \in \mathbb{C}$ be real-collinear complex numbers with argument θ. Let l be a segment in $\mathbb{C}$ with slope θ and length equal to $|a| + |b|$. Denote by P_1 and P_3 the end points of l. Slit $\mathbb{C}$ along l, and denote the resulting sides $l^{\pm}$. More precisely, let l^+ denote the side of l leaving $\mathbb{C}$ on the right, and let l^- denote the side of l leaving $\mathbb{C}$ on the left. On l^+, let P_2 be the point at distance $|a|$ from P_1. In the same fashion, on l^- let P_4 be the point at distance $|a|$ from P_3. We next identify the oriented segment $\overline{P_1 P_2}$ with the oriented segment $\overline{P_4 P_3}$. Finally, identify the oriented segment $\overline{P_2 P_3}$ with the oriented segment $\overline{P_1 P_4}$. The result is a handle with absolute periods a, $b \in \mathbb{C}$ and, by construction, it has volume zero; see volume formula (14) and Figure 5.

We shall need the following lemmas.

Lemma 3.7. *If (X, ω) is a translation surface of even type and (Y, ξ) is a translation surface obtained by bubbling a handle with nonpositive volume, the (Y, ξ) is of even type.*

Proof. Let (X, ω) be a translation surface of even type, and let $\mathcal{P} \subset (X, \omega)$ be an embedded parallelogram. Since (X, ω) is a structure of even type, all vertices of $\mathcal{P}$ are either regular points or zeros of even order. Notice that a point of even order is of the form $(4m+2)\pi$, with $m = 0$ for a regular point. Remove the interior of $\mathcal{P}$, identify the opposite edges and let (Y, ξ) be the resulting structure. The vertices of

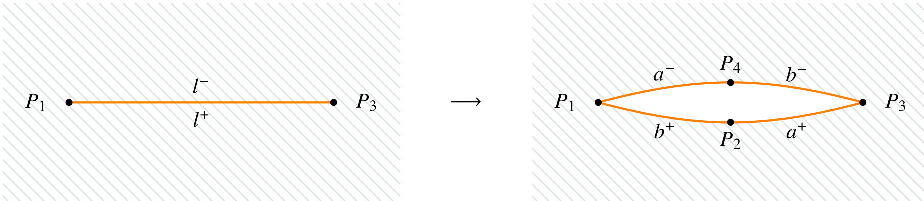

Figure 5. *Bubbling a handle with zero volume.*

$\mathcal{P}$ are identified to a nonregular point, namely a zero of ξ whose angle is $(4n+2)\pi$ for some n. Since the other zeros and all poles are untouched by this surgery, it follows that (Y, ξ) is a structure of even type. □

Lemma 3.8. *If (X, ω) is a translation surface of even type and (Y, ξ) is a translation surface obtained by bubbling a handle with nonpositive volume, then*

$$\varphi(\omega) - \varphi(\xi) = 0 \pmod 2. \tag{27}$$

In particular, bubbling a handle with nonpositive volume does not alter the spin parity.

Proof. Let (X, ω) be a translation surface, and let $\mathcal{P}$ be an embedded parallelogram. Choose a collection of $2g$ oriented simple curves $\{\alpha_i, \beta_i\}_{1\le i\le g}$ on (X, ω) representing a symplectic basis for $\mathrm{H}_1(S_{g,n}, \mathbb{Z})$. Up to deforming the paths inside their isotopy classes, we can make them stay away from some neighborhood, say U, of $\mathcal{P}$. In other words, we can assume that bubbling a handle with nonpositive volume can be performed inside a neighborhood U. In particular, the surgery does not affect the initial collection of paths. We now complete the former collection of curves by adding two simple curves, say α_{g+1}, β_{g+1} obtained after bubbling. These curves can be taken so that they both lie inside the neighborhood U. Since (X, ω) is a structure of even type, the vertices of $\mathcal{P}$ all have even orders. As a consequence, $\mathrm{Ind}(\alpha_{g+1})$ and $\mathrm{Ind}(\beta_{g+1})$ are odd positive integers; see Figure 6.

A direct computation shows that

$$\begin{aligned}
\varphi(\xi) &= \sum_{i=1}^{g+1} \big(\mathrm{Ind}(\alpha_i)+1\big)\big(\mathrm{Ind}(\beta_i)+1\big) \pmod 2\\
&= \sum_{i=1}^{g} \big(\mathrm{Ind}(\alpha_i)+1\big)\big(\mathrm{Ind}(\beta_i)+1\big) + \big(\mathrm{Ind}(\alpha_{g+1})+1\big)\big(\mathrm{Ind}(\beta_{g+1})+1\big) \pmod 2\\
&= \varphi(\omega) + \big(\mathrm{Ind}(\alpha_{g+1})+1\big)\big(\mathrm{Ind}(\beta_{g+1})+1\big) \pmod 2\\
&= \varphi(\omega) \pmod 2
\end{aligned}$$

as desired. □

3.4. Gluing surfaces along rays

In what follows, we shall need to glue translation surfaces along infinite rays. Recall, in fact, that our strategy to prove Theorem A is based on constructing translation surfaces with prescribed data (periods and geometric invariants) by gluing surfaces of lower complexity. We begin by introducing the following.

Definition 3.9 (Gluing translation surfaces along geodesic rays). Let (X_1, ω_1) and (X_2, ω_2) be two translation surfaces, each with at least one pole. Let $r_i \subset (X_i, \omega_i)$, for $i = 1, 2$, be an embedded geodesic ray that starts from a zero of order m_i for ω_i or a regular point (in this case $m_i = 0$) and ends in a pole of order p_i. Moreover, assume that both r_1 and r_2 develop onto *parallel* infinite rays in $\mathbb{C}$. Then we can define a translation surface (Y, ξ) as follows: Slit each ray r_i, and denote the resulting sides by

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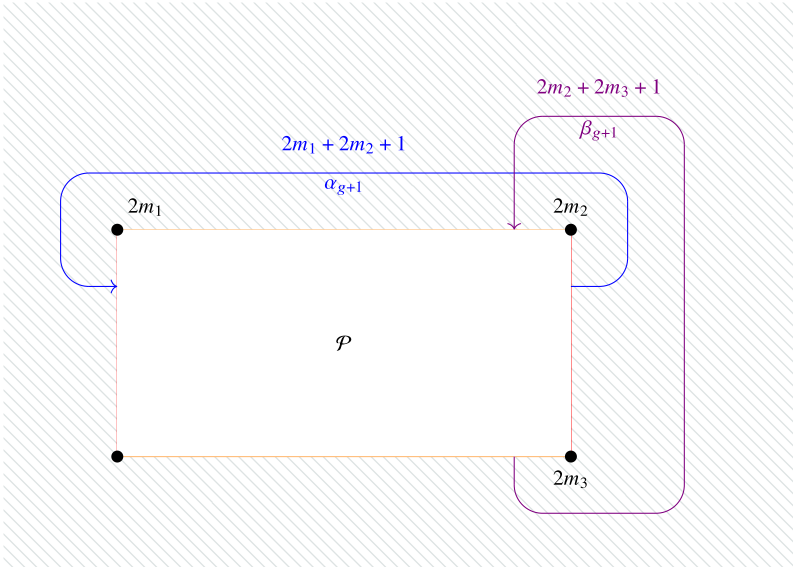

Figure 6. *An embedded parallelogram $\mathcal{P}$ on a translation surface of even type. The vertices of $\mathcal{P}$ are points of orders $2m_i$, for $i = 1, \ldots, 4$ and $m_i = 0$ for regular points. A unit vector field along α_{g+1} winds $2m_1 + 2m_2 + 1$ times and a unit vector field along β_{g+1} winds $2m_2 + 2m_3 + 1$ times. According to Definition 2.12, the indices of α_{g+1}, β_{g+1} are equal to their winding numbers – notice that both curves turn counterclockwise.*

r_i^+ and r_i^-; then identify r_1^+ with r_2^- and r_1^- and r_2^+ by a translation. If the surface X_i is homeomorphic to $S_{g_i,\, k_i}$ for $i = 1, 2$, then the resulting surface Y is homeomorphic to $S_{g_1+g_2,\, k_1+k_2-1}$. The starting points of r_1 and r_2 are identified to a zero point in the resulting translation surface (Y, ξ) with order equal to $m_1 + m_2 + 1$, and the other end points at infinity are identified to a pole of order $p_1 + p_2 - 1$.

Definition 3.10 (Gluing translation surfaces along a bi-infinite rays). Let (X_1, ω_1) and (X_2, ω_2) be two translation surfaces, each with at least one pole. Let $r_i \subset (X_i, \omega_i)$, for $i = 1, 2$, be an embedded bi-infinite geodesic that starts from a pole P_i^1 of order p_i^1 for ω_i and ends in another pole P_i^2 of order p_i^2. Assume that both r_1 and r_2 develop onto *parallel* infinite rays in $\mathbb{C}$. Then we can define a translation surface (Y, ξ) as follows: Slit each ray r_i, and denote the resulting sides by r_i^+ and r_i^-; then identify r_1^+ with r_2^- and r_1^- and r_2^+ by a translation. If the surface X_i is homeomorphic to $S_{g_i,\, k_i}$ for $i = 1, 2$, then the resulting surface Y is homeomorphic to $S_{g_1+g_2,\, k_1+k_2-2}$. Moreover, P_1^1 and P_2^1 are identified to a pole of order $p_1^1 + p_2^1 - 1$, and similarly, P_1^2 and P_2^2 are identified to a pole of order $p_1^2 + p_2^2 - 1$.

Remark 3.11. In both constructions above, when two poles are identified then the corresponding residues sum up.

As special cases, we have the following gluings.

Definition 3.12 (Bubbling a plane). Let (X, ω) be a translation surface with poles, and let $r \subset (X, \omega)$ be an infinite ray starting from a regular point or a zero of ω. Let $\overline{r} \subset \mathbb{C}$ be the developing image of r. By *bubbling a plane* along r, we mean the gluing of (X, ω) and $(\mathbb{C}, dz)$ along r and $\overline{r}$ as described in Definition 3.9. By repeating this surgery, m times is equivalent to glue (X, ω) to $(\mathbb{C}, z^{m-1}dz)$ along parallel rays as described in Definition 3.9.

We can note that in the constructions above at most one new branch point is introduced arising from the extremal points of the rays after identification. Furthermore, bubbling a plane does not change the topology of the underlying surface; in particular, the resulting structure after bubbling has the same periods as the former one. Another special case worth of interest for us is the following.

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Definition 3.13 (Gluing a cylinder along a bi-infinite ray). Let (X, ω) be a translation surface with at least two poles. Let $r_1 \subset (X, \omega)$ be an embedded bi-infinite geodesic joining two distinct poles, and let $v \in \mathbb{S}^1 \subset \mathbb{C}$ be the direction of r once developed via the developing map. Let $C = \mathbb{E}^2/\langle z \mapsto z + w\rangle$ be an infinite cylinder with holonomy $w \in \mathbb{C}$. If v and w are not parallel, then there exists an infinite geodesic line $r_2 \subset C$ with direction v after developing. We can glue C to (X, ω) as follows: Slit (X, ω) along r_1, and denote the resulting sides by r_1^+ and r_1^-. In a similar fashion, slit C along r_2, and denote the resulting sides by r_2^+ and r_2^-. Let (Y, ξ) be the translation surface obtained by gluing back r_1^+ with r_2^- and r_1^- with r_2^+. We shall say that (Y, ξ) is obtained by *gluing a cylinder to* (X, ω) *along a bi-infinite ray*. Moreover, the residues of the resulting poles are given by the original residues ± the residues of the simple poles of the infinite cylinder.

Remark 3.14. Let $r \subset (X, \omega)$ be a geodesic ray leaving from any point of X to a pole of ω with slope θ. In principle, r may hit a zero of ω, say P of order m. Then there are $m + 1$ possible ways to extend r at P so that its slope after developing is still θ. In this case, we shall impose that the ray leaves P with angle exactly π on the left or on the right.

The following results hold.

Lemma 3.15 (Invariance of the rotation number). *Let (X, ω) be a genus-one differential with at least two poles and rotation number k. Suppose there is a bi-infinite geodesic ray, say r, joining two poles. Then gluing a cylinder along r as in Definition 3.13 does not alter the rotation number.*

Lemma 3.16 (Invariance of the spin parity). *Let (X, ω) be a translation surface with poles of even type, and let $\varphi(\omega)$ be the parity of ω. Let (Y, ξ) be the translation surface obtained by bubbling $(\mathbb{C}, z^{2m-1}dz)$, for $m \geq 1$, along a ray as in Definition 3.9. Then $\varphi(\omega) = \varphi(\xi)$.*

The proofs of these lemmas are easy to establish and left to the reader.

4. Finding good systems of generators: actions on the representation space

For a given representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$, our proof of Theorems A, which we will develop from Section §5 to Section §7, relies on a direct construction of genus-g differentials with prescribed invariants such as the rotation number for genus-one differentials and the spin parity or the hyperellipticity for higher genus differentials. In order to perform our constructions, we need to consider a judicious *system of handle generators* which is defined as follows.

Definition 4.1 (Handle, handle-generators). On a surface $S_{g,n}$ of some positive genus $g > 0$, a *handle* is an embedded subsurface Σ that is homeomorphic to $S_{1,1}$, and a *handle-generator* is a simple closed curve that is one of the generators of $\mathrm{H}_1(\Sigma, \mathbb{Z}) \cong \mathrm{H}_1(S_{1,1}, \mathbb{Z})$. A *pair of handle-generators* for a handle will refer to a pair of simple closed curves $\{\alpha, \beta\}$ that generate $\mathrm{H}_1(\Sigma, \mathbb{Z})$; in particular, α and β intersect once.

Definition 4.2 (System of handle generators). On a surface $S_{g,n}$ of some positive genus $g > 0$, we consider a collection of pairwise disjoint g handles $\Sigma_1, \ldots, \Sigma_g$. A *system of handle generators* is a collection of g pairs of handle generators $\{\alpha_i, \beta_i\}_{1\leq i\leq g}$ such that $\{\alpha_i, \beta_i\}$ is a pair of handle generators for Σ_i.

We can immediately notice that every system of handle generators yields a splitting, namely a simple closed separating curve γ homotopic to the product of commutators $[\alpha_i, \beta_i]$. In fact, $S_{g,n}$ splits along γ as the connected sum of a closed genus g surface S_g and a punctured sphere $S_{0,n}$. Conversely, once a splitting is defined, any representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \to \mathbb{C}$ gives rise to a representation χ_g and a representation χ_n as defined in Section §2.1. Recall that χ_n is always well defined as it does not depend on the splitting whereas the representation χ_g does. In particular, χ_g is uniquely determined if and only if χ is of trivial-ends type; see Remark 2.9.

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4.1. Mapping class group action

In order to realize a representation χ as the period character of some translation surface with poles, in what follows it will be convenient to replace, if necessary, a given system of handle generators with a more suitable one. This replacement can be done by precomposing χ with an automorphism induced by a mapping class transformation, that is, an element of the mapping class group $\mathrm{Mod}(S_{g,n})$, where

$$\mathrm{Mod}(S_{g,n}) = \frac{\mathrm{Aut}\big(\pi_1(S_{g,n})\big)}{\mathrm{Inn}\big(\pi_1(S_{g,n})\big)} \cong \frac{\mathrm{Homeo}^+(S_{g,n})}{\mathrm{Homeo}_o(S_{g,n})} = \pi_o\big(\mathrm{Homeo}(\,S_{g,n})\,\big). \tag{28}$$

Remark 4.3. In principle, an element $\phi \in \mathrm{Mod}(S_{g,n})$ may permute punctures; see [FM12, Chapter §2]. In what follows, however, we shall focus on the action of elements that keep the punctures fixed and that may alter the splitting; see Sections §1.3 and §2.1.

Thus, to prove our main Theorem A for a given χ, it is sufficient to construct a genus g meromorphic differential on $S_{g,n}$ for which the induced representation is $\chi \circ \phi$ for some $\phi \in \mathrm{Mod}(S_{g,n})$. It is worth mentioning that this replacement is legitimized by the following:

$$\chi \in \mathrm{Per}\big(\mathcal{H}_g(m_1,\ldots,m_k;-p_1,\ldots,-p_n)\big) \iff \chi \circ \phi \in \mathrm{Per}\big(\mathcal{H}_g(m_1,\ldots,m_k;-p_1,\ldots,-p_n)\big) \tag{29}$$

for any $\phi \in \mathrm{Mod}(S_{g,n})$, and Per is the period mapping defined in Equation (7). In fact, if ϕ is induced by a homeomorphism, say f, then the pull-back of the translation structure by f defines a new translation surface with poles in the same stratum and with period character $\chi \circ \phi$; see also [LF22, Lemma 3.1] for the holomorphic case.

4.2. Nontrivial systems of generators exist

We aim to list a few lemmas about the mapping class group action for finding systems of handle generators such that any element has a nontrivial period. Most of them have already been proved in [CFG22, Section §11], and here, we report a sketch of the proof for the reader's convenience. The following lemmas show that the image of every handle generator under χ can be assumed to be nonzero whenever χ is a nontrivial representation.

Lemma 4.4. *Let $\chi \in Hom\big(\mathrm{H}_1(S_{g,n},\mathbb{Z}),\mathbb{C}\big)$ be a representation, and let $\{\alpha_i,\beta_i\}_{1\le i\le g}$ be a system of handle generators. Suppose that the corresponding χ_g, determined by the induced splitting, is not trivial. Then, there exists $\phi \in Mod(S_{g,n})$ such that $\chi\circ\phi(\alpha_i)$ and $\chi\circ\phi(\beta_i)$ are nonzero for all $1 \le i \le g$.*

Proof. Let χ be a nontrivial representation, and let $\{\alpha_i,\beta_i\}_{1\le i\le g}$ be a system of handle generators. We can assume that $\chi(\alpha_1) \neq 0$. If $\chi(\beta_1) = 0$, then replace β_1 with $\alpha_1\beta_1$ and observe that $\chi(\alpha_1\beta_1) \neq 0$. Suppose there is an index i such that $\chi(\alpha_i) = \chi(\beta_i) = 0$. We consider a mapping class ϕ such that

$$\phi(\alpha_1) = \alpha_1\alpha_i,\ \ \phi(\beta_1) = \beta_1,\ \ \phi(\alpha_i) = \alpha_i,\ \ \phi(\beta_i) = \beta_1\beta_i^{-1}, \tag{30}$$

and ϕ is the identity on the other handle generators. Finally, we replace the generator α_i with $\alpha_i\beta_1\beta_i^{-1}$. Observe that $\{\alpha_i\,\beta_1\beta_i^{-1},\,\beta_1\beta_i^{-1}\}$ is a pair of handle generators and $\chi(\alpha_i\,\beta_1\beta_i^{-1}) = \chi(\beta_1\beta_i^{-1}) \neq 0$ by construction. By iterating this process finitely many times, we get the desired result. □

Furthermore, for a representation χ we can also assume that the χ_g-part induced by a splitting is nontrivial whenever the χ_n-part is nontrivial.

Lemma 4.5. *Let $\chi \in Hom\big(\mathrm{H}_1(S_{g,n},\mathbb{Z}),\mathbb{C}\big)$ be a representation of nontrivial-ends type. Then, there exists a splitting of $S_{g,n}$ with respect to which χ_g is nontrivial.*

Remark 4.6. The claim of Lemma 4.5 is equivalent to the existence of a mapping class $\phi \in \mathrm{Mod}(S_{g,n})$ such that $(\chi\circ\phi)_g$ is nontrivial, where $(\chi\circ\phi)_g$ is the g-part of the representation $\chi\circ\phi$.

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Proof. Let χ be a nontrivial-ends type representation, and let $\{\alpha_i, \beta_i\}_{1\le i\le g}$ be a system of handle generators. Assume $\chi(\alpha_i) = \chi(\beta_i) = 0$ for all $1 \le i \le g$. Let γ_i be a small loop around a puncture such that $\chi(\gamma_i) \neq 0$. Then we can find some handle generator α_j such that we have two curves α_j and α'_j satisfying $\alpha_j\alpha'_j = \gamma_i$ in $\mathrm{H}_1(S_{g,n}, \mathbb{Z})$. This implies that $\chi(\alpha'_j) \neq 0$. Thus, we can take ϕ to be an element of $\mathrm{Mod}(S_{g,n})$, commonly known as a *push transformation*; see [FM12, Section §4.2.1], which takes the generator α_j to α'_j and leaves the other handle generators unchanged. □

4.3. $\mathrm{GL}^+(2,\mathbb{R})$-*action*

We now consider an action on the representation space given by postcomposition with elements of $\mathrm{GL}^+(2,\mathbb{R})$. This action can be combined with the mapping class group action (see Section §4.1 above), and it is easy to check that these actions commute. In the sequel, it will be sometimes useful to consider this action in order to realize representations as the period of meromorphic differentials in a given connected component of a stratum. This is in fact legitimated by the fact

$$\chi \in \mathrm{Per}\big(\mathcal{H}_g(m_1,\dots,m_k;-p_1,\dots,-p_n)\big) \iff A\,\chi \in \mathrm{Per}\big(\mathcal{H}_g(m_1,\dots,m_k;-p_1,\dots,-p_n)\big) \tag{31}$$

for any $A \in \mathrm{GL}^+(2,\mathbb{R})$, and Per is the period mapping defined in Equation (7).

Remark 4.7. Since $\mathrm{GL}^+(2,\mathbb{R})$ acts continuously on every stratum, it follows that the connected components, if any, are preserved under the action.

4.4. *System of handle generators with positive volume*

Notice that a mapping class $\phi \in \mathrm{Mod}(S_{g,n})$ does not need to preserve any splitting in general. Let us now consider again the notion of volume for meromorphic differentials already introduced in Section §2.1. For a representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$, the volume of χ (see Definition 2.10) is well defined if and only if it is of trivial-ends type. In other words, for any representation χ of trivial-ends type and for any mapping class $\phi \in \mathrm{Mod}(S_{g,n})$ the equation

$$\mathrm{vol}(\chi_g) = \mathrm{vol}\big((\chi\circ\phi)_g\big), \tag{32}$$

where χ_g and $(\chi\circ\phi)_g$ are the representations induced by the systems of handle generators $\{\alpha_i, \beta_i\}_{1\le i\le g}$ and $\{\phi(\alpha_i), \phi(\beta_i)\}_{1\le i\le g}$, respectively. For a representation χ of nontrivial-ends type, the volume is no longer well defined, and hence, it is no longer an invariant because the representation χ_g does depend on the splitting and therefore also $\mathrm{vol}(\chi_g)$ depends on it. We shall take advantage of this caveat to prove the following result which will be exploited in the sequel. We begin with introducing the following.

Definition 4.8. A representation χ is called *real-collinear* if the image $\mathrm{Im}(\chi)$ is contained in the $\mathbb{R}$-span of some $c \in \mathbb{C}^*$. Equivalently, χ is real-collinear if, up to replacing χ with $A\,\chi$ where $A \in \mathrm{GL}^+(2,\mathbb{R})$ if necessary, then $\mathrm{Im}(\chi) \subset \mathbb{R}$.

Lemma 4.9. *Let $\chi \in Hom\big(\mathrm{H}_1(S_{g,n}, \mathbb{Z}), \mathbb{C}\big)$ be a representation of nontrivial-ends type. If χ is not real-collinear, then there exists a system of handle generators $\{\alpha_i, \beta_i\}_{1\le i\le g}$ such that* $\mathrm{vol}(\chi_g) > 0$.

Proof. Let χ be a representation of nontrivial-ends type, and let $\{\alpha_i, \beta_i\}_{1\le i\le g}$ be a system of handle generators. From Section §2.1, recall that the curve $\gamma = [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g]$ bounds a subsurface Σ homeomorphic to $S_{g,1}$, the embedding $i\colon \Sigma \hookrightarrow S_{g,n}$ yields an injection $\iota_g\colon \mathrm{H}_1(\Sigma, \mathbb{Z}) \cong \mathrm{H}_1(S_g,\mathbb{Z}) \to \mathrm{H}_1(S_{g,n}, \mathbb{Z})$, and the representation χ_g is defined as $\chi\circ\iota_g$. In the case $\mathrm{vol}(\chi_g) > 0$, there is nothing to prove and we are done, so let us suppose that $\mathrm{vol}(\chi_g) \le 0$. Being χ of nontrivial-ends type, there is a puncture, say p, with nonzero residue and let γ be a simple closed curve around p. Clearly, $\chi(\gamma) = \chi_n(\gamma) \neq 0$. Since χ is not real-collinear, we choose a pair of handle generators $\{\alpha,\beta\}$ in the given system such that $\chi(\beta)$ is not collinear with $\chi(\gamma)$. Replace it with the pair of generators $\{\alpha', \beta'\}$,

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where $\beta' = \beta$ and $\alpha' = \alpha\,\beta^{-n}\,\gamma\,(\beta\,\gamma)^n$ for $n \in \mathbb{Z}$. This is a simple closed nonseparating, curve and we can easily compute that

$$\chi(\alpha\,\beta^{-n}\,\gamma\,(\beta\,\gamma)^n\,) = \chi(\alpha) + (n+1)\chi(\gamma). \tag{33}$$

Up to relabelling all handles, we can assume $\{\alpha, \beta\} = \{\alpha_1, \beta_1\}$. The new system of handle generators will be $\{\alpha', \beta', \alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$. Such a system determines a different splitting, namely a different separating curve γ' that bounds a subsurface Σ' homeomorphic to $S_{g,1}$. The embedding $i'\colon \Sigma' \hookrightarrow S_{g,n}$ yields a representation $\chi'_g = \chi \circ \iota'_g$, where ι'_g is the injection in homology induced by i'. The volume $\mathrm{vol}(\chi'_g)$ can be explicitly computed as follow:

$$\mathrm{vol}(\chi'_g) = \Im\Big(\overline{\chi(\alpha')}\,\chi(\beta')\Big) + \sum_{i=2}^{g}\Im\Big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\Big) \tag{34}$$

$$= \Im\Big(\overline{\chi(\alpha_1) + (n+1)\chi(\gamma)}\,\chi(\beta_1)\Big) + \sum_{i=2}^{g}\Im\Big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\Big) \tag{35}$$

$$= (n+1)\Im\Big(\overline{\chi(\gamma)}\,\chi(\beta_1)\Big) + \sum_{i=1}^{g}\Im\Big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\Big) \tag{36}$$

$$= (n+1)\Im\Big(\overline{\chi(\gamma)}\,\chi(\beta_1)\Big) + \mathrm{vol}(\chi_g). \tag{37}$$

Since $\mathrm{vol}(\chi_g)$ is constant and since $\chi(\gamma)$ and $\chi(\beta_1)$ are not collinear, by choosing $n \in \mathbb{Z}$ judiciously, we can make $\mathrm{vol}(\chi'_g) > 0$ and hence obtain the desired result. □

By combining Lemma 4.9 with Kapovich's results in [Kap20], we can infer the following result on which we shall rely in the sequel.

Corollary 4.10. *Let $\chi \in Hom\big(\mathrm{H}_1(S_{g,n}, \mathbb{Z}), \mathbb{C}\big)$ be a representation of nontrivial-ends type. If χ is not real-collinear, then there exists a system of handle generators $\mathcal{G} = \{\alpha_i, \beta_i\}_{1\le i\le g}$ such that $\Im\big(\overline{\chi(\alpha_i)}\,\chi(\beta_i)\big) > 0$ for any $i = 1, \dots, g$.*

4.5. Discrete and dense representations

In the sequel, we shall distinguish three kinds of representations according to the following.

Definition 4.11. A representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is said to be

- *discrete* if the image of χ is a discrete subgroup of $\mathbb{C}$. Furthermore, we say that χ is discrete of rank *one* if, up to replacing χ with $A\,\chi$, where $A \in \mathrm{GL}^+(2,\mathbb{R})$, then $\mathrm{Im}(\chi) = \mathbb{Z}$. We say that χ is discrete of rank *two* if, up to replacing χ with $A\,\chi$, where $A \in \mathrm{GL}^+(2,\mathbb{R})$, then $\mathrm{Im}(\chi) = \mathbb{Z} \oplus i\,\mathbb{Z}$.
- *semidiscrete* if, up to replacing χ with $A\,\chi$, where $A \in \mathrm{GL}^+(2,\mathbb{R})$, then $\mathrm{Im}(\chi) = U \oplus i\,\mathbb{Z}$, where U is dense in $\mathbb{R}$.
- *dense* if the image of χ is dense in $\mathbb{C}$.

We have the following lemmas.

Lemma 4.12. *Let χ be a discrete representation of nontrivial-ends type. After replacing χ with the representation $A\,\chi$ where $A \in \mathrm{GL}^+(2,\mathbb{R})$ if necessary, there exists a system of handle generators $\{\alpha_i, \beta_i\}_{1\le i\le g}$ such that*

1. *if χ is discrete of rank one then each handle generator has period one, that is, $\chi(\alpha_i) = \chi(\beta_i) = 1$ for all $i = 1, \dots, g$;*

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2. *if χ is discrete of rank two, then the handle generators satisfy the following conditions:*
   - $\chi(\alpha_g) \in \mathbb{Z}_+$ *and* $\chi(\beta_g) = i$,
   - $0 < \chi(\alpha_j) < \chi(\alpha_g)$ *and* $\chi(\alpha_j) = \chi(\beta_j) \in \mathbb{Z}_+$ *for all* $j = 1, \dots, g-1$.

Proof. Up to normalizing the representation with some $A \in \mathrm{GL}^+(2, \mathbb{R})$, we can assume $\mathrm{Im}(\chi) \subset \mathbb{Z} \oplus i\,\mathbb{Z}$. The first case follows from [CFG22, Lemma 12.2]. Let us consider the second case. By Lemma 4.9, there exists a system of handle generators $\{\alpha_i, \beta_i\}_{1 \le i \le g}$ such that $\mathrm{vol}(\chi_g) > 0$. Let us now consider the representation χ_g on its own right. Recall that we can regard this representation as $\chi_g \colon \mathrm{H}_1(S_g, \mathbb{Z}) \longrightarrow \mathbb{C}$. From [Kap20], there exists a system of handle generators such that

- $\chi(\alpha_g) \in \mathbb{Z}_+$ and $\chi(\beta_g) = i$,
- $0 < \chi(\alpha_j) < \chi(\alpha_g)$ and $\chi(\beta_j) = 0$ for all $j = 1, \dots, g-1$.

By replacing $\{\alpha_j, \beta_j\}$ with $\{\alpha_j, \alpha_j \beta_j\}$ for $j = 1, \dots, g-1$, we get the desired result. □

Lemma 4.13. *Let $\chi \in \mathrm{Hom}\big(\mathrm{H}_1(S_{g,n}, \mathbb{Z}), \mathbb{C}\big)$ be a real-collinear representation of nontrivial-ends type. Let $\{\alpha_i, \beta_i\}_{1 \le i \le g}$ be a system of handle generators. If χ is not discrete of rank one, there exists a mapping class $\phi \in \mathrm{Mod}(S_{g,n})$ such that $\chi \circ \phi(\alpha_i)$ and $\chi \circ \phi(\beta_i)$ are both positive and arbitrarily small.*

Proof. Let χ be a representation of nontrivial-ends type such that $\mathrm{Im}(\chi) \subset \mathbb{R}$. Recall that any system of handle generators $\{\alpha_i, \beta_i\}_{1 \le i \le g}$ induces a splitting and hence a representation χ_n. Notice that, since χ is not discrete of rank one, at least one generator of $\mathrm{H}_1(S_{g,n}, \mathbb{Z})$ has an irrational period. If $\mathrm{Im}(\chi_n) \subset \mathbb{Q}$, then the result follows from [CFG22, Lemma 11.5]; notice that this is always the case when $n = 2$ (up to rescaling). Let us now assume $\mathrm{Im}(\chi_n) \not\subset \mathbb{Q}$ and $n \ge 3$. Up to rescaling, we can assume one puncture, say P_1, has a rational residue, say r_1. Necessarily, there is a puncture P_2 with irrational residue r_2. We can assume that at least one handle generator has an irrational period, otherwise we can change the system of handle generators above by replacing α_1 with $\alpha_1\gamma_2$, where γ_2 is a loop around the puncture P_2. Notice that such a replacement can be performed with a mapping class transformation. Therefore, we can assume that α_1 has an irrational period. There exists $\phi_i \in \mathrm{Mod}(S_{g,n})$ such that $\chi \circ \phi_i(\alpha_i)$ is also irrational. This mapping class can be explicitly written as

$$\phi_i(\alpha_i) = \alpha_1\,\alpha_i, \quad \phi_i(\beta_1) = \beta_1\,\beta_i^{-1}, \quad \phi_i(\delta) = \delta \quad \text{for} \quad \delta \notin \{\alpha_i, \beta_1\}. \tag{38}$$

By composing all these ϕ_i's, the resulting mapping class ϕ provides a system of handle generators such that $\chi(\alpha_i)$ is irrational for any $i = 1, \dots, g$. We now proceed as follows. If $\chi(\alpha_i)$ and $\chi(\beta_i)$ are linearly independent over $\mathbb{Q}$, we can assume that both of them are positive after Dehn twists and we apply the Euclidean algorithm for making them arbitrarily small. In the case $\chi(\alpha_i)$ and $\chi(\beta_i)$ are linearly dependent over $\mathbb{Q}$ (hence $\chi(\beta_i)$ is also irrational), we replace β_i with $\beta_i\gamma_1$. Since $\chi(\gamma_1) \in \mathbb{Q}$, then $\chi(\alpha_i)$ and $\chi(\beta_i\,\gamma_1)$ are linearly independent over $\mathbb{Q}$. Again, we can assume that both of them are positive after Dehn twists and we apply the Euclidean algorithm for making them arbitrarily small. □

5. Meromorphic differentials of genus one

In this section, we want to realize a given nontrivial representation $\chi \colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as the period of some translation surface with poles in a prescribed stratum and with prescribed rotation number; see Definition 2.13. More precisely, we shall prove Theorem A for meromorphic differentials of genus one, namely we prove the following:

Proposition 5.1. *Let χ be a nontrivial representation, and suppose it arises as the period character of some meromorphic differential in a stratum $\mathcal{H}_1(m_1, \dots, m_k; -p_1, \dots, -p_n)$. Then χ can be realized as the period character of some translation surfaces with poles in each connected component of the same stratum.*

Remark 5.2. Recall that, for a stratum $\mathcal{H}_1(m_1, \dots, m_k; -p_1, \dots, -p_n)$, the possible values of the rotation number are given by the positive divisors of $\gcd\big(m_1, \dots, m_k; p_1, \dots, p_n\big)$ with the only exception

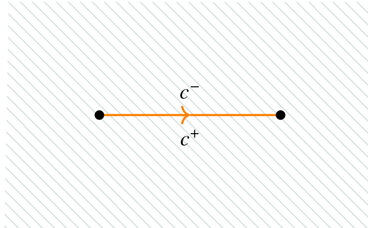

Figure 7. *Labels of a slit*

being the case $k = n = 1$; see Section §2.3 and [Boi15, Chapter 4] for more details. Notice that each stratum of meromorphic differentials in genus one with at least one simple zero or simple pole is connected as in this case the the greatest common divisor of the orders is one. In what follows, we only consider disconnected strata in genus one because for connected strata the problem of realizing period characters in a given stratum has been resolved in [CFG22].

In order to state our results, we introduce the following.

Convention and terminology. Recall that slitting a surface along an oriented geodesic segment s is a topological surgery for which the interior of s is replaced with two copies of itself. On the resulting surface, these two segments form a piecewise geodesic boundary with two corner points, corresponding to the extremal points of s, each of which can be a regular point or a branch point of angle $2(m+1)\pi$ for some $m \geq 1$. We shall denote by s^+ the piece of boundary which bounds the surface on its right with respect to the orientation induced by s. In a similar fashion, we denote by s^- the piece of boundary which bounds the surface on its left with respect to the orientation induced by s. See, for example, Figure 7 for the $\pm$ labeling convention. Sometimes, we shall omit the arrows as the direction is implicitly understood by the signs. In the following constructions, we need to slit and glue surfaces along geodesic segments. In order for this operation to be done, we need to glue along segments which are parallel after developing; see Section §3.4. For any $c \in \mathbb{C}^*$, by *slitting* $(\mathbb{C}, dz)$ *along* c we shall mean a cut along any geodesic segment s of length $|\,c\,|$ and direction equal to $\arg(c)$, that is,

$$\int_s dz = c. \tag{39}$$

In the sequel, we shall need to consider several slit copies of $(\mathbb{C}, dz)$, along different geodesic segments, we make use of an index i to specify on which copy we perform the slit. Finally, given a pair of handle generators α, β and a representation χ, we shall use the Latin letter a for any segment parallel and isometric to $\chi(\alpha)$ and the Latin letter b for any segment parallel and isometric to $\chi(\beta)$.

The proof of Theorem A for meromorphic differentials of genus one is divided as follows. In the next three Sections §5.1–§5.3, we prove the main theorem for strata of genus-one meromorphic differentials with exactly one zero of maximal order. In the final Section §5.4, we derive the most general case as a straightforward corollary of the other sections. The only missing case in the present section is the trivial representation that will be handled in Section §8. Since this a case-by-case proof with several subcases, Appendix §A contains a flow diagram of the proof for the reader's convenience; see Table 3.

5.1. Strata with poles of order two and zero residue

In this section, we shall prove the following lemma for the case where all poles have order exactly two and zero residue. We propose a proof in which the structures are explicitly constructed with prescribed periods and rotation number. As we shall see, all the other cases follow as simple variations of the constructions in this special case.

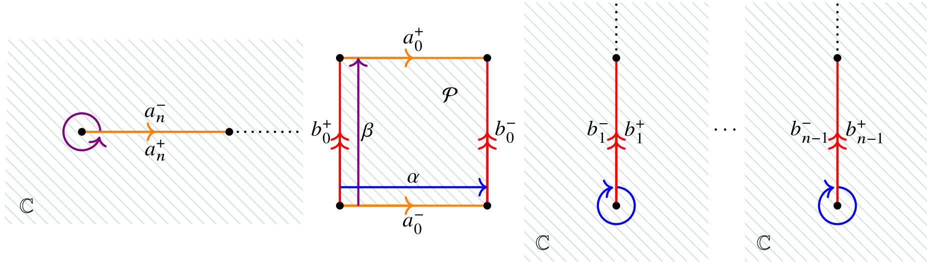

Figure 8. *Realizing a genus-one translation surface with poles of order* 2*, positive volume and rotation number equal to* 1*. In this case, all poles are assumed to have zero residue.*

Lemma 5.3. *Let* $\chi\colon \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ *be a nontrivial representation of trivial-ends type. If* χ *can be realized in the stratum* $\mathcal{H}_1(2n;-2,\dots,-2)$*, then it appears as the period character of a translation surface with poles in each connected component.*

Let $\chi\colon \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation, and assume χ can be realized as the period character of a genus-one meromorphic differential with poles in the stratum $\mathcal{H}_1(2n;-2,\dots,-2)$. Recall that any such a stratum has two connected components according to the rotation number with the only exception being the stratum $\mathcal{H}_1(2,-2)$ which is connected and hence already handled in [CFG22]. Therefore, we assume without loss of generality that $n \geq 2$.

Remark 5.4. Notice that, since χ is a nontrivial representation of trivial-ends type, then for any given pair of handle generators $\{\alpha,\beta\}$ at least one generator, say α, has nonzero absolute period, that is, $\chi(\alpha) \neq 0$. Lemma 4.4 applies, and hence, we can assume that both handle generators have nonzero absolute periods. In the present and next sections, we shall thus assume $\chi(\alpha) \neq 0$ and $\chi(\beta) \neq 0$.

5.1.1. Realizing representations with rotation number one

We begin with realizing χ as the period character of some genus-one meromorphic differential with rotation number *one*. We shall distinguish two cases according to the volume of the representation χ.

5.1.1.1. Positive volume

Let α,β be a pair of handle generators, and let $\mathcal{P} \subset \mathbb{C}$ be the parallelogram defined by the chain

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) = Q_0 \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{40}$$

where $P_0 \in \mathbb{C}$ is any point. According to our convention above, let us denote by a_0^+ (respectively a_0^-) the edge of $\mathcal{P}$ parallel to $\chi(\alpha)$ that bounds the parallelogram on its right (respectively left). In the same fashion, let us denote by b_0^+ (respectively b_0^-) the edge of $\mathcal{P}$ parallel to $\chi(\beta)$ that bounds the parallelogram on its right (resp. left). Recall that $(\mathbb{C},\, dz)$ is a genus-zero translation surface with trivial period character, no zeros and one pole of order 2 at the infinity with zero residue. We slit $(\mathbb{C},\, dz)$ along a segment a_n, and we denote the resulting segments as a_n^+ and a_n^-. Let P_n and $Q_n = P_n + \chi(\alpha)$ be the extremal points of a_n. We next consider other $n-1$ copies of $(\mathbb{C},\, dz)$, and we slit each of them along a segment b. For any $i = 1,\dots,n-1$, let P_i and $Q_i = P_i + \chi(\beta)$ be the extremal points of b_i and denote as b_i^+ and b_i^- the resulting segments after slitting. We are now ready to glue these n copies of $(\mathbb{C}, dz)$ and the parallelogram $\mathcal{P}$ all together. The desired structure is then obtained by identifying the segment a_0^+ with a_n^-, the segment a_0^- with a_n^+, the segment b_i^+ with b_{i+1}^-, where $i = 1,\dots,n-2$ and, finally, b_0^- with b_{n-1}^+ and b_0^+ with b_1^-; see Figure 8.

We will show that the resulting genus-one meromorphic differential, say (X,ω), has rotation number one. We first find a closed loop representing the desired handle generator α in the following way. For any $i = 1,\dots,n-1$, we consider a small metric circle of radius ε centered at P_i. After the cut and paste process just described, these $n-1$ circles define a smooth path with starting point on b_0^- and

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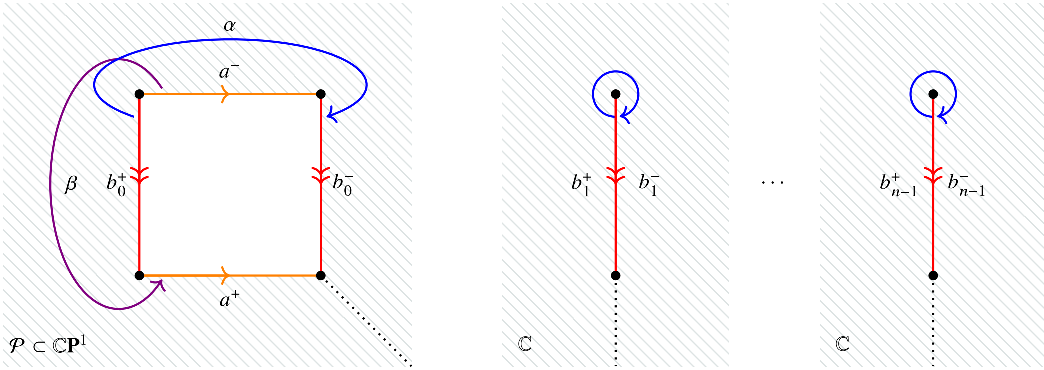

Figure 9. *Realizing a genus-one translation surface with poles of order* 2*, nonpositive volume and rotation number equal to* 1*. In this case, all poles are assumed to have zero residue.*

ending point on b_0^+. These two points on $\mathcal{P}$ differ by $\chi(\alpha)$, so we can join them with a geodesic segment parallel to a. The resulting curve is a simple close curve on (X, ω); see Figure 8. We can see that $\mathrm{Ind}(\alpha) = 1 - n$, with period equal to $\chi(\alpha)$. In a similar way, we can find a closed loop representing the other handle generator β. We consider a small metric circle of radius ε centered at P_n. After the cut and paste process, the extremal points of such an arc are on $a_0^\pm$ and differ by $\chi(\beta)$ on $\mathcal{P}$. We join them with a geodesic segment and the resulting curve is simple and closed in (X, ω). By construction, it has index one with period equal to $\chi(\beta)$. By checking their (self) intersection numbers, the two loops form a pair of handle generators with desired periods. Therefore, according to the formula (2.13) we have that $\gcd(\,1-n,\ 1,\ 2n,\ 2\,) = 1$, and hence, the structure has rotation number one.

5.1.1.2. Nonpositive volume

We now assume that χ has nonpositive volume. This case is similar to the case of positive volume. As above, let α, β be a set of handle generators, and let $\mathcal{P} \subset \mathbb{C}$ be the closure of the *exterior* of the parallelogram

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) = Q_0 \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{41}$$

where $P_0 \in \mathbb{C}$ is any point. Notice that $\mathcal{P}$ itself is a topological quadrilateral on $\mathbb{C}\mathbf{P}^1$. We denote the sides of $\mathcal{P}$ as $a^\pm$ and $b_0^\pm$ according to our convention. We next consider $n-1$ copies of $(\mathbb{C}, dz)$ and we slit each of them along a segment b. For any $i = 1, \ldots, n-1$, we denote as b_i^+ and b_i^- the resulting segments after slitting and let P_i and $Q_i = P_i + \chi(\beta)$ be the extremal points of b_i. The desired structure is then obtained by identifying the segment a^+ with a^-, the segment b_i^- with b_{i+1}^+ for $i = 1, \ldots, n-2$, the segment b_0^- with b_1^+ and the segment b_0^+ with b_{n-1}^-; see Figure 9. In this case, it is still easy to observe that the final structure (X, ω) has rotation number one. We can find a closed loop representing α similarly as before. For β, we can choose any segment in $\mathcal{P}$ parallel to b_0^+ with extremal points R and $R + \chi(\beta)$. We join R (respectively $R + \chi(\beta)$) with any point in a^- (respectively a^+) by means of an embedded arc. The resulting curve close up to a simple closed curve on (X, ω) and has index one. By checking intersection numbers the two loops form a pair of handle generators with desired periods. Therefore, the structure (X, ω) has rotation number one.

5.1.2. Realizing representations with rotation number two

We now realize a nontrivial representation χ as the period character of some genus-one meromorphic differential with rotation number *two*. Recall that poles are supposed to be of order 2 with zero residue. We distinguish four cases according to the volume of χ and the parity of n (the number of punctures).

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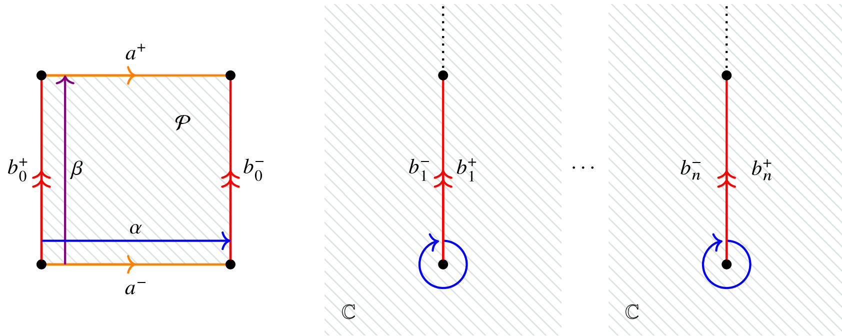

Figure 10. *Realizing a genus-one translation surface with poles of order 2, positive volume and rotation number equal to 2. In this case, there is an even number of punctures corresponding to poles with zero residue.*

All constructions are quite similar to those realized in Section §5.1.1, and in fact, they differ mainly in the way we glue copies of $(\mathbb{C}, dz)$.

5.1.2.1. Positive volume and even number of punctures
Let α, β be a set of handle generators, and let $\mathcal{P} \subset \mathbb{C}$ be the parallelogram defined by the chain

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) = Q_0 \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{42}$$

where $P_0 \in \mathbb{C}$ is any point. Let $a^{\pm}$ denote the edges of $\mathcal{P}$, parallel to $\chi(\alpha)$ that bounds the parallelogram on its right (respectively left). Similarly, let b_0^+ (respectively b_0^-) denote the edge of $\mathcal{P}$ parallel to $\chi(\beta)$ that bounds the parallelogram on its right (resp. left). We next consider n copies of $(\mathbb{C}, dz)$ and we slit each of them along a segment b. For any $i = 1, \ldots, n$, we denote as b_i^+ and b_i^- the resulting segments after slitting and let P_i and $Q_i = P_i + \chi(\beta)$ be the extremal points of b_i. We then glue these n copies of $(\mathbb{C}, dz)$ and the parallelogram $\mathcal{P}$ all together. The desired structure is then obtained by identifying the segment a^+ with a^-, the segment b_i^+ with b_{i+1}^- for $i = 1, \ldots, n-1$ and the segment b_0^- with b_{n-1}^+ and the segment b_0^+ with b_1^-; see Figure 10.

We can find closed loops representing α, β exactly as we have done above for the case discussed in paragraph §5.1.1.1. It is easy to check that $\mathrm{Ind}(\alpha) = -n$ by construction and $\mathrm{Ind}(\beta) = 0$ because β is geodesic. Since in this paragraph n is assumed to be even, we have that $\gcd(-n, 0, 2n, 2) = 2$ and hence (X, ω) has rotation number 2.

5.1.2.2. Positive volume and odd number of punctures
Let α, β be a set of handle generators. Following [Boi15, Section 3.3.2], we shall consider here two half-planes defined as follows. We first consider the broken line defined by the ray corresponding to $\mathbb{R}^-$, two edges corresponding to the chain of vectors $\chi(\alpha)$ and $\chi(\beta)$ in this order with starting point at the origin and, finally, a horizontal ray r_1 from the right end of $\chi(\beta)$ and parallel to $\mathbb{R}$. Define H_1 as the half-plane bounded on the left by this broken line. Similarly, define H_2 as the half-plane bounded on the right by the broken line defined by the ray corresponding to $\mathbb{R}^-$, two edges corresponding to the chain of vectors $\chi(\beta)$ and $\chi(\alpha)$ in this order with starting point at the origin and, finally, a horizontal ray r_2 from the right end of $\chi(\alpha)$ and parallel to $\mathbb{R}$; see Figure 11. We denote by a_0^-, b_0^- the edges of the boundary of H_1 corresponding to $\chi(\alpha)$ and $\chi(\beta)$, respectively. Similarly, we denote by a_0^+, b_0^+ the edges of the boundary of H_2 corresponding to $\chi(\alpha)$ and $\chi(\beta)$, respectively.

We consider a copy of $(\mathbb{C}, dz)$, and we slit it along a segment a, and we denote the resulting segments $a_n^{\pm}$. Next, we consider other $n-2$ copies of $(\mathbb{C}, dz)$, and we slit each of them along a segment b, and we denote the resulting segments $b_i^{\pm}$, for $i = 2, \ldots, n-1$. The sign $\pm$ are according to our convention. We now glue the edges in the usual way as above, and we also glue the ray r_1 with r_2. Notice that this

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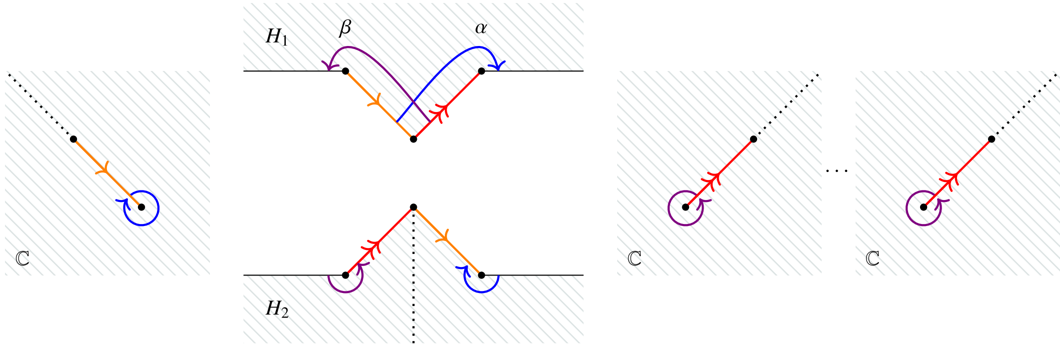

Figure 11. *Realizing a genus-one translation surface with poles of order* 2*, positive volume and rotation number equal to* 2*. In this case, there is an odd number of punctures corresponding to poles with zero residue.*

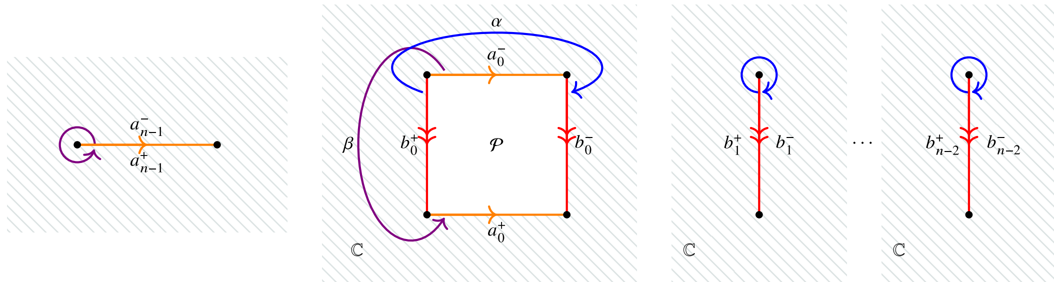

Figure 12. *Realizing a genus-one translation surface with poles of order* 2*, negative volume and rotation number equal to* 2*. In this case, there is an odd number of punctures corresponding to poles with zero residue.*

identification does not affect the set of periods since these two rays differ by a translation $\chi(\alpha)+\chi(\beta)$. The final surface is a genus-one meromorphic differential. It remains to show that it has rotation number 2. We can choose a pair of handle generators as shown in Figure 11. We can notice that the curve α has index -2 by construction whereas the curve β has index $n-1$. Observe that $n-1$ is even because n is assumed to be odd. Therefore, we can conclude that $\gcd(-2,\, n-1,\, 2n,\, -2) = 2$ as desired.

5.1.2.3. Nonpositive volume and odd number of punctures

This case is very similar to the case described in paragraph §5.1.1.2. Notice that $n \geq 3$ in the present case. Given a set of handle generators α, β, we define $\mathcal{P} \subset \mathbb{C}$ to be the closure of the *exterior* of the parallelogram defined by the chain

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) = Q_0 \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{43}$$

where $P_0 \in \mathbb{C}$ is any point and we label the edges $a_0^{\pm}$ and $b_0^{\pm}$ as usual. Consider $n-1$ copies of $(\mathbb{C},\, dz)$, and slit $n-2$ of these copies along a segment b and the remaining one along a segment a; see Figure 12. This is the only difference with respect to the case in paragraph §5.1.1.2. Then we proceed as usual in order to get the desired genus-one meromorphic differential.

We just need to check that this structure has rotation number equal to two. We can find closed loops representing α, β as in Figure 12. The curve α has index $1-n$ which is even because n is supposed to be odd. Since the curve β has index 2 by construction, it follows that $\gcd(1-n,\, 2,\, 2n,\, 2) = 2$ as desired.

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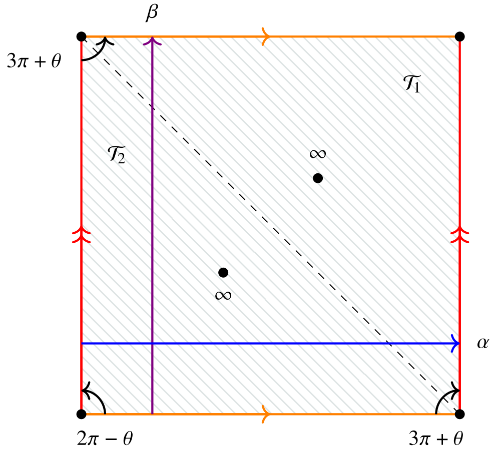

Figure 13. *The topological quadrilateral $\mathcal{Q}$ obtained by gluing the triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ along the edge c. There are two opposite inner angles of magnitude $2\pi-\theta$ and the other two have magnitude $3\pi+\theta$, where $0<\theta<\pi$. The picture shows also how to choose the curves α and β. The curve α might be prolonged in the case the number of punctures is higher than 2; see Figure 14 below (there $\mathcal{Q}$ is drawn in a slightly different way).*

5.1.2.4. Nonpositive volume and even number of punctures

We finally consider the case in which χ has nonpositive volume and the number of puncture is even. For this case, we need a slightly different construction. Let P be any point on $(\mathbb{C}, dz)$, define $Q = P+\chi(\alpha)-\chi(\beta)$, and let c be the geodesic segment joining the points P and Q. Let T_1 be the triangle with vertices P, $P+\chi(\alpha)$ and Q. Notice that, by construction, the sides of T_1 are a, b, c. We define $\mathcal{T}_1$ as the closure in $\mathbb{C}\mathbf{P}^1$ of the exterior of T_1. Notice that $\mathcal{T}_1$ is a triangle in $\mathbb{C}\mathbf{P}^1$. We next consider another copy of $(\mathbb{C}, dz)$ and define T_2 as the triangle with vertices P, $P-\chi(\beta)$ and Q. In this case, the sides of T_2 are still a, b, c. Let $\mathcal{T}_2$ be the triangle in $\mathbb{C}\mathbf{P}^1$ given by the closure of the exterior of T_2. We now glue the triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ along the edge c. The resulting space is a topological parallelogram $\mathcal{P} = \mathcal{T}_1 \cup \mathcal{T}_2$ with two edges parallel to a and two edges parallel to b by construction. Such a parallelogram $\mathcal{P}$ has in its interior two special points corresponding to the points at infinity of the two copies of $\mathbb{C}\mathbf{P}^1$. We set a_0^+ (resp. a_0^-) the edge of $\mathcal{P}$ parallel to a and that bounds the quadrilateral on its right (resp. left). In the same fashion, b_0^+ (resp. b_0^-) is the edge of $\mathcal{P}$ parallel to b and that bounds the quadrilateral on its right (resp. left). See Figure 13.

We consider $n-2$ copies of $(\mathbb{C}, dz)$ each of which slits along a segment b, leaving from P_i to $Q_i = P_i+\chi(\beta)$. We denote the sides as $b_i^\pm$ according to our convention for $i = 1, \dots, n-2$. We finally glue all these structures together as follows. The edges a_0^+ and a_0^- are identified together. Then we glue b_i^- with b_{i+1}^+ for $i = 1, \dots, n-3$, the edge b_{n-2}^- with b_0^+ and, finally, b_0^- with b_1^+. The resulting structure is a genus-one meromorphic differential. See Figure 14.

It remains to show that such a genus-one meromorphic differential has rotation number 2. We can find a closed loop representing α as already done in the previous paragraphs, for example, §5.1.1.1. For any $i = 1, \dots, n-2$, we consider a small metric circle of radius ε around P_i. After the cut and paste process, these $n-2$ curves define a path leaving from a point, say $R \in b_0^+$ to $R+\chi(\alpha) \in b_0^-$. We then join these latter points with a smooth path in $\mathcal{Q}$ in order to get the desired curve α. Such a curve has index $-n$ because it turns clockwise once around any point P_i and then turns by an angle 4π inside $\mathcal{Q}$ (more precisely, the tangent vector to the path turns by an angle 4π). A closed loop representing β can be constructed by joining a point R' to $R'+\chi(\beta)$. Such a curve has index two because it also turns by an angle 4π inside $\mathcal{Q}$. These curves are drawn in Figures 13 and 14. Therefore, it follows that $\gcd(-n,\, 2,\, 2n,\, 2) = 2$ because n is assumed to be even. This concludes the proof of Lemma 5.3.

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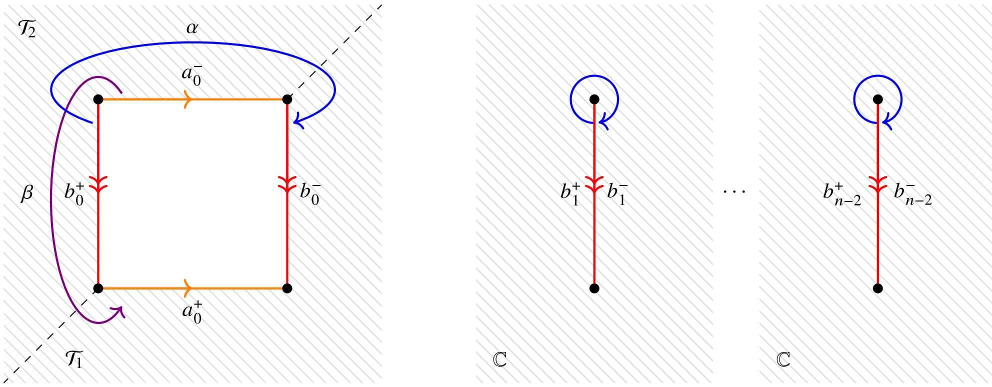

Figure 14. *Realizing a genus-one translation surface with poles of order 2, negative volume and rotation number equal to 2. In this case, there is an even number of punctures corresponding to poles with zero residue.*

5.2. Strata with poles of order higher than two

We next consider strata $\mathcal{H}_1(np; -p, \ldots, -p)$ of genus-one meromorphic differentials where poles are allowed to have orders greater than two. In this section, we still assume that all residues are equal to zero. Recall that, in the present section, we are still under the assumption given in Remark 5.4.

Lemma 5.5. *Let $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation of trivial-ends type. If χ can be realized in the stratum $\mathcal{H}_1(np; -p, \ldots, -p)$, then it appears as the period character of a translation surface with poles in each of its connected components.*

Before moving to the proof of this lemma, we observe that the case $p = 2$ is handled by Lemma 5.3. Therefore, we can assume $p \geq 3$ in the proof of Lemma 5.5.

Let $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation and assume that χ can be realized as the period character of a genus-one meromorphic differential with poles in a stratum $\mathcal{H}_1(np; -p, \ldots, -p)$. Notice that $\gcd(np, p) = p$. Therefore, we want to realize χ as the period of a genus-one meromorphic differential with prescribed rotation number r where r divides p. The cases $n = 1, 2$ are special and they work differently from the generic case $n \geq 3$. Moreover, in all cases, we shall need to distinguish two subcases according to the sign of the volume of χ_g induced by χ. Recall that, since χ is assumed to be of trivial-ends type, the volume of χ is well defined, and it does not depend on any splitting; see Definition 2.10. Therefore, we divide the proof in four subsections according to the cases above. As we shall see, all these cases appear as variations of the preceding constructions.

5.2.1. Positive volume and at most two punctures

Let χ be a representation with positive volume. Let α, β be a pair of handle generators, and let $\mathcal{P}$ be the parallelogram defined by

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) = Q_0 \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{44}$$

where $P_0 \in \mathbb{C}$ is any point. We introduce the following genus-zero meromorphic differential, say (X_1, ω_1). Consider a copy of $(\mathbb{C}, dz)$, and let b_1 be the geodesic segment P_1 to $Q_1 = P_1 + \chi(\beta)$. Let $k \in \{1, \ldots, p-1\}$ be a positive integer, and consider $k-1$ distinct rays starting from P_1 and intersecting b_1 only at P_1. Next, we consider $p-k-1$ distinct rays starting from Q_1 intersecting b_1 only at Q_1. Finally, along every ray we bubble a copy of $(\mathbb{C}, dz)$. The resulting structure is a genus-zero differential with two zeros of orders $k-1$ and $p-k-1$ and a pole of order p.

We glue these structures as follows. We slit (X_1, ω_1) along b_1 and denote the resulting edges by $b_1^{\pm}$, according to our convention. Identify b_j^+ with b_{j+1}^-, where j is taken in $\{0, 1\}$. Finally, identify a_0^+

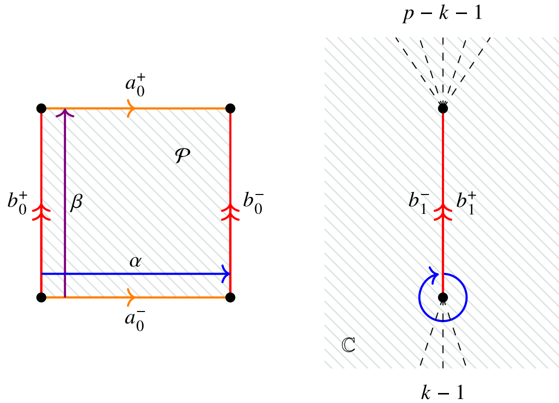

Figure 15. *Realizing a genus-one translation surface with one pole of order p, positive volume and rotation number equal to* $\gcd(k, p)$. *In this case, the pole has (necessarily) zero residue. Dashed lines are drawn to single out copies of* $(\mathbb{C}, dz)$ *glued along rays leaving from* P_1 *and* Q_1. *The same notation will used in several pictures below.*

with a_0^-. The final structure, say (Y, ξ), is a genus-one meromorphic differential with a single zero of order p and one pole of order p. It remains to show that such a structure has rotation number $\gcd(k, p)$. Consider a small metric circle of radius ε centered at P_1. After the cut and paste process, these circles form an oriented smooth path with starting point $R \in b_0^-$ and ending point $R' \in b_0^+$. The points R and R' differ by $\chi(\alpha)$. We choose a representative of α as the smooth oriented curve obtained by joining these points. The unit tangent vector along α turns clockwise by an angle of $2k\pi$. We can choose a representative of β as any waist curve in the cylinder obtained by gluing the parallelogram $\mathcal{P}$. Such a curve can be chosen to be geodesic, and hence, its index is zero. Therefore, (Y, ξ) has rotation number equal to $\gcd(-k, 0, p, p) = \gcd(k, p)$ as desired. Notice that α cannot have index equal to $-p$, and hence, the rotation number cannot be p as expected.

The case with two punctures follows after a small modification of the previous construction. We consider a second copy of $(\mathbb{C}, dz)$ and let b_2 be the geodesic joining the points P_2 and $Q_2 = P_2 + \chi(\beta)$. In this case, we consider $p-2$ distinct rays leaving from Q_2 and intersecting b_2 only at Q_2. We then bubble a copy $(\mathbb{C}, dz)$ along every ray. Equivalently, we consider only one ray, and we bubble along it a copy of $(\mathbb{C}, z^{p-2}dz)$. The resulting structure is a genus-zero meromorphic differential, say (X_2, ω_2) with a single zero of order $p-2$ and a pole of order p. We glue these structures as follows. We slit both (X_i, ω_i) along b_i and denote the resulting edges by $b_i^\pm$, according to our convention. Identify b_j^+ with b_{j+1}^-, where j is taken in $\{0, 1, 2\}$. The final structure, say (Y, ξ), is a genus-one meromorphic differential with a single zero of order $2p$ and two poles of order p. The same argument above shows that (Y, ξ) has rotation number $\gcd(k, p)$ as desired, where $k \in \{2, \ldots, p\}$. See Figure 16.

5.2.2. Positive volume and more than two punctures

Assume that the representation χ has positive volume and at least three punctures. This case is nothing but a variation of the case discussed in the paragraph §5.1.2.2. In order to get the desired structure with prescribed rotation number $\gcd(k, p)$, we shall consider $n-1$ copies of $(\mathbb{C}, dz)$ in this case and we properly modify them to obtain a genus-zero meromorphic differential with a single pole of order p. We begin with these structures.

First of all, we fix a positive integer $k \in \{2, \ldots, p\}$. Notice that, if $k-1 \geq n-2$, then there exist integers $d \geq 1$ and $0 \leq l < n-2$ such that $k-1 = d(n-2)+l$. In the case $k-1 < n-2$, we can always find a positive integer t such that $tk-1 = n-2+l$ and $0 \leq l < n-2$ (here, $d = 1$). Let us consider $(\mathbb{C}, dz)$ and a pair of points P, Q such that $Q = P + \chi(\beta)$. We bubble d copies of $(\mathbb{C}, dz)$ along rays leaving from P and $p-d-2$ copies of $(\mathbb{C}, dz)$ along rays leaving from Q. All rays leaving from

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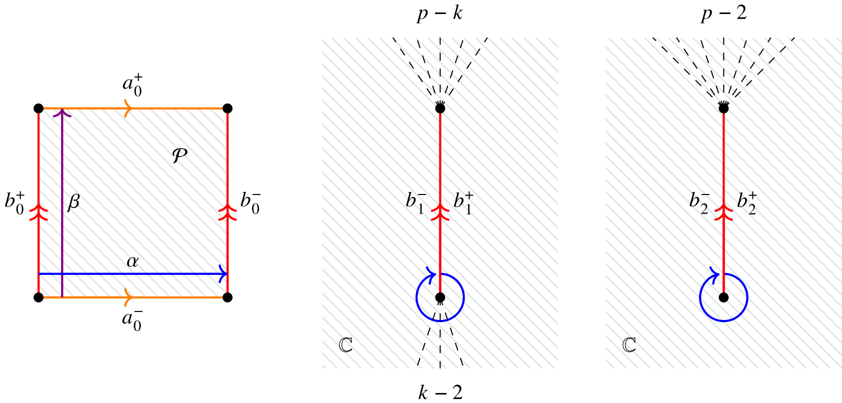

Figure 16. *Realizing a genus-one translation surface with two poles of order p, positive volume and rotation number equal to* $\gcd(k,p)$. *In this case, the number of punctures is exactly two and both poles have zero residue.*

P (resp. from Q) are taken in such a way they do not contain Q (resp. P): None of them contains the geodesic segment joining P and Q. After bubbling, the resulting structure, say (X,ω), is a genus-zero meromorphic differential with two zeros of orders d and $p-d-2$ and a single pole of order p. We consider l copies $(X_1,\omega_1),\dots,(X_l,\omega_l)$ of this structure. Consider again $(\mathbb{C},\, dz)$ and the pair of points P,Q such that $Q=P+\chi(\beta)$. Define (Y,ξ) as the genus-zero differential obtained by bubbling $d-1$ copies of $(\mathbb{C},\, dz)$ along rays leaving from P and $p-d-1$ copies of $(\mathbb{C},\, dz)$ along rays leaving from Q. Even in this case, the rays are taken so that none of them contains the geodesic segment joining P and Q. We consider $n-l-2$ copies $(Y_1,\xi_1),\dots,(Y_{n-l-2},\xi_{n-l-2})$ of this structure. Finally, consider another copy of $(\mathbb{C},\, dz)$ and a segment a with extremal points P_{n-1} and $Q_{n-1}=P_{n-1}+\chi(\alpha)$. We bubble $k-2$ copies of $(\mathbb{C},\, dz)$ along rays leaving from P_{n-1} that do not contain Q_{n-1} and we bubble $p-k$ copies of $(\mathbb{C},\, dz)$ along rays leaving from Q_{n-1} that does not contain P_{n-1}. This genus-zero meromorphic differential, say $(Z,\,\eta)$, has two zeros of order $k-2$ and $p-k$ and a single pole of order p.

Let $\alpha,\,\beta$ be a pair of handle generators, and let $a,\,b\in\mathbb{C}$ the corresponding images via χ. Let H_1 and H_2 be the half-planes as defined in §5.1.2.2. We denote by $a_0^-,\,b_0^-$ the edges of the boundary of H_1 corresponding to $\chi(\alpha)$ and $\chi(\beta)$, respectively. Similarly, we denote by $a_0^+,\,b_0^+$ the edges of the boundary of H_2 corresponding to $\chi(\alpha)$ and $\chi(\beta)$, respectively. Let Q be the end point of $\chi(\alpha)$ (and hence the starting point of $-\chi(\beta)$) on the boundary of H_2. Notice that there are $p-2$ rays, say $r_1,\dots,r_{p-2}$, leaving from Q and pointing at the infinity.

It remains to glue all the structures defined so far together. First of all, we slit (X_i,ω_i) for $i=1,\dots,l$ along the saddle connection b joining the two zeros of ω_i. Denote the resulting edges by $b_i^\pm$. In the same fashion, we slit (Y_j,ξ_j), for $j=1,\dots,n-l-2$, along the saddle connection joining the zeros of ξ_j. We finally slit (Z,η) along a, and we label the edges as $a_n^\pm$. After slitting, we identify a_0^+ with a_n^- and a_0^- with a_n^+. Similarly, we identify b_i^+ with b_{i+1}^-, where $i=1,\dots,l-1$, and b_j^+ with b_{j+1}^-, where $j=1,\dots,n-l-3$. The remaining identifications are: the edge b_0^+ with b_1^-, the edge b_l^+ with b_1^-, the edge b_{n-l-2}^- with b_0^+. The resulting structure is a genus-one meromorphic differential. By bubbling $p-2$ copies of $(\mathbb{C},\, dz)$, each one along a ray leaving from Q, we get the desired structure with rotation number $\gcd(k,p)$. In fact, we can choose representatives of α and β as in Figure 17. The curve α has index

$$\operatorname{Ind}(\alpha)=\begin{cases}-k & \text{if } k-1\geq n-2, \text{ or}\\ -t\,k & \text{if } k-1<n-2\end{cases} \tag{45}$$

by construction. It is straightforward to check that the curve β has index k. Therefore, the rotation number is equal to $\gcd(-tk,k,np,p)=\gcd(k,p)$ as desired, where $t=1$ if $k-1\geq n-2$.

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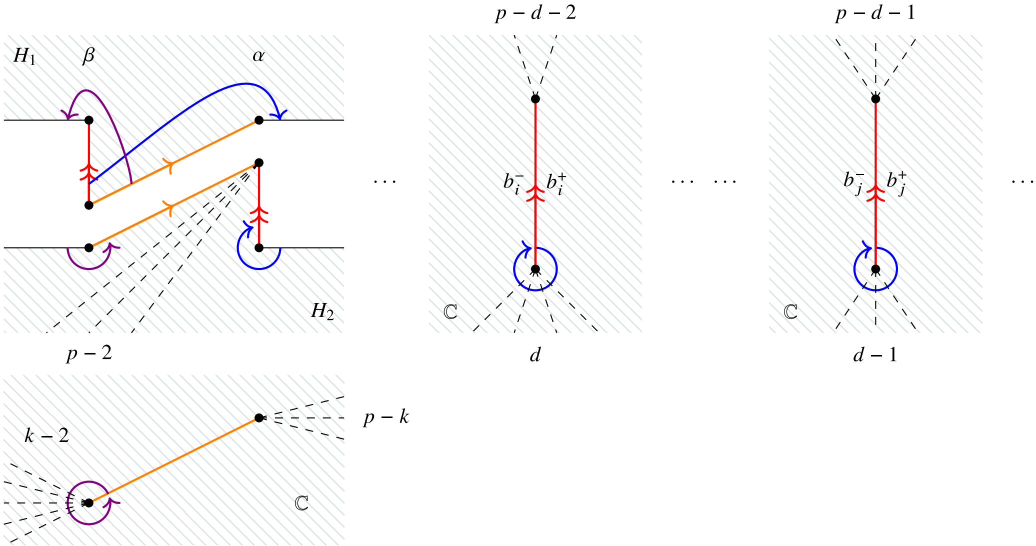

Figure 17. *Realizing a genus-one translation surface with poles of order p, positive volume and rotation number equal to* $\gcd(k, p)$. *In this case, the number of punctures is supposed to be at least three and all the poles have zero residue.*

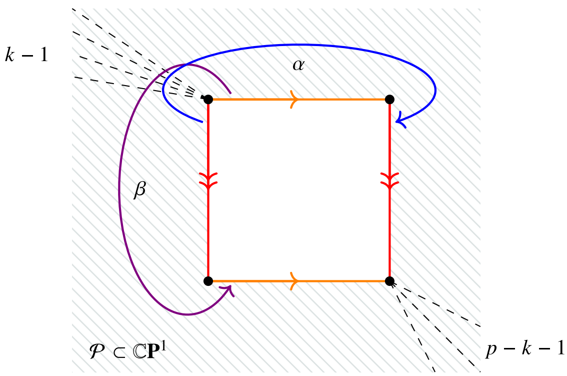

Figure 18. *Realizing a genus-one translation surface with one single pole of order p, nonpositive volume and rotation number equal to* $\gcd(k, p)$.

5.2.3. Nonpositive volume and one puncture

This is an easy case to deal with. Let α, β be a pair of handle generators, and let a, b be their images via χ. Let $\mathcal{P}$ be the exterior of the parallelogram (possibly degenerate) defined by the chain

$$P \mapsto P + a \mapsto P + a + b = Q \mapsto P + b \mapsto P, \tag{46}$$

with sides $a^\pm$, $b^\pm$ according to our convention. Let r_P be a geodesic ray joining P and the pole, and glue along $k-2$ copies of the genus-zero differential $(\mathbb{C}, dz)$ as in Definition 3.9. Similarly, let r_Q be a geodesic ray joining Q and the pole and glue along $p-k$ copies of the genus-zero differential $(\mathbb{C}, dz)$. We finally glue a^+ with a^- and b^+ with b^-. The resulting surface is a genus-one differential in $\mathcal{H}_1(p, -p)$. It remains to show that it has rotation number $\gcd(k, p)$. This can be easily verified by choosing α and β as in §5.1.2.3 above. See Figure 18.

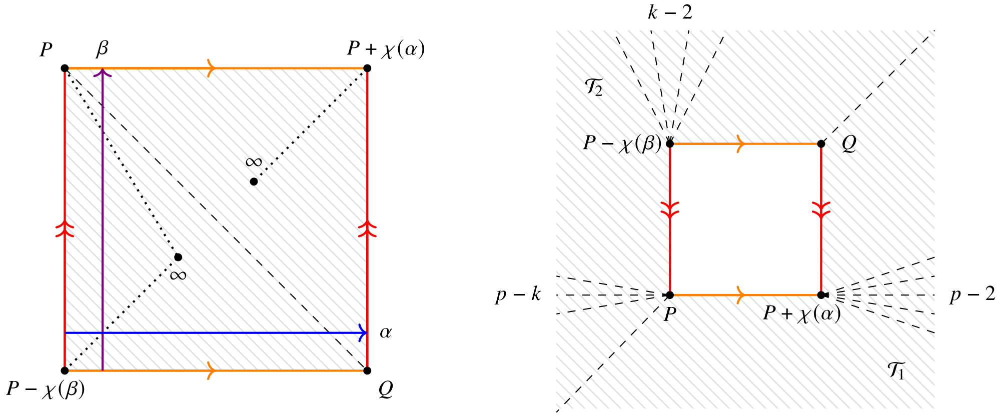

Figure 19. *The quadrilateral $\mathcal{Q}$ from two different perspectives. On the left-hand side, we can see the curves α and β. Any dotted line represents a ray from a vertex of $\mathcal{Q}$ to a point at infinity. These rays are better represented on the right-hand side.*

5.2.4. Nonpositive volume and two punctures

In this case, we consider the quadrilateral $\mathcal{Q}$ already defined in paragraph §5.1.2.4. Recall that $\mathcal{Q}$ is obtained by gluing two triangles, $\mathcal{T}_1$ with vertices P, $P+\chi(\alpha)$, Q and $\mathcal{T}_2$ with vertices P, $P-\chi(\beta)$, Q, where $Q = P+\chi(\alpha)-\chi(\beta)$. There are two pairs of opposite sides one of which differ by a translation $\chi(\alpha)$ and the other by a translation $\chi(\beta)$. See paragraph §5.1.2.4 for more details about this construction.

Given a positive integer $k \in \{2,\dots,p\}$, we consider $k-2$ rays leaving from $P-\chi(\beta)$ and $p-k$ rays leaving from P such that they all point towards the pole contained in $\mathcal{T}_2$. Next, we consider other $p-2$ rays all leaving from $P+\chi(\alpha)$ and pointing towards the pole contained in $\mathcal{T}_1$. By gluing the opposite sides of $\mathcal{Q}$ by using the translations $\chi(\alpha)$ and $\chi(\beta)$, we obtain a genus-one meromorphic differential, say (X,ω) with a single zero of order $2p$ and two poles of order p. It remains to show that (X,ω) has rotation number $\gcd(k,p)$. We can choose the curves α and β as in §5.1.2.42; see Figure 19. By construction, α has index $-k$ and β has index p. Therefore, $\gcd(-k,p,2p,p) = \gcd(k,p)$ as desired.

5.2.5. Nonpositive volume and more than two punctures

This is the last case to handle in this section, and it is similar to the case discussed in §5.2.2. Recall that, for a positive integer $k \in \{2,\dots,p\}$ we have introduced two integers $d, l \in \mathbb{Z}^+$ and used them to define the genus-zero differentials (X,ω), (Y,ξ) and (Z,η); see Section §5.2.2 for these constructions. In this case, we still consider l copies, say $(X_1,\omega_1),\dots,(X_l,\omega_l)$ of (X,ω). Recall that l can be zero. We then consider $n-l-2$ copies of (Y,ξ) that we denote as $(Y_1,\xi_1),\dots,(Y_{n-l-2},\xi_{n-l-2})$ and a unique copy of (Z,η).

Let α, β be a pair of handle generators, and let $\mathcal{P}$ be the closure of the exterior parallelogram in $\mathbb{C}$ defined by the chain

$$P_0 \mapsto P_0+\chi(\alpha) \mapsto P_0+\chi(\alpha)+\chi(\beta) = Q_0 \mapsto P_0+\chi(\beta) \mapsto P_0, \tag{47}$$

where $P_0 \in \mathbb{C}$ is any point. Notice that we can find $p-2$ rays, say $r_1,\dots,r_{p-2}$, leaving from Q pointing at the infinity; see Figure 20. As in Section §5.2.2, we glue all these structures together in the usual way. The resulting structure is a genus-one meromorphic differential. By bubbling a copy of $(\mathbb{C}, dz)$ along r_i, for any $i = 1,\dots,p-2$, we obtain a genus-one meromorphic differential with one zero of order np and n poles of order p.

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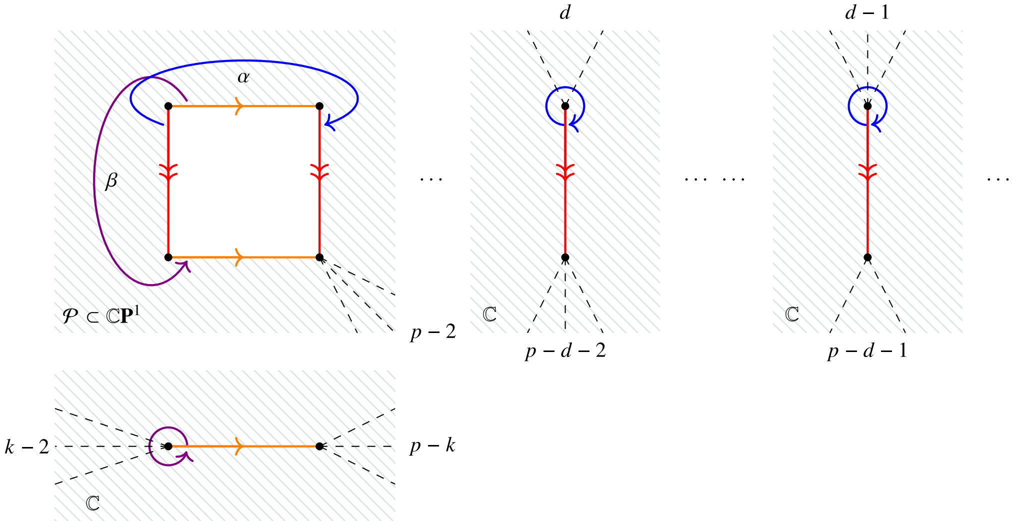

Figure 20. *Realizing a genus-one translation surface with poles of order p, negative volume and rotation number equal to* $\gcd(k, p)$*. In this case, the number of punctures is supposed to be at least three and all the poles have zero residue.*

It just remains to show that this latter structure has the desired rotation number $\gcd(k, p)$. If we choose the curves α and β as shown in Figure 20, we can observe that

$$\operatorname{Ind}(\alpha) = \begin{cases} -k & \text{if } k-1 \geq n-2, \text{ or} \\ -t\,k & \text{if } k-1 < n-2, \end{cases} \tag{48}$$

and $\operatorname{Ind}(\beta) = k$ by construction. Therefore, $\gcd(-tk, k, np, p) = \gcd(k, p)$ as desired. This concludes the proof of Lemma 5.5.

Remark 5.6. It is also possible to realize a genus-one meromorphic differential with poles of order p, a single zero of maximal order and rotation number one by bubbling sufficiently many copies of $(\mathbb{C}, dz)$ along rays on a genus-one differential with poles of order two obtained as in Section §5.1.1. Suppose (X, ω) is a such a structure. We can find n rays, all leaving from the unique zero of ω and such that each one joins a pole of ω. Different rays join different poles. Then we can bubble $p-2$ copies of $(\mathbb{C}, dz)$ along every ray. The resulting structure turns out to be a genus-one differential with n poles each of order p. If these rays are properly chosen, it is also possible to preserve the rotation number. In Figures 8 and 9, the dotted lines are possible candidates. Bubbling along them one gets a genus-one meromorphic differential with poles of order p, and since the indices of the curves α and β are not affected by any bubbling, the rotation number does not change. In a similar fashion in certain cases, it is possible to realize a genus-one meromorphic differential with poles of order p and rotation number two. In fact, it is possible to modify the structures obtained in Figures 10, 11 and 12 by bubbling $p-2$ copies of $(\mathbb{C}, dz)$ along each dotted line. The indices of α and β are not affected in these cases, and therefore, the resulting structures have rotation number two.

5.3. Strata with poles with nonzero residue

We consider the case of characters χ of nontrivial-ends type. Notice that this automatically implies $n \geq 2$. Here, we shall consider $S_{1,n}$ as the surface obtained by gluing the surfaces $S_{1,2}$ and $S_{0,n}$ along a ray joining two punctures as in Definition 3.10 with the only difference that for the moment no geometry is involved; see Figure 21.

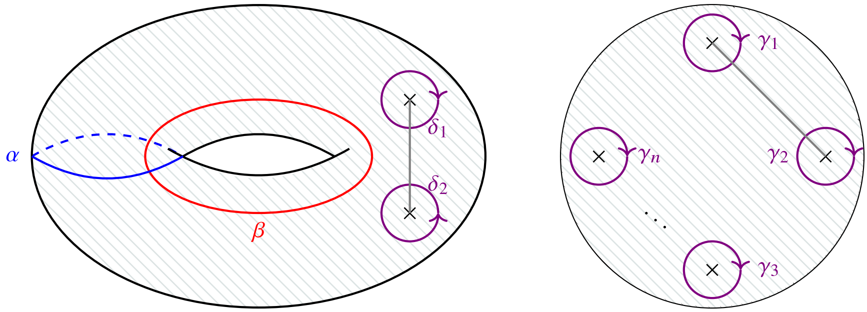

Figure 21. *The surfaces $S_{1,2}$ and $S_{0,n}$. By slitting and pasting along the infinite rays coloured in gray, the resulting surface is homeomorphic to $S_{1,n}$.*

The idea behind the proof of this case is as follow. Starting from the representation χ we shall introduce a new representation $\rho\colon \mathrm{H}_1(S_{1,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ which, in principle, does not need to be the restriction of χ to any subsurface of $S_{1,n}$ homeomorphic to $S_{1,2}$. For any $p \geq 2$ and for any $k \in \mathbb{Z}$ dividing p, we can realize ρ in the stratum $\mathcal{H}_1(2p; -p, -p)$ as the holonomy of a genus-one differential with rotation number k as done in Sections §5.1 and §5.2. Then we properly modify this structure in order to get the desired one. To this latter, in the case $n \geq 3$, we shall finally glue a genus-zero meromorphic differential in order to obtain the desired structure on $S_{1,n}$ with period character χ and prescribed rotation number.

To define an auxiliary representation ρ, we provide a more general definition which we need to use later on.

Definition 5.7 (Auxiliary representation). Let $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation of nontrivial-ends type and $\iota\colon S_{h,m} \hookrightarrow S_{g,n}$ be an embedding with $1 \leq h \leq g$ and $1 \leq m \leq n$. Let $\{\alpha_i, \beta_i\}_{1\leq i\leq h}$ be a system of handle generators for $\mathrm{H}_1(S_{h,m}, \mathbb{Z})$. We define an *auxiliary representation* $\rho\colon \mathrm{H}_1(S_{h,m}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as follows:

$$\rho(\alpha_i) = \chi \circ \iota(\alpha_i), \quad \rho(\beta_i) = \chi \circ \iota(\beta_i), \quad \rho(\delta_1) = \cdots = \rho(\delta_m) = 0, \tag{49}$$

where $\delta_1, \ldots, \delta_m$ are simple closed curves each of which encloses a puncture on $S_{h,m}$ and oriented in such a way that $[\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h] = \delta_1 \cdots \delta_m$. If $h = g$ and $m = n$, then the embedding ι can be taken as the identity map.

Before proceeding, once again, we recall for the reader's convenience that no puncture is assumed to be a simple pole; see Remark 5.2. Moreover, in this section we assume that all poles have the same order $p \geq 2$. We shall distinguish three cases according to the following subsections.

5.3.1. Two poles with nonzero residue

We suppose $n = 2$, and hence, $\chi\colon \mathrm{H}_1(S_{1,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is a representation such that

$$\chi(\delta_1) = \chi(\delta_2^{-1}) = w \in \mathbb{C}^*, \tag{50}$$

where δ_1, δ_2 are nonhomotopic simple closed curves both enclosing a single puncture of $S_{1,2}$. We introduce an auxiliary representation $\rho\colon \mathrm{H}_1(S_{1,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$, as in Definition 5.7, defined as

$$\rho(\alpha) = \chi(\alpha), \qquad \rho(\beta) = \chi(\beta), \qquad \rho(\delta_1) = \rho(\delta_2) = 0. \tag{51}$$

Let (X, ω) be the translation surface with period character ρ and rotation number k realized as in Section §5; more precisely as in §5.1 if $p = 2$ or as in §5.2.1 and §5.2.4 if $p > 2$. The following holds.

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Lemma 5.8. *There is a bi-infinite geodesic ray joining the poles such that its direction after developing, say $v \in \mathbb{C}$, is different from w.*

Suppose Lemma 5.8 holds; then we can find a bi-infinite geodesic ray $r \subset (X, \omega)$ joining the two poles and such that its direction after developing is $v \neq w$. Let $C_w = \mathbb{E}^2/\langle z \mapsto z + w\rangle$, and let $\pi_w : \mathbb{E}^2 \to C_w$ be the covering projection. Since $v \neq w$, we can glue C_w along the bi-infinite ray r as in Definition 3.13. In fact, any straight line with direction v in $\mathbb{E}^2$ projects to a bi-infinite geodesic line which wraps around C_w. Let $\overline{r} \subset C_w$ be a bi-infinite geodesic ray with direction v after developing. Slit (X, ω) along r, and let $r^\pm$ be the resulting edges. Similarly, slit C_w along $\overline{r}$ and call the resulting edges $\overline{r}^\pm$. Notice that $C_w \setminus \overline{r}$ is an infinite open strip whose closure is obtained by adding the edges $\overline{r}^\pm$. We can glue this latter closed strip to the slit (X, ω) by identifying r^+ with $\overline{r}^-$ and r^- with $\overline{r}^+$. The resulting translation surface (Y, ξ) is still a genus-one differential. The differential ξ has a single zero of order $2p$ and two poles of order p with residue $\pm w$ by construction. Therefore, (Y, ξ) has period character χ and rotation number k as a consequence of Lemma 3.15. It remains to show Lemma 5.8.

Proof of Lemma 5.8. Let $\chi\colon \mathrm{H}_1(S_{1,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation, let $\{\alpha, \beta\}$ be a pair of handle generators and let $a = \chi(\alpha)$ and $b = \chi(\beta)$ be their images. We provide a direct proof case by case. For the following cases, we rely on the constructions developed in Section §5.1.1.

- *Case 1: positive volume with $p = 2$ and $k = 1$.* In this case, a genus-one differential with period χ is obtained by gluing together the quadrilateral $\mathcal{P}$ with edges $a_0^\pm$ and $b_0^\pm$ (according to our convention) and two genus-zero differentials, say (X_i, ω_i) for $i = 1, 2$. Notice that $\mathcal{P}$ cannot be degenerate because the volume of χ is positive. From §5.1.1.1, recall that (X_1, ω_1) is slit along a geodesic segment joining P_1 with $Q_1 = P_1 + a$ and recall that (X_2, ω_2) is slit along a geodesic segment joining P_2 with $Q_2 = P_2 + b$. Let δ be the oriented angle between b_0^- and a_0^+. Let r_1 be a geodesic ray on (X_1, ω_1) leaving from Q_1 with angle $0 < \theta < \pi - \delta$ with respect to a^-. Finally, let r_2 be the geodesic ray leaving from Q_2 with angle $\pi - (\delta + \theta)$ with respect to b^+. Once $\mathcal{P}$ is glued with (X_1, ω_1) and (X_2, ω_2) as described in §5.1.1.1, the rays r_1 and r_2 form a bi-infinite geodesic ray, say r, on the final surface passing through the branch point, obtained after identification and such that its developed image is a geodesic straight line in $\mathbb{E}^2$. The construction has been done in such a way that r leaves the branch point with angle π on the left. Notice that a similar construction can be done so that r leaves the branch point with angle π on the right. See Figure 22a.
- *Case 2: nonpositive volume with $p = 2$ and $k = 1$.* In this case, a genus-one differential with period χ is obtained by removing the interior of the quadrilateral $\mathcal{P}$ with edges $a_0^\pm$ and $b_0^\pm$ from a copy $(\mathbb{C}, dz)$ and then glue another copy of $(\mathbb{C}, dz)$ slit along a geodesic segment joining P with $Q = P + b$; see §5.1.1.2. Here, $\mathcal{P}$ could be degenerate. Let r_1 be a geodesic ray on $\mathbb{C} \setminus \mathrm{int}(\mathcal{P})$ leaving from $P + a + b$ with angle $0 < \theta < \pi$ with respect to b_0^-. Then define r_2 as the geodesic ray leaving from Q with angle $\pi - \theta$ with respect to b_1^+. After gluing as described in 5.1.1.2, the rays r_1 and r_2 form a bi-infinite geodesic ray on the final surface, passing through the branch point obtained after identification and such that its developed image is a geodesic line in $\mathbb{E}^2$. The construction has been done in such a way that r leaves the branch point with angle π on the left. A similar construction can be done so that r leaves the branch point with angle π on the right. See Figure 22a.

For the remaining cases, we rely on the constructions made in §5.2.1 and §5.2.4. Recall that both of these constructions extend those made in §5.1.2.1 and §5.1.2.4 for the case $p = 2$ and rotation number two. Therefore, we can handle all these cases together.

- *Case 3: positive volume and $p > 2$ or $p = 2$ with $k = 2$.* In the case $p = 2$ and $k = 2$ or $p > 2$, a genus-one differential with period χ is obtained by gluing together the quadrilateral $\mathcal{P}$ with edges $a_0^\pm$ and $b_0^\pm$ (according to our convention) and two genus-zero differentials, say (X_i, ω_i) for $i = 1, 2$, both slit along a geodesic segment joining P_i with $Q_i = P_i + b$. Recall that if $p = 2$, then (X_i, ω_i) is a copy of $(\mathbb{C}, dz)$ for $i = 1, 2$; see §5.1.2.1. In the case $p > 2$, then (X_1, ω_1) is a genus-zero differential with two zeros of orders $k - 2$ and $p - k$ at P_1 and Q_1, respectively, and one pole of order p, whereas (X_2, ω_2) is a genus-zero differential with one zero of order $p - 2$ at Q_2 and one pole of order p. In

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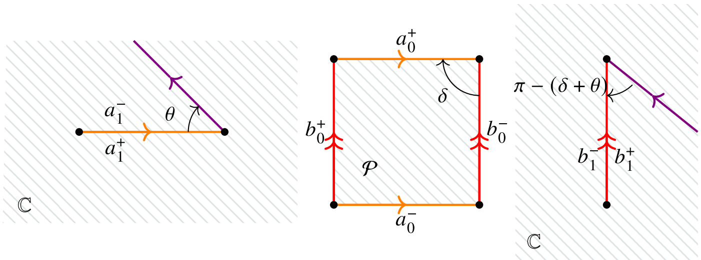
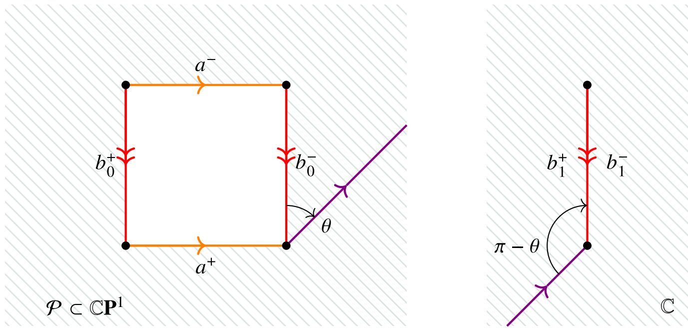


(a) Case 1 of Lemma 5.8.



(b) Case 2 of Lemma 5.8.

Figure 22. *This picture shows how to find a bi-infinite ray as claimed in Lemma 5.8. Once the pieces are all glued together, the purple rays determine a bi-infinite geodesic ray joining the poles of the resulting translation surface.*

both cases, we shall define r_1 as the geodesic ray on (X_1, ω_1) leaving from Q_1 with angle $0 < \theta < \pi$ with respect to b_1^-. Then we define r_2 as the geodesic ray leaving from Q_2 with angle $\pi - \theta$ with respect to b_2^+. Once $\mathcal{P}$ is glued with (X_1, ω_1) and (X_2, ω_2) as described in §5.1.2.1, the rays r_1 and r_2 form a bi-infinite geodesic ray on the final surface, passing through the branch point obtained after identification and such that its developed image is a geodesic straight line in $\mathbb{E}^2$. The construction has been done in such a way that r leaves the branch point with angle π on the left. A similar construction can be done so that r leaves the branch point with angle π on the right. See Figure 23a.

- *Case 4: nonpositive volume and $p > 2$ or $p = 2$ with $k = 2$.* In this last case, we realize a genus-one differential exactly as we have done in §5.1.2.4 if $p = 2$ or as in §5.2.4 if $p > 2$. By adopting the notation used there, we can observe that the angle at Q is $\frac{3\pi}{2}$ in both constructions. There is enough room for finding a bi-infinite ray passing through the branch point such that one of the two angles is π. Moreover, we can choose this bi-infinite ray such that the angle with respect to b^- is $0 < \theta < \pi$. The construction has been done in such a way that r leaves the branch point with angle π on the right. A similar construction can be done so that r leaves the branch point with angle π on the left. See Figure 23a.

We finally observe that in all constructions the direction of the bi-infinite ray r depends on the angle θ which can be chosen with certain flexibility. In particular, θ can be chosen in such a way that r has direction v after developing with $v \neq w$ as desired. This concludes the proof of Lemma 5.8. □

5.3.2. All poles with nonzero residue

Assume $n \geq 3$, and now we deal with a representation of nontrivial-ends type $\chi \colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$. Let $\gamma_i \subseteq S_{1,n}$ denote a simple closed curve enclosing the i-th puncture, and define $w_i = \chi(\gamma_i)$. Since no puncture has zero residue, it follows that $w_i \in \mathbb{C}^*$ for any $i = 1, \ldots, n$. Recall that $w_1 + \cdots + w_n = 0$ as

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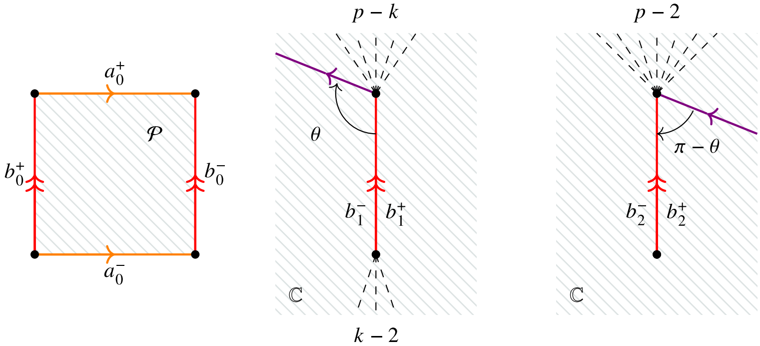
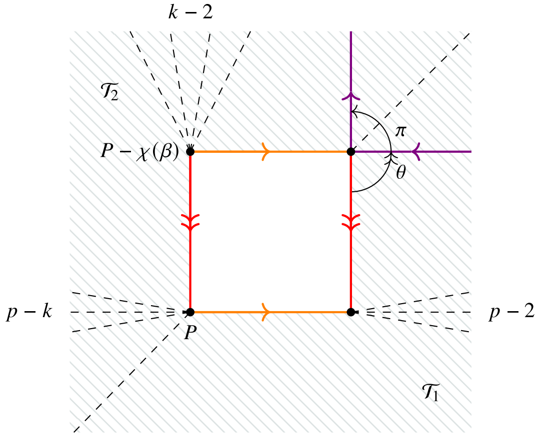


(a) Case 3 of Lemma 5.8.



(b) Case 4 of Lemma 5.8.

Figure 23. *This picture shows how to find a bi-infinite ray as claimed in Lemma 5.8. Once the pieces are all glued together the purple rays determine a bi-infinite geodesic ray joining the poles of the resulting translation surface. Notice the cases $p = 2$ and $k = 2$ are subsumed in 23a and 23b according to the sign of the volume.*

a consequence of the residue theorem. Before proceeding with this case, we introduce some notations and generalities.

Let us split $S_{1,n}$ as in Figure 21, and focus on the surface $S_{0,n} \cong \mathbb{S}^2 \setminus \{P_1, \ldots, P_n\}$ of such a splitting. Let γ_i be the simple closed curve enclosing the puncture P_i on $S_{0,n}$ for $i = 1, \ldots, n$. For the reader's convenience we recall that the χ_n-part of χ is a representation defined as follows:

$$\chi_n \colon \mathrm{H}_1(S_{0,n}, \mathbb{Z}) \longrightarrow \mathbb{C}, \quad \chi_n(\gamma_i) = w_i \text{ for } i = 1, \ldots, n. \tag{52}$$

Notice that, since $w_1 + \cdots + w_n = 0$, the representation χ_n is well defined. We now need to extend our earlier Definition 4.8 as follows:

Definition 5.9. A real-collinear representation $\chi \colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is called *rational* if the image $\mathrm{Im}(\chi)$ is contained in the $\mathbb{Q}$-span of some $c \in \mathbb{C}^*$. A real-collinear representation χ is not rational otherwise.

Definition 5.10 (Reordering property). Let $w_1, \ldots, w_n \in \mathbb{R}^*$ be nonzero real numbers with zero sum, namely they satisfy $w_1 + \cdots + w_n = 0$. We shall say that $\{w_1, \ldots, w_n\}$ satisfy the *reordering property* if there is a permutation $\sigma \in \mathfrak{S}_n$ and an integer smaller than n such that

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- $w_{\sigma(i)} > 0$ for any $1 \le \sigma(i) \le h$,
- $w_{\sigma(i)} < 0$ for any $h+1 \le \sigma(i) \le n$, and
- the following condition:

$$\sum_{\sigma(i)=1}^{s} w_{\sigma(i)} = -\sum_{\sigma(j)=t+1}^{n} w_{\sigma(j)} \tag{53}$$

  holds only for $s = t = h$.

Notice that this definition can naturally extend to sets of real-collinear complex numbers with zero sum.

This definition is slightly different from those used in [CFG22]. As we shall see, the reordering property plays an important role in this section. The following lemma holds, and the proof can be found in [CFG22, Appendix B].

Lemma 5.11. *Let $W = \{w_1, \dots, w_n\} \subset \mathbb{R}^*$ be a set of nonzero real numbers with zero sum. Suppose there exists a pair of numbers in W with irrational ratio. Then W satisfies the reordering property.*

From [CFG22], we also need to invoke the following proposition about holonomy representations of genus-zero differentials with prescribed order of zeros and poles.

Proposition 5.12. *Let $\chi_n\colon \mathrm{H}_1(S_{0,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation. Let $p_1, p_2, \dots, p_n \in \mathbb{Z}^+$ be positive integers satisfying the following properties:*

- *Either*
  - *χ_n is not real-collinear; that is, $\mathrm{Im}(\chi_n)$ is not contained in the $\mathbb{R}$-span of some $c \in \mathbb{C}^*$,*
  - *χ_n is real-collinear but not rational; that is, $\mathrm{Im}(\chi_n)$ is contained in the $\mathbb{R}$-span but not in the $\mathbb{Q}$-span of some $c \in \mathbb{C}^*$,*
  - *at least one of $p_1, p_2, \dots, p_n$ is different from 1, and*
- *$p_i \ge 2$ whenever $\chi(\gamma_i) = 0$.*

Then χ appears as the holonomy of a translation structure on $S_{0,n}$ determined by a meromorphic differential on $\mathbb{C}\mathbf{P}^1$ with a single zero of order $m = p_1 + \cdots + p_n - 2$ and a pole of order p_i at the puncture enclosed by the curve γ_i, for each $1 \le i \le n$.

We shall need to consider two subcases depending on whether χ_n is rational. We begin with the following case:

5.3.2.1. The representation χ_n is not rational

Let α, β be a pair of handle generators for $\mathrm{H}_1(S_{1,2}, \mathbb{Z})$. By means of an auxiliary representation ρ defined as in Equation (51) (see also Definition 5.7), we can realize a genus-one differential $(X_1, \omega_1) \in \mathcal{H}_1(2p; -p, -p)$ with rotation number k and period character defined as

$$\rho(\alpha) = \chi(\alpha), \quad \rho(\beta) = \chi(\beta), \quad \rho(\delta_1) = \rho(\delta_2^{-1}) = 0. \tag{54}$$

We apply Proposition 5.12 to our representation χ_n and realize this latter as the period character of a genus-zero differential $(X_2, \omega_2) \in \mathcal{H}_0\big((n-2)p; -1, -1, -p, \dots, -p\big)$.

In order to realize the desired surface, we shall glue them along a bi-infinite geodesic ray. Recall that, according to Lemma 5.8, we can always find a bi-infinite geodesic ray on (X_1, ω_1) with prescribed direction (hence slope) once developed such that:

(i) it passes through the zero of ω_1 only once with angle π on its *right*, and
(ii) it joins the poles of ω_1.

Let $r \subset (X_1, \omega_1)$ be a ray as above with direction v once developed. What remains to do is to find a proper bi-infinite ray on (X_2, ω_2) with the same direction and passing through the unique zero of ω_2

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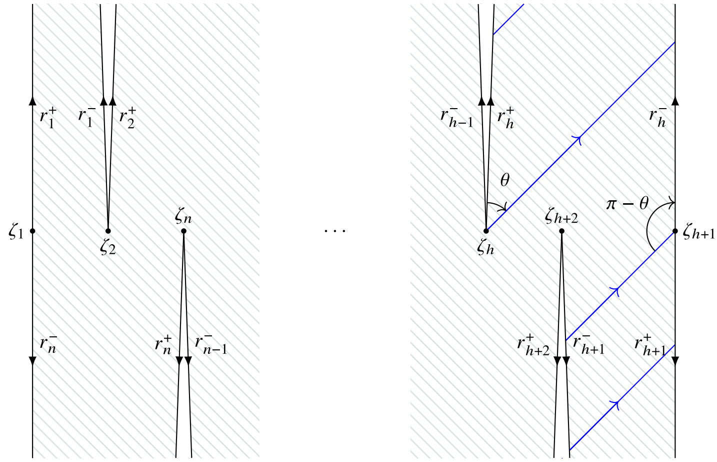

Figure 24. *The picture shows the construction in the case* $\mathrm{Im}(\chi_n)$ *is contained in the* $\mathbb{R}$*-span of some* $c \in \mathbb{C}^*$ *but not in the* $\mathbb{Q}$*-span. The blue line represents a ray with slope* $\frac{\pi}{2} - \theta$ *along which we can glue a genus-one differential.*

with angle π on its *left*. Here, $\mathrm{Im}(\chi_n)$ is assumed to be not contained in the $\mathbb{Q}$-span of any $c \in \mathbb{C}^*$, and hence, according to [CFG22, Proof of Proposition 10.1], there are two possible situations:

1. The first one arises if $\mathrm{Im}(\chi_n)$ is not contained in the $\mathbb{R}$-span of any $c \in \mathbb{C}^*$. Whenever this is the case, we can reorder the punctures in such a way that the points $\{\arg(w_i)\} \subset \mathbb{S}^1$, with $w_i = \chi(\gamma_i)$, are cyclically ordered. With respect to this ordering, the vectors w_i yield a convex polygon $\mathcal{P}$ with n sides. Denote the edges of $\mathcal{P}$ as e_i^-, for $i = 1, \ldots, n$. To any side, we can glue a half-infinite strip $\mathcal{S}_i$ bounded by the edge e_i^+ and two infinite parallel rays $r_i^\pm$ oriented so that $r_i^+ = r_i^- + w_i$. The quotient space $\mathcal{P} \cup \bigcup \mathcal{S}_i$ obtained by identifying the vertices of $\mathcal{P}$ and the rays $r_i^\pm$, for $i = 1, \ldots, n$, turns out to be an n-punctured sphere with a translation structure with all simple poles and a single zero of maximal order. In this case, it is an easy matter to see that for almost any slope $\theta \in \mathbb{S}^1$ (hence almost any directions), there is a bi-infinite geodesic ray with slope θ. Therefore, we can find a bi-infinite geodesic ray, say $\overline{r}$, with direction v and such that it passes through the zero of ω_2 by leaving an angle of magnitude π on its left. Such a ray joins two adjacent poles say P_j and P_{j+1}. We define (X_2, ω_2) as the translation structure obtained by bubbling p copies of the standard differential $(\mathbb{C}, dz)$ to all poles with the only exceptions being P_j and P_{j+1}. Finally, glue (X_1, ω_1) and (X_2, ω_2) as in Definition 3.9 by slitting them along the rays r and $\overline{r}$, respectively. The resulting surface is homeomorphic to $S_{g,n}$ and carries a translation surface with poles $(Y, \xi) \in \mathcal{H}_1(np; -p, \ldots, -p)$ with rotation number k by construction.
2. The second possibility arises in the case χ_n is real-collinear but not rational. In order to recall the construction, we assume for a moment that $\arg(w_i) \in \{0, \pi\}$, that is, $w_i \in \mathbb{R}$, because the general case of $\arg(w_i) \in \{\delta, \delta + \pi\}$ follows by rotating the following construction by δ. Up to relabelling the punctures, there is a positive integer, say h less than n, such that $\{w_1, \ldots, w_h\}$ are all positive real numbers and $\{w_{h+1}, \ldots, w_n\}$ are all negative real numbers. Since $\mathrm{Im}(\chi_n)$ is not contained in the $\mathbb{Q}$-span of any $c \in \mathbb{C}^*$ it is possible to find a pair of reals in $\{w_1, \ldots, w_n\}$ with irrational ratio; hence, Lemma 5.11 applies and the reals $w_1, \ldots, w_n$ satisfy the reordering property. Let $\zeta_1 = 0$, and let us define $\zeta_{l+1} = \zeta_l + w_l$. Consider the infinite strip $\{z \in \mathbb{C} \mid 0 < \Re(z) < \zeta_{h+1}\}$. In this infinite strip, we make half-infinite vertical slits pointing upwards at the points $\zeta_2, \ldots, \zeta_h$ and half-infinite vertical slits pointing downwards at the points $\zeta_{h+2}, \ldots, \zeta_n$; see Figure 24. By gluing the rays r_i^+ and r_i^- for $i = 1, \ldots, n$, we obtain a translation surface (Z, η) with all simple poles and a single zero of maximal order.

   In this case, for any $\theta \neq 0$ we can find a bi-infinite geodesic ray, say $\overline{r}$ joining two punctures and passing through the branch point by leaving an angle π on its left. In fact, we can consider a ray leaving

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from ζ_h with angle θ with respect to r_h^+. Such a ray points towards the puncture P_h. Then consider a ray leaving from ζ_{h+1} with angle $\pi - \theta$ with respect to r_h^- pointing towards the puncture P_{h+1}. These rays determine the desired bi-infinite geodesic ray, say $\overline{r}$, on (Z, η). In fact, by construction it joins two punctures and leaves an angle of magnitude π on its left at the unique branch point of (Z, η). We define (X_2, ω_2) as the structure obtained by bubbling p copies of the differential $(\mathbb{C}, dz)$ to all poles with the only exceptions being P_h and P_{h+1}. As above, we glue (X_1, ω_1) and (X_2, ω_2) as in Definition 3.9 by slitting them along r and $\overline{r}$, respectively. The final surface is homeomorphic to $S_{g,n}$ and carries a translation structure with poles in the stratum $\mathcal{H}_1(np; -p, \dots, -p)$ with rotation number k by construction.

For more details about these constructions, the reader can consult the proofs of [CFG22, Propositions 6.1 and 10.1].

5.3.2.2. The representation χ_n is rational

It remains to deal with the case of χ_n being rational. The strategy developed in paragraph §5.3.2.1 does not always apply because for a rational representation (Definition 5.9) the reordering property (Definition 5.10) fails in general. Whenever a rational representation χ_n satisfies the reordering property, then the construction developed for a nonrational real-collinear representation as in paragraph §5.3.2.1 applies *mutatis mutandis*. Therefore, in what follows, we shall restrict ourselves to rational representations for which the reordering property fails. In this very special situation, we shall adopt a strategy which is a blend of our construction so far with [CFG22, Proof of Proposition 10.1 - Case 3]. In particular, we do not rely on the splitting introduced at the beginning of §5.3.

Let $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation with rational χ_n-part. Let $\Sigma \subset S_{1,n}$ be any handle, let α, β be a pair of handle generators for $\mathrm{H}_1(\Sigma, \mathbb{Z}) \subset \mathrm{H}_1(S_{1,n}, \mathbb{Z})$ and define a, b as their respective images via χ. The restriction of χ to Σ is a representation of trivial-end type, say χ_1, and hence, the notion of volume for χ_1 is well defined. According to the sign of $\mathrm{vol}(\chi_1)$, we shall distinguish two constructions. We shall treat the positive volume case in detail; the nonpositive volume one will follow after a simple modification of the first case. Let $\mathcal{P}$ be the parallelogram defined by the chain

$$P_0 \mapsto P_0 + \chi(\alpha) \mapsto P_0 + \chi(\alpha) + \chi(\beta) \mapsto P_0 + \chi(\beta) \mapsto P_0, \tag{55}$$

where $P_0 \in \mathbb{C}$ is any point. As usual, we label the sides of $\mathcal{P}$ with $a_0^\pm$ and $b_0^\pm$ according to our convention.

We now consider the χ_n-part of χ. The representation $\chi_n\colon \mathrm{H}_1(S_{0,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ is a nontrivial representation, and we denote $\chi_n(\gamma_i) = w_i \in \mathbb{C}^*$ for any i, where γ_i is a simple closed curve around the i-th puncture. We assume $W = \{w_1, \dots, w_n\} \subset \mathbb{C}^*$ is a set of real-collinear complex numbers. Up to permutation of the labels of the punctures, we can assume that all numbers in $\{w_1, \dots, w_h\}$ have the same argument $\delta \in]-\pi, \pi]$ and all numbers in $\{w_{h+1}, \dots, w_n\}$ have the same argument $\pi + \delta$ for some $1 \le h \le n-1$. Assume that W does *not* satisfy the reordering property. According to Definition 5.10, there are two positive integers $1 \le s < h$ and $h + 1 \le t < n$ such that

$$\sum_{i=1}^{s} w_i = -\sum_{j=1}^{n-t} w_{t+j} = -\sum_{j=t+1}^{n} w_j. \tag{56}$$

The indices s, t yield the following two partitions: $W_1^\delta \cup W_2^\delta = \{w_1, \dots, w_s\} \cup \{w_{s+1}, \dots, w_h\}$ of $\{w_1, \dots, w_h\}$ and the partition $W_2^{\pi+\delta} \cup W_1^{\pi+\delta} = \{w_{h+1}, \dots, w_t\} \cup \{w_{t+1}, \dots, w_n\}$ of $\{w_{h+1}, \dots, w_n\}$. We can observe that the collections $W_1 = W_1^\delta \cup W_1^{\pi+\delta}$ and $W_2 = W_2^\delta \cup W_2^{\pi+\delta}$ both have zero sum by construction. Generally, the indices s, t are not uniquely determined, and hence, there can be different partitions. On the other hand, the following construction does not depend on the choice of the partition wherever there are more than one. Finally, we can assume without loss of generality that δ has magnitude such that $-\arg(b) < \delta \le \arg(b)$. Let us proceed with our construction.

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Let $(\mathbb{C}, dz)$ be a copy of the standard genus-zero meromorphic differential, and slit it along a geodesic segment b_1 with extremal points P_1 and $Q_1 = P_1 + \chi(\beta)$. Consider the collection W_1 above. According to our notation, we set $\zeta_1 = Q_1$ and thus define the point $\zeta_{l+1} = \zeta_l + w_l$, where $w_l \in W_1$ $(l = 1, \dots, s, t+1, \dots, n)$. Define $e = \overline{\zeta_1\, \zeta_{s+1}}$, and notice that $e \cap b_1 = \zeta_1$ because $\delta \neq -\arg(b)$. We next slit the structure $(\mathbb{C}, dz)$ along the edge e and label the resulting sides as $e^{\pm}$, where the sign is taken according to our convention. We partition the edge e^+ as follow: We define $e_l^+ = \overline{\zeta_l\, \zeta_{l+1}}$ for $l = 1, \dots, s$. Notice that these segments are pairwise adjacent or disjoint. Moreover, by construction, $e^+ = e_1^+ \cup \cdots \cup e_s^+$. In the same fashion, we partition the edge e^- as follows: We define $e_l^- = \overline{\zeta_l\, \zeta_{l+1}}$ for $l = t+1, \dots, n$. By construction, $e^- = e_{t+1}^- \cup \cdots \cup e_n^-$. We eventually slit $(\mathbb{C}, dz)$ along the ray $\overline{r}_1$ starting from ζ_1, orthogonal to e with oriented angle $\frac{\pi}{2}$ with respect to e. Finally, we introduce $n + s - t$ half-strips as follows: For any $l = 1, \dots, s$, we define $\mathcal{S}_l$ as an infinite half-strip bounded by the geodesic segment e_l^- and by two infinite rays $r_l^{\pm}$ pointing in the direction $\delta + \frac{\pi}{2}$; and for any $l = t+1, \dots, n$, the strip $\mathcal{S}_l$ is bounded by the geodesic segment e_l^+ and by two infinite rays $r_l^{\pm}$ pointing in the direction $\delta - \frac{\pi}{2}$.

We next consider another copy of $(\mathbb{C}, dz)$; we slit it along a geodesic segment b_2 with extremal points P_2 and $Q_2 = P_2 + \chi(\beta)$. Here, we consider the collection $W_2 = \{w_{s+1}, \dots, w_t\}$ above and then we similarly proceed as above: We set $\zeta_{s+1} = Q_2$ and thus define the point $\zeta_{l+1} = \zeta_l + w_l$, where $w_l \in W_2$. Define $\overline{e} = \overline{\zeta_{s+1}\, \zeta_{h+1}}$ and notice that $\overline{e} \cap b_2 = \zeta_{s+1}$ because $\delta \neq -\arg(b)$. We then slit the structure $(\mathbb{C}, dz)$ along the edge $\overline{e}$ and label the resulting sides as $\overline{e}^{\pm}$. As above the edge $\overline{e}^+$ is partitioned by segments $e_l^+ = \overline{\zeta_l\, \zeta_{l+1}}$ for $l = s+1, \dots, h$. By construction, $\overline{e}^+ = e_{s+1}^+ \cup \cdots \cup e_h^+$. In the same fashion, the edge e^- is partitioned by segments $e_l^- = \overline{\zeta_l\, \zeta_{l+1}}$ for $l = h+1, \dots, t$. It follows by construction that $\overline{e}^- = e_{h+1}^- \cup \cdots \cup e_t^-$. We eventually slit $(\mathbb{C}, dz)$ along $\overline{r}_{s+1}$, the ray starting from ζ_{s+1}, orthogonal to $\overline{e}$ with oriented angle $\frac{\pi}{2}$ with respect to $\overline{e}$. We introduce $t-s$ half-strips as follows: For any $l = s+1, \dots, h$, the infinite half-strip $\mathcal{S}_l$ is bounded by the geodesic segment e_l^- and by two infinite rays $r_l^{\pm}$ pointing towards the direction $\delta + \frac{\pi}{2}$; for any $l = h+1, \dots, t$, let $\mathcal{S}_l$ be an infinite half-strip bounded by the geodesic segment e_l^+ and by two infinite rays $r_l^{\pm}$ pointing in the direction $\delta - \frac{\pi}{2}$.

We can finally glue all the pieces together and then obtain the desired structure. We begin by some usual identifications, namely we identify the pair of edges b_j^- and b_{j+1}^+, for $j = 0, 1, 2$ and $b_3^+ = b_0^+$, and then the pair of edges a^+ and a^-. We next proceed with gluing the strips $\mathcal{S}_l$, for $l = 1, \dots, n$, in the following way. For any $l \notin \{1, s+1\}$, we paste the strip $\mathcal{S}_l$ by identifying e_l^+ with e_l^- and then the rays r_l^+ and r_l^- together. In the case $l \in \{1, s+1\}$, we paste the strip $\mathcal{S}_l$ in a different way, by first identifying the edges e_l^- with e_l^- and then identifying r_l^+ with $\overline{r}_l^-$ and r_l^- with $\overline{r}_l^+$; see Figure 25.

The resulting surface is a genus-one differential (Y, ξ) with one single zero, two double poles and $n-2$ simple poles. It remains to bubble sufficiently many copies of $(\mathbb{C}, dz)$ in order to get the desired pole orders and rotation number. Once identified together, the rays r_l^+ and r_l^- determine an infinite ray $\tilde{r}_l \subset (Y, \xi)$ joining the unique zero of ξ and a simple pole for any $l \notin \{1, s+1\}$. We bubble along any such a ray a copy of the genus-zero differential $(\mathbb{C}, z^{p-2}dz)$. Equivalently, we bubble $p-1$ copies of $(\mathbb{C}, dz)$. All simple poles are turned to higher-order poles of order p with nonzero residue. By construction, we can always find an infinite ray leaving from Q_2 and pointing towards the infinity. We bubble along such a ray a copy of $(\mathbb{C}, z^{p-2}dz)$. Finally, we can find an infinite ray leaving from P_1 and another leaving from Q_1 which are disjoint and both point towards the infinity. We bubble a copy of $(\mathbb{C}, z^{k-2}dz)$ along the first ray and a copy of $(\mathbb{C}, z^{p-k}dz)$ along the second ray. The final surface is a genus-one differential $(X, \omega) \in \mathcal{H}_1(np; -p, \dots, -p)$ where each pole has nonzero residue. By choosing α and β as in §5.2.1, one can show that (X, ω) has rotation number equal to $\gcd(k, p)$ as desired.

So far, we have focused on the case that the representation χ_1 has positive volume. The nonpositive volume case works in the same fashion with the only exception being that the interior of the parallelogram $\mathcal{P}$ defined by the above chain (55) is cut out from the first copy of $(\mathbb{C}, dz)$ considered above. The rest of the construction works *mutatis mutandis* for the case of nonpositive volume; see Figure 26. Alternatively, since χ is of nontrivial-ends type, Lemma 4.9 applies and one can find another pair of handle generators such that the respective volume is positive.

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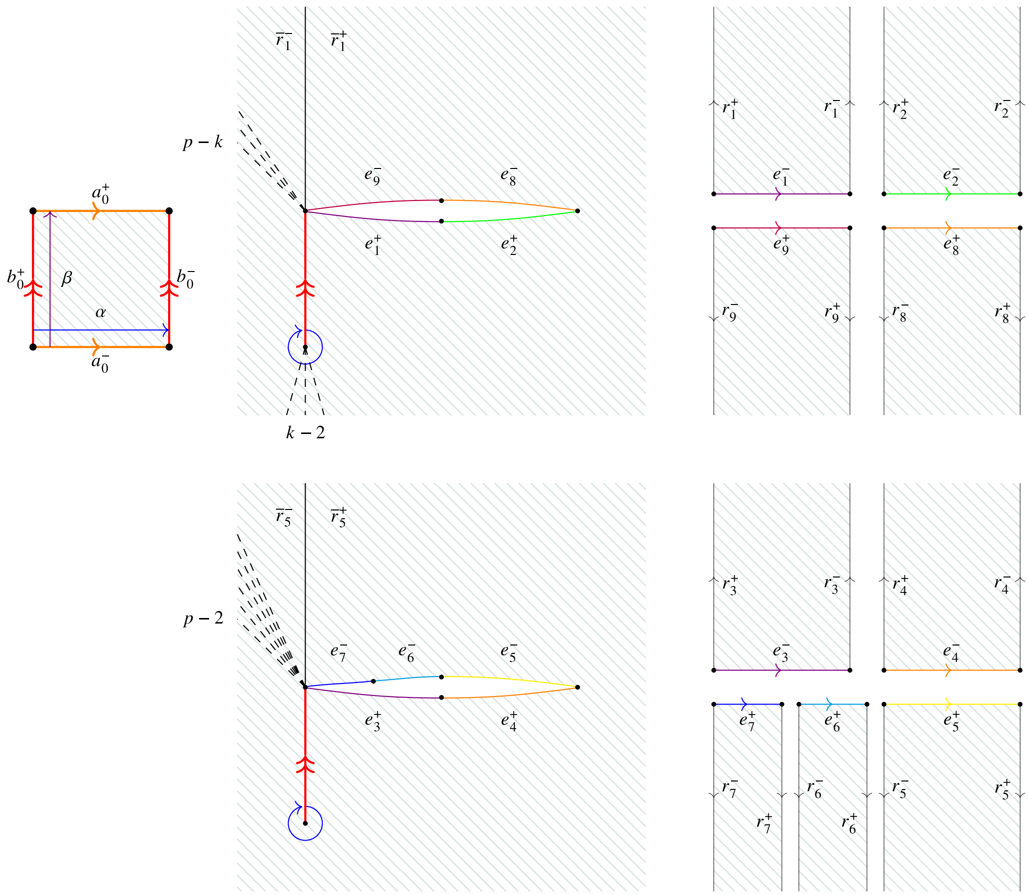

Figure 25. *An example to illustrate how to realize a representation χ of nontrivial-ends type with rational χ_n-part as the holonomy of a translation surface with poles in $\mathcal{H}_1(np; -p, \ldots, -p)$ with prescribed rotation number k. In this picture, $n = 9$ with $h = 4$, $s = 2$ and $t = 7$.*

Remark 5.13. There is an exceptional case not covered by the construction above, which is a representation of nontrivial-ends type $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as the holonomy of some genus-one differential in the connected component of the stratum $\mathcal{H}_1(2n, -2, \ldots, -2)$ of translation surfaces with rotation number one. Recall that here we assume that χ has rational χ_n-part and $\mathrm{Im}(\chi_n)$ does not satisfy the reordering property. Nevertheless, a slight modification of the previous construction permits to realize χ even in this special case. In short, the modification consists in pasting all the strips $\mathcal{S}_l$ in one copy of $(\mathbb{C}, dz)$. This can be simply done as follows. We define $e = \overline{\zeta_1\, \zeta_{h+1}}$ and let $e^{\pm}$ be the edges we obtain by slitting $(\mathbb{C}, dz)$ along e. Then we partition $e^+ = e_1^+ \cup \cdots \cup e_h^+$ and $e^- = e_{h+1}^- \cup \cdots \cup e_n^-$. The rest of the construction is essentially the same.

5.3.3. At least one pole has zero residue

Let $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation of nontrivial-ends type. As above, let γ_i denote a simple closed curve enclosing the i-th puncture. In this subsection, we assume that $\chi(\gamma_i) \neq 0$ for $i = 1, \ldots, m < n$ and $\chi(\gamma_i) = 0$ for $i = m+1, \ldots, n$. The representation χ naturally yields a new representation $\overline{\chi}\colon \mathrm{H}_1(S_{1,m}, \mathbb{Z}) \longrightarrow \mathbb{C}$ of nontrivial-ends type obtained by ‘filling’ the punctures with labelling $i = m+1, \ldots, n$.

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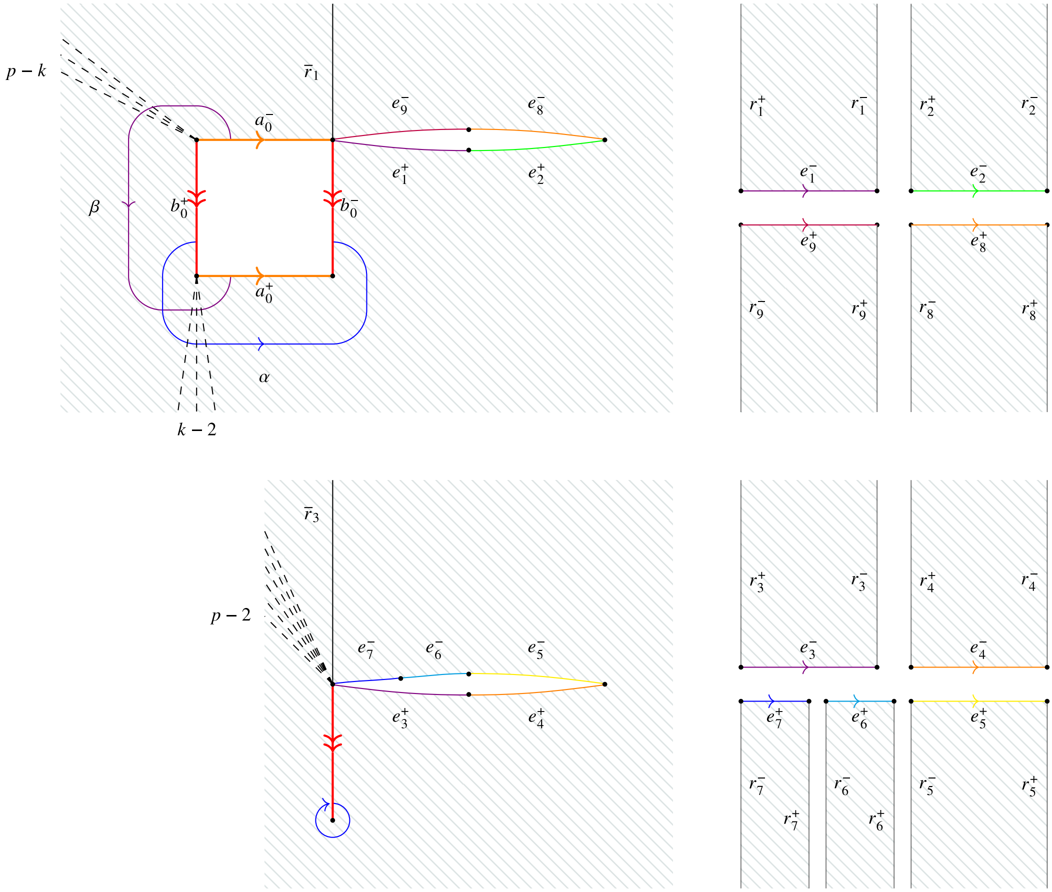

Figure 26. *An example to illustrate how to realize a representation χ of nontrivial-ends type with negative χ_1-part and with rational χ_n-part as the holonomy of a translation surface with poles in $\mathcal{H}_1(np;-p,\dots,-p)$ with prescribed rotation number k. In this picture, $n=9$ with $h=4$, $s=2$ and $t=7$.*

Let $\chi_m\colon \mathrm{H}_1(S_{0,m},\mathbb{Z}) \longrightarrow \mathbb{C}$ be the χ_m-part of the representation $\overline{\chi}$. We distinguish two cases as follows. If the representation χ_m is not real-collinear (see Definition 5.9), or it is real-collinear and satisfies the reordering property (see Definition 5.10), we need to introduce an auxiliary representation as in Definition 5.7:

$$\rho(\alpha)=\chi(\alpha),\quad \rho(\beta)=\chi(\beta),\quad \rho(\delta_1)=\cdots=\rho(\delta_{n-m+1})=0, \tag{57}$$

where $\{\alpha,\beta\}$ is a pair of handle generators of $\mathrm{H}_1(S_{1,n-m+1},\mathbb{Z})\subset \mathrm{H}_1(S_{1,n},\mathbb{Z})$. According to the constructions developed in Sections §5.1 and §5.2, we can realize ρ as the holonomy of some translation structure (X_1,ω_1) in the stratum $\mathcal{H}_1\big((n-m+1)p;-p,\dots,-p\big)$ with $n-m+1$ poles of order p with zero residue and prescribed rotation number k. Notice that, from our constructions it is always possible to find an infinite ray r with fixed direction, say v, starting from the zero of ω_1 and pointing towards a pole. For instance, such a ray can be taken as any ray leaving from any point labelled with 'Q' in our constructions and pointing towards a direction v. Next, we can realize χ_m as the holonomy of some translation surface $(X_2,\omega_2)\in\mathcal{H}_0\big(m-2;-1,\dots,-1\big)$. Up to changing the direction of v a little if needed, we can find an infinite ray, say $\overline{r}$, leaving from the zero of ω_2 and pointing towards one of the poles with direction v. We glue (X_1,ω_1) and (X_2,ω_2) by slitting and glue the rays r and $\overline{r}$ as described

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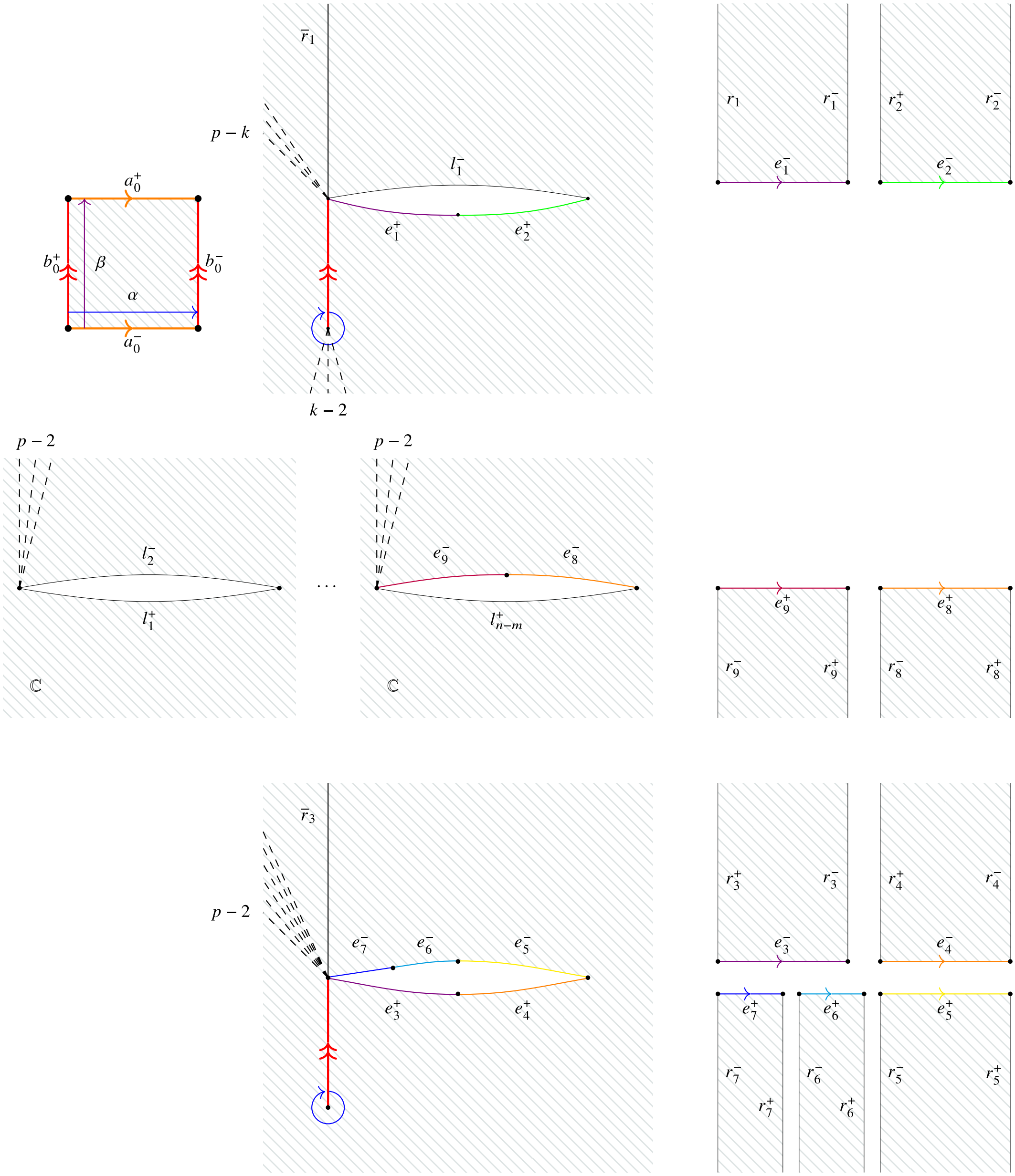

Figure 27. *How to realize a representation χ as the period character of some translation structure in $\mathcal{H}_1(np; -p, \dots, -p)$ when the χ_n-part is rational and some (not all) punctures have zero residue.*

in Definition 3.9. Notice that the resulting translation structure is homeomorphic to $S_{1,n}$, and it has one zero, $n-m$ poles of order p with zero residue, one pole of order p with nonzero residue and $m-1$ simple poles. We can finally find $m-1$ rays joining the zero of the resulting translation structure and the simple poles. We bubble along each one of these rays a copy of $(\mathbb{C}, z^{p-2}dz)$. This final translation structure lies in $\mathcal{H}_0(np; -p, \dots, -p)$. By construction it has period character χ and rotation number k as desired.

In the case that χ_m is rational and it does not satisfy the reordering property, we need to modify a little our construction in paragraph §5.3.2.2 as shown in Figure 27. We adopt the same notation as above with the only difference being that n is now replaced by m. Let $w_i = \chi(\gamma_i)$ for $i = 1, \dots, m$, and consider the collection of $W = \{w_1, \dots, w_m\} \subset \mathbb{C}^*$. Since χ_m does not satisfy the reordering property, there are two integers $s < t$ such that $W = W_1 \cup W_2 = \{w_1, \dots, w_s, w_{t+1}, \dots, w_m\} \cup \{w_{s+1}, \dots, w_t\}$. We introduce the same collections of (possibly unbounded) polygons that comprise the following pieces. A parallelogram $\mathcal{P}$ determined by the chain (55). A copy of $(\mathbb{C}, dz)$ slit along the edges $b = \chi(\beta)$ and e and along an infinite ray $\overline{r}_1$. A second copy of $(\mathbb{C}, dz)$ slit along the edges $b = \chi(\beta)$ and $\overline{e}$ and along an infinite ray $\overline{r}_{s+1}$ and, finally, m strips $\mathcal{S}_l$ for $l = 1, \dots, m$. As in paragraph §5.3.2.2, we slit e and we partition the edge e^+ into s subsegments each of length $|w_l|$ for $l = 1, \dots, s$. Unlike above, here we denote $e^- = l_1^-$. We next introduce $n - m$ new pieces, each of which is a copy of $(\mathbb{C}, dz)$. We slit them along e, that is, a segment congruent and with the same direction as e, and we denote the resulting edges l_i^+ and l_{i+1}^- for $i = 1, \dots, n - m$. Finally, we partition l_{n-m+1}^- into $n - t$ segments, say e_l^-, each of which of length $|w_l|$ for $l = t + 1, \dots, n$. Now, we paste all the pieces as done above by identifying the edges with the same label (and opposite sign) and eventually bubbling copies of $(\mathbb{C}, dz)$ in order to have all poles of order p. The resulting translation surface lies in $\mathcal{H}_1(np; -p, \dots, -p)$. In particular, bubbling copies of $(\mathbb{C}, dz)$ properly, we can realize a structure so that its rotation number is k as desired. The case of nonpositive volume works *mutatis mutandis*.

Remark 5.14. Since our construction relies on the discussion in paragraph §5.3.2.2, similarly in this case there is an exceptional case not covered by the construction above, which is to realize a representation of nontrivial-ends type $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as the holonomy of some genus-one differential in the connected component of the stratum $\mathcal{H}_1(2n, -2, \dots, -2)$ of translation surfaces with rotation number one. However, this issue can be solved by using the same construction mentioned in Remark 5.13 by adding $n - m$ copies of $(\mathbb{C}, z^{p-2}dz)$ each of which is slit along e.

5.4. General cases

We finally consider strata of genus-one differentials with poles of different orders and possibly multiple zeros. The following statements are now corollaries of the lemmas proved in the previous Sections §5.1–§5.3.

Corollary 5.15. *Let $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be any nontrivial representation. If χ can be realized in the stratum $\mathcal{H}_1(m; -p_1, \dots, -p_n)$, then it appears as the period character of a translation surface with poles in each connected component.*

Proof. We begin with noticing that $m = p_1 + \cdots + p_n$, and hence, the following condition holds

$$\gcd(m, p_1, \dots, p_n) = \gcd(p_1, \dots, p_n) = p. \tag{58}$$

For any k dividing p, we can realize the representation χ as the holonomy of some translation structure (X, ω) in the stratum $\mathcal{H}_1(np; -p, \dots, -p)$ with rotation number k as done in the previous sections. According to our constructions, we can always find n rays r_i joining the zero of ω with the puncture P_i. By bubbling $p_i - p$ copies of $(\mathbb{C}, dz)$ along the ray r_i, we can realize a translation surface $(Y, \xi) \in \mathcal{H}_1(m; -p_1, \dots, -p_n)$. In particular, we can choose these rays in such a way that the rotation number remains unaffected after any bubbling – for instance, by using the notation in Sections §5.1–§5.2, choose the ray r_i as any ray leaving the point Q_i; see the dotted lines in Figures 8, 9, 10 and 11. This completes the proof. □

Remark 5.16. Recall that our approach is to use an inductive foundation having as the base case strata with poles of order 2, as the intermediate step the case of strata with all poles of order $p \geq 3$ and then the general case. This motivates the reason why the proof of Corollary 5.15 relies on the previous constructions which are explicit.

Corollary 5.17. *Let $\chi\colon \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ be any nontrivial representation. If χ can be realized in the stratum $\mathcal{H}_1(m_1,\dots,m_k;-p_1,\dots,-p_n)$, then it appears as the period character of a translation surface with poles in each connected component.*

Proof. It is sufficient to observe that if $d = \gcd(m_1,\dots,m_k,p_1,\dots,p_n)$, then d divides $m = m_1 + \cdots + m_k$. Therefore, we can realize χ as the period character of a translation surface with a single zero of order m in the stratum $\mathcal{H}_1(m;-p_1,\dots,-p_n)$. Then we can break the single zero as described in Section §3.1 to get the desired structure. □

6. Higher genus meromorphic differentials with hyperelliptic structure

We begin to prove Theorem A for surfaces of genus at least two and we complete the proof in Section §7. Given a nontrivial representation $\chi\colon \mathrm{H}_1(S_{g,n},\,\mathbb{Z}) \longrightarrow \mathbb{C}$, in the present section we shall determine whether χ appears as the period character of a hyperelliptic translation surface with poles, that is, a translation surface admitting a special symmetry of order two already introduced in Section §2.2.2; see Definition 2.14. More precisely, our aim is to prove the following:

Proposition 6.1. *Let χ be a nontrivial representation and suppose it arises as the period character of some meromorphic genus g differential in a stratum admitting a hyperelliptic component. Then χ can be realized as the period character of some hyperelliptic translation surfaces with poles in the same stratum.*

According to Boissy (see [Boi15, Proposition 5.3] and Section §2.3 above), a stratum admits a connected component of hyperelliptic translation surfaces if and only if it is one of the following:

$$\mathcal{H}_g(2m;-2p),\ \ \mathcal{H}_g(m,m;-2p),\ \ \mathcal{H}_g(2m;-p,-p),\ \ \mathcal{H}_g(m,m;-p,-p), \tag{59}$$

for some $1 \le p \le m$. Therefore, in what follows we assume that χ can be realized as the period character of some translation surface in one of those strata. We shall consider two cases according to whether the representation χ is or is not of trivial-ends type. In Sections §6.2, §6.3 and §6.4, we prove Proposition 6.1 for strata of meromorphic differentials with exactly one zero of maximal order and we derive the general case by breaking zeros (see Section §3.1), as similarly done previously for genus-one differentials. Finally, in Section §6.5, we shall derive Theorem A for strata of genus-two meromorphic differentials listed above. We first premise the following.

6.1. *Change of basis strategy*

Let $\chi\colon \mathrm{H}_1(S_{g,n},\,\mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation. Let $\mathcal{G} = \{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g\}$ be a system of handle generators for $\mathrm{H}_1(S_{g,n},\,\mathbb{Z})$, and define pairs $\{\,a_i,\ b_i\,\}_{1\le i\le g}$ as

$$\chi(\alpha_i) = a_i \ \text{ and }\ \chi(\beta_i) = b_i. \tag{60}$$

Notice that the image of χ is a $\mathbb{Z}$–module $M_\chi = \mathbb{Z}[\,a_1,\ b_1,\dots,a_g,\ b_g\,] \subset \mathbb{C}$ generated by the ordered basis $\mathcal{B} = \{a_1,b_1,\dots,a_g,b_g\}$. Two remarks are in order.

Remark 6.2. M_χ as a module on its own right, it appears as the image of different representations not necessarily in the same $\mathrm{Mod}(S_{g,n})$-orbit.

Remark 6.3. A representation χ is real-collinear if there exists an element $A \in \mathrm{GL}(1,\,\mathbb{C}) < \mathrm{GL}^+(2,\mathbb{R})$ such that $A\cdot M_\chi \subset \mathbb{R}$.

We may notice that M_χ is totally real if and only if χ is real-collinear. Being $M_\chi \cong \mathbb{Z}^{2g}$, we notice that $\mathrm{Aut}(M_\chi)$ identifies with $\mathrm{SL}(2g,\mathbb{Z})$. In what follows, it will be convenient to define new pairs $\{u_i, v_i\}_{1\le i\le g}$ of complex numbers such that

$$\begin{array}{llllll} & -v_1 & & & & = a_1 \\ u_1 & & & & & = b_1 \\ u_1 & +v_1 & & -v_2 & & = a_2 \\ u_1 & +v_1 & +u_2 & & & = b_2 \\ & & & & & \vdots \\ u_1 & +v_1 & +u_2 & +v_2 & \quad -v_g & = a_g \\ u_1 & +v_1 & +u_2 & +v_2 & +u_g & = b_g \end{array} \tag{61}$$

Notice that the matrix A_g associated to the linear system above has the form

$$A_g = \begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 1 & & & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & A_{g-1} & & \\ \vdots & \vdots & & & & & \\ 1 & 1 & & & & & \end{pmatrix} \tag{62}$$

which is recursively defined where $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Remark 6.4. It is an easy matter to check that $A_g \in \mathrm{SL}(2g,\mathbb{Z})$ for every $g \ge 1$, and hence, the linear system (61) has a unique solution.

In order to prove Proposition 6.1, the gist of the idea is to create (possibly noncompact) polygons whose boundary contains segments of the form $P \longmapsto P + u_i$ and $P \longmapsto P + v_i$ and such that, once glued together in a proper way, the resulting translation surface admits a hyperelliptic involution and period character χ. For this purpose, it will be convenient to assume that $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. However, this is not guaranteed unless we replace the system of handle generators $\mathcal{G}$ with a new one. More precisely, the following holds.

Lemma 6.5 (Technical Lemma). *Given a period character $\chi\colon \mathrm{H}_1(S_{g,n},\mathbb{Z}) \longrightarrow \mathbb{C}$, there exists a system of handle generators $\mathcal{G} = \{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g\}$ with periods $(a_1,b_1,\dots,a_g,b_g)$ such that the linear system*

$$\begin{cases} \quad -y_1 & = a_1 \\ x_1 & = b_1 \\ x_1 + y_1 \quad -y_2 & = a_2 \\ x_1 + y_1 + x_2 & = b_2 \\ & \\ x_1 + y_1 + x_2 + y_2 \quad -y_g & = a_g \\ x_1 + y_1 + x_2 + y_2 + x_g & = b_g \end{cases} \tag{63}$$

admits a solution $(u_1,v_1,\dots,u_g,v_g) \in \mathbb{C}^{2g}$, where $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Moreover, such a system $\mathcal{G}$ can be found starting from any given system and then applying Dehn twists recursively to pairs $\{\alpha_i,\beta_i\}$.

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Proof. Let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be any system of handle generators with periods $(a_1, b_1, \ldots, a_g, b_g) \in \mathbb{C}^{2g}$, and let $(u_1, v_1, \ldots, u_g, v_g) \in \mathbb{C}^{2g}$ be the solution of Equation (63). If the condition $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ holds for every $i = 1, \ldots, g$, we are done. Next, we assume this is not the case and use induction to show the desired result. Let $k = 1$. If the pair (u_1, v_1) does not satisfy the desired property then replace the pair $\{\alpha_1, \beta_1\}$ with either $\{\alpha_1^{-1}, \beta_1^{-1}\}$, $\{\beta_1, \alpha_1^{-1}\}$ or $\{\beta_1^{-1}, \alpha_1\}$ – notice that this can be done by using a mapping class element. The first two equations of Equation (63) are then replaced with either

$$\begin{cases} & -y_1 = -a_1 \\ x_1 & \quad\; = -b_1 \end{cases}, \quad \begin{cases} & -y_1 = \;\; b_1 \\ x_1 & \quad\; = -a_1 \end{cases} \quad \text{or} \quad \begin{cases} & -y_1 = -b_1 \\ x_1 & \quad\; = \;\; a_1 \end{cases} \tag{64}$$

that have solutions $(-u_1, -v_1)$, $(v_1, -u_1)$ and $(-v_1, u_1)$, respectively.

Since the condition $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ holds for exactly one of these pairs, the result clearly holds for $k = 1$. With a small abuse of notation, denote let us denote this pair by (u_1, v_1). Now, we assume the result holds for the first $k - 1$ pairs $(u_i, v_i) \in \mathbb{C}^2$. Notice that the vector w_{k-1} defined as

$$w_{k-1} = \sum_{i=1}^{k-1} u_i + v_i$$

satisfies $\arg\left(w_{k-1}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Consider the following subsystem of Equation (63)

$$\begin{cases} w_{k-1} & \quad -y_k = a_k \\ w_{k-1} & +x_k \quad\; = b_k \end{cases} \iff \begin{cases} x_k = +b_k - w_{k-1} \\ y_k = -a_k + w_{k-1} \end{cases}. \tag{65}$$

We claim there is an element

$$M_k = \begin{pmatrix} p_k & q_k \\ s_k & t_k \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z}) \tag{66}$$

such that the following subsystem of Equation (63)

$$\begin{cases} w_{k-1} & \quad -y_k = p_k\, a_k + s_k\, b_k \\ w_{k-1} & +x_k \quad\; = q_k\, a_k + t_k\, b_k \end{cases} \tag{67}$$

admits a solution (u_k, v_k) with $\arg(u_k), \arg(v_k) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

In the case χ is real collinear, we may assume $M_\chi \subset \mathbb{R}$ and $w_{k-1} > 0$ at each step. It is easy to see that up to applying Dehn twists, we can make both $u_i, v_i > 0$ and the result easily follows in this case. Assume χ to be not real-collinear, so at least one handle generator is not real. There always exists a mapping class $\varphi \in \mathrm{Mod}(S_{g,n})$ such that every handle generator is not real and we assume to already be in this case.

Since χ is not real collinear, we can use a mapping class in $\mathrm{Sp}(2, \mathbb{Z})$ to assume that $\Re(\,a_k\,) < 0$, where $\Re(\cdot)$ denotes the real part. Since both $-a_k$ and w have positive real part, it follows that

$$\Re(y_k) = \Re\big(-a_k + w_{k-1}\big) > 0; \tag{68}$$

notice that it cannot be zero. We next replace β_k with $\alpha_k^{-n}\,\beta_k$ by using a mapping class in $\mathrm{Sp}(2, \mathbb{Z})$ and we may observe that

$$\arg(x_k) = \arg\big(-n\, a_k + b_k - w_{k-1}\big) \longrightarrow \arg\big(-a_k\big) \tag{69}$$

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for $n \to \infty$. Since $\Re(a_k) < 0$, for $|n|$ big enough it follows that $\Re(x_k) = \Re(-n\,a_k + b_k + w_{k-1}) > 0$. Finally, M_k is then the compositions of all mapping class elements applied along the proof and the desired claim follows. □

Remark 6.6. We may finally notice that the system of handle generators $\mathcal{G}$ just defined is far from being unique. In fact, for every $k \geq 2$, at the k−th step there is $n_k \in \mathbb{Z}$ such that $2\arg(-n\,a_k + b_k + w_{k-1}) \in (-\pi, \pi)$ for all $n \geq n_k$.

Given any point $P_o \in \mathbb{C}$, we now define a chain $\mathcal{C}$ as follows

$$\begin{aligned} P_0 \mapsto P_0 + u_1 = P_1 \mapsto P_0 + u_1 + v_1 = P_2 \mapsto P_0 + u_1 + v_1 + u_2 \\ = P_3 \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g}(u_i + v_i) = P_{2g}. \end{aligned} \tag{70}$$

Notice that, since $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$, the chain just defined is embedded in $\mathbb{C}$.

Let r be the straight line passing through P_0 and P_{2g}, and let $\mathcal{C}_{\text{ext}}$ be the extended chain of edges $\mathcal{C} \cup \overline{P_{2g}\,P_0}$. Notice that $\mathcal{C}_{\text{ext}}$ always bounds a possibly self-overlapping polygon in $\mathbb{C}\mathbf{P}^1$ on its right, but, in principle, it may not bound a polygon in $\mathbb{C}$ on its *left*. The following claim states that we can always change the pair (u_g, v_g) above so that $\mathcal{C}_{\text{ext}}$ bounds a polygon both on the left and on the right. This is equivalent to say that $\mathcal{C}$ lies on the right of r oriented from P_0 to P_{2g}. More precisely, we can refine Lemma 6.5 as follows.

Corollary 6.7. *With the same notation above, if χ is not real-collinear, then there exists a system of handle generators $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ such that the chain $\mathcal{C}_{\text{ext}}$ bounds a polygon on its left, that is, $\mathcal{C}$ entirely lies on the right of the line r.*

Proof. If $\mathcal{C}$ already lies on the right of r, there is nothing to prove, and we are done. If $\mathcal{C}$ does not entirely lie on the right of r, there is no such a polygon bounded on the left of $\mathcal{C}_{\text{ext}}$. We show that, by changing the last pair $\{\alpha_g, \beta_g\}$ of handle generators, the chain $\mathcal{C}_{\text{ext}}$ bounds a compact embedded polygon on its left. Therefore, by following the proof of Lemma 6.5, suppose we have already found $(u_1, v_1, \ldots, u_{g-1}, v_{g-1}) \in \mathbb{C}^{2g-2}$ and let us show how to determine the last pair $(u_g, v_g) \in \mathbb{C}^2$ such that the desired conclusion holds.

Since χ is not real-collinear, up to replacing the pair $\{\alpha_g, \beta_g\}$ with another one by using a judicious mapping class, we may assume

$$\max_{1 \leq j \leq 2g-2}\left\{0, \arg\left(\overline{P_0\,P_j}\right)\right\} < \arg(b_g) \leq \frac{\pi}{2}, \tag{71}$$

where $\overline{P_0\,P_j}$ is the segment joining the points P_0 and P_j of $\mathcal{C}$. Assume this to be in this case. We then apply the following mapping class

$$\begin{pmatrix} 1 & -1 \\ -m & m+1 \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{Z}) \tag{72}$$

to replace $\{\alpha_g, \beta_g\}$ with $\{\alpha_g\,\beta_g^{-m}, \alpha_g^{-1}\,\beta_g^{m+1}\}$. Notice that

$$\arg\left(a_g - mb_g\right) \longrightarrow -\arg(b_g) \quad \text{and} \quad \arg\left(-a_g + (m+1)b_g\right) \longrightarrow \arg(b_g) \tag{73}$$

for $m \longrightarrow +\infty$. Following the proof of Lemma 6.5, let us focus on the pair (u_g, v_g). We have that

$$u_g = -a_g + (m+1)b_g - w_{g-1} \quad \text{and} \quad v_g = -a_g + mb_g + w_{g-1}. \tag{74}$$

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Notice that the following chain of inequalities holds:

$$\max_{1\le j\le 2g-2}\left\{0,\ \arg\left(\overline{P_0\,P_j}\right)\right\} < \arg(u_g),\ \arg(v_g) \approx \arg(b_g) \tag{75}$$

and $\left|-a_g + mb_g + w_{g-1}\right| \longrightarrow \infty$ for $m \longrightarrow \infty$. As a consequence, there exists m such that the chain

$$P_0 \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g}(u_i + v_i) = P_{2g} \mapsto P_0 \tag{76}$$

bounds a polygon on its left as desired. □

From now on, in every forthcoming subsection, we shall assume $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Moreover, we assume $\mathcal{C}_{\text{ext}}$ always bounds a polygon on both sides; this is a strong condition for the most of constructions, but it will be necessary in §6.4.1.1 and §6.4.2.

6.2. *Poles with zero residue*

In the present subsection, we shall consider representations of trivial-end type (see Definition 2.8), and we distinguish two cases depending on whether $n = 1$ or $n = 2$.

6.2.1. One higher order pole

According to Section §8.2, a representation $\chi\colon \mathrm{H}_1(S_{g,1},\mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized in a stratum $\mathcal{H}_g(2m;-2p)$, for any $m \ge p \ge 1$, as long as χ is nontrivial. Therefore, our aim here is to realize any such a representation as the period character of some hyperelliptic translation surface with poles. Let $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a system of handle generators for $\mathrm{H}_1(S_{g,1},\mathbb{Z})$; see Definition 4.2. Given a nontrivial representation $\chi\colon \mathrm{H}_1(S_{g,1},\mathbb{Z}) \longrightarrow \mathbb{C}$, we define

$$\chi(\alpha_i) = a_i \quad\text{and}\quad \chi(\beta_i) = b_i. \tag{77}$$

Since χ is nontrivial, Lemma 4.4 applies, and we can assume that both a_i and b_i are nonzero for all $i = 1, \ldots, g$. We now define new pairs $\{u_i, v_i\}_{1\le i\le g}$ of complex numbers as in §6.1 that we shall use to realize the representation χ as the period character of the desired structure. Let $P_0 \in \mathbb{C}$ be any point and consider the chain, say $\mathcal{C}_1$, of segments defined as in Equation (70), that is,

$$\begin{aligned} P_0 \mapsto P_0 + u_1 = P_1 \mapsto P_0 + u_1 + v_1 = P_2 \mapsto P_0 + u_1 + v_1 + u_2 \\ = P_3 \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g}(u_i + v_i) = P_{2g} \end{aligned} \tag{78}$$

and recall that it is not self-intersecting because of our assumptions from §6.1. Let r_1 be a half-ray leaving from P_0 and parallel to $\mathbb{R}^- = \{x \in \mathbb{R} \,|\, x < 0\}$ with the usual orientation, hence pointing rightwards. Similarly, we define r_2 as the half-ray leaving from P_{2g} and parallel to $\mathbb{R}^+$, hence pointing rightwards. We define H_1 as the (broken) half-plane bounded by the half-rays r_1 and r_2 and the chain $\mathcal{C}_1$ on their left; that is, H_1 is bounded by r_1^-, r_2^- and $\mathcal{C}_1^-$ (see Figure 28), where the sign is used according to the usual convention.

On another copy of $(\mathbb{C}, dz)$, let us now define a second chain, say $\mathcal{C}_2$ as follows:

$$\begin{aligned} P_0 \mapsto P_0 + v_g = P_1' \mapsto P_0 + v_g + u_g = P_2' \mapsto P_0 + v_g + u_g + v_{g-1} \\ = P_3' \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g}(u_i + v_i) = P_{2g}. \end{aligned} \tag{79}$$

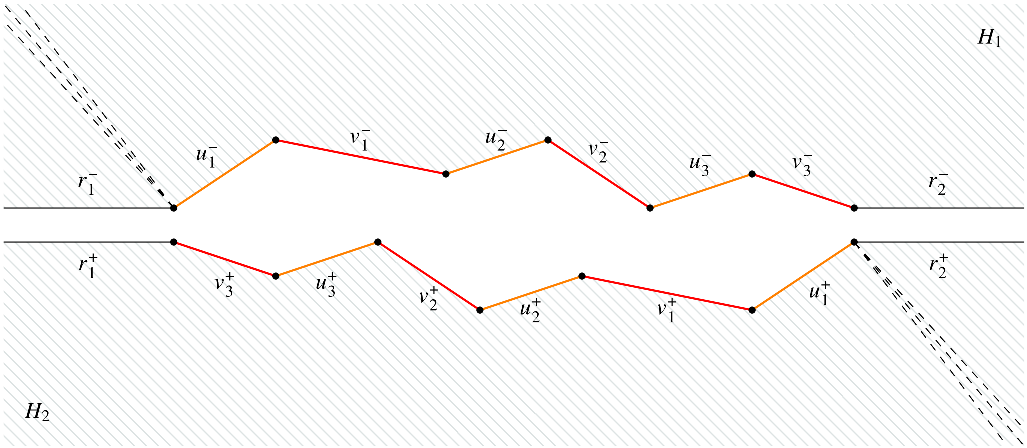

Figure 28. *Realization of a hyperelliptic genus g meromorphic differential with a single zero of order* $2m = 2g + 2p - 2$*, a single pole of order* $2p$ *and prescribed periods. The figure depicts the case* $g = 3$.

Notice that it differs from $\mathcal{C}_1$ by a rotation of order two about the midpoint of the segment $\overline{P_0 P_{2g}}$. Moreover, the half-rays r_1 and r_2 are swapped by this rotation. Define H_2 as the (broken) half-plane bounded by the rays r_1 and r_2 and the chain $\mathcal{C}_2$ on their right; that is, H_2 is bounded by the rays r_1^+ and r_2^+ and $\mathcal{C}_2^+$.

Next, we identify the rays r_1^+ and r_1^- together. In the same fashion, we also identify the rays r_2^+ and r_2^- together. The resulting is a $4g$-gon bounding a topological immersed disk on the Riemann sphere $\mathbb{C}\mathbf{P}^1$ punctured at the infinity. We finally identify the edges with the same label; that is, we identify u_i^+ with u_i^- and v_i^+ with v_i^- for all $i = 1, \dots, g$. The resulting space is a topological genus g surface with one puncture equipped with a translation structure (X, ω) with one branch point of order $2g$ and one pole of order 2 and period character χ. By construction, such a structure admits an involution ι induced by the rotation of order two that exchanges C_1 with C_2. Notice that ι has $2g + 2$ fixed points corresponding to the midpoints of the edges u_i, v_i – these are $2g$ points – the unique zero of ω and the point at infinity. Therefore, the resulting structure is hyperelliptic as desired, and lies in the stratum $\mathcal{H}_g(2g; -2)$.

It remains to show that the resulting structure has the prescribed periods. More precisely, we now determine close loops representing the desired handle generators $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$. These are shown in Figure 29, and we can easily see that

$$\begin{array}{lllllllll}
\chi(\alpha_1) = & -v_1 & & & & & & = a_1 \\
\chi(\beta_1) = & u_1 & & & & & & = b_1 \\
\chi(\alpha_2) = & u_1 & +v_1 & & -v_2 & & & = a_2 \\
\chi(\beta_2) = & u_1 & +v_1 & +u_2 & & & & = b_2 \\
\quad\vdots & & & & & & & \\
\chi(\alpha_g) = & u_1 & +v_1 & +u_2 & +v_2 & & -v_g & = a_g \\
\chi(\beta_g) = & u_1 & +v_1 & +u_2 & +v_2 & +u_g & & = b_g
\end{array} \tag{80}$$

as desired.

Remark 6.8. Such a system of handle generators can be found in all constructions we shall realize in the following. However, we avoid to draw pictures similar to Figure 29.

In order to get a structure in the stratum $\mathcal{H}_g(2m; -2p)$, we need to modify the broken half-planes as follows. Consider on H_1 a ray leaving from P_0 with angle $0 < \theta < \frac{\pi}{2}$ with respect to r_1^- and bubble $p - 1$ copies of $(\mathbb{C}, dz)$; see Definition 3.12. Similarly, in H_2 consider a ray leaving from P_{2g} with angle θ with respect to r_2^+ and then bubble $p - 1$ copies of $(\mathbb{C}, dz)$. Then proceed as above. The resulting space

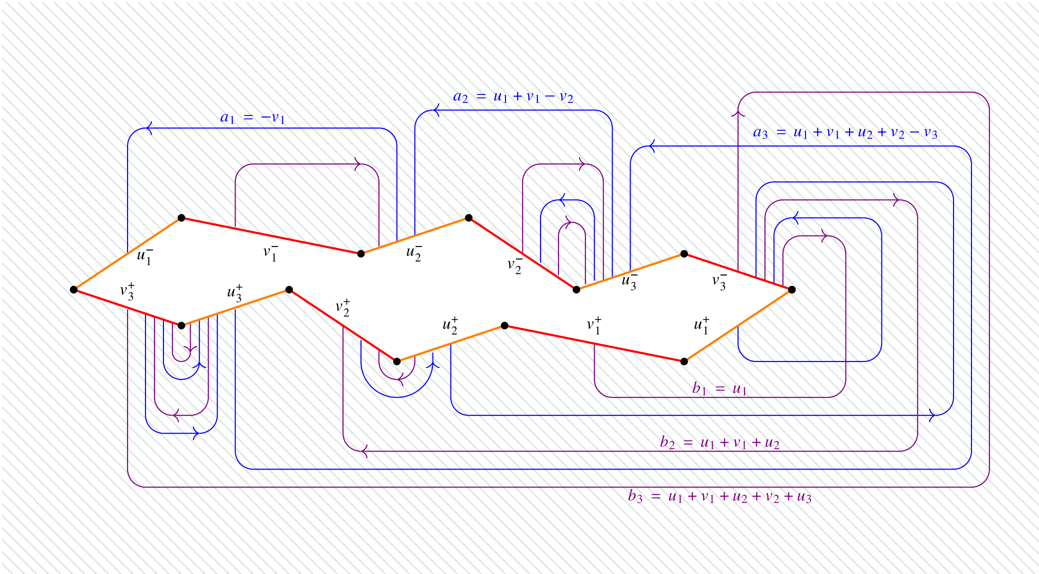

Figure 29. *System of handle generators with prescribed periods for a genus g meromorphic differential with hyperelliptic involution. The figure depicts the case $g = 3$.*

is again a genus g surface equipped with a translation structure with period character χ in the stratum $\mathcal{H}_g(2m; -2p)$ by construction, where $m = g + p - 1$.

By breaking the zero of the structure obtained as above as two zeros both of order m, Lemma 3.4 implies that the resulting translation surface lies in the hyperelliptic component of $\mathcal{H}_g(m, m; -2p)$, and it has period character χ as desired.

6.2.2. Two higher-order poles

Let $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation of trivial-ends type. We aim to realize χ as the period character of some hyperelliptic translation surface in the stratum $\mathcal{H}_g(2m; -p, -p)$, where $p > 1$. We assume χ to be nontrivial because, as we shall in §8.2, the trivial representation cannot be realized in any strata of the form $\mathcal{H}_g(2m; -p, -p)$ and $\mathcal{H}_g(m, m; -p, -p)$.

This case follows from a modification of the construction developed in Section §6.2.1. We shall adopt the same notation as above. Given a nontrivial representation χ as above, we define

$$\chi(\alpha_i) = a_i \quad \text{and} \quad \chi(\beta_i) = b_i. \tag{81}$$

Once again, since χ is nontrivial, Lemma 4.4 applies, and hence, we can assume that both a_i and b_i are nonzero. We use again the change of basis strategy (see §6.1) to define new pairs $\{u_i, v_i\}_{1 \le i \le g}$ such that $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ for all $i = 1, \ldots, g$. Given any point $P_0 \in \mathbb{C}$, we define the broken half-plane H_1 exactly as in Section §6.2.1, that is, a half-plane bounded by two half-rays r_1 and r_2 and a chain of segments, say $\mathcal{C}_1$, defined as in Equation (70), joining P_0 with $P_{2g} = P_0 + \sum(u_i + v_i)$. Notice that $r_1 \cup \mathcal{C}_1 \cup r_2$ bounds H_1 on its left.

In the same fashion, we define the broken half-plane H_2 exactly as in Section §6.2.1, that is, a half-plane bounded by two half-rays r_1 and r_2 and a chain of segments, say $\mathcal{C}_2$, defined as in Equation (79), joining the points P_0 and P_{2g}. Notice that, in this case, $r_1 \cup \mathcal{C}_2 \cup r_2$ bounds H_2 on its right.

Next, we single out two broken half-planes, say H_3 and H_4, defined as follows. Consider first the unique segment, say e, joining P_0 and P_{2g}. Recall that $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$, so $\Re(P_{2g}) \ge \Re(P_0)$. We next extend e with the half-ray s_1 leaving from P_0 and parallel to $\mathbb{R}^-$ and the half-ray s_2 leaving from P_{2g} and parallel to $\mathbb{R}^+$. The chain $s_1 \cup e \cup s_2$ is embedded in $\mathbb{C}$; in fact, it bounds an embedded triangle

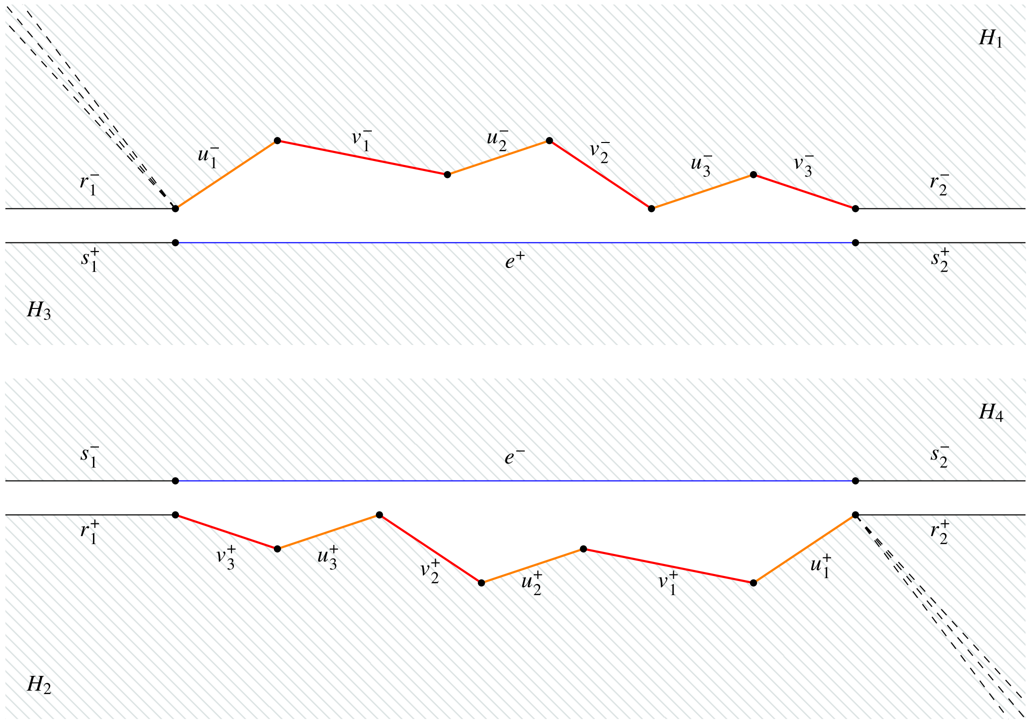

Figure 30. *Realization of a hyperelliptic genus g meromorphic differential with a single zero of order* $2m = 2g + 2p - 2$*, two poles each of order* $p \geq 2$ *and prescribed periods. The figure depicts the case* $g = 3$.

with one vertex at the infinity and splits the complex plane in two half-planes. Let H_3 be the broken half-plane on the right of $s_1 \cup e \cup s_2$, and let H_4 be the broken half-plane on the left. By design, the following identification hold $s_1 = r_1$ and $s_2 = r_2$.

We finally glue these broken half-planes as follows; see Figure 30. The edges $e^+ \subset H_3$ and $e^- \subset H_4$ are identified. Then, for $j = 1, 2$, we identify $r_j^- \subset H_1$ with $s_j^+ \subset H_3$. Similarly, for $j = 1, 2$, we identify the rays $r_j^+ \subset H_2$ and $s_j^- \subset H_4$. Finally, we identify u_i^+ with u_i^- and v_i^+ with v_i^- for all $i = 1, \ldots, g$. The resulting object is a translation surface (X, ω) with a single zero of order $2g + 2$ and two poles both of order 2 and zero residue. The resulting structure admits a hyperelliptic involution in the sense of Definition 2.14 that swaps $H_1 \cup H_3$ with $H_2 \cup H_4$. In fact, even in this case there is an involution ι that fixes $2g + 2$ points corresponding to the midpoints of the edges, the unique zero of ω and the midpoint of the edge e.

In order to get a structure in the stratum $\mathcal{H}_g(2m; -p, -p)$, we can modify the broken half-planes as in Section §6.2.1. On H_1, we consider a ray leaving from P_0 with angle $0 < \theta < \frac{\pi}{2}$ with respect to r_1^- and bubble along it $p - 1$ copies of $(\mathbb{C}, dz)$. Similarly, on H_2 we consider a ray leaving from P_{2g} with angle θ with respect to r_2^+ and bubble $p - 1$ copies of $(\mathbb{C}, dz)$. Then proceed as above. The resulting object is a genus g surface equipped with a translation structure with period character χ in the stratum $\mathcal{H}_g(2m; -p, -p)$ by construction, where $m = g + p - 1$.

Again, by breaking the zero of the structure obtained as above as two zeros both of order m, Lemma 3.4 implies that the resulting translation surface lies in the hyperelliptic component of $\mathcal{H}_g(m, m; -p, -p)$, and it has period character χ as desired.

6.3. Higher-order poles with nonzero residue

Suppose $p \geq 2$, and let $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation of *non*trivial-ends type. We aim to realize χ as the period character of some hyperelliptic translation surface in the stratum

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$\mathcal{H}_g(2m;-p,-p)$. In this case, we realize a hyperelliptic translation surface with poles by extending the construction developed in §6.2.2. Let $u_i, v_i \in \mathbb{C}^*$ be defined as in §6.2.2, and recall that $\arg(u_i), \arg(v_i) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Let γ be a simple loop around a puncture, say P, since χ is of nontrivial-ends type; let $\chi(\gamma) = w \in \mathbb{C}^*$. Notice that we can assume $\arg(w) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$ if necessary. In fact, if Q is the second puncture and δ is a simple loop around oriented so that $\chi(\delta) = -w$ – notice that the minus sign appears because of the residue theorem. We shall distinguish two cases depending on whether w is parallel to $v = \sum_i (u_i + v_i)$. Notice that, according to our choices, $\arg(v) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. We begin with the general case handled in the following subsection.

6.3.1. Generic case: *v* and *w* are not parallel

Given a representation χ of nontrivial-ends type, we can define four, possibly broken, half-planes H_1, H_2, H_3, H_4 as done in Section §6.2.2. Since we need to change a little the notation, we briefly summarize it as follows; see Figure 31. Let $P_0 \in \mathbb{C}$ be any point, and set as above the point $P_{2g} = P_0 + v$. Let $\theta = \arg(v)$, and let R_θ be the counterclockwise rotation of angle θ about the origin on $\mathbb{C}$. In order to realize the desired regions, in this subsection we shall agree that all rays leaving from P_0 are parallel to $R_\theta \cdot \mathbb{R}^-$, and similarly, all rays leaving from P_{2g} are parallel to $R_\theta \cdot \mathbb{R}^+$. The regions are:

1. H_1 is a broken half-plane bounded by chain of edges defined as in Equation (70) joining P_0 and P_{2g} and two rays, say r_1^- and r_2^-, respectively leaving from P_0 and P_{2g};
2. H_2 is a half-plane bounded by a geodesic edge s_0^+ joining P_0 and P_{2g} and two rays, say r_3^+ and r_2^+, respectively leaving from P_0 and P_{2g};
3. H_3 is a half-plane bounded by a geodesic edge r_0^- joining P_0 and P_{2g} and two rays, say r_5^- and r_4^-, respectively leaving from P_0 and P_{2g}; finally,
4. H_4 is a broken half-plane bounded by a chain of edges as in Equation (79) joining P_0 and P_{2g} and two rays, say r_5^+ and r_6^+, respectively leaving from P_0 and P_{2g}.

We next realize a cylinder with period w as follows. We join the points P_0 and P_{2g} by an edge, say s_0^-. Extend s_0^- with two rays, say r_3^- and r_6^- leaving from P_0 and P_{2g}, respectively. Notice that s_0^- has slope θ, and therefore, $r_3^- \cup s_0^- \cup r_6^-$ is a straight line, say l_1 by design. Define $r_1^+ = r_3^- + w$ and $r_4^+ = r_6^- + w$. Since w is not parallel to v, the chain $r_1^+ \cup r_0^+ \cup r_4^+$ is a parallel straight line, say l_2. We define C as the strip bounded by the lines l_1 and l_2. By construction, the cylinder obtained by identifying them has period w.

All the necessary pieces are now introduced and defined. We glue all rays with the same label together; that is, we glue r_i^+ with r_i^- for $i = 1, \ldots, 6$. Then glue the edges r_0^+ and r_0^- as well as the edges s_0^+ and s_0^- together. Finally, glue the edges u_i^+ and v_i^+ with u_i^- and v_i^-, respectively. The resulting object is a translation surface with two poles of order 2 and a single zero of order $2g+2$. By construction, this translation surface admits an involution ι that fixes the cylinder C and maps the half-planes H_1, H_2 to H_4, H_3, respectively. This involution is hyperelliptic because it fixes $2g+2$ points corresponding to the midpoints of the edges, the unique zero of ω and one point in the cylinder C. Therefore, the structure just defined lies in the hyperelliptic component of the stratum $\mathcal{H}_g(2g+2;-2,-2)$, and by breaking up the zero into two zeros each of order $m = g+1$, we get a structure in $\mathcal{H}_g(g+1, g+1;-2,-2)$. To get a structure in the stratum $\mathcal{H}_g(2m;-p,-p)$, we eventually bubble $p-1$ copies of $(\mathbb{C}, dz)$ along a ray leaving from $P_0 \in H_1$ – not necessarily parallel to $R_\theta \cdot \mathbb{R}^-$ – and, similarly, bubble $p-1$ copies along a ray leaving from $P_{2g} \in H_4$ – not necessarily parallel to $R_\theta \cdot \mathbb{R}^+$. Again, we apply Lemma 3.4 above in order to obtain a hyperelliptic translation structure in $\mathcal{H}_g(m,m;-p,-p)$ with period character χ as desired.

6.3.2. Exceptional case: *v* and *w* are parallel

In the above construction, the cylinder C has nonempty interior because v is not parallel to w, and thence, the lines $r_1^+ \cup r_0^+ \cup r_4^+$ and $r_3^- \cup s_0^- \cup r_6^-$ are parallel but disjoint. In the case v is parallel

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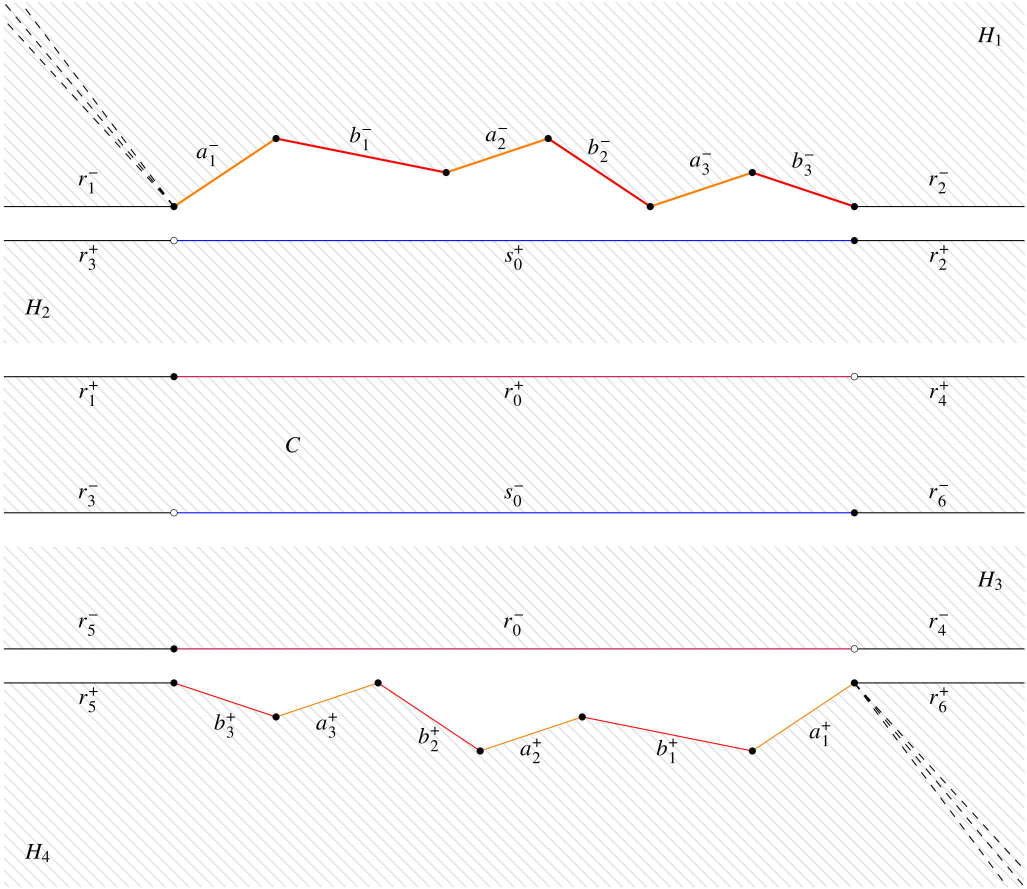

Figure 31. *Realization of a hyperelliptic genus g meromorphic differential with two zeros each of order $m = g + p - 1$, arising from the identification of all black bullets, two poles each of order $p \geq 2$ with nonzero residue and prescribed periods. The figure depicts the case $g = 3$.*

to w, the cylinder C degenerates to a straight line. However, this case can be handled as follows. Let $\rho\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be an auxiliary representation; see Definition 49, defined as

$$\rho(\alpha_i) = \chi(\alpha_i) \quad \rho(\beta_i) = \chi(\beta_i) \ \text{ and } \ \rho(\gamma_1) = \rho(\gamma_2) = 0, \tag{82}$$

where γ_1, γ_2 are simple loops each around one of the punctures. We can realize ρ as the period character of some translation surface, say (X, ω), with poles in the hyperelliptic component of $\mathcal{H}_g(2m; -p, -p)$ as done in §6.2.2. Adapting the notation therein, we claim that, on (X, ω) there always exists a bi-infinite ray, say l_0, joining the poles and orthogonal to the saddle connection e, arose from the identification of e^+ with e^-, on its midpoint. The slope of e, after developing, is equal to $\arg(v) = \arg(w) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Let $l_1^+ \subset \mathbb{C}$ be a straight line with slope $\arg(w) + \frac{\pi}{2}$, and let $l_1^- = l_1^+ + w$. Define C as the strip between the lines l_1^+ and l_1^-. Notice that the lines l_0, l_1^+, l_1^- all have the same slope by design. Slit (X, ω) along l_0 and call l_0^+, l_0^- the resulting edges. Then glue l_0^- with l_1^+ and l_0^+ with l_1^-; see also Definition 3.13. The resulting structure, say (Y, ξ), is a translation surface with poles in the stratum $\mathcal{H}_g(2m; -p, -p)$ having period character χ. By construction, the straight line l_0 is invariant under the hyperelliptic involution of (X, ω). As a consequence, (Y, ξ) also admits a hyperelliptic involution that keeps the strip C (after gluing) invariant.

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Finally, we apply again Lemma 3.4. By breaking the unique zero of (Y,ξ), we can realize χ as the period character of some structure in the hyperelliptic component of $\mathcal{H}_g(m,m;-p,-p)$ as desired.

6.4. *Simple poles*

Throughout this section, we always assume $p = 1$ and we aim to realize a representation $\chi\colon \mathrm{H}_1(S_{g,2},\mathbb{Z}) \longrightarrow \mathbb{C}$ as the period character of some translation surface in the hyperelliptic component of the following strata:

$$\mathcal{H}_g(2g;-1,-1) \text{ and } \mathcal{H}_g(g,g;-1,-1). \tag{83}$$

We shall distinguish two cases depending on whether χ is discrete; see Definition 4.11. For a given system of handle generators $\{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g\}$, let $a_i,\, b_i \in \mathbb{C}^*$ be the images of α_i, β_i via χ, respectively. Once again, we use the change of basis stratagem to define new pairs $\{u_i,v_i\}_{1\le i\le g}$; see Section §6.1. We begin with the following case:

6.4.1. The representation χ is not discrete

Recall that any system of handle generators yields a splitting, and hence, a well-defined representation $\chi_g\colon \mathrm{H}_1(S_g,\mathbb{Z}) \longrightarrow \mathbb{C}$. Since χ is a representation of nontrivial-ends type, Lemma 4.10 applies and we can assume $\mathrm{vol}(\chi_g) > 0$. We distinguish two subcases depending on whether χ is or is not real-collinear; see Definition 4.8.

6.4.1.1. χ is not real-collinear

Assume here that χ is not a real-collinear representation. Let γ be a simple loop around a puncture, and let $w = \chi(\gamma)$, and let $v = \sum_i (u_i + v_i)$. According to the following remark, we can assume that v and w are not parallel.

Remark 6.9. In principle, v and w could be parallel. However, since χ is not real-collinear then Corollary 6.7 applies. Therefore, there is a handle generator, say β_g, such that a stronger version of Equation (71) holds; more precisely

$$\max_{1\le j\le 2g-2}\left\{0,\, \arg(w),\, \arg\left(\overline{P_0\,P_j}\right)\right\} < \arg(b_g) \le \frac{\pi}{2}; \tag{84}$$

in particular, b_g and w are not parallel. Following the proof of Corollary 6.7, in Equation (74) we may replace (u_g,v_g) with (u_g+nb_g, v_g+nb_g) with $n>0$ so that w and $v+2nb_g$ are no longer parallel.

Let $P_0\in\mathbb{C}$ be any point, and let $P_{2g} = P_0 + v$. Let r be the unique straight line passing through these points. Up to rotating the whole construction with an appropriate rotation in $\mathrm{SO}(2,\mathbb{R}) < \mathrm{GL}^+(2,\mathbb{R})$, let us assume for simplicity that r is horizontal parallel to the real line; in this case, $\arg(v) = 0$ and $w\in\mathbb{C}\setminus\mathbb{R}$. Define a chain, say $\mathcal{C}_1$, as in Equation (70) starting from P_0 and necessarily ending at P_{2g}.

According to Corollary 6.7, we can assume that the chain $\mathcal{C}_1$ entirely lies on the right of r with respect to the orientation induced by v. Let r_1^- as the subray of r leaving from P_0 and not passing through P_{2g}. Similarly, define r_2^- as the subray of r leaving from P_{2g} and not passing through P_0. Let $P_0' = P_0 + w$, set $P_{2g}' = P_0' + v = P_0 + v + w$, and let r' be the unique straight line passing through these points. Clearly, r' is parallel to r. We define a chain $\mathcal{C}_2$ exactly as in Equation (79) starting from P_0' and ending at P_{2g}'. By design, $\mathcal{C}_2$ lies on the left of r' with respect to the orientation induced by v. Let r_1^+ be the subray of r' leaving from P_0' and not passing through P_{2g}', and similarly, define r_2^+ as the subray of r leaving from P_{2g}' and not passing through P_0'. Define $C\subset\mathbb{C}$ as the region bounded by $r_1^-\cup\mathcal{C}_1\cup r_2^-$ and $r_1^+\cup\mathcal{C}_2\cup r_2^+$; see Figure 32.

Finally, glue the half-rays r_i^- and r_i^+ together and the edges u_i^-, v_i^- with u_i^+, v_i^+, respectively, for $i = 1,\dots,g$. The resulting object is a topological surface homeomorphic to $S_{g,2}$ equipped with a translation structure with poles having period character χ. Notice that C is invariant with respect to a

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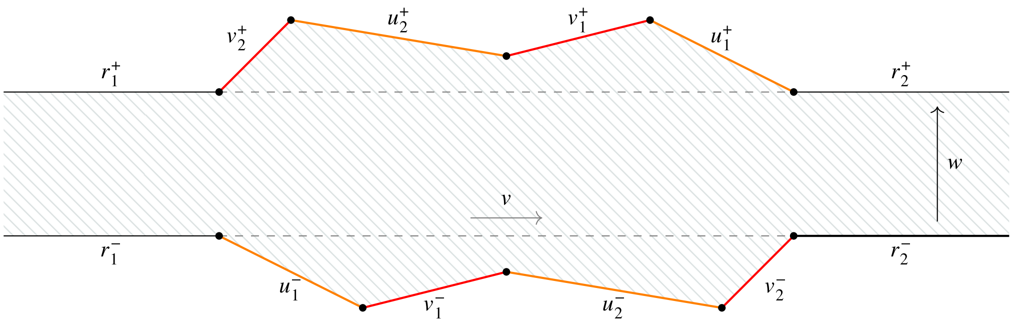

Figure 32. *Realization of a hyperelliptic translation surface of genus two with a single zero and two simple poles having nondiscrete and nonreal-collinear period character χ.*

rotation of order 2 about a point $O \in \mathbb{C}$. Such a point can be explicitly determined; in fact, it turns out to be the intersection point of the diagonals of the parallelogram with vertices P_0, P_{2g}, P'_{2g}, P'_0. As a consequence, the resulting structure above admits an involution, and it can been seen this is hyperelliptic. In particular, χ can be realized in the hyperelliptic component of $\mathcal{H}_g(2g;-1,-1)$. By breaking the zero into two zeros each of order g, we obtain a hyperelliptic structure in the stratum $\mathcal{H}_g(g,g;-1,-1)$ as a consequence of Lemma 3.4.

6.4.1.2. χ is real-collinear

We now assume χ to be real-collinear. Up to replacing χ with $A\,\chi$, for some appropriate $A \in \mathrm{GL}^+(2,\mathbb{R})$, we can assume $\mathrm{Im}(\chi) \subset \mathbb{R}$. Let γ be a simple loop around a puncture, and let $w = \chi(\gamma)$. Let us assume $w > 0$ for simplicity.

Lemma 4.13 applies, and we can find a system of handle generators with absolute periods $(a_1, b_1, \ldots, a_g, b_g)$ such that the linear system (63) admits a solution $(u_1, v_1, \ldots, u_g, v_g)$ such that all entries are positive reals such that

$$v := \sum_i (u_i + v_i) < w \tag{85}$$

and we set $2\delta = w - v$.

Let $l_1 \subset \mathbb{C}$ be a straight line with slope $\frac{\pi}{2}$, let $l_2 = l_1 + w$ and define $\mathcal{S}$ as the vertical strip bounded by l_1^+ and l_2^-. Let $P_0 \in \mathcal{S} \subset \mathbb{C}$ be any point at distance δ from l_1, and let $P_{2g} = P_0 + v$. Notice that $P_{2g} \in \mathcal{S}$ at distance δ from l_2.

Let e be the edge joining P_0 and P_{2g}. Slit $\mathcal{S}$ along e, and define $e^{\pm}$ the resulting edges according to our convention. We partition e^- into $2g$ subedges $\{e_i^-\}_{1\le i\le 2g}$ of lengths $u_1, v_1, \ldots, u_g, v_g$, respectively. In the same fashion, we partition the edge e^+ into $2g$ subedges $\{e_i^+\}_{1\le i\le 2g}$ of lengths $v_g, u_g, \ldots, v_1, u_1$, respectively; see Figure 33. By gluing the lines l_1^- and l_2^+ together, we get an infinite cylinder with period w slit along the edge e. We next glue the edges e_i^- with e_{2g-i+1}^+ together for any $i = 1, \ldots, 2g$. The resulting object is homeomorphic to $S_{g,2}$, and it is equipped with a translation structure (X,ω) with period χ. We can observe that $\mathcal{S} \in \mathbb{C}$ is invariant under a rotation of order 2 about the midpoint of the segment $\overline{P_0\,P_{2g}}$. As a consequence, (X,ω) admits a hyperelliptic involution meaning that χ can be realized in the hyperelliptic component of $\mathcal{H}_g(2g;-1,-1)$. By breaking the zero into two zeros each of order g we obtain a translation surface with period χ in the hyperelliptic component of the stratum $\mathcal{H}_g(g,g;-1,-1)$ as a consequence of Lemma 3.4.

6.4.2. The representation χ is discrete of rank two

Now, assume that χ is a discrete representation of rank two; see Definition 4.11. The idea of this case is mostly subsumed in paragraph §6.4.1.1. Up to $\mathrm{GL}^+(2,\mathbb{R})$, we assume without loss of generality that

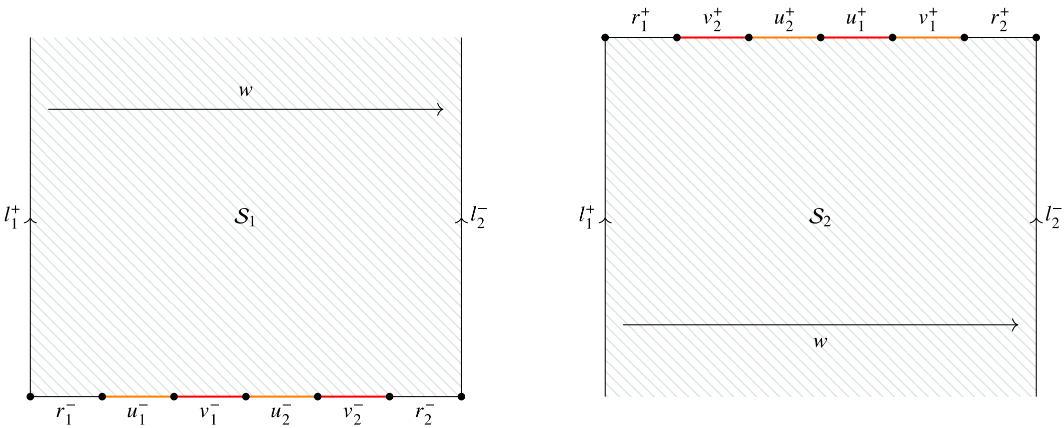

Figure 33. *The half-strips $\mathcal{S}_1$ and $\mathcal{S}_2$ represent the top and bottom sides of the strip $\mathcal{S}$ cut along a horizontal segment of length w containing e. The edges e_i^-'s are in the ascending order from the left to the right, and the e_i^+'s are in the descending order from the left to the right.*

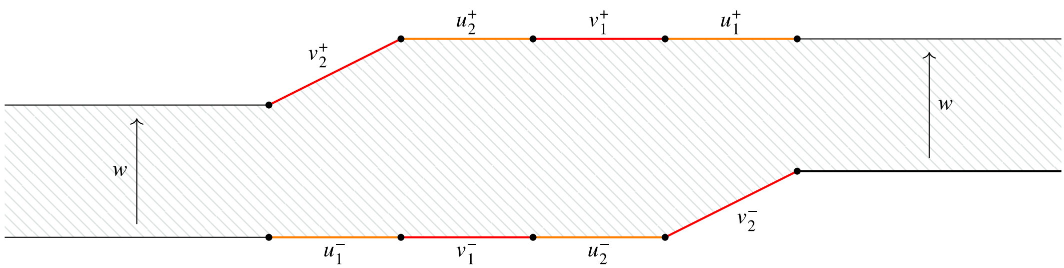

Figure 34. *Realization of a translation surface of genus two with poles admitting a hyperelliptic involution and having discrete period character of rank two. The edges labelled with $u_i^\pm$ and $v_i^\pm$ are obtained from the absolute periods $a_1, b_1, \ldots, a_g, b_g$ prescribed by the given representation.*

$\mathrm{Im}(\chi) = \mathbb{Z} \oplus i\,\mathbb{Z}$. Lemma 4.12 above applies in this case and it guarantees the existence of a system of handle generators such that

- $\chi(\alpha_g) = a \in \mathbb{Z}$ and $\chi(\beta_g) = i$,
- $0 < \chi(\alpha_j) < \chi(\alpha_g)$ and $\chi(\alpha_j) = \chi(\beta_j)$ for all $j = 1, \ldots, g-1$.

Since χ is not real-collinear, it is an easy matter to check that Lemmas 6.5 and 6.7 hold in this case, and hence, we can proceed as done in §6.4.1.1 in order to get a translation surface with poles in the hyperelliptic component of the stratum $\mathcal{H}_g(2g; -1, -1)$ with period character χ; see Figure 34. As above, by breaking the single zero into two zeros of order g, we can also realize χ as the period character of some hyperelliptic translation surface in $\mathcal{H}_g(g, g; -1, -1)$.

6.4.3. The representation χ is discrete of rank one

We finally consider the case of discrete representation of rank one; see Definition 4.11. For any such a representation $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ of rank one, we can replace χ with $A\,\chi$, for some appropriate $A \in \mathrm{GL}^+(2, \mathbb{R})$, and assume that $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ is surjective. Let γ be a simple loop around a puncture, and let $w = \chi(\gamma) \in \mathbb{Z}$ be its period; that is, the residue (up to $2\pi i$). [CFG22, Theorem D] provides necessary and sufficient conditions for such representations to appear as the holonomy of some translation surfaces with simple poles at the punctures. For the reader's convenience, we review here a simplified version of that result.

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Proposition 6.10 ([CFG22, Theorem D] for $n = 2$). *Let $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ be a discrete representation of nontrivial-ends type. Then χ appears as the period character of some translation surface with simple poles at the punctures and zeros of prescribed orders $(m_1, m_2, \ldots, m_k)$ that satisfy the degree condition $m_1 + \cdots + m_k = 2g$ if and only if*

$$|w| > \max\{m_1, m_2, \ldots, m_k\}, \tag{86}$$

where $w, -w \in \mathbb{Z}$ are the residues of the simple poles.

According to Proposition 6.10 above, we need to distinguish two subcases according to the value of $w \in \mathbb{Z}$. In the case $w > 2g$, then χ appears as the period character of some translation surface with poles in both strata $\mathcal{H}_g(2g; -1, -1)$ and $\mathcal{H}_g(g, g; -1, -1)$. In paragraph §6.4.3.1 below, we shall prove that χ can be realized in the hyperelliptic component of these strata. In paragraph §6.4.3.2, we shall consider the case $g < w \leq 2g$, then χ can be realized only in the stratum $\mathcal{H}_g(g, g; -1, -1)$. Notice that this case requires an ad-hoc construction because we can no longer realize a structure with a single zero and then break it into two zeros of order g. Finally, if $0 < w \leq g$, then χ cannot be realized in both strata, and hence, it cannot appear as the period character of some translation surface with simple poles and hyperelliptic involution. Before continuing, we state the following crucial remark:

Remark 6.11. For χ being discrete of rank one, Lemma 4.12 applies, so there exists a system of handle generators $\mathcal{G}_1 = \{\alpha_1', \beta_1', \ldots, \alpha_g', \beta_g'\}$ such that $a_i' = b_i' = 1$ for all i. Notice that $z_1 = (1, \ldots, 1) \in \mathbb{Z}^{2g}$ is a primitive vector – the greatest common divisor of all entries is one. We next consider the following primitive vector $z_2 = (-1, 1, 1, 3, 3, \ldots, 2g-3, 2g-3, 2g-1) \in \mathbb{Z}^{2g}$. Since they are both primitive, there is an element $A \in \mathrm{Sp}(2g, \mathbb{Z})$ such that $A(z_1) = z_2$ because $\mathrm{Sp}(2g, \mathbb{Z})$ acts transitively on primitive integer vectors in $\mathbb{Z}^{2g}$. Therefore, there is another system of handle generators $\mathcal{G}_2 = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ such that

$$a_1 = -1,\ b_1 = a_2 = 1,\ \ldots\ b_i = a_{i+1} = 2i-1,\ \ldots,\ b_{g-1} = a_g = 2g-3,\ b_g = 2g-1. \tag{87}$$

The crucial fact is that the linear system

$$\begin{cases} \quad -y_1 & = & -1 \\ x_1 & = & 1 \\ x_1 \ +y_1 \quad -y_2 & = & 1 \\ x_1 \ +y_1 \ +x_2 & = & 3 \\ \\ x_1 \ +y_1 \ +x_2 \ +y_2 \quad -y_g & = & 2g-3 \\ x_1 \ +y_1 \ +x_2 \ +y_2 \ +x_g & = & 2g-1 \end{cases} \tag{88}$$

has solutions $(u_i, v_i) = (1, 1)$ for all $i = 1, \ldots, g$. Therefore, we can use a_i, b_i in place of u_i, v_i in this case. In what follows, we shall rely on this observation.

6.4.3.1. Case: $w > 2g$

This case is completely subsumed in paragraph §6.4.1.2. In fact, as a consequence of Lemma 4.12 there is a system of handle generators $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ such that $\chi(\alpha_i) = \chi(\beta_i) = 1$ for all $i = 1, \ldots, g$. Let $v = \sum_i (a_i + b_i)$ as usual. Since $w > 2g$, then $w > v = 2g$, and hence, we can proceed exactly as in §6.4.1.2. Therefore, χ appears as the period character of some hyperelliptic translation surface with simple poles in $\mathcal{H}_g(2g; -1, -1)$. By breaking the zero, we get a translation surface with simple poles in the hyperelliptic component of $\mathcal{H}_g(g, g; -1, -1)$ as desired.

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6.4.3.2. Case: $g < w \leq 2g$

We finally deal with this last case. Let $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ be a representation and assume that, for a simple loop γ around a puncture, we have that $g < w = \chi(\gamma) \leq 2g$. Since $w \in \mathbb{Z}$, we can write it as $w = g + 1 + l$, where l is a nonnegative integer. We now define two half-infinite strips, say $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{C}$ of width w and pointing in opposite directions. More precisely, let $P_0 \in \mathbb{C}$ be any point, and let $P_{2g+2} = P_0 + w$. Let s be the segment joining them, and define r_1, $r_2 = r_1 + w$ as the straight lines orthogonal to P_0 and P_{2g+2} respectively. Define $\mathcal{S}_1$ as the strip bounded by $r_1^+ \cup s^- \cup r_2^-$, and in a similar fashion, define $\mathcal{S}_2$ as the strip bounded by $r_1^+ \cup s^+ \cup r_2^-$. Consider these half-strips separately. We partition s^- as the union of segments $s_1^-, \ldots, s_{2g+2}^-$, from left to right, where each s_i is of length $\frac{1}{2}$, except that the last one is of width $\frac{1}{2} + l$, so the total length amounts to $\frac{2g+2}{2} + l = w$. Similarly, we partition s^+ as the union of segments $s_{2g+2}^+, \ldots, s_1^+$, from the left to the right, where each s_i is of length $\frac{1}{2}$, except that the first one is of width $\frac{1}{2} + l$. Again, the total length amounts to $\frac{2g+2}{2} + l = w$. Glue the half-strips $\mathcal{S}_1, \mathcal{S}_2$ by identifying the edges with the same label and the rays r_1^+ with r_2^- together. The resulting structure is a translation surface with two simple poles and two zeros each of order g. These two zeros appear in an alternating way as the end points of the s_i. Notice that any absolute period is an integer as it consists of an even number of the s_i. Finally, the structure is hyperelliptic because the half-strips are symmetric with respect to a rotation of order two by design.

6.5. *Nonhyperelliptic translation surfaces in genus two*

For certain strata of genus-two meromorphic differentials, we are now ready to complete the proof of Theorem A. In fact, by performing appropriate modifications to our construction developed in Sections §6.2 and §6.4 we can prove the following propositions.

Proposition 6.12. *Let $\chi\colon \mathrm{H}_1(S_{2,1}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation. Assume χ appears as the period character of some translation surface with poles in a stratum $\mathcal{H}_2(4; -2)$ or $\mathcal{H}_2(2, 2; -2)$, possibly both. Then χ can be realized in the nonhyperelliptic component of the same stratum.*

Proposition 6.13. *Let $\chi\colon \mathrm{H}_1(S_{2,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation. Assume χ appears as the period character of some translation surface with poles in a stratum $\mathcal{H}_2(4; -1, -1)$ or $\mathcal{H}_2(2, 2; -1, -1)$, possibly both. Then χ can be realized in the nonhyperelliptic component of the same stratum.*

Sketch of the proofs of Propositions 6.12 and 6.13. Let $\chi\colon \mathrm{H}_1(S_{2,1}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation that appears as the period character of some translation surface in $\mathcal{H}_2(4; -2)$ or $\mathcal{H}_2(2, 2; -2)$. In Section §6.2, we have realized χ as the period character of some hyperelliptic translation surface by gluing broken half-planes designed in such a way that the resulting structure turned out to be hyperelliptic. The gist of the idea was to realize these broken half-planes so that they were invariant under a rotation of order 2 about a certain point in $\mathbb{C}$. For this purpose, in Section §6.1 we introduced g pairs of complex numbers $\{u_i, v_i\}$ and then found unbounded regions of $\mathbb{C}$ bounded by chains as in Equations (70) and (79).

However, it is also possible to design broken half-planes so that they are no longer symmetric with respect to a rotation of order two. Given a system of handle generators $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, we define broken half-planes as noncompact regions bounded by chains that are now defined by using the absolute periods $(a_1, b_1, \ldots, a_g, b_g)$. It is an easy matter to show that, up to replacing a pair $\{\alpha_i, \beta_i\}$ with either $\{\alpha_i^{-1}, \beta_i^{-1}\}$, $\{\beta_i, \alpha_i^{-1}\}$ or $\{\beta_i^{-1}, \alpha_i\}$, we may assume $\Re(a_i) > 0$ and $\Re(b_i) > 0$. In all construction done in the previous sections, in place of the chain (70) we can use the following one:

$$\begin{aligned} P_0 \mapsto P_0 + a_1 = P_1 \mapsto P_0 + a_1 + b_1 = P_2 \mapsto P_0 + a_1 + b_1 + a_2 \\ = P_3 \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g} (a_i + b_i) = P_{2g} \end{aligned} \tag{89}$$

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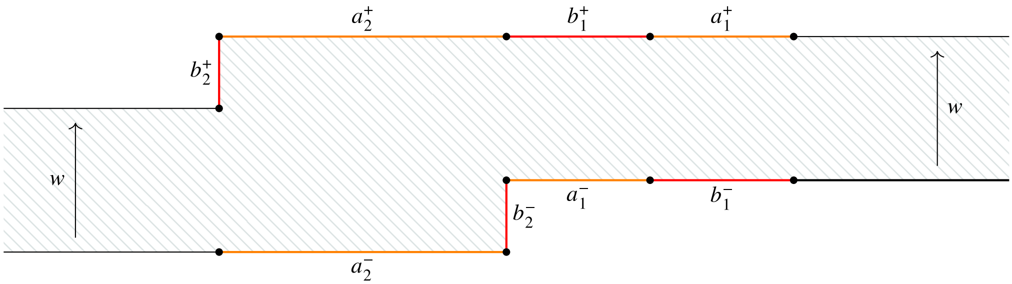

Figure 35. *Realization of a genus-two translation surface with poles and discrete period character of rank two. In this case the bottom 'zig-zag' line is obtained by sorting the edges in a different way with respect the order used in Figure 34. Notice that in this case the shadow area is no longer invariant under a rotation of order 2 of $\mathbb{C}$. As a consequence, the structure obtained by gluing the edges according to the labels is no longer hyperelliptic.*

and, in place of the chain (79) use

$$P_0 \mapsto P_0 + b_1 = P_1 \mapsto P_0 + b_1 + a_1 = P_2 \mapsto P_0 + b_1 + a_1 + b_2$$
$$= P_3 \mapsto \cdots \mapsto P_0 + \sum_{i=1}^{g}(a_i + b_i) = P_{2g} \tag{90}$$

along with the half-rays r_1, r_2 (we adopt the same notation). Notice that these chains differ because the edges a_i, b_i are sorted in a different way. Glue the half-planes as usual. The resulting translation surface still lies in $\mathcal{H}_2(4;-2)$ or $\mathcal{H}_2(2,2;-2)$, but it is no longer hyperelliptic. We finally recall that for these genus-two strata under considerations, the connected components are distinguished by the hyperellipticity as well as by the spin parity; see Section §2.3. A direct computation shows that all hyperelliptic structures realized in §6.2 and §6.4 have even spin parity whereas all structures realized by defining the broken half-plane H_2 as above have odd spin parity. In Section §6.4, we have adopted the same strategy for representations $\chi\colon \mathrm{H}_1(S_{2,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$, and hence, the same discussion holds for them; see Figure 35. □

Corollary 6.14. *For a partition κ of 4 and a partition ν of 2, define $\mathcal{H}_2(\kappa;-\nu)$ as the stratum of meromorphic genus-two differentials with zeros and poles of orders prescribed according to κ and $-\nu$, respectively. Then Theorem A holds for any stratum $\mathcal{H}_2(\kappa;-\nu)$ thus defined.*

Proof. We begin with the following observation. Let κ be a partition of 4, and let ν be a partition of 2. According to Boissy (see [Boi15, Theorem 1.2]), a stratum $\mathcal{H}_2(\kappa;-\nu)$ of genus-two meromorphic differentials admits at most two connected components. More precisely, the stratum is connected whenever $\kappa \neq \{4\}, \{2, 2\}$. For $\kappa = \{4\}$ or $\kappa = \{2, 2\}$, it admits two connected components one of which is hyperelliptic and the other is not. Suppose a representation χ can be realized in a certain stratum $\mathcal{H}_2(\kappa;-\nu)$. If the stratum is connected then the claim follows from [CFG22, Theorems C and D]. We next assume that the stratum is not connected. This is one of the strata $\mathcal{H}_2(4;-2)$, $\mathcal{H}_2(2,2;-2)$, $\mathcal{H}_2(4;-1,-1)$ and $\mathcal{H}_2(2,2;-1,-1)$. Now, Proposition 6.1 says that χ can be realized in the hyperelliptic component of each stratum. On the other hand, Propositions 6.12 and 6.13 say that χ can be realized in the nonhyperelliptic component of each stratum. The claim thus follows. □

Remark 6.15. According to Boissy, [Boi15], the connected components of strata in Corollary 6.14 are also distinguished by the spin parity; see §2.2.3. In fact, by a direct computation, one can see that hyperelliptic translation surfaces have even parity and nonhyperelliptic ones have odd parity. See Figures 36 and 37 for an example related to Figures 34 and 35.

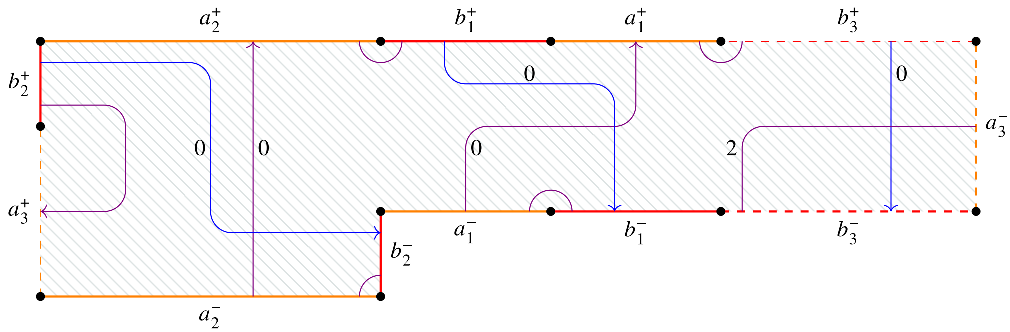

Figure 36. *Computation of the spin parity for the translation surface* (X, ω) *in Figure 35. According to Remark 2.18, the structure depicted here can be obtained from* (X, ω) *in Figure 35 by truncating the cylindrical ends along waist geodesic curves. The dashed edges correspond to those obtained after truncation. The colored lines represent a symplectic base, and the separate labels denote the indices of the respective curves. According to formula (23), it is easy to check that* $\varphi(\omega) = 1 \pmod 2$*; hence, the structure is not hyperelliptic.*

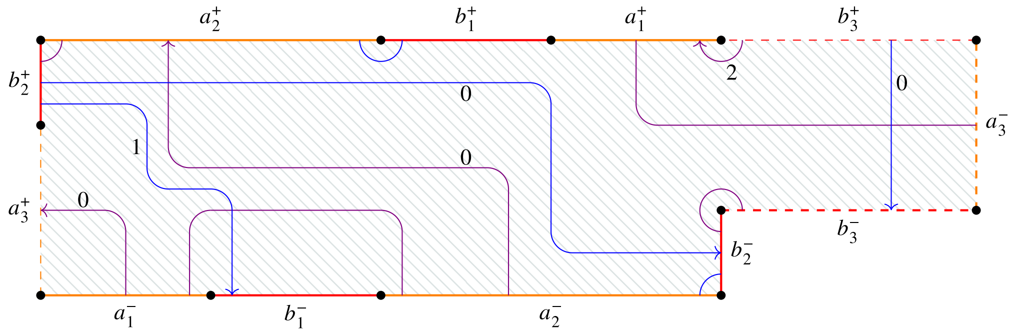

Figure 37. *Computation of the spin parity for the translation surface* (X, ω) *in Figure 34. According to Remark 2.18, the structure depicted here can be obtained from* (X, ω) *in Figure 34 by truncating the cylindrical ends along waist geodesic curves. By using formula (23), it is easy to check that* $\varphi(\omega) = 0 \pmod 2$*; hence, the structure is hyperelliptic.*

7. Higher genus meromorphic differentials with prescribed parity

We aim to determine whether a representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized as the period character of some translation surface with prescribed spin parity; see §2.2.3. In this section, we assume the representation χ to be nontrivial and we shall handle the trivial representation in Section §8. Recall that a genus g meromorphic differential ω on a Riemann surface X determines a well-defined spin structure if and only if the set of zeros and poles is of even type; see Definition 2.20. Our aim is to prove the following.

Proposition 7.1. *Let* χ *be a nontrivial representation and suppose it arises as the period character of some meromorphic genus g differential in a stratum admitting two connected components distinguished by the spin parity. Then* χ *can be realized in both components of the same stratum as the period character of some translation surfaces with poles.*

7.1. Inductive process

The strategy we shall adopt in the present section is based on an inductive foundation on the genus g of surfaces. We begin with an explanation of our strategy which we shall develop in Section §7.2. In the

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explanation below, we shall mainly consider strata of differentials with a single zero of maximal order because the general case follows by breaking a zero.

7.1.1. Higher-order poles

Let $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation, possibly of nontrivial-ends type, and let $\mathcal{H}_g(2m+2g-2; -2p_1, \dots, -2p_n)$ be a stratum of genus g differentials of even type, where $n \geq 2$ and $m = p_1 + \cdots + p_n$. According to [CFG22, Theorem C], χ can be realized in such a stratum. Let $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ be a system of handle generators for $S_{g,n}$ (see Definition 4.2), and define an auxiliary representation $\rho\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as follow:

$$\rho(\alpha) = \chi(\alpha_1), \quad \rho(\beta) = \chi(\beta_1), \quad \rho(\delta_i) = \chi(\delta_i), \tag{91}$$

where $\{\alpha, \beta\}$ is a pair of handle generators for $\mathrm{H}_1(S_{1,n}, \mathbb{Z})$ and δ_i is a peripheral loop around the i-th puncture. Notice that this auxiliary representation differs from that defined in Definition 49. Since χ is a nontrivial representation, Lemmas 4.4 and 4.5 apply, and we can assume $\chi(\alpha_1), \chi(\beta_1) \in \mathbb{C}^*$. As a consequence, the auxiliary representation ρ is also nontrivial. Next, we realize ρ as the period character of some translation surface with poles in the stratum $\mathcal{H}_1(2n; -2, \dots, -2)$ as in Section §5. This structure serves as the base case for an inductive foundation. The inductive process consists in showing that at each step we always obtain a translation surface with poles with enough room to bubble a handle so that the resulting structure has the desired parity. Notice that, once the polar part $\nu = (2, \dots, 2)$ is fixed, the genus determines the order of the zero uniquely; this is the Gauss–Bonnet condition; see Remark 2.3. Therefore, bubbling a handle yields a sequence of mapping between strata as follows:

$$\begin{aligned} &\mathcal{H}_1(2n; -\nu) \longmapsto \mathcal{H}_2(2n+2; -\nu) \longmapsto \cdots \\ &\qquad \cdots \longmapsto \mathcal{H}_g(2n+2g-2; -\nu) \longmapsto \mathcal{H}_{g+1}(2n+2g; -\nu) \longmapsto \cdots. \end{aligned} \tag{92}$$

According to Lemmas 3.5 and 3.8, at each step bubbling a handle does not alter the spin parity. Therefore, as we shall see in Section §7.2, the spin parity is completely determined by the rotation number of the initial genus-one differential. The stratum $\mathcal{H}_1(2n; -2, \dots, -2)$ has exactly two connected components: One of these comprises genus-one differentials with rotation number $k = 1$, and the other one comprises differentials with rotation number $k = 2$. In the former case, each bubbling gets the access to the connected component of translation surfaces with even spin parity. In the latter case, each bubbling gets the access to the connected component of translation surfaces with odd spin parity.

Once a structure in a stratum $\mathcal{H}_g(2n+2g-2; -2, \dots, -2)$ is realized with prescribed parity and period character χ, then we can get the access to all other strata of genus g differentials by bubbling copies of the differential $(\mathbb{C}, z\,dz)$ along suitable rays joining the single zero with the punctures. Note that these copies of $(\mathbb{C}, z\,dz)$ flat geometrically can be represented by gluing entire Euclidean planes; hence, the residues at the original poles are unchanged in the process. Moreover, Lemma 3.16 ensures that the parity remains unaltered. Finally, by breaking a zero (see §3.1), we get the desired result for all possible strata $\mathcal{H}_g(\kappa; -\nu)$, where $\kappa = (2m_1, \dots, 2m_k) \in 2\,\mathbb{Z}_+^k$ and κ, ν satisfy the Gauss–Bonnet condition (13).

7.1.2. Two exceptional cases

The strategy above does not apply for strata of differentials with a single pole of order 2. The problem is due to the connectedness of the stratum $\mathcal{H}_1(2; -2)$; see [Boi15, Theorem 1.1] or Section §2.3. We bypass this issue by using genus-two differentials as the base case for the induction. More precisely, given a representation $\chi\colon \mathrm{H}_1(S_{g,1}, \mathbb{Z}) \longrightarrow \mathbb{C}$ and a system of handle generators $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$, we define an auxiliary representation $\rho\colon \mathrm{H}_1(S_{2,1}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as

$$\rho(\alpha_1) = \chi(\alpha_1), \quad \rho(\beta_1) = \chi(\beta_1), \quad \rho(\alpha_2) = \chi(\alpha_2), \quad \rho(\beta_2) = \chi(\beta_2). \tag{93}$$

Next, we realize ρ as the period character of some translation surface in $\mathcal{H}_2(4;-2)$. We shall use this structure as the base case for the induction and then we can rely on the same kind of process described in Section §7.1.1. Bubbling a handle yields a sequence of mapping similar to Equation (92), that is,

$$\mathcal{H}_2(4;-2) \longmapsto \mathcal{H}_3(6;-2) \longmapsto \cdots \longmapsto \mathcal{H}_g(2g;-2) \longmapsto \mathcal{H}_{g+1}(2g+2;-2) \longmapsto \cdots . \tag{94}$$

Recall that bubbling does not alter the spin parity of the structure (whenever it is defined). Since $\mathcal{H}_2(4;-2)$ has two connected components distinguished by the spin parity, by bubbling a genus-two differential with even parity we get a translation surface with even parity in each stratum $\mathcal{H}_g(2g;-2)$. Similarly, by bubbling a genus-two differential with odd parity we get a structure with odd parity in each stratum $\mathcal{H}_g(2g;-2)$. We next induct on the order of the pole. More precisely, by bubbling $p-1$ copies of $(\mathbb{C},\, z\,dz)$ along an infinite ray joining the single zero and the pole we can access to the stratum $\mathcal{H}_g(2g+2p-2;-2p)$. Notice that, again, Lemma 3.16 ensures that the spin parity only depends on the parity of the initial structure. By breaking a zero, we get the access to all strata of the form $\mathcal{H}_g(\kappa;-2p)$, where $\kappa = (2m_1,\ldots,2m_k) \in 2\,\mathbb{Z}_+^k$.

In the light of Remark 2.18, for two-punctured surfaces there is an additional case to take into account. This is the second exceptional case. Let $\chi\colon \mathrm{H}_1(S_{g,2},\, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation of nontrivial-ends type and let $\mathcal{H}_g(2g;-1,-1)$ be a stratum of genus g meromorphic differentials with two simple poles. A nonrational representation (see Definition 5.9) can be realized in that stratum; see [CFG22, Theorem C]. However, if χ is rational, then necessary and sufficient conditions for the realization are given by [CFG22, Theorem D]; see also Proposition 6.10 above. Once again, we rely on an inductive foundation and, since the stratum $\mathcal{H}_1(2;-1,-1)$ is connected, genus-two differentials will serve as the base case for the induction. In fact, the stratum $\mathcal{H}_2(4;-1,-1)$ has two connected components distinguished by the spin parity. We shall realize an auxiliary representation in this stratum with prescribed parity and then bubbling will provide the access to connected components of all the other strata $\mathcal{H}_g(2g,-1,-1)$ according to the sequence of mapping

$$\begin{aligned}&\mathcal{H}_2(4;-1,-1) \longmapsto \mathcal{H}_3(6;-1,-1) \longmapsto \cdots \\ &\qquad\qquad \cdots \longmapsto \mathcal{H}_g(2g;-1,-1) \longmapsto \mathcal{H}_{g+1}(2g+2;-1,-1) \longmapsto \cdots .\end{aligned} \tag{95}$$

Once again, by breaking a zero we get the access to all strata of the form $\mathcal{H}_g(\kappa;-1,-1)$, with $\kappa \in 2\,\mathbb{Z}_+^k$.

In both exceptional cases, the basic cases for the inductive process have already been realized in Section §6; see Remark 6.15. In the next section, we move to develop our inductive process. We have already mentioned above that bubbling copies of $(\mathbb{C},\, z\,dz)$ along rays and breaking a zero into zeros of even order do not alter the spin parity and do not change the existing residues. Since these operations provide the access to all other strata with poles of order greater than 2 and multiple zeros, in what follows we can reduce to consider strata with poles of order 2 and then prove Proposition 7.1 for these strata. The generic case immediately follows.

7.2. Into the process

We discuss the inductive foundation by distinguishing two cases according to Sections §7.1.1 and §7.1.2 above.

7.2.1. Generic case

Let $n \geq 2$, and let $\chi\colon \mathrm{H}_1(S_{g,n},\, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation, possibly of nontrivial-ends type. Let $\mathcal{G} = \{\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g\}$ be a system of handle generators, and let $\rho\colon \mathrm{H}_1(S_{1,n},\, \mathbb{Z}) \longrightarrow \mathbb{C}$ be an auxiliary representation defined as in Equation (91). Assume without loss of generality that ρ is a nontrivial representation. According to our constructions in Section §5, ρ can be realized as the period character of some translation surface with poles, say (X,ω), in $\mathcal{H}_1(2n;-2,\ldots,-2)$ with rotation number $k=1$ or $k=2$.

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More precisely, we realize χ according to the following list:

- If χ is of trivial-ends type, then (X, ω) is realized as in Section §5.1, otherwise
- If χ is of nontrivial-ends type and
  - $p = 2$, then (X, ω) is realized as in Section §5.3.1,
  - $p \geq 3$, then (X, ω) is realized as in Section §5.3.2 or §5.3.3.

In order to be consistent in what follows we adopt the notation of these sections. Notice that all constructions but one developed in these sections just mentioned contain an entire copy of $(\mathbb{C}, dz)$; that is, a copy of $(\mathbb{C}, dz)$ has been glued along an infinite ray. Moreover, we can always find rays starting the single zero of ω the punctures along which we can glue copies of $(\mathbb{C}, dz)$ to increase the order of poles and then get the access to all other stratum.

Remark 7.2. Among those constructions mentioned above, only one does not contain a whole copy of $(\mathbb{C}, dz)$. In fact, if $n = 2$ and we aim to realize ρ as the period character of $(X, \omega) \in \mathcal{H}_1(4; -2, -2)$ with rotation number 2, then we proceed as in §5.1.2.4. In this case, (X, ω) is obtained by gluing the closures of the exteriors of two isometric triangles each one in a different copy of $(\mathbb{C}, dz)$; see §5.1.2.4 for details.

Nevertheless, we always have enough room for bubbling handles with positive or nonpositive volumes. We proceed in a recursive way as follows. Choose an initial point O_1 according to the following rule:

- If (X, ω) contains an entire copy of $(\mathbb{C}, dz)$, then pick O_1 as any of the points Q_i for $i = 1, \dots, n-1$; otherwise
- we are in the special case mentioned in Remark 7.2 above. The starting point O_1 can be taken as $P + \chi(\alpha)$; see Figure 14.

In both case,s we can find a straight line l_1 passing through O_1 so that one of the two sides is an embedded half-plane, say H_1. Without loss of generality, we can orient l_1 so that H_1 lies on the left of l_1. We will show that all bubbling can be done within H_1.

Remark 7.3. An alternative approach may be the following. If a pair (X, ω) is a meromorphic differential with all poles of order at least 2, then any saddle connection in the boundary of its convex core is contained in such a line. For more about this notion, the reader may consult [Fil24, Section §3.3]. On the other hand, since our approach is to show that higher genus differentials with prescribed parity can be obtained from genus-one meromorphic differentials with prescribed rotation number, we shall proceed with an inductive foundation as follows.

We are now ready to implement the recursion we just alluded above, namely we will show that all bubbling can be done within H_1. For $j = 2, \dots, g$, let l_{j-1} be a straight line parallel and with the same orientation with respect to l_1 and passing through a point O_{j-1} which will be recursively determined step by step. Finally, define H_{j-1} as the half-plane on the left side of l_{j-1}. Next, consider the j-th pair of handle generators $\{\alpha_j, \beta_j\} \subset \mathcal{G}$. Then, depending on the value $\Im\left(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\right)$, we proceed as follows:

- If $\Im\left(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\right) > 0$, then we bubble a handle with positive volume. Up to replacing $\{\alpha_j, \beta_j\}$ with their inverses and renaming the curves if needed, we can assume that the edge, say e_j, joining O_{j-1} with $O_{j-1} + \chi(\alpha_j)$ entirely lies in H_{j-1}. Let $\mathcal{P}_j \subset \mathbb{C}$ be the parallelogram bounded by the chain

$$O_{j-1} \mapsto O_{j-1} + \chi(\alpha_j) \mapsto O_{j-1} + \chi(\alpha_j) + \chi(\beta_j) \mapsto O_{j-1} + \chi(\beta_j) \mapsto O_{j-1}.$$

  According to our convention, let us denote by a_j^+ (respectively a_j^-) the edge of $\mathcal{P}_j$ parallel to $\chi(\alpha_j)$ that bounds the parallelogram on its right (respectively left). Similarly, we denote by b_j^+ (respectively b_j^-) the edge of $\mathcal{P}_j$ parallel to $\chi(\beta_j)$ that bounds the parallelogram on its right (resp. left). Next, we slit H_{j-1} along e_j and denote $e_j^\pm$ the resulting edges. Then identify the edge b_j^+ with b_j^-, the edge a_j^+ with e_j^- and the edge a_j^- with e_j^+. The resulting structure is a genus j surface and the newborn handle has periods $\chi(\alpha_j)$ and $\chi(\beta_j)$. Finally, define $O_j := O_{j-1} + \chi(\alpha_j)$.

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- If $\Im\left(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\right) = 0$, then we bubble a handle with null volume. Up to replacing $\{\alpha_j, \beta_j\}$ with their inverses, we can assume that the edge, say e_j, joining O_{j-1} with $O_{j-1} + \chi(\alpha_j) + \chi(\beta_j)$ entirely lies in $\overline{H}_{j-1}$. Notice that here we need to consider the closure of H_{j-1} because $\chi(\alpha_j)$ and $\chi(\beta_j)$ can be parallel to l_1. Slit e_j, and denote $e_j^{\pm}$ the resulting sides. On e_j^+, define a_j^+ the subedge joining O_{j-1} with $O_{j-1} + \chi(\alpha_j)$, and define b_j^+ the subedge joining $O_{j-1} + \chi(\alpha_j)$ with $O_{j-1} + \chi(\alpha_j) + \chi(\beta_j)$. On e_j^-, define b_j^- the subedge joining O_{j-1} with $O_{j-1} + \chi(\beta_j)$, and define a_j^+ the subedge joining $O_{j-1} + \chi(\beta_j)$ with $O_{j-1} + \chi(\alpha_j) + \chi(\beta_j)$. Identify the edge a_j^+ with a_j^- and the edge b_j^+ with b_j^-. The resulting structure is a genus j surface, and the newborn handle has periods $\chi(\alpha_j)$ and $\chi(\beta_j)$. Finally, define $O_j := O_{j-1} + \chi(\alpha_j)$. In the case O_j lies in l_{j-1}, then $l_{j-1} = l_j$ and $H_{j-1} = H_j$.
- If $\Im\left(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\right) < 0$, then we bubble a handle of negative volume. Up to replacing $\{\alpha_j, \beta_j\}$ with their inverses, we can assume that the edge, say e_j, joining O_{j-1} with $O_{j-1} + \chi(\alpha_j) + \chi(\beta_j)$ entirely lies in H_{j-1}. Let $\mathcal{Q}_j \subset H_{j-1}$ be the quadrilateral bounded by the chain

$$O_{j-1} \mapsto O_{j-1} + \chi(\alpha_j) \mapsto O_{j-1} + \chi(\alpha_j) + \chi(\beta_j) \mapsto O_{j-1} + \chi(\beta_j) \mapsto O_{j-1}.$$

  We can assume without loss of generality that $\mathcal{Q}_j \cap l_{j-1} = \{O_{j-1}\}$. In fact, if one of the edges of $\mathcal{Q}_j$ would lie on l_{j-1}, then we can replace $\{\alpha_j, \beta_j\}$ with a new set of handle generators obtained by applying suitable Dehn twists so that the resulting quadrilateral enjoys the desired property. Remove the interior of $\mathcal{Q}_j$, and denote by a_j^+, respectively a_j^-, the edge of $\mathcal{Q}_j$ parallel to $\chi(\alpha_j)$. Denote by b_j^+, respectively b_j^-, similarly as before. Identify the edge a_j^+ with a_j^- and the edge b_j^+ with b_j^-. The resulting structure is a genus j surface, and the newborn handle has periods $\chi(\alpha_j)$ and $\chi(\beta_j)$ as desired. Finally, define $O_j := O_{j-1} + \chi(\alpha_j)$.

It remains to show that the translation surface, say (Y, ξ), obtained after g steps has the desired parity. This can be seen with a direct computation as follows:

$$\begin{aligned}\varphi(\xi) &= \sum_{j=1}^{g} \big(\mathrm{Ind}(\alpha_j) + 1\big)\big(\mathrm{Ind}(\beta_j) + 1\big) \pmod 2 \\ &= (\mathrm{Ind}(\alpha_1) + 1)(\mathrm{Ind}(\beta_1) + 1) \pmod 2 \;\; = \begin{cases} 0, & \text{if } k = 1 \\ 1, & \text{if } k = 2, \end{cases}\end{aligned} \tag{96}$$

because bubbling handles does not alter the spin parity; see Lemmas 3.5 and 3.8. The third equality holds because the parity of $\gcd(\mathrm{Ind}(\alpha_1), \mathrm{Ind}(\beta_1), 2)$ is always the opposite of $(\mathrm{Ind}(\alpha_1) + 1)(\mathrm{Ind}(\beta_1) + 1)$ and bubbling handles preserves the spin parity.

7.2.2. Exceptional case: single pole of order 2

Let $n = 1$, and let $\chi\colon \mathrm{H}_1(S_{g,1}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a nontrivial representation. Fix a set of handle generators $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, define an auxiliary representation as in Equation (93) and finally realize ρ as the period character of some translation surface (X, ω) as in Section §6.2.1. In order to be consistent and facilitate reading, we adopt the same notation.

Remark 7.4. Since X has genus two, the extremal points of the broken chain (70) are P_0 and P_4.

Let $O_1 = P_4$ be an initial point, and let l_1 be a straight line passing through O_1, orthogonal to r_2 and oriented so that the broken chains (70) and (79) lie on the left. In this case, we set E_1 as the half-plane on the right of l_1. We can now implement the same recursion as in §7.2.1, and hence, after $g - 2$ steps, we get a genus g differential ξ on a Riemann surface Y. A straightforward computation shows that the parity of ξ is determined by the parity of ω can be computed as in Equation (96).

Since (X, ω) can be realized with both even or odd parity, Proposition 7.1 follows in this case.

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7.2.3. Exceptional case: two simple poles

Let $n = 2$, and let $\chi\colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation of nontrivial-ends type. Let $w \in \mathbb{C}^*$ such that $\mathrm{Im}(\chi_2) = \langle w \rangle$, where χ_2 is the representation encoding the polar part of χ; see Section §2.1. Let $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a system of handle generators, and define an auxiliary representation $\rho\colon \mathrm{H}_1(S_{2,2}, \mathbb{Z}) \longrightarrow \mathbb{C}$ as

$$\rho(\alpha_1) = \chi(\alpha_1), \quad \rho(\beta_1) = \chi(\beta_1), \quad \rho(\alpha_2) = \chi(\alpha_2), \quad \rho(\beta_2) = \chi(\beta_2), \tag{97}$$

$$\rho(\delta_1) = \rho(\delta_2^{-1}) = w \in \mathbb{C}^*. \tag{98}$$

We already know from §6.4 how to realize ρ as the period character of some translation surfaces with poles, say $(X, \omega) \in \mathcal{H}_2(4; -1, -1)$, and prescribed parity. For simplicity, we shall adopt the same notation as therein, and we proceed with a discussion case by case as follows.

7.2.3.1. χ is not real-collinear.

Consider first nonreal-collinear representations. By Corollary 4.10, we can assume that any pair $\{\alpha_j, \beta_j\} \subset \mathcal{G}$ has positive volume, that is, $\Im\Big(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\Big) > 0$. Without loss of generality, we can also assume that the auxiliary representation ρ is also not real-collinear. Notice that ρ can be discrete of rank two even if the overall representation χ is not (it cannot be discrete of rank one because all pairs of handle generators have positive volume). If ρ is discrete of rank two, then Lemma 4.12 applies and we renormalize the handle generators $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ so that

$$\rho(\alpha_1) = \rho(\beta_1) \in \mathbb{Z}_+ \quad \text{and} \quad \rho(\alpha_2) \in \mathbb{Z}_+,\ \rho(\beta_2) = i.$$

Under these conditions, we realize (X, ω) as in §6.4.1.1 or §6.4.2 depending on whether ρ is discrete or not. Let $O_1 = P_4$ be the initial point. For any $j = 3, \ldots, g$, consider the pair of handle generators $\{\alpha_j, \beta_j\}$. Notice that least one between $\chi(\alpha_j)$ and $\chi(\beta_j)$ is not parallel to w, otherwise $\Im\Big(\overline{\chi(\alpha_j)}\,\chi(\beta_j)\Big) = 0$.

Remark 7.5. By replacing $\{\alpha_j, \beta_j\}$ with either $\{\alpha_j^{-1}, \beta_j^{-1}\}$, $\{\beta_j, \alpha_j^{-1}\}$ or $\{\beta_j^{-1}, \alpha_j\}$ and then renaming the new pair of handle generators as $\{\alpha_j, \beta_j\}$ (with a little abuse of notation), we can assume that $\chi(\alpha_j)$ is not parallel to w and points rightwards, that is, $\arg\big(\chi(\alpha_j)\big) \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big[$. Notice that this changing does not alter the volume of the handle.

For $j = 3, \ldots, g$, let $O_{j-1} \coloneqq O_{j-2} + \chi(\alpha_j)$, and let e_{j-2} be the edge joining O_{j-2} with O_{j-1}; see Figure 38. Notice that e_j is a geodesic segment that 'wraps' around the cylindrical ends on (X, ω) without overlapping itself because $\chi(\alpha_j)$ is not parallel to w by design. Let $\mathcal{P}_j \subset \mathbb{C}$ be the parallelogram bounded by the chain

$$O_{j-2} \mapsto O_{j-2} + \chi(\alpha_j) \mapsto O_{j-2} + \chi(\alpha_j) + \chi(\beta_j) \mapsto O_{j-2} + \chi(\beta_j) \mapsto O_{j-2}.$$

As usual, according to our convention, we denote by a_j^+, respectively a_j^-, the edge of $\mathcal{P}_j$ parallel to $\chi(\alpha_j)$ that bounds the parallelogram on its right, respectively left. Similarly, we denote by b_j^+, respectively b_j^-, the edge of $\mathcal{P}_j$ parallel to $\chi(\beta_j)$ that bounds the parallelogram on its right, respectively left. Next, we slit (X, ω) along e_{j-2} and denote by $e_{j-2}^\pm$ the resulting sides. Identify the edge e_{j-2}^- with a_j^+ and e_{j-2}^+ with a_j^-. Finally, identify the edge b_j^+ with b_j^-. After repeating this process $g - 2$ times, we obtain a translation surface (Y, ξ) with period character χ and spin parity $\varphi(\xi)$ equal to $\varphi(\omega)$ because bubbling handles with positive volume, as in Section §3.2, does not alter the spin parity. Since (X, ω) can be realized with prescribed parity, both even and odd parity are achievable, and the desired conclusion follows in this case.

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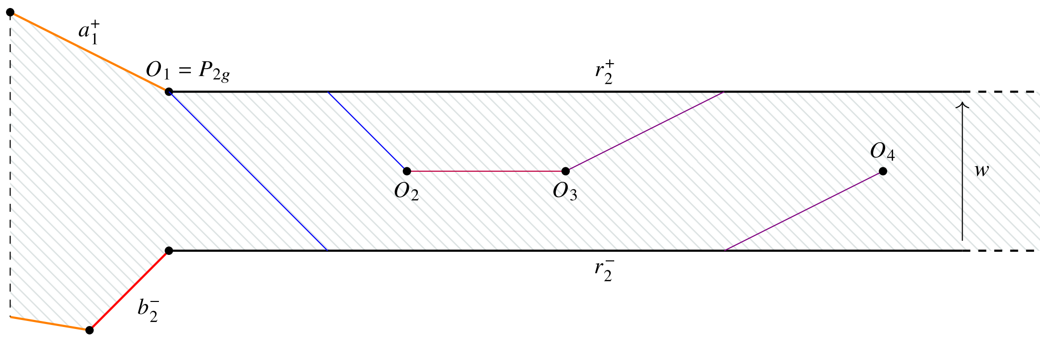

Figure 38. *Bubbling handles of positive volume on a genus-two differential constructed as in §6.4.1.1. All bubbling is performed inside a cylinder. Each coloured edge corresponds to a slit along which we bubble a handle with positive volume. Notice that, if (X, ω) is realized in the hyperelliptic component, then the labels a_1^+ and b_2^- should be replaced with u_1^+ and v_2^-, respectively.*

7.2.3.2. χ is real-collinear but not discrete.
We now assume χ to be real-collinear but not discrete. Up to replacing χ with $A\chi$, for some $A \in \mathrm{GL}^+(2, \mathbb{R})$, we can assume that $\mathrm{Im}(\chi) \subset \mathbb{R}$. Lemma 4.13 applies, and hence, we can find a system of handle generators $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ such that $a_j = \chi(\alpha_j)$ and $b_j = \chi(\beta_j)$ satisfy the inequality

$$v = \sum_{j=1}^{g} a_j + b_j < w. \tag{99}$$

By introducing an auxiliary representation ρ as in Equation (97), we first realize a translation surface (X, ω) in the stratum $\mathcal{H}_2(4; -1, -1)$ with period character ρ and prescribed parity as in §6.4.1.2. If we aim to realize ρ in the hyperelliptic component of $\mathcal{H}_2(4; -1, -1)$, then recall that our construction relies on §6.1 and hence involve the introduction of two pairs (u_1, v_1), $(u_2, v_2) \in \mathbb{C}^2$. By construction, there is a closed saddle connection, say s, of length

$$w - \sum_{j=1}^{2} u_j + v_j \quad \text{or} \quad w - \sum_{j=1}^{2} a_j + b_j$$

depending on whether ρ is realized in the hyperelliptic component of $\mathcal{H}_2(4; -1, -1)$ or not. In both cases, since $v < w$ we have enough room on s for bubbling $g - 2$ handles with zero volume. After $g - 2$ steps, we obtain a translation surface, say (Y, ξ), with period character χ and spin parity $\varphi(\xi)$ equal to $\varphi(\omega)$. Since the initial structure (X, ω) can be realized with either even or odd parity, the desired conclusion follows in this case.

7.2.3.3. χ is discrete of rank one.
We are left to consider discrete representations of rank one that require a deeper discussion. We can assume $\chi \colon \mathrm{H}_1(S_{g,2}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ without loss of generality. Recall that realizing such a representation in a certain stratum $\mathcal{H}_g(2m_1, \ldots, 2m_k; -1, -1)$ depends only on whether the condition

$$\max\{2m_1, \ldots, 2m_k\} < w = \chi(\gamma) \in \mathbb{Z} \tag{100}$$

holds, where γ is a simple loop around a puncture.

In the case $w > 2g$, the auxiliary representation ρ defined as in Equation (97) can be realized as in §6.4.3.1 as the holonomy of some translation surface with poles and with prescribed parity. On the other hand, this latter construction can be done as in §6.4.1.2. Therefore, we can proceed exactly as in §7.2.3.2.

Let us assume $w \leq 2g$, and let $\mathcal{H}_g(2m_1, \ldots, 2m_k; -1, -1)$ be a stratum such that the condition (100) holds. It is easy to observe that $m_1 + \cdots + m_k = g$ for this stratum; therefore, $m_i \leq g$ and the equality holds if and only if $k = 1$. Without loss of generality, assume that

$$2m_1 \geq 2m_2 \geq \cdots \geq 2m_k \tag{101}$$

holds. Consider the auxiliary representation $\rho\colon \mathrm{H}_1(S_{m_1,2}, \mathbb{Z}) \longrightarrow \mathbb{Z}$ defined as follows:

$$\rho(\alpha_i) = \rho(\beta_i) = 1, \quad \text{for } i = 1, \ldots, m_1, \quad \text{and} \quad \rho(\delta_1) = \rho(\delta_2^{-1}) = w. \tag{102}$$

Since $w > 2m_1$, then ρ can be realized as the period character of some translation surface, say (X_1, ω_1), in the stratum $\mathcal{H}_{m_1}(2m_1; -1, -1)$ with prescribed parity; see paragraph §6.4.3.2. If $k = 1$, we are done; otherwise, we proceed as follows.

From paragraph §6.4.3.2, we recall that (X_1, ω_1) is obtained by gluing two infinite half-strips $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{C}$. Let us focus on $\mathcal{S}_1$. This region is bounded by a segment s^- of length w and two half-rays r_1^+ and r_2^-. The sign as usual denote on which side the region is bounded according to their orientation. Since $w > 2m_2$, the interior of $\mathcal{S}_1$ contains a segment, say s_2, of length $2m_2$ and parallel to s^-. On the other hand, the interior of $\mathcal{S}_1$ is embedded in (X_1, ω_1), and hence, there is an isometric copy of s_2 inside (X_1, ω_1). With a little abuse of notation, this latter is also denoted by s_2. Next, we divide $s_2 \subset (X_1, \omega_1)$ into $2m_2$ subsegments, say $s_{2,1}, s_{2,2}, \ldots, s_{2,2m_2-1}, s_{2,2m_2}$, each of length 1. Slit all of them, and reglue as follows: $s_{2,2i-1}^-$ is identified with $s_{2,2i}^+$ and $s_{2,2i-1}^+$ is identified with $s_{2,2i}^-$. The resulting space, say (X_2, ω_2), is a surface of genus $m_1 + m_2$. As a consequence of Lemma 3.8, (X_1, ω_1) and (X_2, ω_2) have the same parity. Again, if $k = 2$ we are done; otherwise, there is always a segment, say s_3 of length $2m_3$, that lies in the interior of $\mathcal{S}_1$. If this is the case, we proceed as we have just done. After k steps, we obtain a surface $(X_k, \omega_k) = (Y, \xi)$ of genus $m_1 + \cdots + m_k = g$ with period character χ. Moreover, since bubbling a handle with zero volume does not alter the spin parity (see Lemma 3.8), the resulting structure has the same parity as (X_1, ω_1). This latter, according to Section §6 can be realized with even or odd parity (see also Remark 6.15); hence, Proposition 7.1 also holds for discrete representations of rank one. Notice that the same argument would have been valid if we have chosen $\mathcal{S}_2$ in place of $\mathcal{S}_1$. Since there are no other cases to consider, this concludes the proof of Proposition 7.1 and indeed the proof of Theorem A which is specific for nontrivial representations.

8. Meromorphic exact differentials

We finally consider the trivial representation, and we aim to prove Theorem B. On a compact Riemann surface $\overline{X}$, any nonconstant rational function $f\colon \overline{X} \longrightarrow \mathbb{C}\mathbf{P}^1$ yields a finite degree branched covering and the meromorphic differential $\omega = df$ has trivial absolute periods; that is, ω determines a trivial period character. Let us denote by $X = \overline{X} \setminus \{\text{ poles of } \omega \}$. Then the couple (X, ω) is a translation surface with poles in the sense of Definition 2.1. Conversely, if (X, ω) is a translation surface with poles on $S_{g,n}$ and with zero absolute periods, then the developing map (see Section §2) boils down to a holomorphic mapping $X \longrightarrow \mathbb{C}$ that extends to a rational function $f\colon \overline{X} \longrightarrow \mathbb{C}\mathbf{P}^1$ and $\omega = df$.

Definition 8.1. A meromorphic differential ω on a compact Riemann surface is called an *exact differential* if all absolute periods of ω are equal to zero. On a Riemann surface X of finite type (g, n), we say that a holomorphic differential ω with finite-order poles at the punctures is *exact* if all absolute periods are zero.

8.1. Bubbling handles with trivial periods

We describe here how to glue handles with trivial periods. More precisely, we provide a surgery to add a handle with trivial periods on a genus-zero differential in order to obtain a genus-one differential with

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prescribed rotation number; see §8.1.1. We next provide an alternative construction (see §8.1.2), which will be useful later on.

Let (X,ω) be any translation structure on a surface $S_{g,n}$, and let $\mathrm{dev}\colon \widetilde{S}_{g,n} \longrightarrow \mathbb{C}$ be its developing map. We introduce the following terminology.

Definition 8.2 (Twin paths). On a translation surface (X,ω), let P be any branch point of order m. Consider a collection of $m+1$ embedded path $c_i\colon [0,1] \longrightarrow (X,\omega)$ such that $c_i(0) = P$ for $i = 0,\dots,m$, each of which is injectively developed, and all of which overlap once developed; that is, there is a determination of the developing map around $c_0 \cup \cdots \cup c_m$ which injectively develops $c_0, c_1, \dots, c_m$ to the same arc $\widehat{c} \subset \mathbb{C}$. For any $2 \le k \le m$, the paths $c_{i_1}, \dots, c_{i_k}$ are called *twin paths*. For any pair c_i, c_j, we may notice that the angle at P between them is a multiple of 2π.

Convention. In what follows, we shall agree that all twin paths $c_0, \dots, c_m$ at P as defined in the above are counterclockwise ordered if not otherwise specified.

The following technical lemma is straightforward and the proof is left to the reader.

Lemma 8.3. *On a translation surface (X,ω), let P be any branch point of order m, and let $B_{4\varepsilon}(P)$ be a simply connected open metric ball centered at P. Break P into two zeros, say P_1 and P_2 of orders m_1 and m_2, respectively, such that P_1 and P_2 are joined by a saddle connection, say s, of length ε. Then there are m_1 paths, say $c_1, \dots, c_{m_1}$, all leaving from P_1, such that s and c_i are twins for every $i = 1, \dots, m_1$. Similarly, there are m_2 paths, say $c_{m_1+1}, \dots, c_m$, all leaving from P_2, such that c_{m_1+j} and s are twins for every $j = 1, \dots, m_2$.*

Remark 8.4. Notice that by choosing sufficiently small ε so that the open metric ball $B_{4\varepsilon}(P)$ is simply connected then all twins are contained inside it.

8.1.1. Handles with trivial periods

Let (X,ω) be a translation surface with poles, let P be a branch point of order $m_1 + m_2$ and let $B_{4\varepsilon}(P)$ be a simply connected open ball. Finally, assume the open ball $B_{4\varepsilon}(P)$ does not contain branch points other than P. Break P into two branch points, say P_1 and P_2, of orders m_1 and m_2, respectively, so that they are joined by a saddle connection s of length ε. After breaking P into P_1 and P_2, the resulting open ball does not contain branch points other than them.

Fix an orientation on s, say from P_1 to P_2. The saddle connection s has m_1 twins leaving from P_1, and it determines on each of them an obvious outbound orientation. In the same fashion, s has m_2 twin paths leaving from P_2, and it determines on each of them an obvious inbound orientation to P_2. Let c_1 be the twin of s leaving from P_1 that forms an angle of 2π on its right. Define c_2 as the twin of s leaving from P_2 that forms an angle of $2\pi(r+1)$ on its left. Next, slit both c_1 and c_2, and we denote the resulting sides $c_1^{\pm}$ and $c_2^{\pm}$ where the signs are taken according to our convention. Then, identify c_1^{+} with c_2^{-} and similarly identify c_1^{-} with c_2^{+}. If the initial translation surface (X,ω) has genus g, the resulting surface (Y,ξ) has genus $g+1$.

Let α be a simple closed curve around c_2 that winds around P_2 but not around the other branch point. It can be checked that α has index equal to m_2+1 because P_2 has order m_2. Next, Let $\beta \subset (X,\omega)$ be a smooth path starting from the midpoint of c_1^{-} to the midpoint of c_2^{+} that crosses s, and it does not contain any branch point in its interior (hence it misses both P_1 and P_2). On (Y,ξ), the smooth curve β closes up to a simple closed curve such that along with α they provide a basis of handle generators; see Definition 4.2 for the newborn handle. It can be checked that β has index equal to r because it turns of an angle 2π around P_1 counterclockwise, and then it turns of an angle $2\pi(r+1)$ around P_2 clockwise. See Figure 39.

By denoting P_1' the extremal point of c_1 other than P_1 and by P_2' the extremal point of c_2 other than P_2, we can notice that they are both regular. Next, identify P_1 with P_2' and P_2 with P_1'. Once the identification is done, we get two branch points of angles $2\pi(m_1+2)$ and $2\pi(m_2+2)$. The following holds.

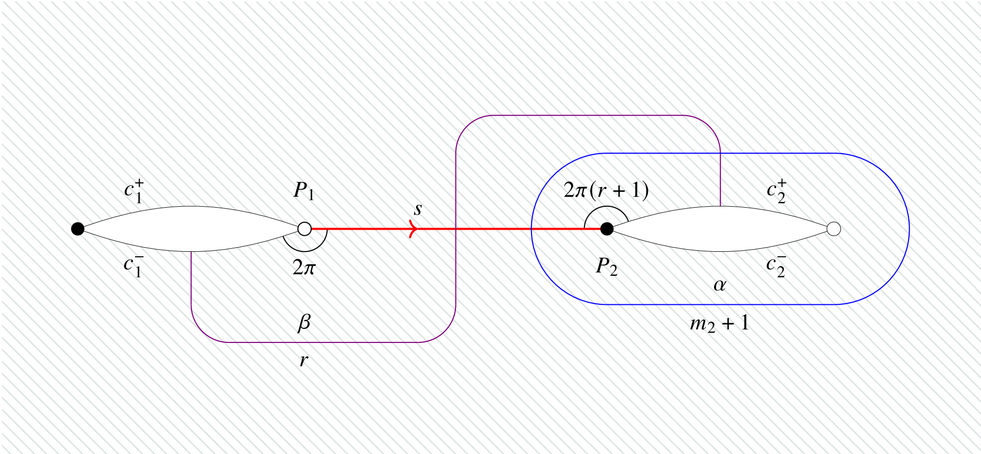

Figure 39. *Adding a handle with trivial periods and handle generators with prescribed indices. The orange segment is a saddle connection joining two zeros of odd orders. The blue curve α has index m_2+1 whereas the violet curve β has index r.*

Lemma 8.5. *Let $(X,\omega) \in \mathcal{H}_0(m_1,\dots,m_{k-1}-1,\ m_k-1;\ -p_1,\dots,-p_n)$ be a genus-zero differential with trivial periods. Let P_{k-1} and P_k be the zeros of orders $m_{k-1}-1$ and m_k-1, respectively, and assume there is a saddle connection joining them. Assume we can bubble a handle with trivial periods as described above, and let (Y,ξ) be the resulting translation surface. Then (Y,ξ) is a genus-one differential with rotation number equal to* $\gcd(r,m_1,\dots,m_k,p_1,\dots,p_n)$.

Proof. The fact that (Y,ξ) is a genus-one differential directly follows from the construction. By adopting the notation above, α has index equal to m_k and β has index r. Then the desired conclusion follows. □

8.1.2. Alternative construction

We now introduce an alternative way for adding a handle with trivial periods. We shall use this version later on for the inductive foundation in Section §8.6. Let $(X,\omega) \in \Omega\mathcal{M}_{g,n}$ be a translation surface with poles, and let $P \in (X,\omega)$ be a branch point of even order $2m$; that is, the angle at P is $(4m+2)\pi$. Let $B_{4\varepsilon}(P)$ be an open ball of radius 4ε at P. Consider three geodesic twin paths c_1, c_2, c_3 at P, all leaving from P with length ε and counterclockwise ordered as in our convention such that the angle between c_1 and c_2 and the angle between c_2 and c_3 are both 2π. Then the angle between c_3 and c_1 is $(4m-2)\pi$. Let Q_1, Q_2, Q_3 denote the extremal regular points of c_1, c_2, c_3, respectively, other than P. For each $i=1,2,3$, assume that c_i is oriented from P to Q_i. By slitting all of these paths, the branch point P splits into three points P_1, P_2, P_3, and we get a surface of genus g with piecewise geodesic boundary. We can assume that P_1, Q_1, P_2, Q_2, P_3, Q_3 are cyclically ordered as shown in Figure 40. Then the corner angles at all vertices but P_3 is 2π and the angle at P_3 is $(4m-2)\pi$.

Notice that, by prolonging each c_i to a geodesic ray r_i of length 4ϵ, the open ball $B_{4\varepsilon}(P)$ is divided into three sectors, say S_1, S_2, S_3. We assume S_i is the sector containing P_i after slitting c_1, c_2, c_3. We next consider the following smooth arcs on the surface with boundary obtained after slitting. See Figure 40.

- Let α be a smooth arc joining a point $A_1 \in P_2Q_2$ at distance $\varepsilon/3$ from P_2 and a point $A_2 \in P_1Q_3$ at distance $\varepsilon/3$ from P_1. We can take this arc such that it lies in the sectors S_1 and S_2 without crossing the sector S_3.
- Let β_1 be a smooth arc joining a point $B_1 \in P_1Q_1$ at distance $2\varepsilon/3$ from P_1 and a point $B_2 \in P_1Q_3$ at distance $2\varepsilon/3$ from P_1. We can assume that β_1 lies entirely in the sector S_1. Notice that, by construction, α and β_1 cross.

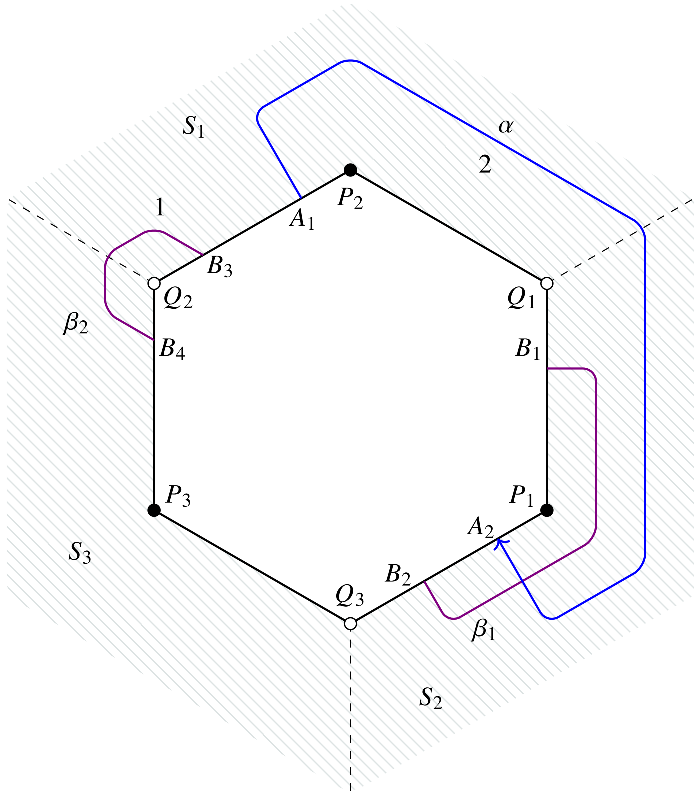

Figure 40. *Adding a handle with trivial periods. The blue curve* α *has index* 2*, and violet curve* β *has index* 0*.*

- Finally, let β_2 be a smooth arc joining a point $B_3 \in P_3Q_3$ at distance $2\varepsilon/3$ from P_2 and a point $B_4 \in P_3Q_2$ at distance $2\varepsilon/3$ from P_3. We can take β_2 such that it does not cross the sector S_1 and is disjoint from α.

We next glue the edges of the hexagonal boundary as follows. First, identify P_1Q_1 with P_3Q_2, then P_2Q_1 with P_3Q_3 and, finally, P_2Q_2 with P_1Q_3. The resulting surface, say (Y, ξ), has genus $g+1$. By construction, the arc α closes up to a simple closed curve of index 2. Moreover, B_1 identifies with B_4 and B_2 identifies with B_3. Therefore, $\beta_1 \cup \beta_2$ yields a simple closed curve β of index 2. The pair of curves $\{\alpha, \beta\}$ determines a set of handle generators for the newborn handle. The points P_1, P_2, P_3 are identified according to our construction and they determine a branch point of angle $(4m+2)\pi$. Similarly, Q_1, Q_2, Q_3 are also identified and they determine a branch point of angle 6π because the extremal points are all assumed to be regular. The following holds.

Lemma 8.6. *Let* $(X, \omega) \in \mathcal{H}_g(2m_1, \dots, 2m_k; -2p_1, \dots, -2p_n)$ *be a translation surface with poles and trivial periods. Let* (Y, ξ) *be the translation surface obtained by adding a handle with trivial periods as described above. Then* $(Y, \xi) \in \mathcal{H}_{g+1}(2, 2m_1, \dots, 2m_k; -2p_1, \dots, -2p_n)$ *which is a stratum of even type, it has trivial periods, and its parity is given by* $\varphi(\xi) = \varphi(\omega) + 1$*.*

Proof. The fact that (Y, ξ) belongs to a stratum of even type directly follows from the construction. By adopting the notation above, since α has index 2 and β has index 0, then

$$(\mathrm{Ind}(\alpha) + 1)(\mathrm{Ind}(\beta) + 1) = 3 \cdot 1 \equiv 1 \pmod 2.$$

Therefore, $\varphi(\xi) = \varphi(\omega) + 1$. □

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We shall use the surgeries above to realize translation surfaces with poles and trivial periods in a given stratum with prescribed parity.

8.2. Hurwitz type inequality

In [CFG22, Theorem B], the authors provided necessary and sufficient conditions for realizing the trivial representation in a given stratum $\mathcal{H}_g(m_1, \dots, m_k; -p_1, \dots, -p_n)$. One of these conditions relates the order of zeros with the order of poles as follows:

$$m_j \leq \sum_{i=1}^{n} p_i - n - 1. \tag{103}$$

In what follows, we shall make a strong use of this constraint which led to consider several exceptional cases. We shall refer to it as *Hurwitz type inequality*.

8.3. Genus-zero exact differentials

We briefly recall the strategy adopted in [CFG22, Section §8] to realize the trivial representation as the period character of some exact differentials on $\mathbb{C}\mathbf{P}^1$ with prescribed zeros and poles. The aim is to make the autonomous explanation of the next subsections as self-contained as possible.

Suppose we want to realize the trivial representation in a stratum $\mathcal{H}_0(m_1, \dots, m_k; -p_1, \dots, -p_n)$. We consider n translation surfaces $(\mathbb{C}, \omega_i)$ for $1 \leq i \leq n$, where the differential ω_i has a pole of order $p_i \geq 2$ at the infinity and a zero of order $p_i - 2$ at $0 \in \mathbb{C}$. We consider some, possibly all, of these translation surfaces and we glue them, by slitting along a segment of finite length, to define a sequence of translation surfaces (Y_j, ξ_j). Here, each (Y_j, ξ_j) is a sphere with an exact meromorphic differential with poles of orders $p_1, \dots, p_t$ for some $1 \leq t \leq n$ and zeros of orders $m_1, \dots, m_{s-1}$ and $\widetilde{m}_s$ for some $1 \leq s \leq k$, where $\widetilde{m}_s \geq m_s$ and $\widetilde{m}_s \leq m_s + m_{s+1} + \cdots + m_k$. Since the degree of any meromorphic differential on the sphere is -2, we have

$$\sum_{l=1}^{s} m_l + (\widetilde{m}_s - m_s) = \sum_{i=1}^{t} p_i - 2. \tag{104}$$

The sequence (Y_j, ξ_j) is constructed such that, as j increases, s and t increase until $t = n$. The process runs until all the $(\mathbb{C}, \omega_i)$'s are exhausted. We finally break the last zero to get the desired differential.

Remark 8.7. The key observation here is for the translation surface (Y_1, ξ_1) obtained after the first step with poles $p_1, \dots, p_t$ and two zeros $m_1, \widetilde{m}_2$, where $\widetilde{m}_2 \geq m_2$. In the case $t = n$, then only the following cases arise:

- $\widetilde{m}_2 = m_2$, and hence, the desired genus-zero differential belongs to $\mathcal{H}_0(m_1, m_2; -p_1, \dots, -p_n)$; otherwise
- $\widetilde{m}_2 > m_2$, and hence, the desired genus-zero differential belongs to a certain stratum $\mathcal{H}_0(\kappa; -\nu)$, where $\kappa = (m_1, m_2, \dots, m_k)$ and $\nu = (p_1, \dots, p_n)$. In this case, the zero of order $\widetilde{m}_2$ is broken into $k - 1$ zeros of orders $m_2, \dots, m_k$.

This constriction can be performed in such a way that if P_i and P_j are two zeros such that $|\, i - j \,| = 1$, then there exists at least one saddle connection joining P_i and P_j, that is, a geodesic segment with no zeros in its interior. Finally, we can arrange the zeros so that two different zeros have different developed images. As a consequence, if s_i is the saddle connection joining P_i with P_{i+1}, all twins of s_i leaving from P_i do not contain any zero other than P_i and P_{i+1} themselves. See [CFG22, Section §8.3.4] for more details.

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8.4. *Trivial periods and prescribed rotation number*

The aim of this section is to prove the following proposition concerning genus-one differentials.

Proposition 8.8. *Suppose the trivial representation $\chi\colon \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized in a stratum $\mathcal{H}_1(m_1,\ldots,m_k;\ -p_1,\ldots,-p_n)$. Then it can be realized in each connected component of the same stratum with the only exceptions being the strata $\mathcal{H}_1(2,2;\ -4)$, $\mathcal{H}_1(2,2,2;\ -2,-2,-2)$ and $\mathcal{H}_1(3,3;-3,-3)$.*

The proof of this proposition is based on the construction done before in subsection §8.1.1. We recall for the reader's convenience that the connected components of $\mathcal{H}_1(m_1,\ldots,m_k;\ -p_1,\ldots,-p_n)$ are distinguished by the divisors of $\gcd(m_1,\ldots,m_k,p_1,\ldots,p_n)$. Clearly, if the greatest common divisor is one there is nothing to prove and the realization is subject to [CFG22, Theorem B]. Throughout the present subsection, for a generic stratum let $p = p_1+\cdots+p_n \ = \ m_1+\cdots+m_k$ and let r be a divisor of $\gcd(m_1,\ldots,m_k;p_1,\ldots,p_n)$. We shall need to distinguish several cases according to certain mutual relationships among p, r, n and k.

8.4.1. Strata with two zeros

It is not hard to show that the trivial representation $\chi : \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ cannot be realized in every stratum with a single zero of maximal order because the Hurwitz type inequality (103) never holds for these strata. Therefore, strata with two zeros appear as the simplest minimal strata in which the trivial representation can be realized. Recall that our aim is to realize r as the prescribed rotation number for a trivial representation. The first case we consider is the following:

8.4.1.1. All poles have order r

Let us realize the trivial representation in the stratum $\mathcal{H}_1(m_1,m_2;-r^n)$. In this case, $m_1+m_2 = nr$ and m_1, m_2 are both divisible by r by assumption. Moreover, m_1 and m_2 are also subject to the Hurwitz type inequality (103); hence,

$$nr = m_1+m_2 \le 2nr-2n-2 \qquad \Longleftrightarrow \qquad r \ge 3. \tag{105}$$

A few remarks are in order.

Remark 8.9. We may also notice that $n \ge 2$. In fact, let us assume that $n = 1$, and hence, $m_1+m_2 = r$. Since m_1, m_2 are both divisible by r we would get the following equality:

$$\frac{m_1}{r}+\frac{m_2}{r} = k_1+k_2 = 1 \tag{106}$$

with k_1, $k_2 \in \mathbb{Z}^+$. This is possible if and only if $k_1 = 1$ and $k_2 = m_2 = 0$, up to relabelling. Since the trivial representation cannot be realized in the strata of type $(m;\ -m)$, we get the desired contradiction.

The present paragraph is entirely devoted to show the following:

Lemma 8.10. *Let $\mathcal{H}_1(m_1,m_2;\ -r^n)$ be a stratum of genus-one differentials with $r \ge 3$. Then the trivial representation can be realized in the connected component with rotation number r. Furthermore, the trivial representation can be realized in the connected component with rotation number* 1 *with the only exception being the stratum $\mathcal{H}_1(3,3;\ -3,-3)$ (that corresponds to the case $n = 2$ and $r = 3$).*

In this case, we first realize the trivial representation in $\mathcal{H}_0(m_1-1,m_2-1;-r^n)$ by using the saddle connection configuration description in [EMZ03, Section §9.3]. More precisely, we choose n pairs of positive integers, say (a_i,b_i), such that $a_i+b_i = r$ for $i = 1,\ldots,n$, and moreover, $a_1+\cdots+a_n = m_1$ and $b_1+\cdots+b_n = m_2$.

Remark 8.11. The existence of these pairs (a_i,b_i) can be explained as follows. We first notice that, the Hurwitz type inequality (103) for $\mathcal{H}_1(m_1,m_2;-r^n)$ implies $n < m_1,m_2 < (r-1)n$. We want pairs of positive integers a_i,b_i to satisfy the conditions above. In particular, $0 < a_i < r$ ensures b_i to be positive.

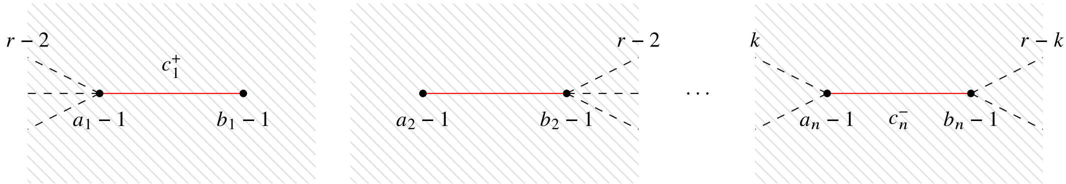

Figure 41. *Realization of a genus-zero differential in* $\mathcal{H}_0(m_1-1, m_2-1; -r^n)$.

The extreme case happens when $m_1 = n+1$, where we can take $a_1 = 2$ and $a_2 = \cdots = a_n = 1$ – recall that r is at least 3; see Equation (105) above. If m_1 increases, we increase a_1, until it reaches $r-1$, then we increase a_2, and so on.

For every $i = 1, \dots, n$, consider a copy of $(\mathbb{C}, z^{r-2}dz)$, and fix any vector $c \in \mathbb{C}^*$. Let $(\mathbb{C}\mathbf{P}^1, \omega_i)$ be the translation surface with poles obtained by breaking the zero of $(\mathbb{C}, z^{r-2}dz)$ into two zeros, say P_i and Q_i, of orders $a_i - 1$ and $b_i - 1$, respectively, so that the resulting saddle connection joining them, say c_i, is parallel to c with the same length, that is, $|\,c\,| = |\,c_i\,|$. See Figure 41.

Slit every $(\mathbb{C}\mathbf{P}^1, \omega_i)$ along c_i, and denote by $c_i^{\pm}$ the resulting edges according to our convention. Identify c_i^- with c_{i+1}^+ for every $i = 1, \dots, n-1$, and finally glue c_n^- with c_1^+. The resulting space is now a pair $(\mathbb{C}\mathbf{P}^1, \omega)$, where ω is a meromorphic differential with two zeros, say P_o and Q_o, of orders $m_1 - 1$ and $m_2 - 1$ and n poles of order r. Let s_1 be the saddle connection resulting from the identification of c_1^+ with c_n^-. We extend the notation by defining s_{i+1} to be the saddle connection resulting from the identification of c_i^- with c_{i+1}^+ for every $i = 1, \dots, n-1$. In the resulting space, mark the saddle connection arising from the identification of c_1^+ with c_n^-, and define it as the *beginning saddle connection* as the saddle connection. This is purely a convention as in principle any other saddle connection arising from the identification of c_i^+ with c_{i+1}^- could be equally entitled as the beginning saddle connection. We shall use this terminology later on.

In order to bubble the desired handle, we shall need to slit two paths, say s_o and s_{n+1}, at P_1 and Q_n, respectively, with the same holonomy as the saddle connections; that is, we want to find s_o and s_{n+1} such that any pair s_i, s_j in the collection $s_o, s_1, \dots, s_n, s_{n+1}$ are twins in $(\mathbb{C}\mathbf{P}^1, \omega)$. We need to make sure that s_o and s_{n+1} exists, and they do not coincide with any preexisting saddle connection in $(\mathbb{C}\mathbf{P}^1, \omega)$.

Remark 8.12. For this purpose, we need to choose the pairs (a_i, b_i) in a proper way. Let $A_i = a_1 + \cdots + a_i$ and $B_i = b_1 + \cdots + b_i$, for $i = 1, \dots, n$. The gist of idea is to find pairs of positive integers (a_i, b_i) as above such that there exist two positive integers, say A and B, such that one of the following conditions holds:

1. $A \equiv B \pmod r$ and $A \neq A_i$ and $B \neq B_i$ for all $i = 1, \dots, n$ if we aim to get a structure with rotation number r, or
2. $A + 1 \equiv B \pmod r$ or $A - 1 \equiv B \pmod r$, where $A \neq A_i$ and $B \neq B_i$ for all $i = 1, \dots, n$ if we aim to get a structure with rotation number 1.

Following Remark 8.12, we may pick s_o that forms an angle $2\pi A$ away from the beginning saddle connection at P_o. This is equivalent to pick s_o that forms an angle $2\pi A$ away from c_1^+ in $(\mathbb{C}\mathbf{P}^1, \omega_1)$. In the same fashion, we pick s_{n+1} that forms an angle $2\pi B$ away from the beginning saddle connection at Q_o, that is, that forms an angle $2\pi B$ away from c_n^- in $(\mathbb{C}\mathbf{P}^1, \omega_n)$. Then the two slits s_o and s_{n+1} do not coincide with any existing saddle connections on $(\mathbb{C}\mathbf{P}^1, \omega)$ by the assumption on A and B.

Case 1: rotation number r. We begin with realizing a genus-one differential in the given stratum with rotation number r. We shall discuss this case in details because the other case of interest follows after small modifications.

We consider first the extreme case $m_1 = n+1$; see Remark 8.11. Then we can choose the pairs (a_i, b_i) so that $a_1 = 2$ and $a_2 = \cdots = a_n = 1$, $b_1 = r-2$ and $b_2 = \cdots = b_n = r-1$. Since $r - 1 \geq 2$, then $A = B = 1$ works. If m_1 increases, we increase a_1 until it reaches $r-1$, then we increase a_2, and so on. For all of them we can use $A = B = 1$. Once we reach $a_1 = \cdots = a_{n-1} = r-1$, that means

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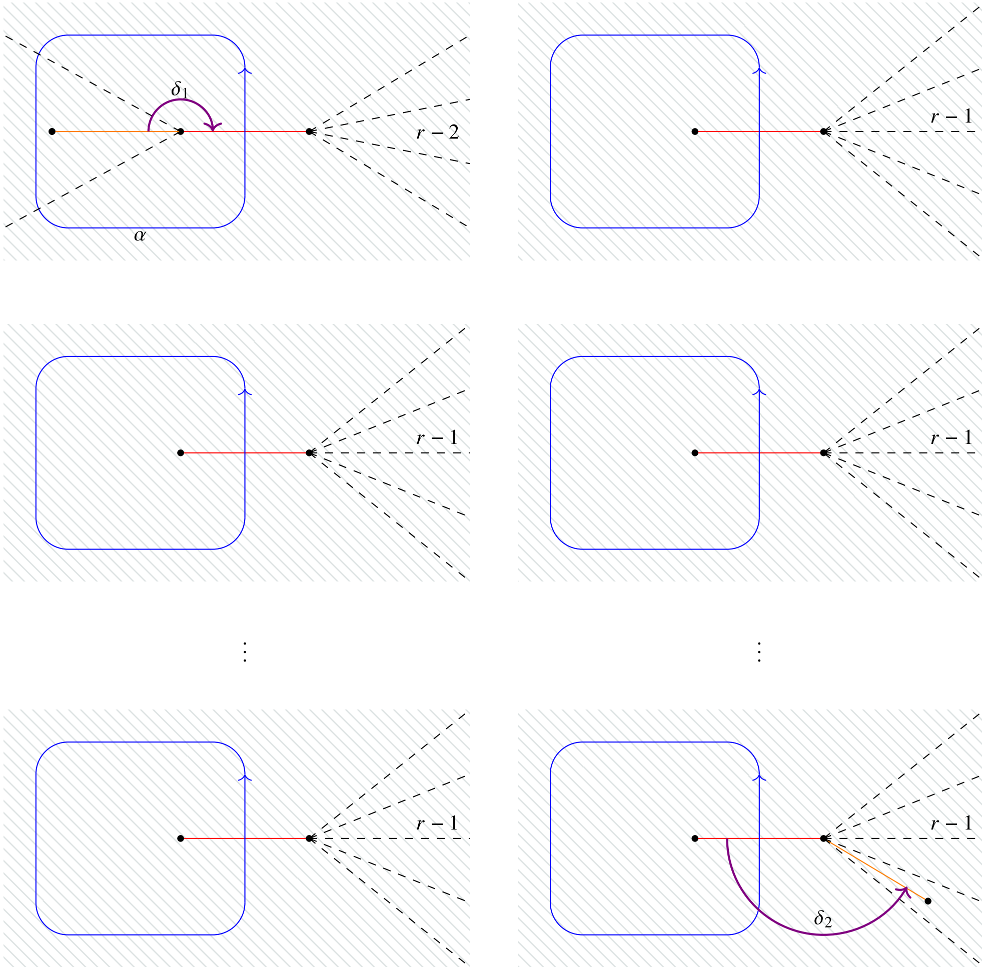

Figure 42. *Realization of the trivial representation in the stratum* $\mathcal{H}_1(n+1, nr-n-1; -r^n)$*. This figure depicts the extreme case in which* $m_1 = n+1$*, that is,* $a_1 = 2$*,* $a_2 = \cdots = a_n = 1$ *and* $b_1 = r-2$ *and* $b_2 = \cdots = b_n = r-1$*. The close curve* α *has index* $n+1$ *whereas the close* β *has index* 0*. In this case,* $n = kr - 1$ *for some* $k \in \mathbb{Z}^+$*.*

$b_1 = \cdots = b_{n-1} = 1$, m_1 might still increase, and then we need to increase a_n, that is, decrease b_n, until it drops to 2 – in this case $m_2 = n+1$ – and $A = B = 1$ still works.

We then choose the slits s_o and s_{n+1} to be of angle 2π away from the beginning saddle connection because $A = B = 1$, and hence, they do not coincide with any saddle connection $s_1, \ldots, s_n$. Let P_{n+1} and Q_{n+1} be the extremal points of s_{n+1} and s_o, respectively. Slit both, and denote the resulting sides as $s_o^\pm$ and $s_{n+1}^\pm$ according to our convention. If any saddle connection s_i is oriented from P_o to Q_o, then s_o is oriented from P_o to Q_{n+1}, and similarly, s_{n+1} is oriented from P_{n+1} to Q_o. Identify s_o^+ with s_{n+1}^-, and identify s_o^- with s_{n+1}^+. Notice that the points P_o and P_{n+1} as well as Q_o and Q_{n+1} are now identified. The resulting space is a genus-one surface equipped with a meromorphic differential. By construction, this structure lies in $\mathcal{H}_1(m_1, m_2; -r^n)$ and has trivial period character. See Figure 42.

It remains to show that such a structure has rotation number r as desired. First, we can find a closed loop representing the desired handle generator α, with index m_1 in the following way – recall that m_1 is divisible by r by assumption. For every $i = 1, \ldots, n$, let M_i be the midpoint of c_i. Then consider a closed loop around P_i based at M_i that turns counterclockwise. For every i, we may take the corresponding loop

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so that it does not contain Q_i in its interior. For $i = 1, n$, we may also assume that it does not intersect s_o and s_{n+1}, respectively. After the cut and paste process just described, these n loops define a smooth closed path of index m_1 as desired. We then find the handle generator β. On $(\mathbb{C}\mathbf{P}^1, \omega_1)$, consider a circle of radius ε centered at P_1, and denote by δ_1 the subarc representing the angle $2\pi A$ between s_o and the beginning saddle connection c_1^+. In the same fashion, on $(\mathbb{C}\mathbf{P}^1, \omega_n)$, consider a circle of radius $|c| - \varepsilon$ around Q_n, and then denote by δ_2 the subarc representing the angle $2\pi B$ between the beginning saddle connection c_n^- and s_{n+1}. On $(\mathbb{C}\mathbf{P}^1, \omega)$, the path $\delta_1 \cup \delta_2$ is not closed, but it closes up to a simple loop β once s_o and s_{n+1} are identified as described above. By construction, β has index zero because $A = B = 1$, and along with α it provides the desired system of handle generators. As a consequence, the resulting structure has rotation number equal to $\gcd\big(\mathrm{Ind}(\alpha), \mathrm{Ind}(\beta), m_1, m_2, r\big) = \gcd\big(0, m_1, m_2, r\big) = r$ as desired.

Remark 8.13. More generally, we could pairs (A, B) other than $(1, 1)$ such that $A \equiv B \pmod r$. In that case, the curve β have index equal to $B - A$. Since this latter is divisible by r we eventually get the same result.

***Case 2: rotation number*1.** We now discuss the realization of genus-one differentials in the given stratum with rotation number 1. This case follows after the same construction with only exception being to take $A = 1$ and $B = 2$. This works as soon as $b_n > 2$. In the case $b_n = 2$, then we may take $A = 2$ and $B = 1$. In both cases, β has $|\mathrm{Ind}(\beta)| = 1$, and hence, the resulting structure necessarily has rotation number one.

8.4.1.2. Al least one pole has order bigger than r – premise

From now on, until the end of Section §8.4.1, in the forthcoming paragraphs we shall always assume $p > nr$ which means at least one pole has order bigger than r, say p_1, without loss of generality. We aim to extend Lemma 8.10 as follows:

Lemma 8.14. *Let $\mathcal{H}_1(m_1, m_2; -p_1, \ldots, -p_n)$ be a stratum of genus-one differentials. Let $r > 1$, then the trivial representation can be realized in the connected component with rotation number r. Moreover, the trivial representation can be also realized in the connected component with rotation number* 1 *with the only exception being the strata $\mathcal{H}_1(2, 2; -4)$ and $\mathcal{H}_1(3, 3; -3, -3)$.*

In this case, the gist of the idea is to use the result obtain in Section §8.4.1 as the base case for an inductive foundation. This approach, however, has two issues: The first is that $r \geq 3$ if $p = nr$, that is, the case $r = 2$ is still uncovered;, the second is that an inductive arguments works if and only if at least one zero, say m_1, has order less that $p - n - r$. Moreover, in both cases $n \geq 2$, and hence, we need an ad-hoc argument for $n = 1$. This led us to split the proof in different subcases as follows:

1. $n = 1$. In this case $r \geq 1$; see paragraph §8.4.1.3.
2. $r \geq 3$ and at least one zero has order less that $p - n - r$. In this case, we shall use an inductive argument; see paragraph §8.4.1.4.
3. $r = 2$ and at least one zero has order less that $p-n-r$. In this case, we provide an ad-hoc construction; see paragraph §8.4.1.5.

Before mentioning the remaining cases, let us now assume that both zeros have order at least $p - n - r$. In this case, we may observe that $2p - 2r - 2n \leq m_1 + m_2 = p$; that means

$$(n+1)r \leq p \leq 2r + 2n \iff (n-1)(r-2) \leq 2. \tag{107}$$

Remark 8.15. Equation (107) may hold even when one zero has order less that $p - n - r$; for example, the stratum with signature $\mu = (3, 6; -3, -6)$. It also holds for every stratum with a single pole.

Therefore, in the case both zeros have order at least $p-n-r$, there are several isolated cases to consider which we define as special, and we shall treat them separately. The remaining cases are as follows:

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4. $r \geq 3$ and both zeros have order at least $p - n - r$; see paragraph §8.4.1.6.
5. $r = 2$ and both zeros have order at least $p - n - r$; see paragraph §8.4.1.7.

In both cases above, $n \geq 2$. The forthcoming paragraphs are all devoted to the proof of Lemma 8.14, case by case according to the list above.

8.4.1.3. Proof of Lemma 8.14–case 1
In the present paragraph, we assume $n = 1$, that is, we consider strata with signature $\mu = (m_1, m_2; -p)$ with $m_1 + m_2 = p$. Notice that $p > nr$ is automatically fulfilled. We aim to show the following lemmas that, combined together, cover all possible strata with two zeros and one pole. We begin with:

Lemma 8.16. *Let $\mathcal{H}_1(m_1, m_2; -p)$ be a stratum of genus-one differentials. Let r be a divisor of* $\gcd(m_1, m_2)$ *such that $m_1, m_2 \geq p - r - 1$. Then $m_1 = m_2 = r$, and the trivial representation can be realized in the connected component of the same stratum with rotation number r if and only if $r \geq 2$.*

Proof. In the first place, we show that $r \geq 2$ is a sharp condition. By assumption, $m_i \geq p - r - 1$ that means

$$m_i \geq m_1 + m_2 - r - 1 \quad \Longrightarrow \quad r + 1 + m_i \geq m_1 + m_2 \tag{108}$$

being $m_1 + m_2 = p$ for both $i = 1, 2$. As a consequence, $r + 1 \geq \max\{m_1, m_2\}$. Since r divides $\gcd(m_1, m_2)$, it follows that $m_1 = r\,k_1$ and, similarly, $m_2 = r\,k_2$, and hence,

$$r \max\{k_1, k_2\} \leq r + 1 \quad \Longleftrightarrow \quad \max\{k_1, k_2\} \leq \frac{r+1}{r}. \tag{109}$$

Now, it is an easy matter to see that

1. If $r \geq 2$ then $\max\{k_1, k_2\} = 1$, and hence, $m_1 = m_2 = r$ and $p = 2r$.
2. If $r = 1$, then $\max\{k_1, k_2\} \leq 2$. In this case, we have the following possibilities:
   2.1 $m_1 = m_2 = 1$ and $p = 2$,
   2.2 $m_1 = 1, m_2 = 2$ and $p = 3$, and
   2.3 $m_1 = m_2 = 2$ and $p = 4$.
   Among these three cases, $(1, 1; -2)$ is the only signature having the desired form. The trivial representation cannot be realized in $\mathcal{H}_1(1, 1; -2)$ because the Riemann–Hurwitz condition does not hold, and hence, $r \geq 2$ is a sharp condition as stated. The other two cases do not have the desired form and so are ruled out from our discussion.

The remaining part of the lemma can be proved exactly as Lemma 8.18. In fact, by choosing e_P and e_Q such that their turning angles are 2π their difference is zero, and hence, the same construction yields the desired structure in $\mathcal{H}_1(r, r; -2r)$ for all $r \geq 2$. □

Some comments about the other signatures that appeared in the proof of Lemma 8.16 are in order. In the first place, we remark that the trivial representation cannot be realized in the stratum $\mathcal{H}_1(1, 2; -3)$ because the Riemann–Hurwitz condition does not hold. The stratum $\mathcal{H}_1(2, 2; -4)$ is a more interesting case to discuss. Lemma 8.16 states that the trivial representation can be realized in the connected component with rotation number $r = 2$ and leaves open the realization of the trivial representation in the other connected component, that is, the one with rotation number one.

Remark 8.17. Notice that $(2, 2; -4)$ is the only signature such that $\gcd(m_1, m_2, p)$ has nontrivial divisors $r > 1$, and for all of them, $m_i > p - r - 1$ holds.

It turns out that the trivial representation *cannot* be realized in the connected component of $\mathcal{H}_1(2, 2; -4)$ with rotation number one, and this is handled by Lemma 8.27 in §8.4.1.8. It remains to assume that at least one zero has order less than $p - r - 1$.

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Lemma 8.18. *Let $\mathcal{H}_1(m_1, m_2; -p)$ be a stratum of genus-one differentials. Suppose that r is a divisor of $\gcd(m_1, m_2)$ such that $m_i < p - r - 1$ for at least one zero. Then the trivial representation can be realized in the connected component of the same stratum with rotation number r.*

Proof. Observe that $p > r$ because r divides $\gcd(m_1, m_2)$. We first realize the trivial representation as the period character of some genus-zero differential in the stratum $\mathcal{H}_0(m_1 - 1, m_2 - 1; -p)$. Let $(\mathbb{C}, dz)$ be the complex plane seen as a translation surface. Pick two distinct points, say P and Q, and let s denote the edge joining them. Let r_P be a half-ray leaving from P, and let r_Q be a half-ray leaving from Q such that $r_P \cap r_Q = \phi$; for example, pick r_P and r_Q so that $r_P \cup s \cup r_Q$ are aligned. Bubble $m_1 - 1$ copies of $(\mathbb{C}, dz)$ along the ray r_P and then bubble $m_2 - 1$ copies of $(\mathbb{C}, dz)$ along the ray r_Q. Since $m_1 + m_2 - 2 = p$, the resulting space is now a genus-zero meromorphic differential in the stratum $\mathcal{H}_0(m_1 - 1, m_2 - 1; -p)$ as desired. Orient s from P to Q, and denote by s^- and s^+ the left and right side of s, respectively, according to our convention. Before bubbling, let us observe the following. Suppose $m_1 < p - r - 1$. Then a simple manipulation shows that

$$r + 1 < m_2 \tag{110}$$

where $p = m_1 + m_2$.

In order to bubble a handle with trivial periods as described in §8.1.1 above, choose a segment e_P from P so that its angle with respect to s^- is 2π and choose a segment e_Q from Q so that its angle with respect to s^+ is $2\pi(r+1)$. Notice that, as a consequence of Equation (110), we have that $2\pi(r+1) < 2\pi m_2$, and hence, there is enough room for finding such a segment for every divisor r of $\gcd(m_1, m_2)$ such that at least one zero satisfies $m_i < p - r - 1$.

Slit the edges e_P and e_Q, and denote the resulting sides $e_P^\pm$ and $e_Q^\pm$, according to our convention. Then glue e_P^+ with e_Q^- and similarly glue e_P^- with e_Q^+. The resulting space is now a genus-one differential with trivial periods in the stratum $\mathcal{H}_1(m_1, m_2; -p)$ with rotation number r as desired. In fact, consider an arc joining the midpoint of e_P with the midpoint of s that winds clockwise around P. Next, prolong this arc with another one joining the midpoint of s with the midpoint of e_Q that winds counterclockwise around Q. After the cut and paste described above, this arc closes up to a simple close curve α with index r by construction. Then pick β as any simple loop around e_P (or e_Q). This procedure is doable as long as $r+1 < \max(m_1, m_2)$, that is, if and only if there is at least one zero whose order is less that $p-r-1$. □

8.4.1.4. Proof of Lemma 8.14–case 2

Assume $m_1 < p - n - r$. Suppose we want to realize the trivial representation in the stratum $\mathcal{H}_1(m_1, m_2; -p_1, \dots, -p_n)$. Of course, r divides p_i, hence we can write this latter as $p_i = r + h_i r$, for some $h_i \geq 0$ for every $i = 1, \dots, n$. Since $p > nr$, it necessarily follows that $h_i \geq 1$ for some i. Suppose $h_1 = 1$ and $h_i = 0$ for $i = 2, \dots, n$. Then we can realize the trivial representation as the period character of some translation surface in the stratum $\mathcal{H}_1(m_1, m_2 - r; -r^n)$ as done in the previous paragraph §8.4.1.1. Notice that this is possible because Hurwitz type inequality (103) still holds due to the assumption that $m_1 < (p - r) - n$.

Remark 8.19. Notice that, since $m_1 \leq p - n - r - 1$ and $m_1 + m_2 = nr + r > nr$, because we are assuming $h_1 = 1$, it directly follows that

$$m_2 > nr - m_1 \geq nr - p + n + r + 1 = r + 1. \tag{111}$$

Therefore, $m_2 - r \geq 1$.

Next, we bubble r planes along a ray from the zero of order $m_2 - r$ to any pole of order r as described in Definition 3.12. Notice from the previous construction that this is always possible. If a closed curve crosses the ray before bubbling, then its index alters by r after, hence the rotation number remains unchanged. The resulting structure lies by design in the stratum $\mathcal{H}_1(m_1, m_2; -p_1, -r^{n-1})$ as desired. The most general case follows by applying a double induction. More precisely, we first induct on h_1 by

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keeping $h_2 = \cdots = h_n = 0$, and then we induct on the number of poles with order greater than r that is on the number of poles whose orders p_i satisfy $h_i \geq 1$.

Remark 8.20. Notice that, if $p_i = 2$ for all $i = 1, \dots, n$, then the trivial representation cannot be realized in the stratum $\mathcal{H}_1(m_1, m_2;\, -2^n)$ because m_1 or m_2 is at least $n > p - n - 1 = n - 1$, and we would have a contradiction with Hurwitz type inequality (103).

8.4.1.5. Proof of Lemma 8.14–case 3

Let $\mu = (2m_1, 2m_2; -2p_1, \dots, -2p_n)$ be a signature of even type such that $p > 2n$, compare with Remark 8.20. We first observe that, if $2m_1 = 2$ then $2m_2 = p - 2$ and the Hurwitz type inequality (103) readily implies that

$$2m_2 + 1 \;=\; p - 1 \;\leq p - n \quad\implies\quad n \leq 1. \tag{112}$$

Since this latter case has already been handled in paragraph §8.4.1.3 we assume $m_1, m_2 \geq 2$. The most general case now follows after a refinement of the construction developed in paragraph §8.4.1.1. In fact, as a preliminary remark we notice that we get the access

$$\mathcal{H}_o(2m_1 - 1, 2m_2 - 1; -2p_1, \dots, -2p_n) \;\longmapsto\; \mathcal{H}_1(2m_1, 2m_2; -2p_1, \dots, -2p_n) \tag{113}$$

to the desired stratum from a lower stratum of genus-zero differentials having the same polar part. Let us assume first that $p_i \geq 2$, which means $2p_i \geq 4$ for all $i = 1, \dots, n$. In this situation, the construction used in §8.4.1.1 works *mutatis mutandis*. For simplicity, let us recall how to find the pairs (a_i, b_i) in this setting. In the first place, we observe that $n + 1 \leq 2m_1,\ 2m_2$, for otherwise we would get a contradiction with the Hurwitz type inequality (103). By adopting the same notation, we may take $a_1 = 2$ and $a_2 = \cdots = a_n = 1$, that means $b_1 = 2p_1 - 2$ and $b_2 = \cdots = b_n = 2p_i - 1$. Since $2p_i - 1 \geq 3 > 2$, if m_1 increases, we increase a_1 until it reaches $2p_1 - 1$, then we increase a_2, and so on. In this case, once we reach $a_1 = 2p_1 - 1, \dots, a_{n-1} = 2p_{n-1} - 1$, that means $b_1 = \cdots = b_{n-1} = 1$, the value of m_1 might still increase, and then we need to increase a_n, that is, decrease b_n, until it drops to 2, that is, $m_2 = n + 1$. Recall that $r = \gcd(2m_1, 2m_2; -2p_1, \dots, -2p_n)$. We thus take A, B such that $A - B \equiv 2 \pmod r$ in order to realize a structure in $\mathcal{H}_1(\mu)$ with rotation number two, whereas we may take $A = B \pm 1$ in order to realize a structure with rotation number one.

Next, suppose $p_i = 1$, which means $2p_i = 2$, for some $i = 1, \dots, l < n$; recall that $l = n$ cannot happen because $p > 2n$. Since $2m_i \geq n + 1$, it follows that $2m_i - l > 0$ for $i = 1, 2$ and the trivial representation can be realized in the stratum with signature $\mu' = (2m_1 - l, 2m_2 - l; -2p_{l+1}, \dots, -2p_n)$ because the Hurwitz type inequality (103) holds for both zeros. Let (X_1, ω_1) be the translation surface with poles and trivial absolute periods in $\mathcal{H}_1(\mu')$. If l is even, then μ' is also a signature of even type and the corresponding stratum has at least two connected components. On the other hand, if l is odd, the corresponding stratum may very well be connected. In both situations, we take $A = B$ or $A = B \pm 1$. In the latter case, we always get a structure with rotation number one. Recall that, by construction, there is a marked saddle connection s, the *beginning saddle connection*.

We next realize a translation surface (X_2, ω_2) with trivial absolute periods in the stratum $\mathcal{H}_o(l - 1,\, l - 1; -2^l)$. In particular, we can realize (X_2, ω_2) so that any saddle connection, say s', joining the zeros has relative period equal to that of s, that is,

$$\int_s \omega_1 \;=\; \int_{s'} \omega_2. \tag{114}$$

We finally glue these structures along the beginning saddle connection to get a translation surface with poles, say $(Y, \xi) \in \mathcal{H}_1(\mu)$, with trivial absolute periods. Let $\{\alpha,\, \beta\}$ be the pair of handle generators of (X_1, ω_1) obtained as in §8.4.1.1. After gluing, the indices of α, β are altered by $\pm l$. As a consequence, the rotation number of (Y, ξ) may change according to the parity of l as follows:

1. If l is even, the gluing does not change the parities of $\mathrm{Ind}(\alpha)$ and $\mathrm{Ind}(\beta)$. Hence, the rotation number (Y, ξ) has the same parity of the rotation number of (X_1, ω_1).
2. If l is odd, the gluing changes the parities of $\mathrm{Ind}(\alpha)$ and $\mathrm{Ind}(\beta)$. Hence, the rotation number (Y, ξ) has the opposite parity of the rotation number of (X_1, ω_1).

Since the rotation number of (X_1, ω_1) is one or even, we get the desired result, and case 3 is now completed.

8.4.1.6. Proof of Lemma 8.14–case 4
In this paragraph, we still assume that $p > nr$ and that both zeros have order at least $p - n - r$. Let us observe the following:

Remark 8.21. If both zeros have order at least $p - n - r$, then $r \geq 5$ in Equation (107) readily implies that $n = 1$, and therefore, $p \geq 6$. In other words, under the conditions above, $r \geq 5$ holds only for strata with a single pole.

Since strata with a single pole have already been handled in §8.4.1.3, we may assume $n \geq 2$ and, as a consequence, $r \leq 4$. On the other hand, if both zeros have order at least $p - n - r$, then Equation (107) holds if $n \leq 3$. Therefore, $n \in \{2, 3\}$, and we have only a few isolated cases to discuss. In order to avoid a further proliferation of paragraphs and subparagraphs, we list all these special cases here, and we handle them one by one by using the following lemmas. Under the assumptions of the present paragraph, there are three families of strata to consider according to the following possibilities:

i. $3 \leq r \leq 4$ and $n = 2$, and
ii. $r = 3$ and $n = 3$.

Since the total pole order p is bounded between $(n+1)r \leq p \leq 2(n+r)$ (see inequality (107)), there are a few isolated cases to consider as follows:

1. $n = 2$ and $r = 3$. In this case, we consider strata of the form $\mathcal{H}_1(3k_1, 3k_2; -3h_1, -3h_2)$ for some positive integers k_1, k_2, h_1, h_2. A direct check shows that $9 \leq p \leq 10$. Since $3h_1 + 3h_2 = 10$ has no integral solutions, we only need to consider $p = 9$. This forces $k_1 = h_1 = 1$ and $k_2 = h_2 = 2$ – up to relabelling. Notice that h_1 and h_2 cannot be zero because we consider strata with two poles. Therefore, we only need to consider the stratum $\mathcal{H}_1(3, 6; -3, -6)$. Since
$$m_1 = 3 < 4 = 9 - 2 - 3 = p - n - r, \tag{115}$$
the argument given in paragraph §8.4.1.4 applies, and hence, we are done in this case.
2. $n = 2$ and $r = 4$. This case is quite similar. We consider strata of the form $\mathcal{H}_1(4k_1, 4k_2; -4h_1, -4h_2)$ for some positive integers k_1, k_2, h_1, h_2. A direct check shows that $p = 12$ and the Diophantine equation $4h_1 + 4h_2 = 12$ has $(h_1, h_2) = (h_2, h_1) = (1, 2)$ as the only possible solutions with positive integers. Assume, without loss of generality that $(h_1, h_2) = (1, 2)$ – the other pair leads to the same stratum, namely $\mathcal{H}_1(4, 8; -4, -8)$. Since
$$m_1 = 4 < 6 = 12 - 2 - 4 = p - n - r, \tag{116}$$
we fall once again in the case considered in paragraph §8.4.1.4, and hence, in this case, we are done.
3. $n = 3$ and $r = 3$. We consider strata of the form $\mathcal{H}_1(3k_1, 3k_2; -3h_1, -3h_2, -3h_3)$ for some positive integers k_1, k_2, h_1, h_2, h_3. A direct check shows that $p = 12$, and hence, $(1, 1, 2)$ is the only triple of positive integers that satisfies $3(h_1 + h_2 + h_3) = 12$, up to permutation. There are two strata to consider in this case according to the following partitions of 4 as the sum of two positive integers:
   i. $k_1 = k_2 = 2$ that leads to $\mathcal{H}_1(6, 6; -3, -3, -6)$, and
   ii. $k_1 = 1$ and $k_2 = 3$ that leads to $\mathcal{H}_1(3, 9; -3, -3, -6)$.

   In the case *ii* above, it is easy to check that $m_2 = 9 > 8 = p - n - 1$, and hence, the trivial representation cannot be realized in this stratum because the Hurwitz type inequality (103) does not

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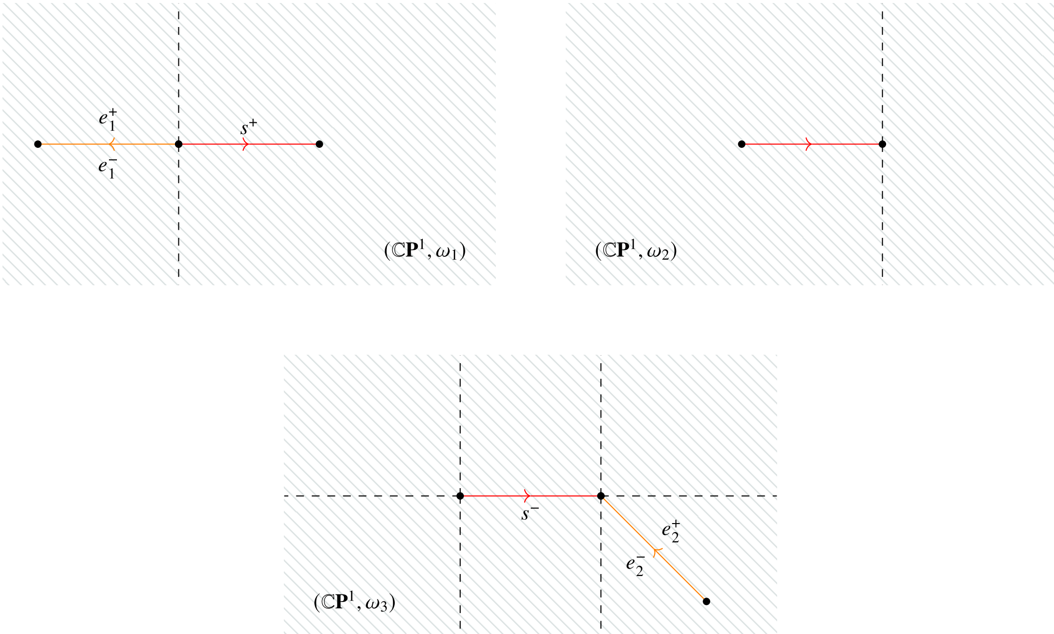

Figure 43. *Realization of the trivial representation in the connected component of the stratum* $\mathcal{H}_1(6, 6; -3, -3, -6)$ *with rotation number* $r = 3$. *The simple closed curves* α, β *can be drawn exactly as in Figure 42 above. It is easy to check that the resulting genus-one differential has rotation number* 3.

hold. The case i above, however, needs a special treatment. Notice that we cannot apply the induction, as in §8.4.1.4, to the stratum $\mathcal{H}_1(3, 6; -3, -3, -3)$ because the trivial representation cannot be realized in the latter; in fact, the Hurwitz type inequality (103) does not hold.

Lemma 8.22. *The trivial representation can be realized in the connected component of* $\mathcal{H}_1(6, 6; -3, -3, -6)$ *with rotation number* $r = 3$.

Sketch of the proof. The proof is very similar to the argument given in §8.4.1.4, and hence, we provide it briefly. We first realize the trivial representation in $\mathcal{H}_0(5, 5; -3, -3, -6)$ by using the saddle connection configuration description in [EMZ03] with $(a_1, b_1) = (2, 1)$, $(a_2, b_2) = (1, 2)$ and $(a_3, b_3) = (3, 3)$. Notice that $a_1 + a_2 + a_3 = m_1 + 1 = 6$ and $b_1 + b_2 + b_3 = m_2 + 1 = 6$.

For every $i = 1, \ldots, n$, consider a copy of $(\mathbb{C}, z^{r-2}dz)$ and fix any vector $c \in \mathbb{C}^*$. Let $(\mathbb{C}\mathbf{P}^1, \omega_i)$ be the translation surface with poles obtained by breaking the zero of $(\mathbb{C}, z^{r-2}dz)$ into two zeros, say P_i and Q_i, of orders $a_i - 1$ and $b_i - 1$, respectively, so that the resulting saddle connection joining them, say c_i, is parallel to c with the same length, that is, $|c| = |c_i|$.

Next, we slit every $(\mathbb{C}\mathbf{P}^1, \omega_i)$ along c_i and denote by $c_i^\pm$ the resulting edges according to our convention. Identify c_i^- with c_{i+1}^+ for every $i = 1, 2$, and glue c_3^- with c_1^+. The resulting space is now a pair $(\mathbb{C}\mathbf{P}^1, \omega)$, where ω is a meromorphic differential with two zeros, say P_o and Q_o, of orders 5, two poles of order 3 and one pole of order 6. Let s be the saddle connection resulting from the identification of c_1^+ with c_n^-. Orient s from P_o to Q_o.

In $(\mathbb{C}\mathbf{P}^1, \omega)$, there is a segment, say e_1, leaving from P_o such that s and e_1 are twins (see Definition 8.2), and the angle on the left is 2π. Similarly, there is only segment, say e_2, leaving from Q_o such that s and e_2 are twins and the angle on the right is 2π. Slit both e_1 and e_2, and denote the resulting sides $e_1^\pm$ and $e_2^\pm$ according to our convention. Glue e_1^+ with e_2^-, and glue e_1^- with e_2^+. The resulting space is a genus-one surface equipped with a translation structure with trivial periods. By construction, it lies in the connected component of $\mathcal{H}_1(6, 6; -3, -3, -6)$ with rotation number $r = 3$; see Figure 43. □

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We finally notice that for $n \geq 4$ the inequality (107) holds if and only if $r \leq 2$. Therefore, there are no other cases to consider for $r \geq 3$.

8.4.1.7. Proof of Lemma 8.14–case 5

We continue to assume that $p > nr$ and that both zeros have order at least $p - n - r$. It remains to consider the cases $r = 1, 2$. For both of them, we no longer have any upper bound on the number of poles because Equation (107) holds for every $n \geq 1$.

If $r = 1$, the assumption implies that $m_1, m_2 \geq (p-1) - n$, but the Hurwitz type inequality (103) then implies that $m_1 = m_2 = p - n - 1$. Hence, $p = m_1 + m_2 = 2p - 2n - 2$, so $p = 2n + 2$ and $m_1 = m_2 = n + 1$. Since each $p_i \geq 2$, the only cases are

1. $\mathcal{H}_1(n+1, n+1; -2^{n-1}, -4)$. This stratum is connected whenever n is even whereas it has exactly two connected components whenever n is odd. Therefore, this latter case is worth of interest for us. See Lemmas 8.25 and 8.27.
2. $\mathcal{H}_1(n+1, n+1; -2^{n-2}, -3, -3)$. It is an easy matter to check that such a stratum is connected for every $n \geq 3$ because $\gcd(2, 3, n+1) = 1$. On the other hand, for $n = 2$ we get $\mathcal{H}_1(3, 3; -3, -3)$ which has two connected components. See Lemma 8.28 for this special case which turns out to be an exceptional stratum.

If $r = 2$, the assumption implies that $m_1, m_2 \geq (p-2) - n$. Hence, $p = m_1 + m_2 \geq 2p - 2n - 4$ and $p \leq 2n + 4$. The Hurwitz inequality is that $m_1, m_2 \leq p - n - 1$. Since each p_i is even, the only cases are

3. $\mathcal{H}_1(n+1, n+1; -2^{n-1}, -4)$. If n is even the stratum is connected, and it has two connected components if n is odd. See Lemmas 8.25 and 8.27.
4. $\mathcal{H}_1(n+2, n+2; -2^{n-1}, -6)$. If n is odd the stratum is connected, and hence, the interesting cases arise for n even. See Lemma 8.23. Finally,
5. the last family of strata is given by $\mathcal{H}_1(n+2, n+2; -2^{n-2}, -4, -4)$. For n odd the stratum is connected, and it has two connected components if n is even. See Lemma 8.24.

We have the following lemmas.

Lemma 8.23. *Let $n = 2k$ be an even positive integer. For every $k \geq 1$, the trivial representation can be realized in both connected components of the stratum $\mathcal{H}_1(2k+2, 2k+2;\, -2^{2k-1}, -6)$.*

Proof. In this case, we only need to show that the trivial representation can be realized in the connected component of $\mathcal{H}_1(2k+2, 2k+2;\, -2^{2k-1}, -6)$ with rotation number 2. In fact, since

$$n + 2 < 2(n-1) + 6 - n - 1 = n + 3 \tag{117}$$

for every $n \geq 0$, it follows that $m_1 = m_2 < p - n - 1$. We can proceed as in §8.4.1.4, and hence, we are already done if $r = 1$. In order to realize the trivial representation as the period character of some exact differential with rotation number 2, we proceed as follows. We use the saddle connection configuration description as in [EMZ03]. We need to find pairs (a_i, b_i) such that $a_1 + b_1 = 6$ and $a_i + b_i = 2$ for all $2 \leq i \leq n$, that satisfy $a_1 + \cdots + a_n = b_1 + \cdots + b_n = n + 1$. We choose these pairs to be $(3,3),\ (1,1),\ \ldots,\ (1,1)$. More precisely, consider the differential $z^4\,dz$ on $\mathbb{C}$, break the sole zero into two zeros of the same order and denote by c_1 the resulting saddle connection. Denote the resulting translation surface as $(\mathbb{C},\ \omega)$. Next, we consider $n-1$ copies of $(\mathbb{C},\ dz)$, and we consider on each of them a marked segment c_i, for $2 \leq i \leq n$ so that $c_1, \ldots, c_n$ are all parallel with the same length. The desired handle with trivial periods can be realized in $(\mathbb{C}, \omega)$. Let P, and Q denote the zeros of order two of $(\mathbb{C},\ \omega)$, and let c_1 be the saddle connection joining them. According to our Lemma 8.3, there are two edges e_1 and e_2 leaving from P such that $s,\ e_1,\ e_2$ are pairwise twins. Similarly, there are two edges $e_3,\ e_4$ leaving from Q such that $c_1,\ e_3,\ e_4$ are pairwise twins. Assume without loss of generality that c_1, e_1, e_2 and s, e_3, e_4 are ordered counterclockwise around P and Q, respectively. Slit the edges e_1 and e_4, and denote the resulting sides $e_1^{\pm}$ and $e_4^{\pm}$ according to our convention. Then glue the edges e_1^{+} with e_4^{-},

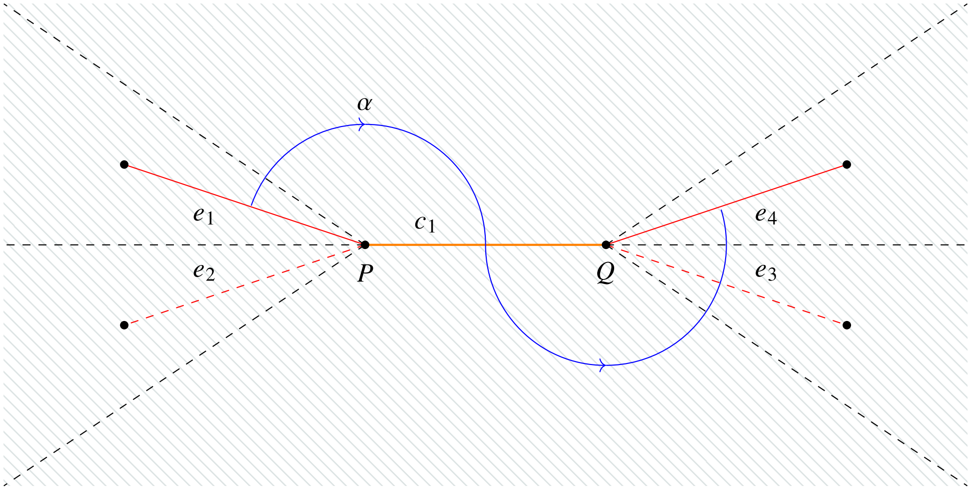

Figure 44. *Realizing the intermediate structure* $(Y, \xi) \in \mathcal{H}_1(3, 3; -6)$. *The orange edge is the saddle connection joining P and Q. The red edges* e_1, e_2 *are the twins of* c_1 *leaving from P, and the edges* e_3 *and* e_4 *are the twins of* c_1 *leaving from Q. The dashed edges are those not involved in the construction. The curve* α *has index* 1 *because it winds clockwise around P once and counterclockwise around Q twice. Therefore, the resulting structure has rotation number one. If we had chosen* e_3 *instead of* e_4, *then its index would have been* 0 *because* α *would wind around Q only once. In this way, we would obtain a structure in* $\mathcal{H}_1(3, 3; -6)$ *with rotation number* 3.

and similarly glue e_1^- with e_4^+. The resulting space is a genus-one differential, say $(Y, \xi) \in \mathcal{H}_1(3, 3; -6)$ with trivial periods and rotation number 1; see Figure 44.

In fact, on $(\mathbb{C}, \omega)$, we consider an arc joining the midpoint of e_1 with the midpoint of c_1 that winds clockwise around P. Next, we prolong the arc above with another arc leaving from the midpoint of c_1 with the midpoint of e_4 that winds counterclockwise around Q. Notice that, since c_1, e_3, e_4 are ordered counterclockwise, this second arc winds twice around Q. In the resulting space (Y, ξ), the union of these arc just described close up to a simple close curve, say α of index one; see Figure 44. We define β as any simple close loop around e_1. Notice that its index is equal to 3.

In the resulting space (Y, ξ), there is by construction a saddle connection, say c_1 with a small abuse of notation, joining the two zeros of order 3 which is still parallel to $c_2, \dots, c_n$. We glue the remaining $n-1$ copies of $(\mathbb{C}, dz)$ by slitting the saddle connections c_i as already explained in Paragraph §8.4.1.1. In this process, the indices of α and β are both altered by $\pm(n-1)$. Since n is even it follows that $n-1$ is odd. In particular, both $\operatorname{Ind}(\alpha) \pm (n-1)$ and $\operatorname{Ind}(\beta) \pm (n-1)$ are even. Therefore, the resulting translation surface lies in $\mathcal{H}_1(n+2, n+2; -2^{n-1}, -6)$, and it has rotation number 2 as desired. □

Notice that, if we had chosen e_3 in place of e_4 we would have got a translation surface in the same stratum with rotation number one. Finally,

Lemma 8.24. *Let* $n = 2k$ *be an even positive integer. For every* $k \geq 1$, *the trivial representation can be realized in both connected components of the stratum* $\mathcal{H}_1(2k+2, 2k+2; -2^{2k-2}, -4, -4)$.

Proof. This proof is quite similar to that of Lemma 8.23. In fact, in this case we only need to show that the trivial representation can be realized in the connected component of $\mathcal{H}_1(2k+2, 2k+2; -2^{2k-2}, -4, -4)$ with rotation number 2. Since

$$n + 2 < 2(n-2) + 8 - n - 1 = n + 3 \tag{118}$$

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for every $n \geq 0$, it follows that $m_1 = m_2 < p - n - 1$ we can proceed as in §8.4.1.4, and hence, we are already done if $r = 1$. In order to realize the trivial representation as the period character of some exact differential with rotation number 2, we proceed as follows. We use the saddle connection configuration description as in [EMZ03]. We need to find pairs (a_i, b_i) such that $a_1 + b_1 = a_2 + b_2 = 4$ and $a_i + b_i = 2$ for all $3 \leq i \leq n$, and $a_1 + \cdots + a_n = b_1 + \cdots + b_n = n + 2$. We choose these pairs to be $(2, 2)$, $(2, 2)$, $(1, 1)$, $\dots$, $(1, 1)$. More precisely, in this case consider two copies of $(\mathbb{C},\, z^2dz)$ and break on each of them the sole zero into two zeros each of order one. In both cases, the sole zero of z^2dz can be broken so that the resulting saddle connections, say c_1 and c_2 are parallel with the same length. Next, we consider $n - 2$ copies of $(\mathbb{C},\, dz)$, each of which with a marked segment c_i, for $3 \leq i \leq n$ so that $c_1, c_2, c_3, \dots, c_n$ are all parallel with the same length. The desired handle with trivial periods can be bubbled inside the copy $(\mathbb{C}, \omega_1)$. We then proceed as in the proof of Lemma 8.23 to realize the desired structure in the stratum $\mathcal{H}_1(2k + 2, 2k + 2;\, -2^{2k-2}, -4, -4)$ with rotation number 2 and trivial periods. More precisely, we first realize an intermediate structure $(Y, \xi) \in \mathcal{H}_1(2, 2; -4)$ with rotation number 2, and then we glue the remaining translation surfaces by slitting the saddle connection as done in paragraph §8.4.1.1. □

Lemma 8.25. *Let $n = 2k + 1$ be an odd positive integer. For every $k \geq 1$, the trivial representation can be realized in both connected components of the stratum $\mathcal{H}_1(2k + 2, 2k + 2;\, -2^{2k}, -4)$.*

Proof. This is the most delicate case to consider. Once again, we wish to realize the trivial representation by using the saddle connection configuration description as in [EMZ03]. We need to find n pairs (a_i, b_i) such that $a_1 + b_1 = 4$ and $a_i + b_i = 2$ for all $2 \leq i \leq n$, that satisfy $a_1 + \cdots + a_n = b_1 + \cdots + b_n = n + 1$. The only possible case is given by the pairs $(2, 2)$, $(1, 1), \dots, (1, 1)$. We shall need to distinguish two cases depending on which connected component of $\mathcal{H}_1(2k + 2, 2k + 2;\, -2^{2k}, -4)$ we aim to realize the trivial representation.

Suppose we aim to realize a translation surface with trivial periods and rotation number two in that stratum. Then we can proceed exactly as in Lemma 8.24 above. The only difference is that there is only one pair $(2, 2)$, but this is irrelevant to the construction because the handle with trivial periods is realized in $(\mathbb{C}, \omega_1)$ – in the above notation. Therefore, this case is essentially subsumed in the proof of Lemma 8.24.

It remains to realize a translation surface with trivial periods in $\mathcal{H}_1(2k + 2, 2k + 2;\, -2^{2k}, -4)$ with rotation number one. Let $(\mathbb{C}, \omega)$ be a translation surface with trivial periods and two zeros, say P and Q, each of order one and a single pole of order 4. Denote by c_1 the saddle connection joining them. Lemma 8.3 states that there is a segment e_P at P such that e_P and c_1 are twins. For the same reason, there is a segment e_Q at Q such that e_Q and c_1 are twins.

Pick $n - 1$ copies of $(\mathbb{C},\, dz)$, and denote them by $(\mathbb{C}, \omega_i)$ for $2 \leq i \leq n$. On each such copy, we thus consider an edge, say c_i, such that c_i, c_j are parallel and with the same length. On $(\mathbb{C},\, \omega)$ slit c_1, e_P, and e_Q, and denote by $c_1^\pm$, $e_P^\pm$ and $e_Q^\pm$ the resulting edges according to our convention. Next, for every $2 \leq i \leq n$, slit $(\mathbb{C},\, \omega_i)$ along c_i, and denote the resulting sides as $c_i^\pm$. We glue the sides as follows: For $1 \leq i \leq n - 2$ identify c_i^+ with c_{i+1}^- and identify c_{n-1}^+ with c_1^-. Then identify the edge e_P^+ with e_Q^-, next identify e_P^- with c_n^+ and finally identify e_Q^+ with c_n^-. The resulting space is a translation surface in the stratum $\mathcal{H}_1(2k + 2, 2k + 2;\, -2^{2k}, -4)$ as desired. It remains to show it has rotation number 1. We can define α as the simple closed curve shown in Figure 45; since n is odd, it follows that $\operatorname{Ind}(\alpha) = \pm(n - 2)$ is also odd, and hence, the rotation number is necessarily one. □

8.4.1.8. Exceptional strata

We finally consider the exceptional cases. The following one corresponds to the case $k = 0$ of Lemma 8.25 above. We first introduce the following definition.

Definition 8.26. Let $\mathcal{H}_1(m_1, \dots, m_k;\, -p_1, \dots, -p_n)$ be a stratum of genus-one differentials. We define the *primitive component* of this stratum as the unique connected component with rotation number (equivalently torsion number) one.

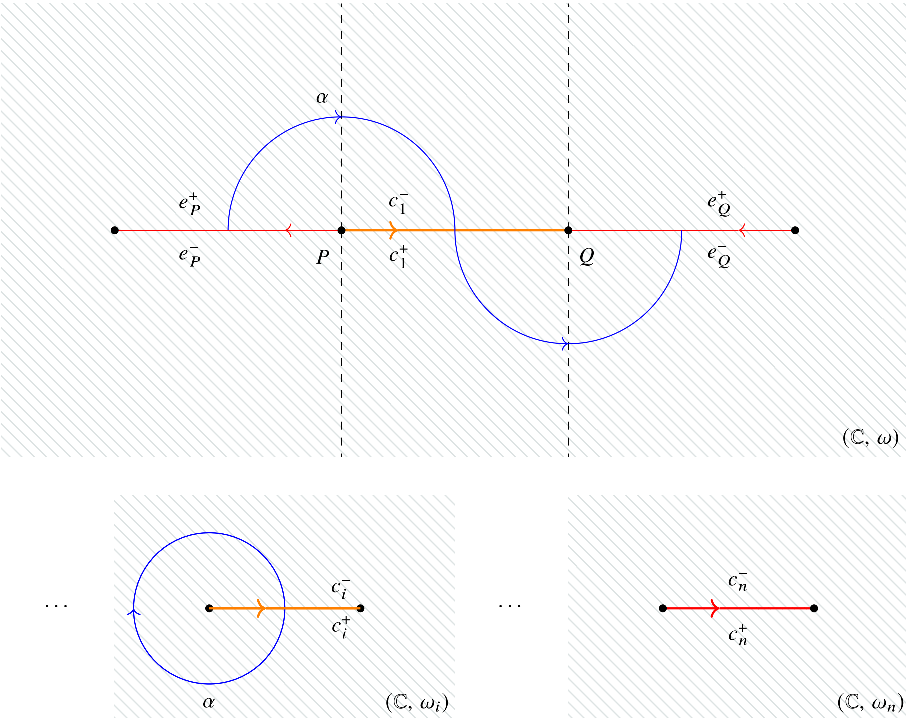

Figure 45. *Realizing a translation surface with trivial periods and rotation number one in the stratum $\mathcal{H}_1(2k+2, 2k+2; -2^{2k}, -4)$. The blue curve α closes up to a simple closed curve in the resulting space and has index $\mathrm{Ind}(\alpha) = \pm(n-2)$. Notice that in $(\mathbb{C}, \omega)$ the red segments point leftwards and hence labelled accordingly. This happens because the angles at P and Q are both 4π, and the angles at these points between the red and orange segment is 2π.*

Notice that, if a stratum is connected then it coincides with its primitive component. In Lemma 8.27, Lemma 8.28 and Lemma 8.36, we shall consider strata with exactly two connected components; one of which is primitive and the other is *nonprimitive*.

Lemma 8.27. *The trivial representation can only be realized in the connected component of $\mathcal{H}_1(2, 2; -4)$ with rotation number 2, that is, the nonprimitive component.*

Proof. Suppose that (X, ω) is a translation surface in $\mathcal{H}_1(2, 2; -4)$ with trivial periods. Let Z_1, Z_2 and P be the two zeros and the unique pole of ω, respectively. The trivial period condition is equivalent to the existence of a triple cover of X to $\mathbb{C}\mathbf{P}^1$ totally ramified at Z_1, Z_2 and P, where ω arises from differentiating the rational function associated to the cover. This implies the linear equivalence relation $3Z_1 \sim 3Z_2 \sim 3P$. On the other hand, by assumption $2Z_1 + 2Z_2 \sim 4P$. It follows that $Z_1 + Z_2 \sim 3Z_1 + 3Z_2 - (2Z_1 + 2Z_2) \sim 2P$, which implies by definition that (S, ω) belongs to the nonprimitive component of $\mathcal{H}_1(2, 2; -4)$. □

Lemma 8.28. *The trivial representation can only be realized in the connected component of $\mathcal{H}_1(3, 3; -3, -3)$ with rotation number 3, that is, the nonprimitive component.*

Proof. Suppose that (X, ω) is a translation surface in $\mathcal{H}_1(3, 3; -3, -3)$ with trivial periods. Let Z_1, Z_2, P_1 and P_2 be the zeros and poles of ω, respectively. The trivial period condition is equivalent to the existence of a quadruple cover of X to $\mathbb{C}\mathbf{P}^1$ totally ramified at Z_1, Z_2, and having $2P_1 + 2P_2$ as the fiber over infinity. This implies the linear equivalence relation $4Z_1 \sim 4Z_2 \sim 2P_1 + 2P_2$. On the other hand, by assumption $3Z_1 + 3Z_2 \sim 3P_1 + 3P_2$. It follows that $Z_1 + Z_2 \sim 4Z_1 + 4Z_2 - (3Z_1 + 3Z_2) \sim P_1 + P_2$, which implies by definition that (S, ω) belongs to the nonprimitive component of $\mathcal{H}_1(3, 3; -3, -3)$. □

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8.4.2. Strata with more than two zeros

We now consider the general case of $\mathcal{H}_1(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ with $k \geq 3$. We begin with the following observation reported here as:

Lemma 8.29. *Let $\kappa = (m_1, \ldots, m_k)$ and $\nu = (p_1, \ldots, p_n)$ be tuples such that $m_1 + \cdots + m_k = p_1 + \cdots + p_n$. Suppose there is a pair m_i, m_j such that $m_i + m_j < p - n$. Let $\overline{\kappa} = (m_1, \ldots, \widehat{m}_i, \widehat{m}_j, \ldots, m_i + m_j, \ldots, m_n)$. Finally, let r be a divisor of $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n)$. If the trivial representation can be realized in the connected component of $\mathcal{H}_1(\overline{\kappa}; -\nu)$ with rotation number r, then it can also be realized in the connected component of $\mathcal{H}_1(\kappa; -\nu)$ with rotation number r.*

Explicitly, if there exist a pair m_i, m_j such that $m_i + m_j < p - n$, then we can do induction by splitting a zero of order $m_i + m_j$ as the Hurwitz type inequality (103) still holds for the merged zero order. It remains to determine whether the trivial representation can be realized in the stratum $\mathcal{H}_1(\overline{\kappa}; -\nu)$ above. In the case there is another pair of zeros, say m_h, m_l such that $m_h + m_l < p - n$, then Lemma 8.29 applies once again. This process iterated finitely many times leads to one of the following situations:

1. We fall in a stratum with two zeros, and hence, we may apply the results of subsection §8.4.1, or
2. We end up with a stratum $\mathcal{H}_1(m_1, \ldots, m_k; -\nu)$ in which $m_i + m_j \geq p - n$ for all pairs i, j.

In paragraph §8.4.1.7, we have shown the existence of two strata with two zeros that are exceptional in the sense the trivial representation cannot be realized in every connected component. These are $\mathcal{H}_1(2, 2; -4)$ and $\mathcal{H}_1(3, 3; -3, -3)$. We need to make sure that, by iterating the process just described, if we fall in a stratum with two zeros, then this stratum cannot be exceptional. It is easy to check that, by merging zeros, one gets the access to $\mathcal{H}_1(2, 2; -4)$ only from strata $\mathcal{H}_1(1, 1, 1, 1; -4)$ and $\mathcal{H}_1(1, 1, 2; -4)$ which are connected. Similarly, by merging zeros one gets the access to $\mathcal{H}_1(3, 3; -3, -3)$ from $\mathcal{H}_1(1, 2, 3; -3, -3)$ and $\mathcal{H}_1(1, 1, 2, 2; -3, -3)$ which are also connected. Since we begin with strata $\mathcal{H}_1(m_1, \ldots, m_k; -\nu)$ with more than one connected component, we never fall in those exceptional cases.

Suppose it to be in the second case above. If $m_i + m_j \geq p - n$ for all pairs i, j, then summing over all pairs we obtain that

$$(k-1)p \geq \frac{k(k-1)}{2}(p-n) \tag{119}$$

that is $(k-2)p \leq kn$. Since

$$(k-2)nr \leq (k-2)p \tag{120}$$

and $k \geq 3$, it necessarily follows that $r \leq 3$.

Remark 8.30. Let $l = \gcd(p_1, \ldots, p_n)$, and notice that $n\, l \leq p$. It is easy to check that the equality holds if and only if $p_i = l$ for all $i = 1, \ldots, n$. Equation (120) above also holds if we replace r with l, in fact

$$(k-2)nl \leq (k-2)p, \tag{121}$$

and, since $(k-2)p \leq kn$ and $k \geq 3$, we get that $l \leq 3$. On the other hand,

$$\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n) \leq \gcd(p_1, \ldots, p_n) \tag{122}$$

holds and, as a consequence, $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n)$ cannot exceed $l \leq 3$. Therefore, for a given stratum $\mathcal{H}_1(\kappa; -\nu)$ that satisfies the condition $m_i + m_j \geq p - n$ for all pairs i, j, it necessarily follows $\gcd(\kappa, \nu)$ cannot be greater than 3.

Assume in the first place $\gcd(p_1, \ldots, p_n) = r = 3$. In this case, the inequalities (119) and (120) above readily implies $k \leq 3$; on the other hand, k is supposed to be at least 3 by assumption, and hence, the only possible case is $k = 3$. For $n = 3h$ and $h \geq 1$, the collection of strata $\mathcal{H}_1(3h, 3h, 3h; -3^{3h})$ is the only one that satisfies all these conditions. In principle, one might need to deal with strata of the form

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$\mathcal{H}_1(3h_1, 3h_2, 3h_3; -3^{h_1+h_2+h_3})$ with $h_1 \leq h_2 \leq h_3$. If we assume $3h_1 + 3h_2 \geq p - n = 2(h_1 + h_2 + h_3)$, we readily obtain $h_1 + h_2 \geq 2h_3$, and hence, $h_1 + h_2 = 2h_3$. This implies $h_1 = h_2 = h_3$. We have the following:

Lemma 8.31. *The trivial representation can be realized in each connected component of the stratum of differentials* $\mathcal{H}_1(3h, 3h, 3h;\ -3^{3h})$.

Proof. Let $n = 3h \geq 3$. We can first realize the trivial representation in the stratum $\mathcal{H}_0(2n-2, n; -3^n)$. This is possible because the Hurwitz type inequality (103) is satisfied. Then split the first zero into two nearby zeros each of order $n-1$. By construction, there is a saddle connection, say s, joining the newborn zeros, say P_1 and P_2. Moreover, our Lemma 8.3 applies, and hence, there are $n-1$ paths $c_1, \dots, c_{n-1}$, all leaving from P_1 such that s and c_i are twins. Similarly, there are $n-1$ paths $c_{n+1}, \dots, c_{2n-1}$, all leaving from P_2 such that s and c_{n+i} are twins for $i = 1, \dots, n-1$. We may suppose without loss of generality that both collections of paths $c_1, \dots, c_{n-1}, s$, all based at P_1, and $s, c_{n+1}, \dots, c_{2n-1}$, all based at P_2, are in positive cyclic order. For any $i = 1, \dots, n-1$, slit c_i and c_{n+i} and denote $c_i^{\pm}$ and $c_{n+i}^{\pm}$ the resulting edges. Then glue c_i^+ with c_{n+i}^-, and glue c_i^- with c_{n+i}^+. The resulting space is a genus-one differential with rotation number 3. Notice that the choice of the slit is crucial here because the angle difference divisible by 3 modulo 2π. A different choice of paths so that the angle difference is no longer divisible by 3 modulo 2π would have produced a genus-one differential with rotation number 1. □

Since there are no other strata $\mathcal{H}_1(m_1, \dots, m_k; -p_1, \dots, -p_n)$ with $\gcd(m_1, \dots, m_k; p_1, \dots, p_n) = 3$, we next consider the case of strata with $\gcd(m_1, \dots, m_k; p_1, \dots, p_n) = 2$.

We now assume $r = \gcd(m_1, \dots, m_k; p_1, \dots, p_n) = 2$ and let $\mathcal{H}_1(m_1, \dots, m_k;\ -p_1, \dots, -p_n)$ be a stratum of genus-one differentials. The key observation here is the following reduction process: If there is some pole of order $p_i > 2$ and if there is at most one zero, say P_j, of order $m_j \geq p-n-2$, then we can do induction from the connected component of the reduce stratum $\mathcal{H}_1(\dots, m_j-2, \dots; \dots, -p_i+2, \dots)$ with the rotation number 1 or 2, by bubbling 2 copies of $(\mathbb{C}, dz)$ along a ray from P_j to the pole of order $p_i - 2$.

The reduced stratum now satisfies only of the following conditions:

1. In the reduced stratum there is at most one zero of order $m_h = m_j - 2 \geq p - n - 4$. In this case, we can reduce the stratum one more time.
2. There are now at least two zeros of orders $m_h,\ m_l \geq p - n - 4$.

Suppose to be in the second case; that is, suppose there are two zeros of orders $m_1, m_2 \geq p - n - 2$. Then

$$2p - 2n - 4 \leq m_1 + m_2 \leq p - 2(k-2), \tag{123}$$

where the second inequality holds because $m_i \geq 2$ for all $i = 1, \dots, k$. There is only one case we need to consider given by the stratum $\mathcal{H}_1(n, n, 2; -2, \dots, -2, -4)$ with n even. The following holds:

Lemma 8.32. *Let $n = 2h$. The trivial representation can be realized in each connected component of the stratum* $\mathcal{H}_1(2h,\ 2h, 2; -2^{2h-1}, -4)$.

Proof. We begin by realizing the trivial representation as the period character of genus-zero differential in the stratum $\mathcal{H}_0(n, n;\ -2^{n-1}, -4)$. Notice that this is possible because the Hurwitz type inequality (103) holds and the realization is guaranteed by [CFG22, Theorem B]. Let (Y, ξ) such a structure. Let Q be a zero of order n, and let $\varepsilon > 0$ such that the open ball $B_{4\varepsilon}(Q)$ does not contain the other zero. Break Q into two zeros of order $n-1$ and 1 as described in §3.1. Since $n = 2h > 0$, both zeros have order at least one. Denote the newborn zeros as P_1 and P_2, respectively. Moreover, we can break Q so that the resulting saddle connection, say s, joining P_1 and P_2 has length ε. According to our Lemma 8.3, there are $n-1$ edges, say $e_1, \dots, e_{n-1}$, leaving from P_1 so that $s,\ e_1, \dots, e_{n-1}$ are pairwise twins. By the same lemma, there is an edge e_n leaving from P_2 so that $s,\ e_n$ are twins. Orient s from P_1 to P_2, and denote by s^+ and s^- the right and left, respectively. Let e_i be any edge from P_1, slit the edges e_i and e_n and denote

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the resulting sides as $e_i^\pm$ and $e_n^\pm$ according to our convention. Then identify e_i^+ with e_n^-, and identify e_i^- with e_n^+. The resulting space is a genus-one differential in the stratum $\mathcal{H}_1(2h, 2h, 2; -2^{2h-1}, -4)$. Suppose, without loss of generality that $e_1, \ldots, e_{n-1}$ are ordered counterclockwise around P_1. If i is odd, then it is easy to check that the resulting structure has rotation number 2 whereas, if i is even the resulting structure has rotation number one. This construction works as long as $n = 2h \geq 4$ and the case $n = 2$; that is, $h = 1$, needs a special treatment as follows.

We provide explicit constructions for realizing the trivial representation in both connected components of the stratum $\mathcal{H}_1(2, 2, 2; -2, -4)$. Let us start by realizing a translation surface in this stratum with rotation number 1. Let P, Q, R be three not aligned points on the complex plane equipped with the standard differential dz. Let e be the edge joining P and Q, and let r be a half-ray leaving from R such that $e \cap r = \phi$. Consider four copies of $(\mathbb{C}, dz)$, and perform the following surgeries:

1. Slit the first copy along e_1, and denote the resulting sides as $e_1^\pm$,
2. Slit the second copy along r_2, and denote the resulting sides as $r_2^\pm$, finally
3. Slit the third and fourth copy along the edges e_3 and e_4, respectively, and along the rays r_3 and r_4, respectively. Denote the resulting sides as $e_3^\pm$, $e_4^\pm$, $r_3^\pm$ and $r_4^\pm$,

where the signs are always taken according our convention. Identify the edge e_1^+ with e_3^-, then e_3^+ with e_4^-, and finally, identify e_4^+ with e_1^-. Next, identify the half-ray r_2^+ with r_3^-, then r_3^+ with r_4^-, and finally, identify r_4^+ with r_2^-. The resulting surface is a genus-one differential with trivial periods and rotation number 1 as desired. In fact, a close loop around anyone of the e_i's, say α, has index one, and this forces the rotation number to be one as well. See Figure 46.

We finally realize the trivial representation in the connected component of $\mathcal{H}_1(2, 2, 2; -2, -4)$ with rotation number 2. Once again let P, Q, R be three not aligned points on the complex plane equipped with the standard differential dz. Let a be the edge joining P and Q, let b be the edge joining Q and R and, finally, let c the edge joining P and R. As above, let r be a half-ray leaving from R such that it does not intersect none of the segments a, b, c. Consider four copies of $(\mathbb{C}, dz)$ and perform the following surgeries:

1. Slit the first copy along the edges a_1, b_1 and c_1 and the ray r_1. Denote the resulting sides as $a_1^\pm$, $b_1^\pm$, $c_1^\pm$ and $r_1^\pm$.
2. Slit the second copy along b_2 and r_2, and denote the resulting sides as $b_2^\pm$ and $r_2^\pm$,
3. Slit the third copy along the edges c_3 and r_3. Denote the resulting sides as $c_3^\pm$ and $r_3^\pm$, finally
4. Slit the fourth copy along a_4, and denote the resulting sides as $a_4^\pm$,

where the signs are always taken according our convention. Identify the edge a_1^+ with a_4^- and a_1^- with a_4^+. Next, identify b_1^+ with b_2^- and b_1^- with b_2^+. Finally, identify c_1^+ with c_3^- and c_1^- with c_3^+. Then identify the half-ray r_1^+ with r_2^-, then r_2^+ with r_3^- and finally r_3^+ with r_1^-. The resulting surface is a genus-one differential with trivial periods and rotation number 2 as desired. In fact, we can find two simple closed curves α, β such that $\mathrm{Ind}(\alpha) = \mathrm{Ind}(\beta) = 2$. See Figure 47. □

Finally, we still assume $\gcd(m_1, \ldots, m_k, p_1, \ldots, p_n) = 2$, and we suppose that $p_i = 2$ for all $i = 1, \ldots, n$, that is, we consider the case of strata $\mathcal{H}_1(m_1, \ldots, m_k; -2^n)$. In this particular case, since $m_i + m_j \geq n$ for all possible pairs (i, j), we just have a few isolated cases to consider corresponding to $k = 3, 4$. In fact, Equation (119) implies that $2n(k-2) \leq kn$; that is, $k \leq 4$. Since $k \geq 3$, it follows that $3 \leq k \leq 4$.

If $k = 4$, the only case to consider is the stratum $\mathcal{H}_1(m, m, m, m; -2^n)$, where $n = 2m$ and m is even. In principle, one might need to deal with strata of the form $\mathcal{H}_1(2h_1, 2h_2, 2h_3, 2h_4; -2^{h_1+h_2+h_3+h_4})$, where, without loss of generality, $h_1 \leq h_2 \leq h_3 \leq h_4$. If we assume $2h_1 + 2h_2 \geq n = h_1 + h_2 + h_3 + h_4$, we readily obtain that the only possibility is that $h_1 = h_2 = h_3 = h_4$.

Lemma 8.33. *Let $n = 2m$. The trivial representation can be realized in each connected component of the stratum $\mathcal{H}_1(m, m, m, m; -2^n)$.*

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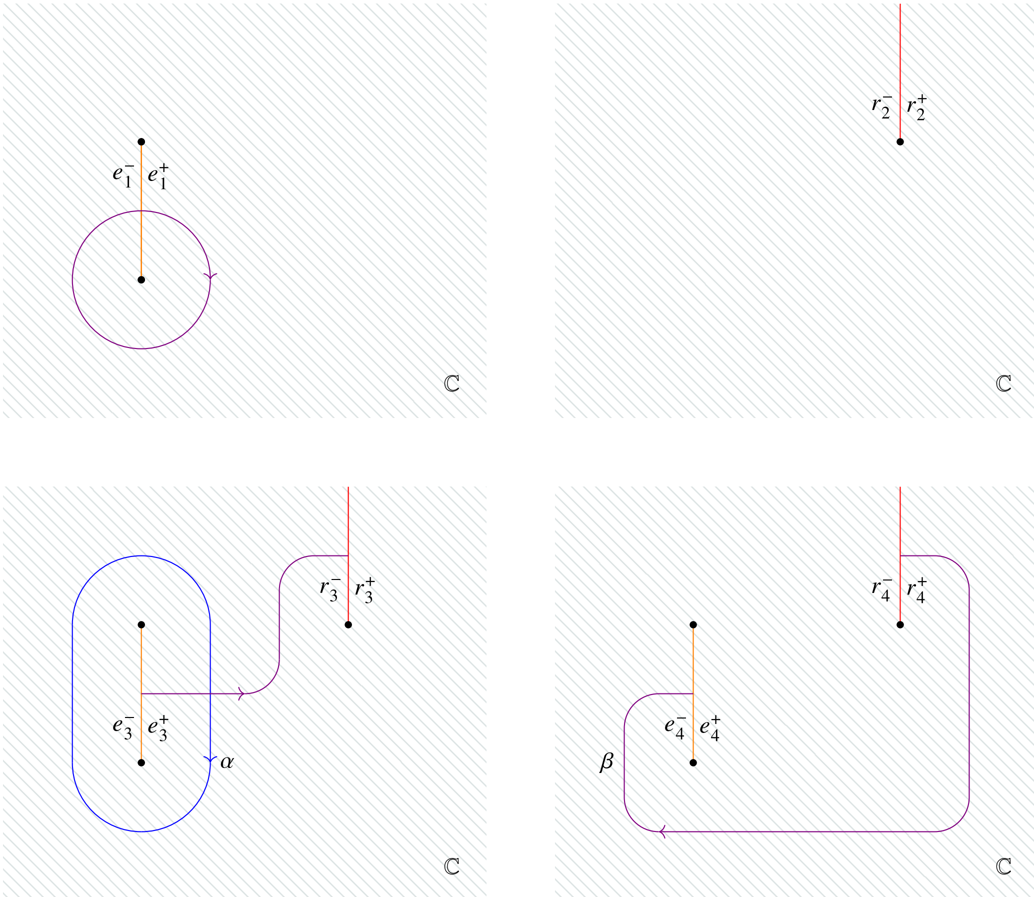

Figure 46. *Realizing a translation surface with trivial periods and rotation number one in the stratum* $\mathcal{H}_1(2,2,2;\,-2,-4)$. *The curve* α *has index one whereas the curve* β *has index two.*

Proof. In the first place, we notice that if m if odd, then $\mathcal{H}_1(m,m,m,m;\,-2^n)$ is connected, and hence, the trivial representation is realisable in this stratum if and only if $m \leq n-1$ (see [CFG22, Theorem B]) that is satisfied for $m \geq 1$ – compare the Hurwitz type inequality (103) with the equation $4m = 2n$. Therefore, in what follows we assume m to be even.

We first provide a generic construction that works as long as $m \geq 4$, and subsequently, we handle the case $m = 2$ in a different way. Therefore, suppose in the first place that $m \geq 4$. We realize the trivial representation as the period character of genus-zerp differential in the stratum $\mathcal{H}_0(2m-2,m,m;\,-2^n)$. Notice that this is possible because the Hurwitz type inequality (103) holds, and the realization is guaranteed by [CFG22, Theorem B]. Let (Y,ξ) such a structure. Let Q be a zero of order $2m-2$, and let $\varepsilon > 0$ such that the open ball $B_{4\varepsilon}(Q)$ does not contain the other zero. Break Q into two zeros of order $m-1$ and $m-1$ as described in §3.1. Since $m > 0$ is even, both zeros have order at least one. Denote the newborn zeros as P_1 and P_2, respectively. Moreover, we can break Q so that the resulting saddle connection, say s, has length ε. Now, we can proceed as in the first part of Lemma 8.32. By Lemma 8.3, there are $m-1$ edges, say $e_1,\dots,e_{m-1}$, leaving from P_1 so that $s,\ e_1,\dots,e_{m-1}$ are pairwise twins and there are $m-1$ edges, say $e_m,\dots,e_{2m-2}$, leaving from P_2 so that $s,\ e_m,\dots,e_{2m-2}$ are twins. Orient s from P_1 to P_2 and denote by s^+ and s^- the right and left, respectively. Let e_i be any edge from P_1, slit the edges e_i and e_m and denote the resulting sides as $e_i^{\pm}$ and $e_m^{\pm}$ according to our convention. Then identify e_i^+ with e_m^-, and identify e_i^- with e_n^+. The resulting space is a genus-one differential in the stratum $\mathcal{H}_1(m,m,m,m;\,-2^n)$ with trivial periods as desired. Suppose, without loss of generality, that $e_1,\dots,e_{m-1}$ are ordered

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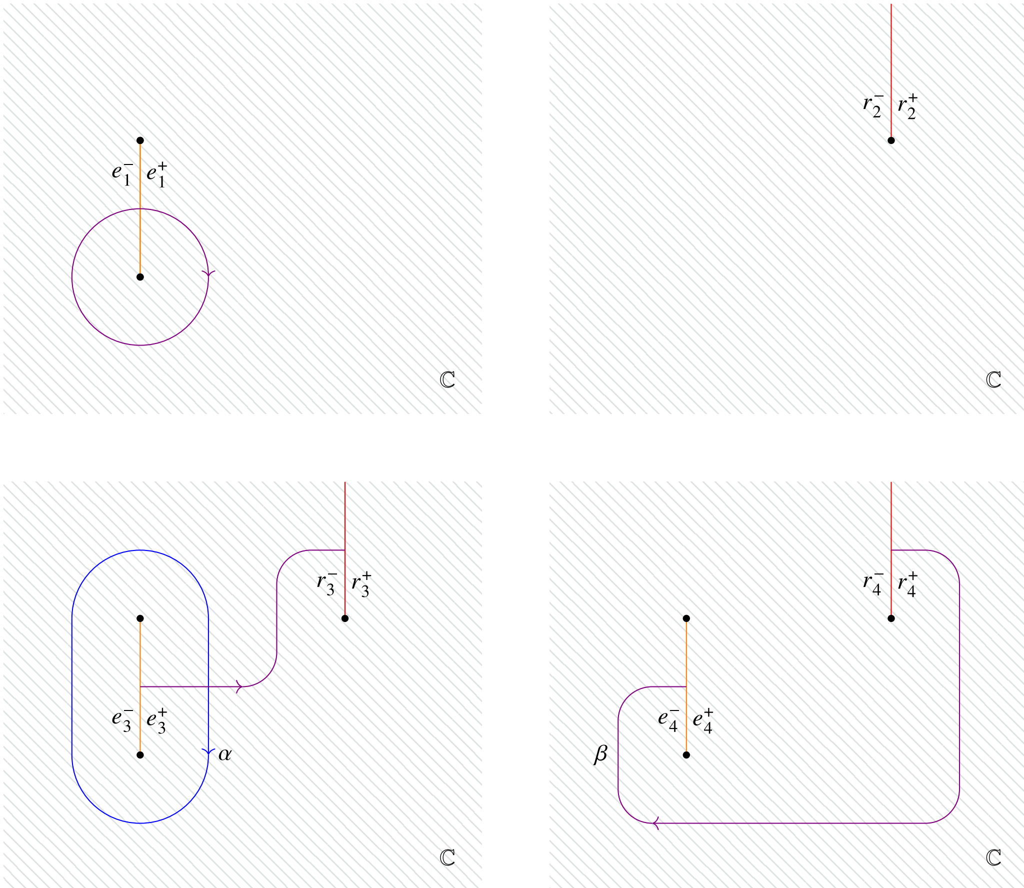

Figure 47. *Realizing a translation surface with trivial periods and rotation number one in the stratum* $\mathcal{H}_1(2, 2, 2; -2, -4)$. *It is not hard to check that both curves* α *and* β *have index two.*

counterclockwise around P_1. If i is odd, then it is easy to check that the resulting structure has rotation number 2 whereas, if i is even the resulting structure has rotation number one.

It remains to deal with the special case $m = 2$. The argument above fails because, for $m = 2$, we do not have any edge for realizing a translation surface with rotation number one in the stratum $\mathcal{H}_1(2, 2, 2, 2; -2^4)$. However, the same proof still works for realizing a translation surface with trivial periods and rotation number two. We only realize the trivial representation in the connected component of $\mathcal{H}_1(2, 2, 2, 2; -2^4)$ with rotation number one. Let A, B, C and D be the vertices of a square in $\mathbb{C}$. Let a, b, c and d denote the edges AB, BC, CD and DA, respectively. Consider four copies of $(\mathbb{C}, dz)$, and perform the following slits:

1. In the first copy, we slit all the edges a_1, b_1, c_1 and d_1,
2. Then slit in the second copy the edges a_2 and c_2,
3. Slit in the third copy the edge d_3 and, finally,
4. Slit in the fourth copy the edge b_4.

Identify the edge a_1^+ with a_2^- and a_1^- with a_2^+. Similarly, identify c_1^+ with c_2^- and c_1^- with c_2^+. Then identify the edge d_1^+ with d_3^- and d_1^- with d_3^+. Finally, identify b_1^+ with b_4^- and b_1^- with b_4^+. The resulting space is a genus-one differential with trivial periods, and it lies in the stratum $\mathcal{H}_1(2, 2, 2, 2; -2^4)$. It remains to show that it has rotation number one as desired. This follows because we can find a pair of curves α, β so that $\mathrm{Ind}(\alpha) = 3$ and $\mathrm{Ind}(\beta) = 1$ as depicted in Figure 48. □

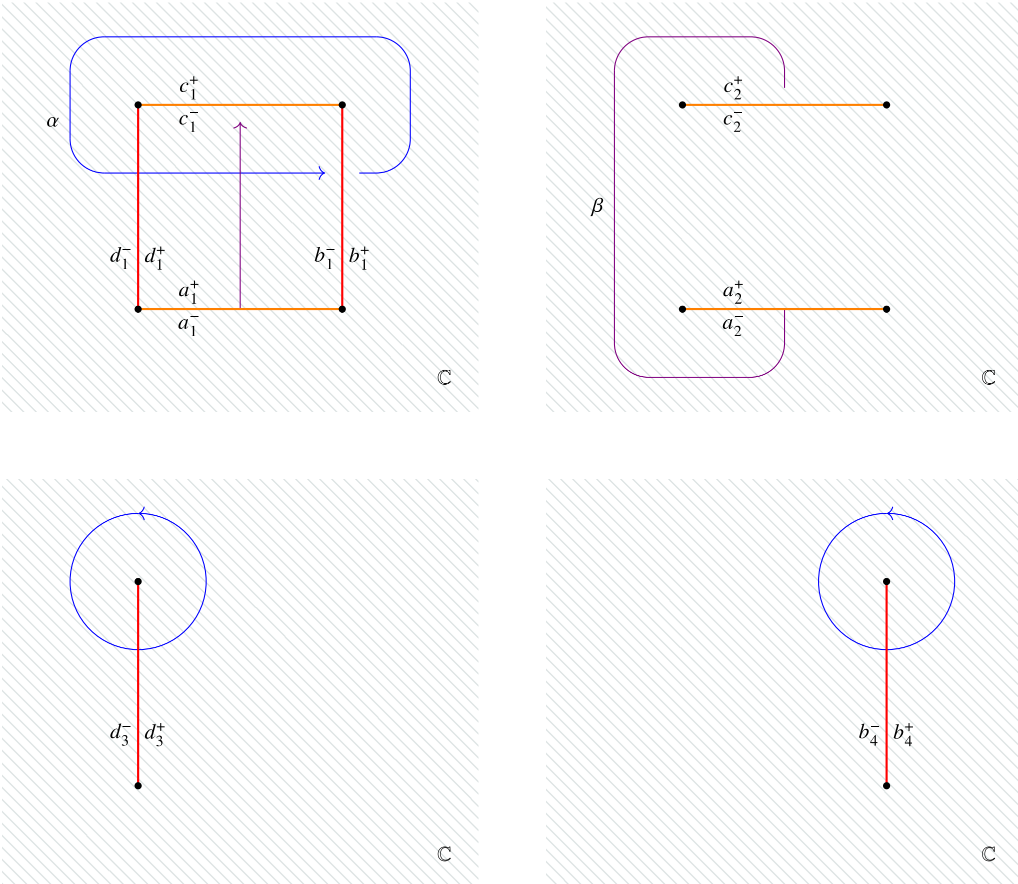

Figure 48. *Realizing a translation surface with trivial periods and rotation number one in the stratum* $\mathcal{H}_1(2, 2, 2, 2; -2^4)$. *It is not hard to check that* α *has index three and* β *has index one.*

It remains the case $k = 3$, that is, the case of strata of the form $\mathcal{H}_1(m_1, m_2, m_3; -2^n)$ with m_i all even. We can do induction from the two components of the lower stratum $\mathcal{H}_1(m_1 - 2, m_2 - 2, m_3; -2^{n-2})$, slit a saddle connection, say s joining two zeros P_1 and P_2, and then add two copies of $(\mathbb{C}, dz)$ with the same slit. If a closed curve crosses s, then after the operation its index alters by 2, hence the rotation number remains unaltered. For this construction, we need to check a few conditions. First, we need

$$m_3 < 2(n-2) - (n-2) = n - 2. \tag{124}$$

Notice that at least one of the three zeros satisfies this bound as soon as $n \geq 7$. In fact, in the case $m_i \geq n - 2$ we get that $2n \geq 3(n-2)$ that implies $n \leq 6$. We check the cases of small n directly. Next, we need to find a saddle connection joining two specified zeros. Indeed, we can assume that there is a saddle connection joining any two of the three zeros to start with, and after adding more slit planes, clearly the statement of existing saddle connections between any two zeros still holds.

So it reduces to check the induction base cases for strata $\mathcal{H}_1(2^3; -2^3)$, $\mathcal{H}_1(2, 4, 4; -2^5)$ and $\mathcal{H}_1(4, 4, 4; -2^6)$. Notice that, combinatorially, the stratum $\mathcal{H}_1(2, 2, 4; -2^4)$ is allowed but the trivial representation cannot be realized inside it because the Hurwitz type inequality (103) fails. We have the following lemmas.

Lemma 8.34. *The trivial representation can be realized in each connected component of* $\mathcal{H}_1(2, 4, 4; -2^5)$.

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Proof. In this case, provide a unique construction, and as we shall see, the realization of the rotation number only depends on the choice of certain slits. The idea is to get the access to such a connected component from the stratum of genus-zero differentials $\mathcal{H}_0(1,3,4;-2^5)$ as follows. Let $A,\ B,\ C \in \mathbb{C}$ be distinct and not aligned points. Notice that the convex hull of $\{A,B,C\}$ is a nondegenerate triangle. Let a,b,c denote the edges AB, BC and CA. Consider five copies of $(\mathbb{C},\ dz)$, and slit them as follows:

1. slit the first copy along the edge b_1;
2. slit the second copy along the edges b_2 and c_2; then
3. slit the remaining copies along c_3, c_4 and c_5, respectively;
4. finally, slit the first copy along a_1 and either
   i. slit the third copy along a_3 if we aim to realize a translation surface with rotation number one, otherwise
   ii. slit the fourth copy along a_4 if we aim to realize a translation surface with rotation number two.

Label the resulting sides with $\pm$ according to our convention. Identify the edges as follows: c_1^+ with c_2^- and c_1^- with c_2^+. Next, identify c_i^- with c_{i+1}^+ for $i=2,3,4$ and c_5^- with c_2^+. The resulting surface is a genus-zero differential in $\mathcal{H}_0(1,3,4;\ -2^5)$. The last step is to identify a_1^+ with a_4^- and a_1^- with a_4^+ in the case we aim to realize a genus-one differential in $\mathcal{H}_1(2,4,4;\ -2^5)$ with trivial periods and rotation number one. Otherwise, identify a_1^+ with a_3^- and a_1^- with a_3^+ in the case we aim to realize a genus-one differential in $\mathcal{H}_1(2,4,4;\ -2^5)$ with trivial periods and rotation number two. See Figure 49. □

Lemma 8.35. *The trivial representation can be realized in each connected component of* $\mathcal{H}_1(4,4,4;\ -2^6)$.

Proof. Even in this case, provide a unique construction, and as we shall see, the realization of the rotation number only depends on the choice of certain slits. Let $A,\ B,\ C \in \mathbb{C}$ be distinct and not aligned points. Notice that the convex hull of $\{A,B,C\}$ is a nondegenerate triangle.

Let a,b,c denote the edges AB, BC and CA. Consider six copies of $(\mathbb{C},\ dz)$, and slit them as follows:

1. slit the first copy along the edge a_1,
2. slit the second copy along the edges a_2, b_2 and c_2,
3. slit the third and fourth copy along the edges a_3 and a_4 and the edges b_3 and b_4,
4. slit the fifth and sixth copy along the edges c_5 and c_6, finally
   i. if we aim to realize a translation surface with rotation number one, then slit the fifth copy along the edge a_5, otherwise
   ii. if we aim to realize a translation surface with rotation number two, slit the sixth copy along the edge a_6.

Label the resulting sides with $\pm$ according to our convention. Identify the edges as follows: a_1^+ with a_2^- and a_1^- with a_2^+. Next, identify the edges b_2^+ with b_3^-, then b_3^+ with b_4^-, and finally, b_4^+ with b_2^-. Similarly, identify the edges c_2^+ with c_5^-, then c_5^+ with c_6^-, and finally, c_6^+ with c_2^-. The resulting surface is a genus-zero differential in the stratum $\mathcal{H}_0(3,3,4;-2^6)$. The last step is to identify a_3^+ with a_5^- and a_3^- with a_5^+ in the case we aim to realize a genus-one differential in $\mathcal{H}_1(4,4,4;\ -2^6)$ with trivial periods and rotation number one. Otherwise identify a_3^+ with a_6^- and a_3^- with a_6^+ in the case we aim to realize a genus-one differential in $\mathcal{H}_1(4,4,4;\ -2^6)$ with trivial periods and rotation number two. See Figure 50. □

Lemma 8.36. *The stratum* $\mathcal{H}_1(2^3;\ -2^3)$ *is exceptional; in fact, the trivial representation can only be realized in the nonprimitive connected component.*

Proof. Suppose that (X,ω) is a translation surface in $\mathcal{H}_1(2^3;\ -2^3)$ with trivial periods. Let Z_i and P_i be the zeros and poles of ω for $i=1,2,3$. The trivial period condition is equivalent to the existence of a triple cover of X to $\mathbb{CP}^1$ totally ramified at the Z_i and having $P_1+P_2+P_3$ as the fiber over infinity. This implies the linear equivalence relation $3Z_1 \sim 3Z_2 \sim 3Z_3 \sim P_1+P_2+P_3$. On the

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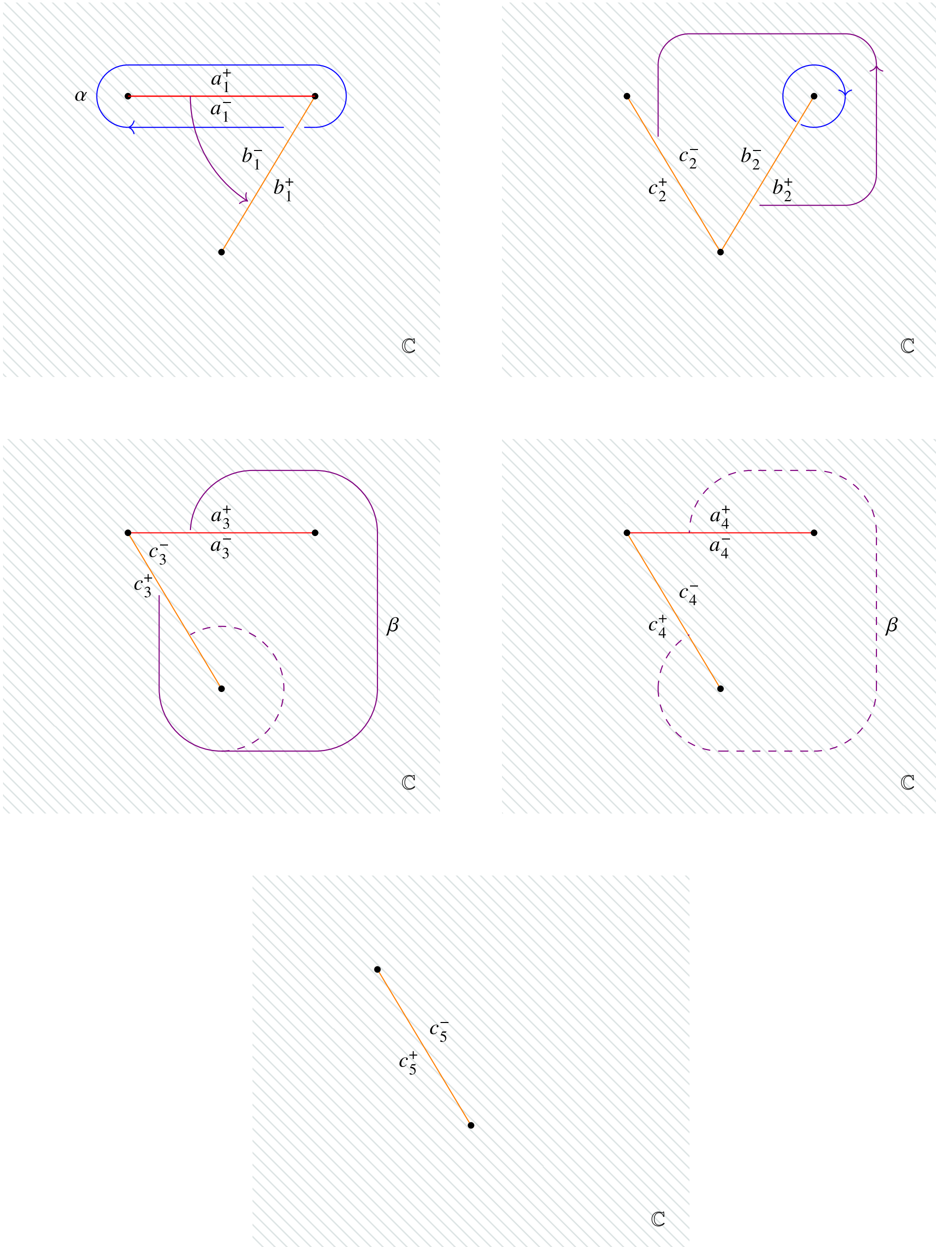

Figure 49. *Realization of a genus-one differential in $\mathcal{H}_1(2, 4, 4;\ -2^5)$ with trivial periods and prescribed rotation number. In both cases, the curve α can be taken as the blue curve depicted. The curve β depends on which edge we decide to slit between a_3 and a_4. In the former case, β has index 2 and the rotation number of the final structure will be two. In the latter case, β has index 3 and the rotation number of the final structure will be one.*

other hand, by assumption $2Z_1 + 2Z_2 + 2Z_3 \sim 2P_1 + 2P_2 + 2P_3$. It follows that $Z_1 + Z_2 + Z_3 \sim 3Z_1 + 3Z_2 + 3Z_3 - (2Z_1 + 2Z_2 + 2Z_3) \sim P_1 + P_2 + P_3$, which implies by definition that (S, ω) belongs to the nonprimitive component of $\mathcal{H}_1(2^3;\ -2^3)$. □

This complete the proof of Proposition 8.8 and indeed the proof of Theorem B for genus-one differentials.

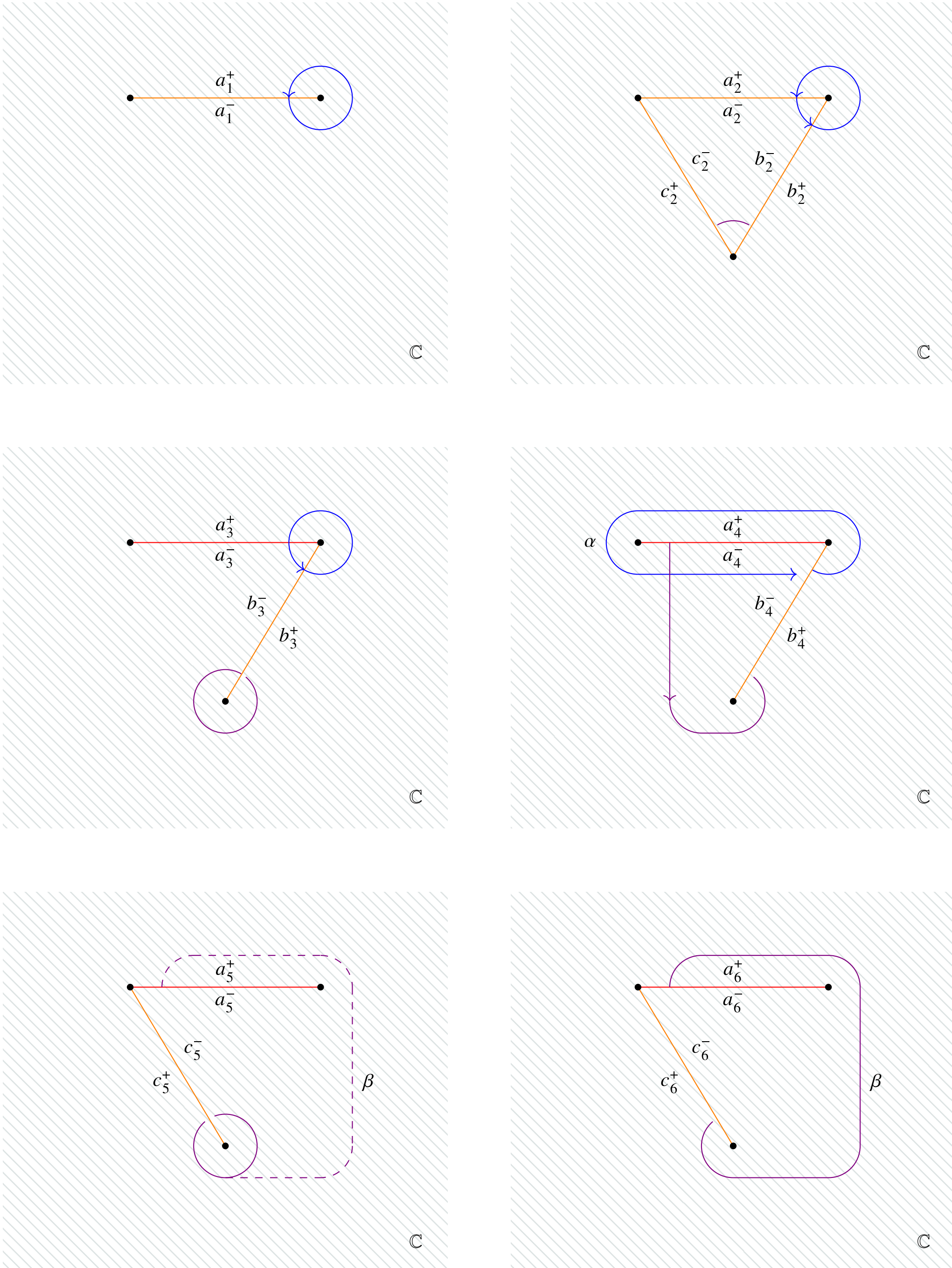

Figure 50. *Realization of a genus-one differential in $\mathcal{H}_1(4,4,4;\,-2^6)$ with trivial periods and prescribed rotation number. In both cases, the curve α can be taken as the blue curve depicted. The curve β depends on which edge we decide to slit between a_5 and a_6. In the former case, β has index 3 and the rotation number of the final structure will be one. In the latter case, β has index 4 and the rotation number of the final structure will be two.*

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8.5. *Hyperelliptic exact differentials*

In subsection §8.2, we have seen that the trivial representation cannot appear as the period character of a hyperelliptic translation surface with a single zero. However, it is still possible to realize trivial representation as the period character of some exact hyperelliptic differentials in some cases. As alluded above, the order of zeros of a meromorphic exact differential ω, on a Riemann surface X, are subject to the constraint provided by the Hurwitz type inequality (103):

$$m_j \leq \sum_{i=1}^{n} p_i - n - 1.$$

Such a formula, for strata $\mathcal{H}_g(m,m;-2p)$ and $\mathcal{H}_g(m,m;-p,-p)$ simplifies to

$$m \leq 2p-2 \qquad m \leq 2p-3, \tag{125}$$

respectively. We also recall that, according to our Remark 2.3, the order of zeros and poles are subject to the so-called Gauss–Bonnet condition

$$2m = 2g + 2p - 2, \ \textit{i.e.} \ \ m = g + p - 1; \tag{126}$$

otherwise, the stratum would be empty. Equations (125) and (126) above combined together readily imply the following:

Lemma 8.37. *The trivial representation appears as the period of a translation surface in* $\mathcal{H}_g(m,m;-2p)$ *if and only if* $g \leq p-1$. *Similarly, the trivial representation appears as the period of a translation surface with poles in* $\mathcal{H}_g(m,m;-p,-p)$ *if and only if* $g \leq p-2$.

We next wonder whether the trivial representation appears as the period of some hyperelliptic translation surface.

Proposition 8.38. *Suppose the trivial representation can be realized in a stratum admitting a hyperelliptic component. Then it can be realized as the period character of some hyperelliptic translation surfaces with poles in the same stratum.*

The remaining part of this subsection is devoted to prove Proposition 8.38. We shall distinguish two cases as follows.

8.5.1. Single higher-order pole

Let (X,ω) be a genus-zero meromorphic differential in $\mathcal{H}_0(p-1,p-1;-2p)$, where $p \geq 2$. This structure can be easily realized as follows. On $(\mathbb{C}, dz)$, we consider two distinct points, say P, $Q \in \mathbb{C}$, and let r be the unique straight line passing through them. Denote by s_0 the segment joining P and Q, and let r_1 and r_2 be the subrays of r leaving from P and Q, respectively. We next bubble $p-1$ copies of $(\mathbb{C}, dz)$ along r_1, and in the same fashion, we bubble $p-1$ copies of $(\mathbb{C}, dz)$ along r_2. The resulting structure (X,ω) is a translation surface with poles in $\mathcal{H}_0(p-1,p-1;-2p)$ having trivial periods. Notice that the straight line r splits the complex plane $\mathbb{C}$ into two half-planes. We orient r so that s_0 is oriented from P to Q. According to this orientation, we denote by H_1 the half-plane bounded by r^- and by H_2 the half-plane bounded by r^+. Around $P \in (X,\omega)$, we can single out $p+1$ sectors, say $H_1, S_1, \dots, S_{p-1}, H_2$, in cyclic order. For $k = 1, \dots, p-1$, each sector S_k comes from a copy of $(\mathbb{C}, dz)$ glued along r_1 by construction. In particular, S_k contains a segment, say s_k, based at P that bounds a wedge of angle $2k\pi$ with s_0. Let Q_k be extremal point of s_k other than P.

Similarly, around $Q \in (X,\omega)$, we can single out other $p+1$ sectors, say $H_2, S_p, \dots, S_{2p-2}, H_1$, in cyclic order. In this case, each sector comes from a copy of $(\mathbb{C}, dz)$ glued along the half-ray r_2. For $k = 1, \dots, p-1$, each sector S_{p+k-1}, for $k = 1, \dots, p-1$, comes from a copy of $(\mathbb{C}, dz)$ glued along r_2 by construction. In this case, S_{p+k-1} contains a segment, say s_{p+k-1}, based at Q that bounds a wedge of angle $2k\pi$ with s_0. Finally, let P_k be extremal point of s_{p+k-1} other than Q.

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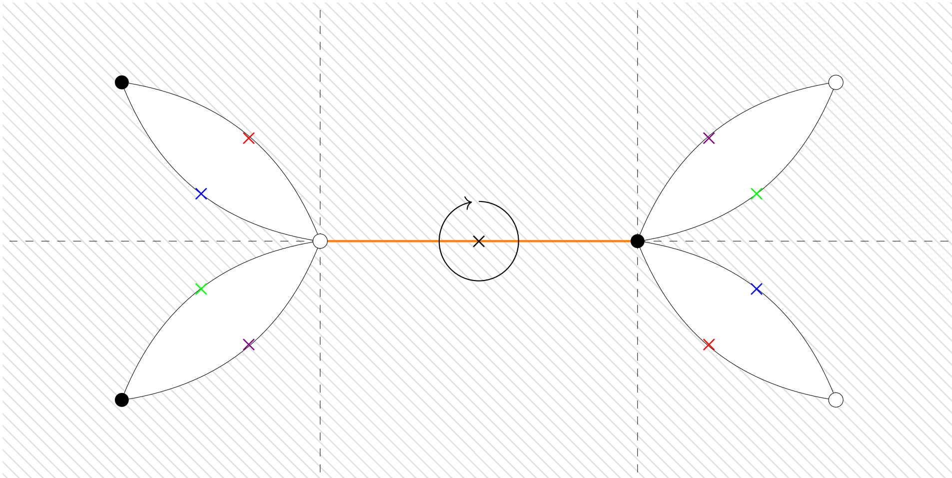

Figure 51. *Realization of an exact hyperelliptic genus-two differential in $\mathcal{H}_2(4,4;-6)$. In this case, the hyperelliptic involution has six fixed points. Three out of four of these points are drawn in the picture with the symbol $\times$. Two symbols with the same color are identified on the final surface, and hence, they need to be counted as a single fix point. The remaining fix point is the puncture corresponding to the pole (in this case $2p=6$).*

Any orientation on s_0 naturally yields a preferred orientation on each segment s_k and s_{p+k-1}, for any k, because they all have the same image under the developing map. Recall that s_0 is oriented from P to Q. As a consequence, s_k is oriented from P to Q_k and s_{p+k-1} is oriented from P_k to Q for all $k=1,\dots,p-1$. Let $1\le g\le p-1$. For $k=1,\dots,g$, slit (X,ω) along s_k and s_{p+k-1}, and denote the resulting edges by $s_k^{\pm}$, $s_{p+k-1}^{\pm}$, where the sign is taken according to our convention. Finally, identify s_k^+ with s_{p+k-1}^- and s_k^- with s_{p+k-1}^+. The resulting space is surface of genus g equipped with a translation structure $(Y,\xi)\in\mathcal{H}_g(m,m;-2p)$, where $m=g+p-1$.

By construction, ξ is an exact differential on Y; that is, (Y,ξ) has trivial absolute periods. Since all sectors are cyclically ordered as $H_1,S_1,\dots,S_{p-1},H_2,S_p,\dots,S_{2p-2}$, the structure (Y,ξ) is also naturally equipped with an involution of order two with $2g+2$ fixed points given by: the midpoint of s_0, the midpoints of $s_k^{\pm}$ and of $s_{p+k-1}^{\pm}$ for $k=1,\dots,g$ and the pole; see Figure 51. Therefore, the structure (Y,ξ) is hyperelliptic with trivial periods as desired.

8.5.2. Two higher-order poles

This case is similar to the previous one, and in fact, the main difference is that we begin with two genus-zero meromorphic differential $(X_i,\,\omega_i)\in\mathcal{H}_0(p-2;-p)$. These structures can be realized as follows: We begin with two copies of $(\mathbb{C},\,dz)$, and we consider on each one two distinct points, say $P_i,\,Q_i\in\mathbb{C}$ such that

$$d\big(P_1,\,Q_1\big)=d\big(P_2,\,Q_2\big),$$

where $d(\cdot\,,\,\cdot)$ denotes the usual Euclidean metric. Let l_i denote the unique straight line passing through P_i and Q_i, and denote by e_i the unique segment joining them. For $i=1,2$, we orient the line l_i so that e_i is oriented from P_i to Q_i. On the first copy of $(\mathbb{C},\,dz)$, we define $r_1\subset l_1$ as the ray leaving from P_1 not passing through Q_1. Similarly, on the second copy of $(\mathbb{C},\,dz)$ we define $r_2\subset l_2$ as the ray leaving from Q_2 and not passing through P_2. Next, we bubble $p-2$ copies of $(\mathbb{C},\,dz)$ along r_1, and similarly, we bubble $p-2$ copies of $(\mathbb{C},\,dz)$ along r_2. Define the resulting structures as $(X_1,\,\omega_1)$ and $(X_2,\,\omega_2)$, respectively. Both structures have a pole of order p and ξ_1 has a zero of order $p-2$ at P_1 whereas ξ_2 has a zero of order $p-2$ at Q_2.

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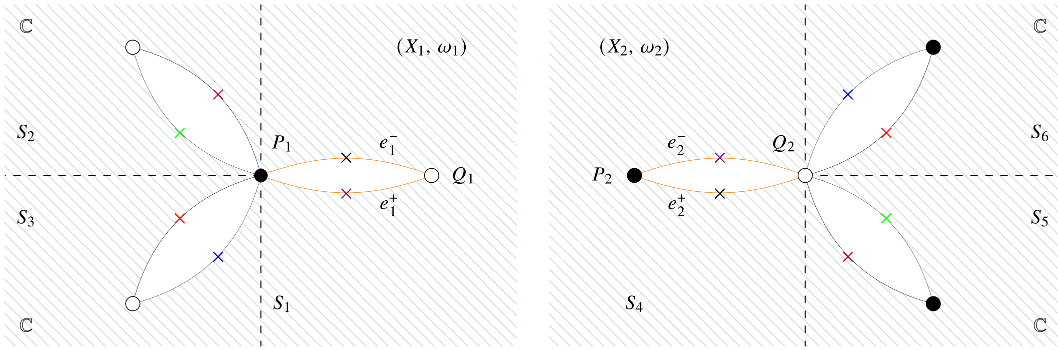

Figure 52. *Realization of a hyperelliptic exact differential in the stratum $\mathcal{H}_2(5, 5; -4, -4)$. The hyperelliptic involution has six fixed points; all of them are drawn in the picture with the symbol $\times$. Two symbols with the same color are identified on the final surface, and hence, they need to be counted as a single fix point.*

Let us focus on (X_1, ω_1). Around P_1, we can single out $p-1$ sectors, say $S_1, \dots, S_{p-1}$ in cyclic positive order, each of which is a wedge of angle 2π. One of these sector, say S_1, is given by the first starting copy of $(\mathbb{C}, dz)$. All the other sectors $S_2, \dots, S_{p-1}$ come from the $p-2$ copies of $(\mathbb{C}, dz)$ bubbled along r_1. For every $k = 2, \dots, p-1$, inside the sector S_k there is a segment, say s_k, based at P_1 that forms a wedge of angle $2k\pi$ with e_1. Let $Q_k{}'$ be the extremal point of s_k other than P_1. Furthermore, the fixed orientation on the edge e_1 determines a preferred orientation on s_k from P_1 to $Q_k{}'$.

Similar considerations hold for (X_2, ω_2). Around Q_2, we can single out $p-1$ sectors, say $S_p, \dots, S_{2p-2}$ in cyclic positive order, each of which is a wedge of angle 2π. One of these sector, say S_p, is given by the second starting copy of $(\mathbb{C}, dz)$. The other sectors $S_{p+1}, \dots, S_{2p-2}$ come from the $p-2$ copies of $(\mathbb{C}, dz)$ bubbled along r_2. For every $k = 1, \dots, p-2$, each sector S_{p+k} contains a segment, say s_{p+k}, based at Q_2 that forms a wedge of angle $2k\pi$ with e_2. Let $P_k{}'$ be the extremal point of s_{p+k} other than Q_2. Even in this case, the already fixed orientation on e_2 determines a preferred orientation on s_{p+k} from Q_2 to $P_k{}'$.

Slit (X_1, ω_1) and (X_2, ω_2) along e_1 and e_2, respectively, denote $e_1^\pm$ and $e_2^\pm$ the resulting sides and identify the edge e_1^+ with e_2^- and the edge e_1^- with e_2^+. The resulting space is a translation surface, say (X, ω), in the stratum $\mathcal{H}_0(p-1, p-1; -p, -p)$. In fact, recall that e_i is oriented from P_i to Q_i for $i = 1, 2$; hence, P_1 is identified with P_2 and, similarly, Q_1 is identified with Q_2 on (X, ω). If we denote P, Q the resulting points, respectively, it is easy to check that ω has two zeros of order $p-1$ at P and Q.

Let $1 \le g \le p-2$. Slit (X, ω) along the segments $s_2, \dots, s_{g+1}$ (all leaving from P), and then slit along the segments $s_{p+1}, \dots, s_{p+g}$ (leaving from Q). Denote the resulting edges as $s_{k+1}{}^\pm$ and $s_{p+k}{}^\pm$ as usual, according to our convention, for $k = 1, \dots, g$. Next, identify s_{k+1}^+ with s_{p+k}^-, and identify s_{k+1}^- with s_{p+k}^+.

The resulting space is a genus g translation surface, say $(Z, \eta) \in \mathcal{H}_g(m, m; -p, -p)$, where $m = g + p - 1$. By construction, (Z, η) has trivial periods and it admits a hyperelliptic involution with $2g + 2$ fixed points given by: the midpoints of e_1, e_2 and the midpoints of $s_{k+1}^\pm$ and of $s_{p+k+1}^\pm$ for all $k = 1, \dots, g$; see Figure 52. This concludes the case with two higher-order poles and indeed the proof of Proposition 8.38.

8.5.3. Realizing the trivial representation

As a consequence of subsections §8.5.1 and §8.5.2, we have the following:

Corollary 8.39. *Suppose the trivial representation can be realized in a stratum admitting two connected components, one of which is hyperelliptic and the other is not. Then it can be realized in each connected component of the same stratum.*

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Sketch of the proof. Suppose a stratum $\mathcal{H}_g(m, m; -\nu)$, where $\nu = \{2m\}$ or $\nu = \{m, m\}$ admits two connected components on of which is hyperelliptic; for example, $\mathcal{H}_2(5, 5; -4, -4)$. We have seen above how to realize the trivial representation as the period of some hyperelliptic translation surface. We adopt the same notation above. In order to get an exact differential which is no longer hyperelliptic, it is sufficient to identify s^+_{k+1} with s^-_{2p-2-k} and s^-_{k+1} with s^+_{2p-2-k}. The resulting space is also a genus g translation surface in $\mathcal{H}_g(m, m; -p, -p)$, but it does not admit any hyperelliptic involution. □

8.6. Meromorphic exact differentials with prescribed parity

We aim to realize the trivial representation as the period character of an exact meromorphic differential with prescribed spin parity. The gist of the idea is, once again, to use the induction on the genus. Similarly to the other cases, the process we shall develop here consists in adding one handle with trivial periods at each step; see subsection §8.6.2. Once again, in order for inductive process to start, we need to treat genus-one surfaces in the first place and then explain how the induction works; see subsections §8.4. The main result of the present section is the following:

Proposition 8.40. *Suppose the trivial representation can be realized in a stratum admitting two connected components distinguished by the spin parity. Then it can be realized as the period character of some translation surfaces with poles in both components of the same stratum with the only exceptions being the strata $\mathcal{H}_g(2^{g+2}; -2^3)$ and $\mathcal{H}_g(2^{g+1}; -4)$ for $g \geq 2$.*

The exceptional cases are handled by the following:

Proposition 8.41. *In the strata $\mathcal{H}_g(2^{g+2}; -2^3)$ and $\mathcal{H}_g(2^{g+1}; -4)$ for $g \geq 2$, the trivial representation can only be realized in the connected component with parity equal to g* (mod 2).

We begin with a few considerations.

8.6.1. Reducing to lower genus surfaces

Suppose the trivial representation appear as the period character of some differential in a stratum $\mathcal{H}_g(2m_1, \dots, 2m_k, -2p_1, \dots, -2p_n)$. For the moment, we do not consider any parity. According to [CFG22, Theorem B], this is possible if and only if

$$2m_j + 1 \leq \sum_{i=1}^{n} 2p_i - n \tag{127}$$

and

$$\sum_{j=1}^{k} 2m_j = \sum_{i=1}^{n} 2p_i + 2g - 2. \tag{128}$$

Let us consider the string $\kappa = (2m_1, \dots, 2m_k)$. We now introduce an algorithm to reduce κ to a new, possibly shorter, string $\lambda = (2d_1, \dots, 2d_l)$ such that

$$\sum_{j=1}^{l} 2d_j = \sum_{i=1}^{n} 2p_i; \tag{129}$$

that is, $\mathcal{H}_1(2d_1, \dots, 2d_l; -2p_1, \dots, -2p_n)$ is a nonempty stratum of genus-one differentials.

Such a reduction process is defined as follow. Let $\{2e_1, \dots, 2e_t\}$ be any string of positive integers indexed in nonincreasing order, we distinguish two cases according to the list below

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- ◦ we reduce as $\{2e_1, \dots, 2e_{t-1}, 2e_t\} \longmapsto \{2e_1, \dots, 2e_{t-1}, 2e_t - 2\}$ in the case the last integer is then 2, that is, $e_t \geq 2$, otherwise
- ◦ if the last integer is equal to 2, then reduce as $\{2e_1, \dots, 2e_{t-1}, 2\} \longmapsto \{2e_1, \dots, 2e_{t-1}\}$.

Starting with a string κ as above, after $g-1$ reductions we end up with a new string $\lambda = (2d_1, \dots, 2d_l)$ such that Equation (129) holds. Notice in the first place that, if κ is indexed in nonincreasing order, then even λ is indexed in the same way. We shall always assume that $2d_1 \geq 2d_2 \geq \cdots \geq 2d_l$ at each step, in particular in the constructions below the assumption $2m_i \geq 2m_j$ for $i \leq j$ plays an important role.

Observe that, if the trivial representation $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized in a stratum $\mathcal{H}_g(\kappa; -\nu)$, then the trivial representation $\mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized in the stratum $\mathcal{H}_1(\lambda; -\nu)$. Therefore, the gist of the proof is to undo the reduction by adding one handle with trivial periods time by time. The way this handle will be glued depends on how the reduction has been performed. We obtain in this way a finite sequence of translation surfaces with the last one being the desired structure.

8.6.2. Inductive process: higher genus surfaces

Let the induction start. The idea is to get the access to a genus $g+1$ stratum starting from a genus g one, similarly to what was done in Section §7, that is,

$$\mathcal{H}_g(2m_1, \dots, 2m_k - 2; -2p_1, \dots, -2p_n) \longmapsto \mathcal{H}_{g+1}(2m_1, \dots, 2m_k; -2p_1, \dots, -2p_n), \tag{130}$$

where $2m_k - 2 \geq 0$. For genus-one surfaces, we have already seen in subsection §8.4 that as soon as the trivial representation $\chi\colon \mathrm{H}_1(S_{1,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ can be realized in a given stratum then it can be realized as the period character of some translation surface with poles in each connected component of the same stratum with only three exceptions. By ignoring the exceptional cases for a moment, we can use these structures as the base case for an inductive foundation. In fact, every nonconnected stratum $\mathcal{H}_1(2\kappa; -2\nu)$ admits at least one component with odd rotation number and at least one component with even rotation number. We shall use the former case for realizing genus g meromorphic differentials with odd spin parity whereas we shall use the latter case for realizing genus g meromorphic differentials with even spin parity.

However, by applying the reduction just described above, it is possible to land to an exceptional stratum that we recall to be $\mathcal{H}_1(2,2;-4)$, $\mathcal{H}_1(2^3; -2^3)$ or $\mathcal{H}_1(3,3; -3,-3)$. Since we consider strata of even type, the polar part is necessarily a string of the form $(2p_1, \dots, 2p_n)$. Since the reduction above preserves the polar part and only alters the string $\kappa = (2m_1, \dots, 2m_k)$, it follows that we never land in the stratum $\mathcal{H}_1(3,3; -3,-3)$ of genus-one differentials.

We next wonder: *from which strata do we land on an exceptional stratum?* Once again, the key observation is that the reduction above preserves the polar part. In particular, we land in the stratum $\mathcal{H}_1(2,2; -4)$ if and only if the reduction starts from a stratum of the form $\mathcal{H}_g(2m_1, \dots, 2m_k; -4)$ with $g \geq 2$. The Gauss–Bonnet condition (see Remark 2.3) implies that $m_1 + \cdots + m_k = g+1$ and the Hurwitz type inequality (103) implies that $2m_i \leq 2$. Therefore, we land in the exceptional stratum $\mathcal{H}_1(2,2; -4)$ if and only if the reduction starts from a stratum of the form $\mathcal{H}_g(2^{g+1}; -4)$ for $g \geq 2$. Finally, we observe that the same argument shows that we land in the exceptional stratum $\mathcal{H}_1(2^3; -2^3)$ if and only if the reduction starts from strata of the form $\mathcal{H}_g(2^{g+2}; -2^3)$ for $g \geq 2$. For $g \geq 2$, we define every stratum $\mathcal{H}_g(2^{g+1}; -4)$ and $\mathcal{H}_g(2^{g+2}; -2^3)$ as an *exceptional stratum of even type*, and we shall consider them in §8.6.3. The remaining part of this subsection is devoted to prove Proposition 8.40 which handles nonexceptional strata of even type.

Proof of Proposition 8.40. Let $\mathcal{H}_g(2m_1, \dots, 2m_k; -2p_1, \dots, -2p_n)$ be a nonexceptional stratum of even type, and suppose the Hurwitz type inequality (103) holds, that means the trivial representation can be realized in this stratum as the period character of some translation surface with trivial periods. Let κ be the string encoding the order of zeros, and reduce it as described in §8.6.1. After $g-1$ steps, we get a new string, say $\lambda = (2d_1, \dots, 2d_l)$ with $l \leq k$. In the reduced stratum $\mathcal{H}_1(\lambda; -\nu)$, the trivial representation can be realized as the period character of some genus-one differential with rotation number 1 or 2, say

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(X_1, ω_1). We then undo the reduction in order to realize an exact genus g meromorphic differential in the initial stratum. Finally, it remains to compute the parity of the spin structure. We bubble a handle with trivial periods as described in subsection §8.1.2 and, as we have already seen, such a surgery alters the spin parity. Therefore, for a given stratum $\mathcal{H}_g(2m_1, \ldots, 2m_k; -2p_1, \ldots, -2p_n)$ we get the access to the even component from the odd component of the lower stratum of differentials $\mathcal{H}_{g-1}(2m_1, \ldots, 2m_k - 2; -2p_1, \ldots, -2p_n)$. Similarly, we get the access to the odd component of the stratum above from the even one of the lower stratum of differentials $\mathcal{H}_{g-1}(2m_1, \ldots, 2m_k - 2; -2p_1, \ldots, -2p_n)$.

We now explain how to undo the reduction. For $1 \leq h \leq g-1$, at each step we bubble a handle with trivial periods by using the alternative construction provided at subsection §8.1.2 and denote by (X_{h+1}, ω_{h+1}) the resulting structure. By using the same notation, suppose first that at the h^{th} step the reduction is of the form $\{2e_1, \ldots, 2e_{t-1}, 2\} \mapsto \{2e_1, \ldots, 2e_{t-1}\}$ – this is the easiest case to handle. Let P_{t-1} be the zero of order $2e_{t-1}$ and let c_1, c_2, c_3 be three paths all leaving from P_{t-1} with length ε and such that the angle between the paths c_1 and c_2 and the angle between the paths c_2 and c_3 are both 2π. The angle between the paths c_3 and c_1 is $(4e_{t-1} - 2)\pi$. Bubble a handle with trivial periods as described in §8.1.2. The resulting structure (X_{h+1}, ω_{h+1}) is a translation surface with poles and trivial periods. From now on, suppose that at the h^{th} step the reduction is of the form $\{2e_1, \ldots, 2e_t\} \mapsto \{2e_1, \ldots, 2e_t - 2\}$ with $2e_t - 2 \geq 2$. We need the following:

Lemma 8.42. *Let $(X, \omega) \in \mathcal{H}_g(m_1, \ldots, m_k; -p_1, \ldots, -p_n)$ be a translation surface with poles and trivial absolute periods. Let P, Q be two zeros that satisfy*

$$\min_{\text{distinct zeros of } \omega} d(P_i, P_k),$$

and let s_1 be a saddle connection joining them. Let s_2 be a geodesic segment leaving from P such that s_1 and s_2 are twins. Then s_2 is an embedded geodesic segment such that $s_1 \cap s_2 \subseteq \{P, Q\}$.

Suppose the lemma holds. Let $(X_h, \omega_h) \in \mathcal{H}_g(2m_1, \ldots, 2m_k - 2; -2p_1, \ldots, -2p_n)$, and let $P_1, \ldots, P_k$ be the zeros of ω of orders $2m_1, \ldots, 2m_{k-1}, 2m_k - 2$ respectively. Let P_i be a zero that realises

$$d = \min_{\text{distinct zeros of } \omega} d(P_i, P_k). \tag{131}$$

Let s_1 be a saddle connection joining them and orient it from P_i to P_k. Let s_o and s_2 be two paths leaving from P_i such that s_o, s_1, s_2 are pairwise twins; s_o forms a wedge of angle 2π with s_1 on its left and s_2 forms a wedge of angle 2π with s_1 on its right. For simplicity, rename momentarily P_k as Q_1, and then denote by Q_o and Q_2 the extremal points of s_o and s_2 other than P_i. There are three mutually disjoint possibilities that we now discuss

1. The extremal points Q_o, Q_1 and Q_2 are pairwise distinct,
2. Two out of three extremal points coincide,
3. Q_o, Q_1 and Q_2 all coincide.

Remark 8.43. Notice that, in principle, the extremal points Q_o and Q_2 might be zeros of ω. In the case where the extremal points Q_o and Q_2 do not coincide with Q_1, they are not regular points. For simplicity, suppose $Q_o = P_j$ for some $j \neq i, k$. We proceed exactly as above by ignoring the fact that Q_o is not regular. The resulting structure lies in the stratum

$$\mathcal{H}_{h+1}(2m_1, \ldots, \widehat{2m_j}, \ldots, 2m_j + 2m_k; -2p_1, \ldots, -2p_n). \tag{132}$$

We eventually break the the zero of order $2m_j + 2m_k$ into two zeros of orders $2m_j$ and $2m_k$ to get the desired result. A similar consideration holds if both Q_o and Q_2 are not regular points. Since the constructions we are going to described do not depend on the nature of the extremal points, we suppose them to be regular for simplicity, unless otherwise specified, as the general case follows from the present remark. As an alternative, one may deform a little the given structure by 'moving' the branch points so that the vertices Q_o and Q_2 are both regular. Topologically, this kind of deformation has been described in [Tan94, Section §6], and it is the geometric counterpart of the well-known Schiffer variations (see [Nag85]), a classical tool in the study of Riemann surfaces and their moduli spaces.

The case 1 is the simplest one to deal with, and it does not have exceptional subcases. We bubble a handle with trivial periods as described in §8.1.2. After slitting, we get an hexagon with vertices $V_1, \ldots, V_6$; see Figure 40 above for a reference. We may define V_1 as the vertex arising from the branch point P_i, and then we label the remaining vertices in cyclic positive order. Two of these vertices, namely V_1 and V_4, have magnitude $(4m_i - 2)\pi$ and $(4m_k - 2)\pi$, respectively. All the other vertices, instead, have magnitude 2π. Once we glue the 'opposite', we get the desired handle with trivial periods, and hence, the resulting space is a translation surface $(X_{h+1}, \omega_{h+1}) \in \mathcal{H}_{h+1}(2m_1, \ldots, 2m_k; -2p_1, \ldots, -2p_n)$ with trivial absolute periods.

We now discuss the second case of the list above. Suppose in the first place that Q_1 coincides with Q_o without loss of generality. Since two twins close up, we cannot immediately bubble a handle with trivial periods as in the previous case. We first need the following intermediate step. Suppose $2m_k - 2 \geq 4$, that is, $m_k \geq 3$. The residual case $m_k = 2$ needs a special treatment we shall consider later. Break Q_1 into two zeros Q_{11} and Q_{12} of orders 2 and $2m_k - 4 \geq 2$, respectively. Once the zero is broken, there are three twins with extremal points given by Q_o, Q_{11} and Q_{12} which are now all distinct. In this way, we have reduced the construction to the case 1 above with the only difference that, after slitting, the vertex V_4 of the resulting hexagon has magnitude $(4m_k - 6)\pi$, and there is now another vertex, say V_2, with magnitude 6π; see Figure 53. Once we glue the 'opposite', we get the desired handle with trivial periods and hence a translation surface with poles $(X_{h+1}, \omega_{h+1}) \in \mathcal{H}_{h+1}(2m_1, \ldots, 2m_k; -2p_1, \ldots, -2p_n)$ with trivial absolute periods. As a second possibility, we next suppose the edges s_o and s_2 close up, that means Q_o and Q_2 now coincide. In this case, Q_o turns out to be a branch point of order $2m_j$, with $j < k$. Since $2m_j \geq 2m_k > 2m_k - 2 \geq 2$, it follows that $2m_j \geq 4$, and hence, we can break it into two zeros of even order each; for example, 2 and $2m_j - 2$. We now proceed as above in order to get the desired structure with trivial absolute periods.

The third case is similar to the second one. Once again we cannot immediately bubble a handle with trivial periods. Suppose $2m_k - 2 \geq 6$, that is, $m_k \geq 4$. The remaining cases $m_k = 2, 3$ need a special treatment and we shall consider them below. Break Q_1 into three zeros of orders 2, $2m_k - 6$ and 2 respectively. Notice that $2m_k - 6 \geq 2$. After slitting, the vertices V_1 and V_4 have magnitude $(4m_i - 2)\pi$ and $(4m_k - 10)\pi$ and there are now two vertices, say V_2 and V_5 with magnitude 6π. Once again, we glue the opposite sides, and the resulting structure is a translation surface (X_{h+1}, ω_{h+1}) with trivial periods in $\mathcal{H}_{h+1}(2m_1, \ldots, 2m_k; -2p_1, \ldots, -2p_n)$.

In order to complete the proof, we treat the residual cases mentioned above. Let us summarise them for the reader's convenience

1. Q_1 coincides with Q_o or Q_2 and $m_k = 2$,
2. Q_o, Q_1 and Q_2 all coincides and $m_k = 2$,
3. Q_o, Q_1 and Q_2 all coincides and $m_k = 3$.

All constructions above were performed by defining s_1 as the saddle connection that joins P_k with the closest zero and then defining s_o and s_2 as the twin 2π far away from s_1 on the left and right, respectively. In order to deal with these residual cases, we just need to find suitable twins on order to perform the desired bubbling.

Let s_1 be a saddle connection that joins P_k with the nearest P_i, and let $s_2, \ldots, s_{2m_i+1}$ be paths leaving from P_i such that s_i, s_j are twins for all $1 \leq i, j \leq 2m_i + 1$. In the case $m_k = 2$, then there are at most three edges having P_i and Q_1 as extremal points. If s_1 is the sole saddle connection joining P_i and Q_1, then there is nothing to prove. Moreover, since P_i is a branch point of even order $2m_i$, and $2m_i \geq 2m_k = 4$, there two adjacent edges, say s_l and s_{l+1} such that

i. their extremal points other than P_i are both different from Q_1, and
ii. they, respectively, form an angle 2π and 4π with respect to a saddle connection joining P_i and Q_1. In fact, this is always possible unless $2m_i = 4$. However, this case has already been covered above because we can find three edges with distinct extremal points.

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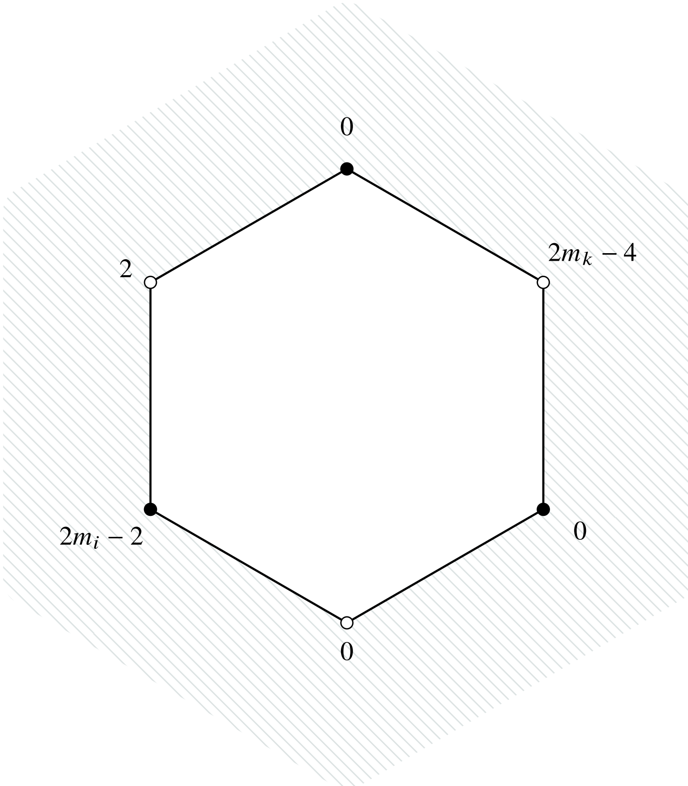

Figure 53. *Magnitude of the vertices $V_1, \ldots, V_6$ in the second case after breaking the zero P_k of order $2m_k - 2$. By drawing a blue and violet curve as in Figure 40, it is possible to verify that both curves have even index, and hence, the spin structure changes after bubbling.*

If s_l and s_{l+1} close up, then the common extremal point other than P_i is a branch point, say P_j, of order $2m_j \geq 2m_k > 2$. Split P_j into two branch points of orders 2 and $2m_j - 2$. Then we can bubble a handle with trivial periods as already described above. The resulting space is a genus-$h+1$ translation surface with trivial absolute periods and prescribed parity as desired. Finally, the case $m_k = 3$ is similar. The only difference here is that, in this case, there are at most five edges having P_i and Q_1 as the extremal points. Once again, since $2m_i \geq 2m_k \geq 6$, there are two adjacent edges s_l and s_{l+1} such conditions *i.* and *ii.* above holds. Then we may proceed as above to bubble a handle with trivial periods and then obtain a translation surface with poles and trivial absolute periods with the desire parity. This last case completes the proof of Proposition 8.40. □

We are left with the following:

Proof of Lemma 8.42. In the first place, we recall that any translation surface with poles and trivial absolute periods has at least two zeros; otherwise, the Hurwitz type inequality (103) does not hold. Let P and Q be a pair of zeros that realize

$$d = \min_{\text{distinct zeros of } \omega} d(\,\cdot\,,\,\cdot\,), \tag{133}$$

where $d(\,\cdot\,,\,\cdot\,)$ denotes the usual Euclidean distance. Let s_1 be a saddle connection joining them, and let s_2 be a twin leaving from P; see Definition 8.2. In principle, any twin path at P may contain a branch

Table 1. *Indices of curves in Figure 54.*

i	$\mathrm{Ind}(\alpha_i)$	$\mathrm{Ind}(\beta_i)$
1	1	1
2	1	1
3	1	1
4	1	1

point in its interior – this is not ruled out by our definition. Nevertheless in our case, both s_1 and s_2 cannot have any branch point in their interior otherwise we would contradict the minimality of d. The extremal point of s_2 cannot be a point in the interior of s_1. In fact, if $s_1 \cap s_2 \subseteq \{P, R\}$ with $R \neq Q$, then $P \to R \to P$ is a simple closed geodesic loop with trivial period. Since the intersection must be transverse – otherwise s_1 and s_2 would coincide – then R must be a branch point but this contradicts the minimality of d. Therefore, the extremal point of s_2 other than P could be any point not in the interior of s_1 nor P. □

8.6.3. Exceptional strata of even type

It remains to consider the exceptional cases of Proposition 8.40, that is, to prove Proposition 8.41. We have the following:

Lemma 8.44. *For every $g \geq 2$, the trivial representation can only be realized in the connected component of $\mathcal{H}_g(2^{g+2}; -2^3)$ with parity equal to g* (mod 2).

Proof. We will prove by induction on g. Denote by $\mathcal{TR}_g$ the locus of exact differentials in $\mathcal{H}_g(2^{g+2}; -2^3)$. Note that $\mathcal{TR}_g$ equivalently parameterises triple covers of $\mathbb{CP}^1$ totally ramified at $Z_1, \dots, Z_{g+2}$ and having $P_1 + P_2 + P_3$ as the fiber over infinity.

First, we show that connected components of $\mathcal{TR}_g$ correspond to integers $0 \leq k \leq (g+2)/2$ such that $2k \equiv g + 2 \pmod 3$. To see this, we use the monodromy description of triple covers parameterized in a connected component of $\mathcal{TR}_g$. Up to permuting the branch points and relabeling the three sheets, we can assume that the monodromy cycles at the first k branch points are given by the permutation $c = (1, 2, 3)$ and the monodromy cycles at the last $g + 2 - k$ branch points are given by the permutation $c^{-1} = (1, 3, 2)$, where $c^k(c^{-1})^{g+2-k} = \mathrm{id}$. This is equivalent to requiring that $2k - g - 2$ is divisible by 3.

Denote by $\mathcal{TR}_g(k)$ the connected component by using k monodromy cycles of c and $g + 2 - k$ monodromy cycles of c^{-1}. Note that relabeling any two sheets of the covers can interchange c and c^{-1}. Consequently, $\mathcal{TR}_g(k) = \mathcal{TR}_g(g + 2 - k)$ for $0 \leq k \leq g + 2$. Since $g \geq 2$, without loss of generality, we can assume that $k \geq 2$. Next we will exhibit a degenerate cover in the boundary of each $\mathcal{TR}_g(k)$ (in the sense of admissible covers [HM98, §3G]) by gluing an element in $\mathcal{TR}_{g-1}(k-2)$ and an element in $\mathcal{TR}_1(3)$ at a separating node. To see this, let two branch points both with monodromy cycle c approach each other in the target $\mathbb{CP}^1$. The resulting (admissible) cover is given by gluing the following two subcovers. One is over $\mathbb{CP}^1$ with the two chosen branch points of monodromy cycle c, and with an extra branch point at the node out of the vanishing cycle enclosing the two merged points whose monodromy cycle is $(c \cdot c)^{-1} = c$. The other is over $\mathbb{CP}^1$ with the remaining branch points of $k-2$ monodromy cycles c and $g + 2 - k$ monodromy cycles of c^{-1}, with an extra branch point at the node whose monodromy cycle is c^{-1}.

Finally, we show that the spin parity of differentials in $\mathcal{TR}_g(k)$ is $g \pmod 2$. By the induction hypothesis, the spin parity of $\mathcal{TR}_{g-1}(k-2)$ is $g - 1 \pmod 2$, and by the proof of Lemma 8.36, the spin parity of $\mathcal{TR}_1(3)$ is given by $h^0(Z_1 + Z_2 + Z_3 - P_1 - P_2 - P_3) = h^0(\mathcal{O}) = 1$. Moreover, the degenerate differential in the boundary of $\mathcal{TR}_g(k)$ described above has the same parity as those in the interior since it is of compact type; that is, the gluing node separates the two subsurfaces. Therefore, by the fact

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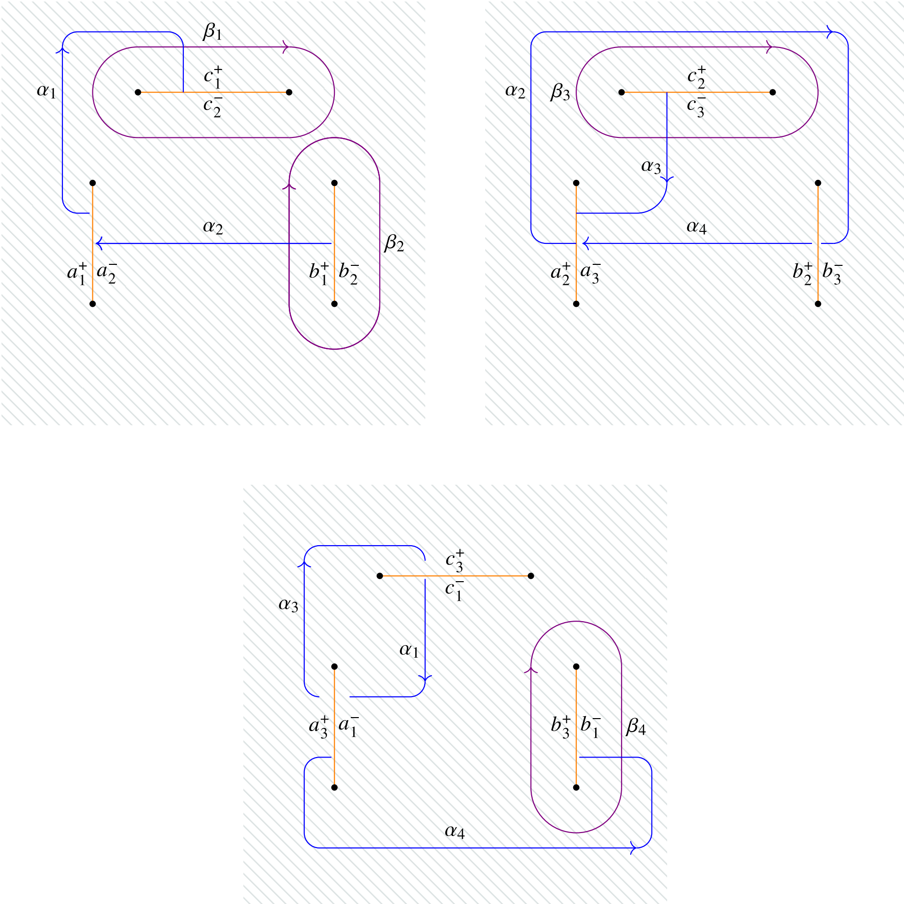

Figure 54. *Realization of a translation surface with poles and even parity in* $\mathcal{H}_4(2^6; -2^3)$*. Here, the monodromy pattern is equal to* $k = 0$*. Unlike the notation used so far, here the edges* a_i^+*,* b_i^+ *and* c_i^+ *are identified with* a_i^-*,* b_i^- *and* c_i^-*, respectively. Moreover, the indices of the curves* α_i *and* β_i *are according to Table 1.*

that the spin parity on a curve of compact type is given by the sum of parities of its components [Cor89, (3.2)], the parity of the interior of $\mathcal{TR}_g(k)$ is equal to $(g-1)+1 = g \pmod 2$. □

We finally show the following:

Lemma 8.45. *For every* $g \geq 2$*, the trivial representation can only be realized in the connected component of* $\mathcal{H}_g(2^{g+1}; -4)$ *with parity equal to* $g \pmod 2$.

Proof. This case essentially follows from the one in the preceding lemma. In both cases, the related triple covers arise from the same description, that is, totally ramified at $g+2$ points. Let $Z_1, \ldots, Z_{g+1}$ and Q be the $g+2$ ramification points. Let $P_1 + P_2 + P_3$ be an unramified fiber divisor. Then in the former case we consider the spin line bundle associated to the divisor $Z_1 + \cdots + Z_{g+1} + Q - P_1 - P_2 - P_3$, while in the latter case it is $Z_1 + \cdots + Z_{g+1} - 2Q$. Since $3Q \sim P_1 + P_2 + P_3$, the two spin line bundle classes are linearly equivalent, hence they have the same parity. Alternatively, one can just apply the same monodromy and degeneration argument as in the preceding lemma. □

In $g = 4$, for instance, there are two distinct monodromy patterns corresponding to $k = 0$ and $k = 3$. That means $\mathcal{TR}_4$ has two connected components in $\mathcal{H}_4(2^6; -2^3)$ as well as in $\mathcal{H}_4(2^5; -4)$.

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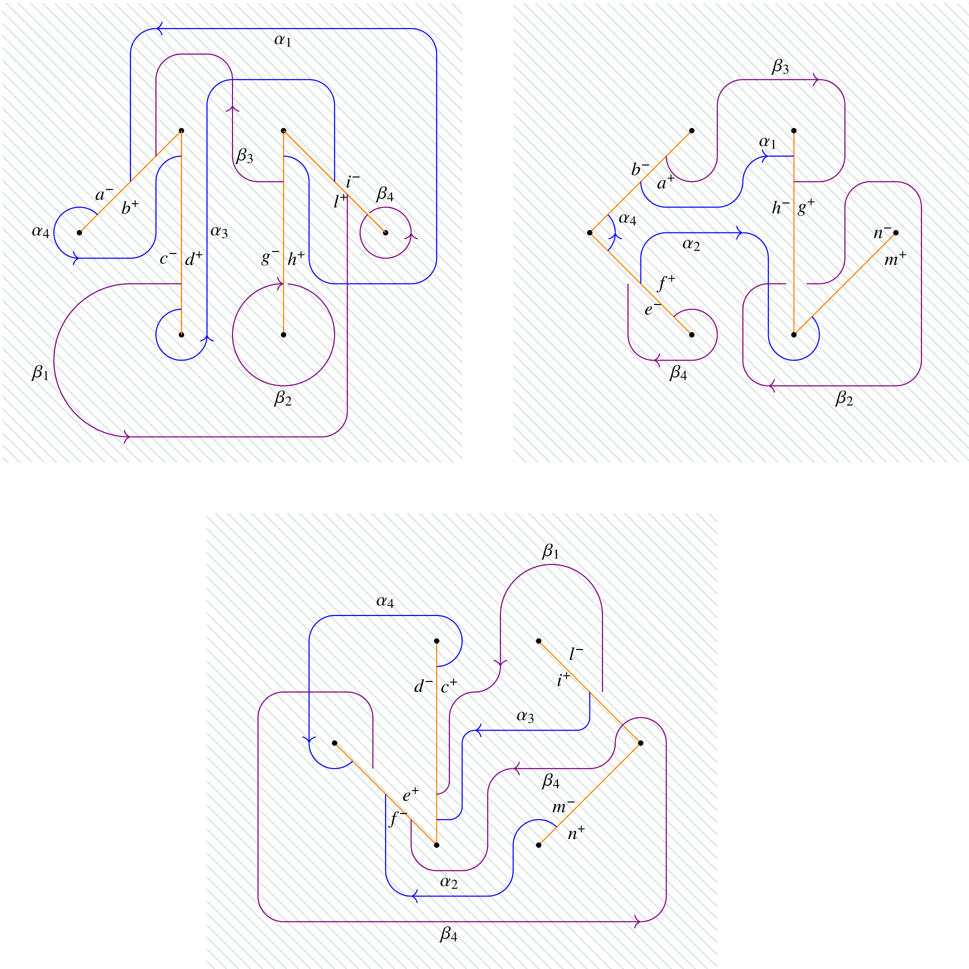

Figure 55. *Realization of a translation surface with poles and even parity in $\mathcal{H}_4(2^6; -2^3)$. Here, the monodromy pattern is equal to $k = 3$. Unlike the notation used so far, here the edges $a^+, \ldots, n^+$ are identified with $a^-, \ldots, n^-$, respectively. Moreover, the indices of the curves α_i and β_i are according to Table 2.*

Table 2. *Indices of curves in Figure 55.*

i	$\mathrm{Ind}(\alpha_i)$	$\mathrm{Ind}(\beta_i)$
1	1	1
2	0	2
3	0	0
4	2	3

Below we depict the realization of translation surfaces with trivial periods in both components of $\mathcal{TR}_4 \subset \mathcal{H}_4(2^6; -2^3)$. A direct computation of the Arf-invariant shows that in both cases the parity (see Section §2.2.3) is even as expected.

These pictures depict the realization of translation surfaces with trivial periods in both components of $\mathcal{TR}_4 \subset \mathcal{H}_4(2^6; -2^3)$. In both cases, by using formula (23), it is possible to see that both structures have even parity and belong to different connected components of $\mathcal{TR}_4$.

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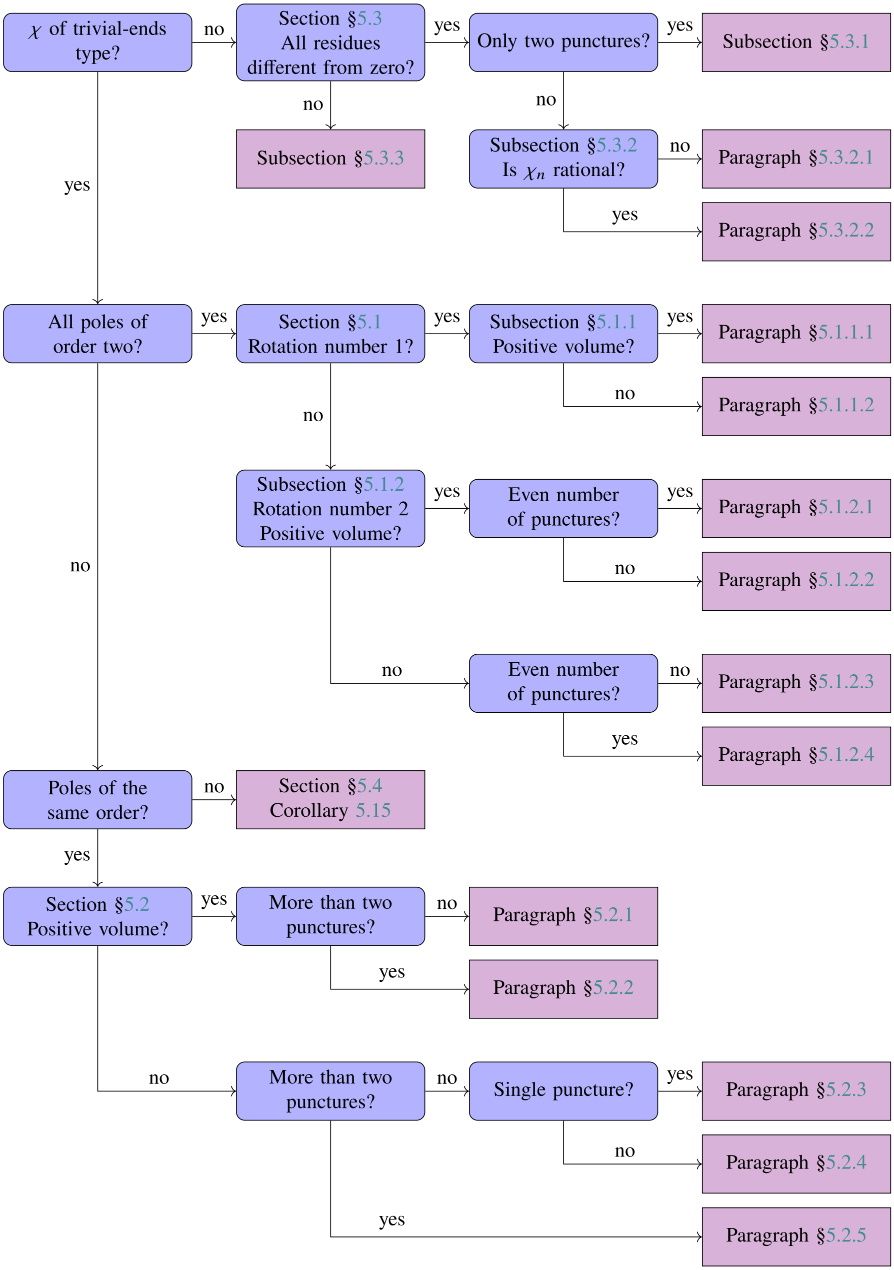
Table 3. *Flowchart for the proof of Theorem A for genus-one differentials with a single zero in Section §5.*



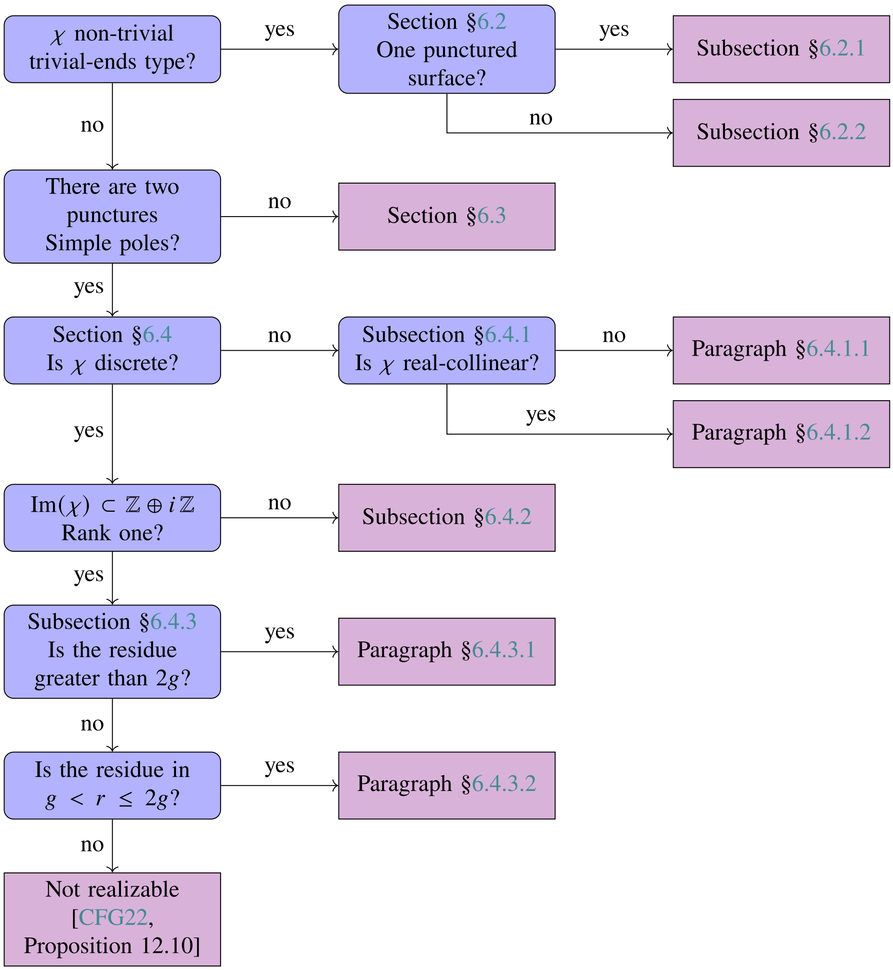
A. Proof strategy flowcharts

The proof of Theorem A is long, and it involves a case-by-case discussion according to the diagrams below.

A.1. Genus-one surfaces

Let $\chi\colon \mathrm{H}_1(S_{1,n},\mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation. Suppose we want to realize χ in a connected component of the stratum $\mathcal{H}_1(m; -p_1, \ldots, -p_n)$, namely as the holonomy of some translation surface with a single

Table 4. *Flowchart for the proof of Theorem A for genus-g differentials in Section §6.*



zero of maximal order. In Section §5, we provide a way for realizing χ as the holonomy of some translation surface with poles in a prescribed stratum with prescribed rotation number. Table 3 provides an outline of the strategy adopted. The multiple zero case is handled by Corollary 5.17.

A.2. Hyperelliptic translation surfaces

Let $\chi\colon \mathrm{H}_1(S_{g,n}, \mathbb{Z}) \longrightarrow \mathbb{C}$ be a representation. We want to realize χ as the holonomy of some translation surfaces with poles and a hyperelliptic involution; see Definition 2.14. Table 4 provides an outline of the strategy adopted.

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Competing interest. The authors have no competing interest to declare.

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