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Dropping plates

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A friend of mine [1] mentioned a problem to me, which he was told had an interesting solution involving an unexpected square root. I have not seen this problem described elsewhere, so I have carried out my own analysis, which I will present here. In fact the solution involves not only square roots, but also higher roots … and a logarithm.

The problem

You have available a limited number of identical objects, which will break if you drop them from above a certain height. You live in a block of flats and you want to establish with certainty the highest floor you can drop the objects from without them breaking. What is the most effective strategy to establish the answer, i.e. the strategy that minimises the expected number of trials required? Any objects that do not break when you drop them will be undamaged and can be reused.

In the version my friend was told, the objects were plates, and my guess is that this problem was dreamed up by a group of inebriated students eating by a window at a party – possibly influenced by jokes about flying saucers. If, like me, you think this problem is trivial because a defenestrated china plate would *always* break, you can substitute a more durable object of your choice, or you can assume that the ground below has a soft surface that cushions the impact to some extent.

Note that the strategy used must establish the answer with *certainty* in all cases.

Example

As an example, suppose I live in a three-storey block with the floors numbered as in Figure 1. (Numbering the ground floor as floor 1, rather than floor 0, avoids negative numbers entering the equations.) I also have a supply of objects to drop. We are assuming that the highest floor they can survive from is the same for each of our objects, but we do not know which that floor is.

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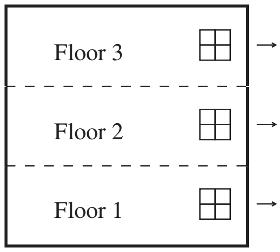

FIGURE 1: Dropping objects from a 3-storey block

Here there are four possibilities for the survivability of the objects:

(1) An object will break even when dropped from Floor 1. Answer = 0.

(2) An object can survive a drop from Floor 1, but no higher. Answer = 1.

(3) An object can survive a drop from Floor 2, but no higher. Answer = 2.

(4) An object can survive a drop from Floor 3 (and possibly higher). Answer = 3.

Note that, although the floors here are numbered from 1 to 3, the outcomes can take values from 0 to 3.

With just one object, the only sure strategy is to work up floor by floor from the bottom. If we were to skip a floor and the object broke, we would have no more objects to establish the precise answer.

To calculate the expected number of throws required with this strategy, I will assume that each of the four outcomes above is equally likely. This assumption has been chosen purely for simplicity. In reality, if the objects are quite delicate, our prior belief would put much higher probabilities on lower floors, whereas if the objects are quite robust, they may well be able to survive a drop from a much higher floor.

With this assumption, each of the four possibilities above has probability $\frac{1}{4}$ and the expected number of trials required to determine the answer is $\frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 3 + \frac{1}{4} \times 3 = 2\frac{1}{4}$.

Mathematical model

To set up a more general model I will use the following notation:

n = total number of floors in the building

r = total number of objects we have available

m = a candidate floor to try next

$\hat{m}$ = the optimal floor to try next

$E(n, r)$ = expected number of trials required using the optimal strategy.

As well as assuming that each of the $n + 1$ possibilities is equally likely, I will also assume that the objects are not weakened by previous trials and that other 'obvious' simplifying assumptions can be made.

One object

As we saw in the example above, when $r = 1$ the only sure strategy is to work up floor by floor from the bottom and the expected number of trials is then

$$\begin{aligned} E(n,\ 1) &= \frac{1}{n+1} \times \left(1 + 2 + \ldots + (n-1) + n + n\right) \\ &= \frac{1}{n+1} \times \left(\{1 + 2 + \ldots + n + (n-1)\} - 1\right) \\ &= \frac{1}{n+1} \times \left(\tfrac{1}{2}(n+1)(n+2) - 1\right) = \frac{n(n+3)}{2(n+1)}. \qquad (1) \end{aligned}$$

Asymptotically, as $n \to \infty$, this becomes $E(n,\ 1) \sim \frac{1}{2}n$.

Several objects

With more than one object we can afford to adopt a more risky strategy until we are down to the last object. Here the optimal strategy is less obvious. Trying higher floors to find the cut-off point more quickly would risk using up too many objects, but playing safe by selecting low floors and moving up slowly would be more time-consuming. To find the answer in this case, we can use the following iterative argument.

With n floors in the full building we can select any floor $m = 1, 2, \ldots, n$ for the next trial. Depending on the outcome, this will reduce the problem to one of pinning down the highest survivable floor in either the top 'half' or the bottom 'half' of the building (i.e. above or below floor m). We can then say that $E(n, r)$ equals 1 (the trial we've just used up) plus the expected number of trials starting from the new 'reduced' position.

To make sure we get the precise details correct, let's use an example with $n = 10$ floors and $r = 3$ objects, and let's suppose that we try floor $m = 4$ first. This will result in one of two outcomes, as illustrated in Figure 2.

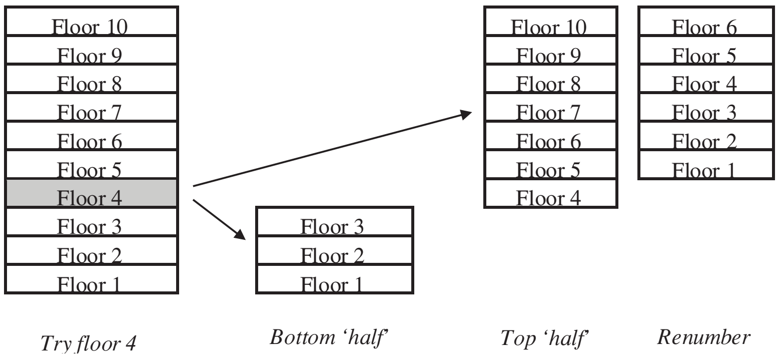

FIGURE 2: Iterative argument for 10 floors if we try Floor 4

Outcome 1: If the object breaks, we have limited the highest 'survivable' floor to 0, 1, 2 or 3, and we can now apply the optimal strategy for a building with 3 floors using the $3 - 1 = 2$ remaining objects, which will take an expected number of trials of $E(3,\ 2)$, i.e. $E(m - 1,\ r - 1)$.

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Outcome 2: If, on the other hand, the object survives the fall, we have limited the highest 'survivable' floor to one from 4 to 10, a total of 7 possible floors. If we subtract 4, we can renumber these floors as 1 to 6 (with 0 corresponding to survival from the original Floor 4). We then need to apply the optimal strategy for the 3 remaining objects and 6 floors, which will take an expected number of trials of $E(6,\ 3)$, i.e. $E(n - m, r)$.

With 10 floors there are 11 possible answers (0, 1, 2, … , 10). So each of the original 10 floors has a probability of $\frac{1}{11}$ of being the correct answer, which means that outcome 1 had a probability of $\frac{4}{11}$ and outcome had a probability of $\frac{7}{11}$.

So, for this choice of m, we would require 1 trial plus an expected additional number of $\frac{4}{11}E(4,\ 2) + \frac{7}{11}E(6,\ 3)$. For the optimal strategy, we need to choose the value of m so as to minimise the overall expected number. So we need to apply a minimum over all the possible values of m running from 1 to 10.

This method leads us to the following general iterative equation

$$E(n,r) = 1 + \min_{1 \leqslant m \leqslant n}\left\{\frac{m}{n+1}E(m-1,r-1) + \left(1 - \frac{m}{n+1}\right)E(n-m,r)\right\}. \quad (2)$$

Equation (2) enables us to calculate the values of $E(n, r)$ for $n = 1, 2, \ldots$ from the corresponding values of $E(n, r - 1)$. The first and last terms in the minimum require values for $E(0, r - 1)$ and $E(0, r)$, which both need to be interpreted as 0 to make the equations correct.

We can reduce the relationship in (2) to a simpler form if we multiply through by $n + 1$ and define $G(n, r) = (n + 1)E(n, r)$ to obtain

$$(n+1)E(n,r) = (n+1) + \min_{1 \leqslant m \leqslant n}\left\{mE(m-1,r-1) + (n-m+1)E(n-m,r)\right\}$$

$$G(n,r) = (n+1) + \min_{1 \leqslant m \leqslant n}\left\{G(m-1,r-1) + G(n-m,r)\right\}, n \geqslant 1, r \geqslant 2. \quad (3)$$

For $r = 1$ we already know from (1) that $G(n,\ 1) = \frac{1}{2}n(n + 3)$. We can now move on to $r = 2$ and apply (3) iteratively to calculate values of $G(n,\ 2)$ for each value of n, and so on. As above, we apply the boundary conditions $G(0, r - 1) = G(0, r) = 0$. Note that all the values of the function G will be integers, since the right-hand side of (3) just involves adding integers.

To find the expected number of trials $E(n, r)$ we can then just divide by $n + 1$.

These functions can be calculated very neatly using the recursive functions feature in *Mathematica* by entering the following commands. Note that it is important here to include the fourth line. Although this line produces no output, it forces *Mathematica* to precalculate all the values in sequence and to store the results, which we can then print out in the final line. Without the fourth line, we would be attempting to evaluate a very large number of heavily nested calculations.

```
g[0,r_]:= 0; g[n_,1]:= n(n+3)/2
g[n_,r_]:= g[n,r] = n+1+Min[Table[g[m-1,r-1]+ g[n-m,r],{m,1,n}]]
e[n_,r_]:= g[n,r]/(n+1); infinity = 10;
Table[Table[g[n,r],{n,1,200}],{r,2,infinity}]; ←Do not omit!
Table[Table[e[n,r],{r,{1,2,3,4,infinity}}],
 {n,{1,2,3,4,5,6,7,8,9,10,100,163,200}}]//TableForm
```

This calculation produces the values shown in Table 1. (The three dots '…' indicate values that repeat across the rows.) I've included $n = 163$ because the building with the highest number of storeys is currently the Burj Khalifa in Dubai, which has 163 floors – coincidentally, the same number of $e^{\pi\sqrt{163}}$ fame!

$E(n, r)$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r \to \infty$
$n = 1$	1.00	…	…	…	…	1.00
$n = 2$	1.67	…	…	…	…	1.67
$n = 3$	2.25	2.00	…	…	…	2.00
$n = 4$	2.80	2.40	…	…	…	2.40
$n = 5$	3.33	2.67	…	…	…	2.67
$n = 6$	3.86	2.86	…	…	…	2.86
$n = 7$	4.38	3.13	3.00	…	…	3.00
$n = 8$	4.89	3.33	3.22	…	…	3.22
$n = 9$	5.40	3.50	3.40	…	…	3.40
$n = 10$	5.91	3.64	3.55	…	…	3.55
$n = 100$	50.99	10.35	7.47	6.83	6.74	6.73
$n = 163$	82.49	12.98	8.65	7.67	7.49	7.44
$n = 200$	101.00	14.28	9.20	8.10	7.77	7.73

TABLE 1: Expected number of trials, $E(n, r)$

As we would expect, as we increase the number of floors n, the expected number of trials we need increases.

As we increase the number of objects r, the expected number of trials we need decreases, since we can afford to break a few more objects along the way. But once the number of available objects reaches n, there is no advantage in having any more, since we could if necessary test every floor without running out of objects. The expected number of trials is then equal to the limiting value shown in the right-hand column. We will see later that we can actually reduce $r \to \infty$ to $r \geqslant \log_2(n + 1)$, as suggested by the entries in the rows for $n = 1$, $n = 3$ and $n = 7$.

The optimal strategy

To determine the optimal strategy, we need to identify the value of $\hat{m}$, the value of m that generates the minimum value in (3). I have used an Excel spreadsheet to examine the individual components of each calculation, which gives the results in Table 2. This shows that for most values of n there is a range of consecutive values of m that all give the same minimum value. Any of these values would offer an equally good strategy to follow. For example, if we are in a 10-storey building with 3 objects, we can optimally try any floor from 4 to 7 first.

$\hat{m}$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r \to \infty$
$n = 1$	1	…	…	…	…	1
$n = 2$	1	1–2	…	…	…	1–2
$n = 3$	1	2	…	…	…	2
$n = 4$	1	2–3	…	…	…	2–3
$n = 5$	1	2–3	2–4	…	…	2–4
$n = 6$	1	3	3–4	…	…	3–4
$n = 7$	1	3	4	…	…	4
$n = 8$	1	3–4	4–5	…	…	4–5
$n = 9$	1	3–4	4–6	…	…	4–6
$n = 10$	1	4	4–7	…	…	4–7
$n = 100$	1	13–14	29–37	42–44	38–57	37–64
$n = 163$	1	17–18	37–46	64–65	57–99	64–100
$n = 200$	1	19–20	46–56	64–93	81–99	73–128

TABLE 2: The optimal floor to try next, $\hat{m}$

Exact formulae

We have already established that when $r = 1$ (when we have just one object), $\hat{m} = 1$ and $E(n, 1) = \dfrac{n(n + 3)}{2(n + 1)}$, but how does this extend to higher values of r?

Two objects ($r = 2$)

Now consider when we have $r = 2$ objects. Figure 3 below shows graphically the results from Table 2. For example, when n is in the range

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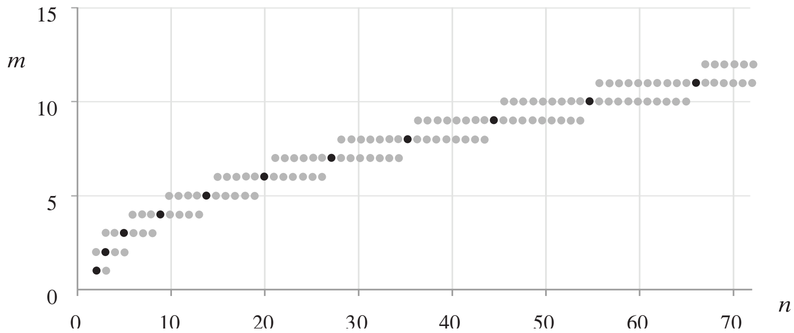
$4 \leqslant n \leqslant 9$, $\hat{m} = 3$ is one of the optimal floors to try last. For most values of n there are two possible values for m, but for some (indicated by the bold dots), e.g. $n = 10$, $\hat{m} = 4$, there is a unique value of $\hat{m}$ that gives the minimum expectation. These unique values are tabulated in Table 3 below.

These values of n are easily recognisable as the triangular numbers and, perhaps not quite so obvious, the values of $G(n, 2)$ are double the tetrahedral numbers [2]. Assuming this pattern continues for higher values of n, the entries for n and $G(n, 2)$ in the i th column of Table 3 will be

$$n = \tfrac{1}{2}i(i + 1) \text{ and } G(n, 2) = \tfrac{1}{3}i(i + 1)(i + 2). \tag{4}$$

(In this case the column number i is the same as $\hat{m}$, but we will see that this is not the case when we consider higher values of r.)



$\hat{n}$	1	3	6	10	15	21	28	36
$\hat{m}$	1	2	3	4	5	6	7	8
$G(n, 2)$	2	8	20	40	70	112	168	240
$E(n, 2)$	1.00	2.00	2.86	3.64	4.38	5.09	5.78	6.49

FIGURE 3: Values of m when $r = 2$ (with unique values shown as black dots)

The equation for n in (4) can be inverted to give $i = \frac{1}{2}\left(\sqrt{8n+1} - 1\right)$, so that

$$G(n,2) = n\left(\tfrac{1}{3}(\sqrt{8n+1} + 1\right) \text{and } E(n,2) = \frac{n}{n+2}\left(\tfrac{1}{3}\sqrt{8n+1} + 1\right). \tag{5}$$

We can use induction to show that these formulae give *upper bounds* for $G(n, 2)$ and $E(n, 2)$ for this subset of 'unique' values. To do this, we start by assuming that $G\left(\frac{1}{2}i(i + 1), 2\right) \leqslant \frac{1}{3}i(i + 1)(i + 2)$ for $1 \leqslant i \leqslant k - 1$. Then, using (3), we can proceed as follows, where the inequality step follows because the minimum of a set of values can never exceed the value of any one individual element.

$$G\left(\tfrac{1}{2}k(k+1)\right) = \tfrac{1}{2}k(k+1) + 1 \min_{1 \leqslant m \leqslant \frac{1}{2}k(k+1)} \left\{G(m-1,1) + G\left(\tfrac{1}{2}k(k+1) - m, 2\right)\right\}$$

$$\leqslant \tfrac{1}{2}k(k+1) + 1 + \left\{G(m-1,1) + G\left(\tfrac{1}{2}k(k+1) - m, 2\right)\right\}\Big|_{m=k}$$

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$$= \tfrac{1}{2}k(k+1) + 1 + \left\{G(k-1,1) + G\left(\tfrac{1}{2}(k-1)k, 2\right)\right\}$$
$$= \tfrac{1}{2}k(k+1) + 1 + \left\{\tfrac{1}{2}(k-1)(k+2) + G\left(\tfrac{1}{2}(k-1)k, 2\right)\right\}$$
$$= k(k+1) + \tfrac{1}{3}(k-1)k(k+1)$$
$$= \tfrac{1}{3}k(k+1)(k+2).$$

This leads us to the correct inequality for k, and so proves the upper bound. To prove that this formula gives the exact value, we would also need to establish that the term corresponding to $m = k$ gives the minimum value, which is not so easy. An examination of my spreadsheet calculations shows that there is a clear pattern and that this is indeed true. However, one thing that I have learned from this project is that proving a pattern that is 'obvious' from a spreadsheet can be far from trivial.

My spreadsheet calculations also appear to show that, when $\frac{1}{2}\left(\sqrt{8n+1} - 1\right)$ is not an integer, the two possible values of $\hat{m}$ are the two neighbouring integers, each of which gives the same value in the minimum calculation. I will give exact formulae below for these values of $\hat{m}$ as part of a more general result.

When $r = 2$, for large values of n that correspond to a unique value of $\hat{m}$, we have

$$\hat{m} = i = \tfrac{1}{2}\left(\sqrt{8n+1} - 1\right) \approx \sqrt{2n} \text{ and } E(n, 2) \approx \frac{2^{3/2}}{3}n^{1/2} = 0.94\sqrt{n}.$$

This confirms that, with 2 objects, and a tall building, the optimal strategy does indeed involve a square root.

Three or more objects ($r \geqslant 3$)

Moving to the case where $r = 3$, and using a similar approach, we find that we get unique values of $\hat{m}$ for the values shown in bold in Figure 4.

In this case the values of $\hat{m}$ are not sequential, i.e. $\hat{m}$ is not equal to the column number as before. A little experimentation suggests that the entries in the i th column of the table within Figure 4 are in fact

$$n = \binom{i}{1} + \binom{i}{2} + \binom{i}{3} = \tfrac{1}{6}i\left(i^2 + 5\right), \hat{m} = \binom{i-1}{0} + \binom{i-1}{1} + \binom{i-1}{2} \tag{6}$$

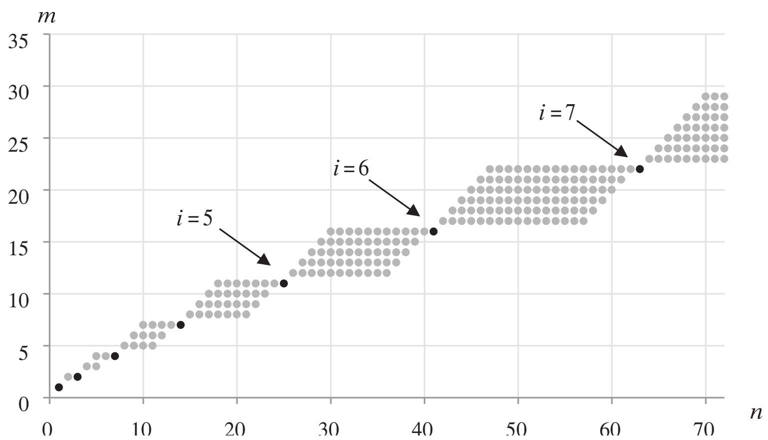
and

$$G(n, 3) = 2\binom{i}{1} + 4\binom{i}{2} + 6\binom{i}{3} + 3\binom{i}{4}. \tag{7}$$

I have not proved these formulae, but I have checked numerically that they work correctly for $n \leqslant 200$.

For higher values of r, my spreadsheet shows similar patterns and suggests the following general formulae for the unique values of $\hat{m}$:

$$n = \sum_{k=1}^{r}\binom{i}{k}, \hat{m} = \sum_{k=0}^{r-1}\binom{i-1}{k}, G(n, r) = \sum_{k=1}^{r}2k\binom{i}{k} + r\binom{i}{r+1}. \tag{8}$$



i	1	2	3	4	5	6	7	8
$\hat{n}$	1	3	7	14	25	41	63	92
$\hat{m}$	1	2	4	7	11	16	22	29
$G(n, 3)$	2	8	24	59	125	237	413	674
$E(n, 3)$	1.00	2.00	3.00	3.93	4.81	5.64	6.45	7.25

FIGURE 4: Values of $\hat{m}$ when $r = 3$ (with unique values shown as black dots)

Ranges for $\hat{m}$

A more detailed examination of the spreadsheet calculations shows that, as we progress through the values of n, the upper and lower bounds of the range of optimal values $\hat{m}$ change gradually, increasing by 1 when the value of n takes certain forms. For example, when $r = 3$, as in Figure 4, the upper bound increases whenever n is of the form $\binom{x+1}{1} + \binom{x+1}{2} + \binom{x+1}{3} + y$, where x and y are integers with $x \geqslant 0$ and $1 \leqslant y \leqslant x + 1$. This leads to the following formulae for $\hat{m}_U$ and $\hat{m}_L$, the upper and lower bounds of $\hat{m}$. Here $I(\dots)$ denotes an indicator function taking the value 1 if its argument is true and 0 if it is false.

$$\hat{m}_U(n, r) = 1 + \sum_{x=0}^{\infty} \sum_{y=1}^{\sum_{p=0}^{r-2}\binom{x}{p}} I\left(\sum_{q=1}^{r}\binom{x+1}{q} + y \leqslant n\right) \qquad (9)$$

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$$\hat{m}_L(n, r) = 1 + \sum_{x=0}^{\infty} \sum_{y=1}^{\sum_{p=0}^{r-2} \binom{x}{p}} I\left(\sum_{q=0}^{r} \binom{x+1}{q} - y \leqslant n\right) \qquad (10)$$

$$G(n, r) = n + \sum_{x=0}^{\infty} \sum_{k=1}^{n} I\left(\sum_{q=0}^{r} \binom{x}{q} \leqslant k\right). \qquad (11)$$

These formulae involve the partial sums of the entries in the rows of Pascal's triangle.

By counting the precise number of terms in the inner sums, we find that equations (9), (10) and (11) can be written in the following equivalent forms, which allow the values of $\hat{m}_U$, $\hat{m}_L$, G and E to be calculated much more efficiently:

$$\hat{m}_U(n, r) = 1 + \sum_{x=0}^{\infty} \max\left[\min\left(n - \sum_{q=1}^{r} \binom{x+1}{q}, \sum_{p=0}^{r-2} \binom{x}{p}\right), 0\right] \qquad (12)$$

$$\hat{m}_L(n, r) = 1 + \sum_{x=0}^{\infty} \max\left[\min\left(n + 1 - \sum_{q=0}^{r} \binom{x+2}{q}, 0\right) + \sum_{p=0}^{r-2} \binom{x}{p}, 0\right] \qquad (13)$$

$$G(n, r) = n + \sum_{x=0}^{\infty} \max\left(n + 1 - \sum_{q=0}^{r} \binom{x}{q}, 0\right). \qquad (14)$$

For example, the lower bound $\hat{m}_L(n, r)$ can now be calculated almost instantaneously in *Mathematica* using the following function definition:

```
mLower[n_,r_]:=
Block[{t = 1,s = 0,x = 0},While[t>0,{t = Sum[Binomial[x,p],{p,0,r−2}]+
Min[n−Sum[Binomial[x+2,q],{q,0,r}]+1,0],s = s+Max[t,0]};x++];1+s]
```

More efficient *Mathematica* formula for $\hat{m}_L(n, r)$

Asymptotic results and approximations

The limiting values of $\hat{m}$ and E for large n and large r are interesting.

Fixed r, large n

When $r \geqslant 3$, the equation for n in (8) cannot easily be inverted to find i, which would allow us to express $\hat{m}$ directly in terms of n. However, we can use the leading terms to deduce some asymptotic relationships, e.g. when $r = 3$, equations (6) and (7) give $n \sim \frac{1}{6}i^3$, $\hat{m} \sim \frac{1}{2}i^2$ and $G(n, 3) \sim \frac{1}{8}i^4$ so that $\hat{m} \sim \frac{6^{2/2}}{2}n^{2/3} = 1.65n^{2/3}$ and $E(n, 3) \sim \frac{6^{4/3}}{8}n^{1/3} = 1.36n^{1/3}$. So, with 3 objects, the optimal strategy involves (approximately) a cube root.

Using the same approach (again based on the values of n where the value of $\hat{m}$ is unique), we can deduce asymptotic relationships for general values of r. This gives $\hat{m} - \dfrac{r}{(r!)^{1/r}} n^{1-1/r}$ and $E(n, r) \sim \dfrac{r(r!)^{1/r}}{r+1} n^{1/r}$ for large n.

	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 10$	$r \to \infty$
$\hat{m}$	$\sim 1.41n^{1/2}$	$\sim 1.65n^{2/3}$	$\sim 1.81n^{3/4}$	$\sim 1.92n^{4/5}$	$\sim 2.21n^{9/10}$	en

TABLE 3: Asymptotic formulae for m for large n

However, these formulae are unsuitable for approximating the values of $\hat{m}$ as they significantly overestimate the true values unless n is *very* large. For example, when $r = 10$, the asymptotic values of m are actually *greater than n* unless $n > 2779$ and so cannot represent the optimal strategy as they exceed n itself. The situation is even worse when $r \to \infty$, since en is always greater than n.

Although the asymptotic formulae in Table 3 cannot be used to give useful approximations for smaller values of n and r, we can use this same form of relationship to find formulae of the form $\hat{m} = \alpha n^{\beta}$ that give accurate values for $\hat{m}$ for smaller values of n. For example, the formulae shown in Table 4, when rounded to the nearest integer, give optimal values of $\hat{m}$ for all values of $n \leqslant 200$. As a check, the values of $\hat{m}$ illustrated here for the case $n = 163$ can all be seen to fall within the optimal ranges shown in Table 2.

	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 10$	$r = \infty$
$\hat{m}$	$1.38n^{0.5}$	$1.07n^{0.73}$	$0.85n^{0.85}$	$0.67n^{0.93}$	$0.53n^{0.99}$	$0.5n$
$n = 163$	18	44	65	76	82	82

TABLE 4: 'Accurate approximations' for $\hat{m}$ when $1 \leqslant n \leqslant 200$

Fixed n, large r

When $r \to \infty$ (or, more specifically, when $r \geqslant n$), we would effectively have an unlimited number of objects available. In this case $\hat{m}_U$ is always equal to $(n + 1) - \hat{m}_L$ since we could equally apply any given strategy counting the floors from the top of the building. This can also be seen from the symmetry of (3) if we use the substitution $p = (n + 1) - m$ to evaluate the minimum in the reverse order:

$$G(n, \infty) = (n + 1) + \min_{1 \leqslant m \leqslant n} \left\{ G(m - 1, \infty) + G(n - m, \infty) \right\}$$

$$\Leftrightarrow G(n, \infty) = (n + 1) + \min_{1 \leqslant p \leqslant n} \left\{ G(m - p, \infty) + G(p - 1, \infty) \right\}.$$

This means that, when $r \geqslant n$, the midpoint of the range $(\hat{m}_L, \hat{m}_U)$ is always $\frac{1}{2}(n + 1)$. This is confirmed by (8), since we now have

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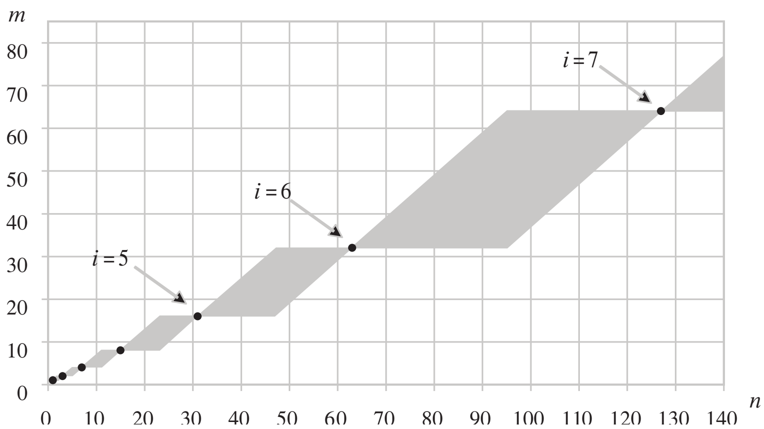
$n = \sum_{k=1}^{i} \binom{i}{k} = 2^i - 1$ and $\hat{m} = \sum_{k=0}^{i-1} \binom{i-j}{k} = 2^{i-1}$, so that $\hat{m} = \frac{1}{2}(n+1)$. Here $G(n, \infty) = i \,.\, 2^i$ and $E(n, \infty) = i$.

The values of $\hat{m}$ when $r \to \infty$, i.e. with an unlimited number of objects, are plotted in Figure 5. As we might expect, the optimal strategy converges to a bisection technique where we go to the midpoint of the remaining section each time. However, there is actually a fair amount of flexibility in the precise point where we make each bisection. The optimal strategy does not need to bisect exactly halfway.

In terms of the number of objects required to correctly identify the value of n, the 'unique value' points represent worst-case scenarios. With these values, after each object is dropped we move directly from i to $i - 1$ and we require a total of i objects. Since $n = 2^i - 1$ at these unique points, we see that for any value of n, the maximum number of objects required is $\log_2(n + 1)$. This is the same maximum as we find when applying a binary search to an ordered list [3].



i	1	2	3	4	5	6	7	8
n	1	3	7	15	31	63	127	255
$\hat{m}$	1	2	4	8	16	32	64	128
$G(n.\infty)$	2	8	24	64	160	384	896	2048
$E(n.\infty)$	1	2	3	4	5	6	7	8

FIGURE 5: Values of $\hat{m}$ when $r \to \infty$ (with unique values shown as black dots)

Other possible areas of application

Although the context of this problem seems frivolous, I can imagine that the results derived here could be relevant to other situations where destructive testing is required and the 'stress' applied comes in discrete

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units. Other analogous problems would be:

- How many people or vehicles can a structure support without breaking?
- How many units of a drug can be given to a patient without causing harm?
- How many adverts can you post before a customer stops looking at them?

References

1. Gary S. Heslop, unpublished communication (September 2010).
2. Eric W. Weisstein, *Concise encyclopedia of mathematics*, Chapman & Hall / CRC 2003. See the entry for Figurate Numbers.
3. Donald E. Knuth, *The Art of Computer Programming, Volume* 3: *Sorting and Searching*, Addison-Wesley (1998).

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The answers to the *Nemo* page from March 2022 on polygons were:

1.	John Buchan	Mr Standfast	Chapter 12
2.	Charles Dickens	The Pickwick Papers	Chapter 53
3.	GK Chesterton	The Man who was Thursday	Chapter 7
4.	Philip Sidney	Our Lady of May	
5.	Stella Gibbons	Cold Comfort Farm	Chapter III
6.	Thomas Hardy	Tess of the d'Urbervilles	Chapter 52

Congratulations to Bryan Thwaites on tracking all of these down. In this section, we celebrate applied mathematics. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd September 2022.

1. Statistics were becoming dry to him, and love was very sweet. Statistics, he thought, might be made as enchanting as ever, if only they could be mingled with love.

2. "Probability is the bane of the age", said M _ , now warming up. "Every Tom, Dick or Harry thinks he knows what is probable."

Continued on page 219.