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CONDITIONAL LOCAL NONDETERMINISM AND HAUSDORFF MEASURE OF LEVEL SETS

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ABSTRACT. Let X be a real stochastic process. We localize S. M. Berman's formulation on the local nondeterminism of X to a fixed level. With this localized idea, we prove that, for large classes of Gaussian and Markov X, at each x the level set $X(t, \omega) = x$ has infinite Hausdorff ϕ-measure (ϕ is certain measure function) for ω in a set of positive probability.

1. Let $X(t), 0 \leq t \leq 1$, be a real stochastic process defined on some probability space $(\Omega, \mathcal{B}, P)$. S. M. Berman formulated the local nondeterminism of X, firstly in the Gaussian case [1] then in the general case [2]. The local nondeterminism will assert the smoothness of the local time, which in turn will assert the irregularity of the path $X(\cdot)$. As an interesting application, Berman ([2, Theorem 6.2 and Examples 7. 1–2], [3, Theorems 4, 6]) proved that: for large classes of Gaussian and Markov X, the "progressive level set" $\{s : X(s, \omega) = X(t, \omega)\}$ has infinite Hausdorff ϕ-measure (ϕ is certain measure function) for almost all t, for a.s. ω. In this note, we shall present the corresponding ones of these results for level sets

$$Z(x, \omega) = \{t \in [0, 1] : X(t, \omega) = x\}.$$

To state our results, we first recall the definition of Hausdorff ϕ-measures. Let $\phi(t)$, $0 \leq t \leq 1$, be an increasing right continuous function such that $\phi(0+) = 0$. Such a function is called a *measure function*. We suppose that ϕ satisfies the additional condition:

$$\phi(t)/t \text{ is decreasing and } \phi(t)/t \longrightarrow \infty \text{ as } t \downarrow 0. \tag{1.1}$$

Now, define the Hausdorff ϕ-measure of a subset E of $[0, 1]$ by

$$\phi(E) = \lim_{\delta \downarrow 0} inf\{ \textstyle\sum \phi(\text{ length of } I_i) : E \subset \cup I_i,$$
$$I_i\text{'s are subintervals of } [0,1] \text{ whose lengths are at most } \delta\}.$$

THEOREM 1.1. *Let $X(t), 0 \leq t \leq 1$, be a real continuous Gaussian process with mean* 0, *and define*

$$b^2(t) = \inf_{|s-s'| \geq t} E(X(s) - X(s'))^2. \tag{1.2}$$

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Suppose that the process is locally nondeterministic (see Berman [1]). If, for some $\varepsilon > 0$ and some integer $m \geq 2$,

$$\sum_{n=0}^{\infty} \left(\frac{\int_0^{2^{-n}} (b(s))^{-1}\, ds}{[\phi(2^{-n})]^{1+\varepsilon}} \right)^m < \infty, \tag{1.3}$$

then, at each x, $P\{\omega : \phi(Z(x,\omega)) = \infty\} > 0$.

Theorem 1.1 is derived from Theorem 2.1 in Section 2; the latter is a result for general stochastic processes. Theorem 2.1 is also applicable to Markov processes. Let $X(t)$, $0 \leq t \leq 1$, be a real time-homogenous Markov process, having a transition density function $p(t; x, y)$ which is continous in (x, y). Assume also that the paths $X(\cdot)$ are right continuous and have left limits everywhere and that

$$\lim_{t \downarrow 0} \sup_{x,y \in K} \int_0^t p(s; x, y)\, ds = 0$$

for all compact K.

THEOREM 1.2. *Let X be given above and $X(0) = x_0$. At a level x, if*

$$\int_0^1 p(t; x_0, x)\, dt > 0, \qquad \int_0^1 p(t; x, x)\, dt > 0,$$

and for some $\varepsilon > 0$ and some integer $m \geq 2$

$$\sum_{n=0}^{\infty} \left(\frac{\int_0^{2^{-n}} p(t; x, x)\, dt}{[\phi(2^{-n})]^{1+\varepsilon}} \right)^m < \infty,$$

then, $P\{\omega : \phi(Z(x,\omega)) = \infty\} > 0$.

The processes and the measure function considered above are those in Berman [2,3]. Note that local time (as an occupation density) arguments in [2,3] fails to have meaning when we consider a fixed level. Our novel idea here is to "localize" Berman's formulation on the local nondeterminism in [2] and the ingredients of the arguments in [2,3] to the (fixed) level x. This idea is developed in the next two sections. In Section 2, we state Theorem 2.1, which is a result for the Hausdorff ϕ-measure of level sets of general stochastic processes and corresponds to Berman's [2, Theorem 6.2] and [3, Theorem 2]. Theorems 1.1 and 1.2 are direct consequences of it. For the clearness of the context, we separate the proof of Theorem 2.1 to Section 3. In final Section 4, we remark a question on the comparison between Theorem 1.2 and the viewpoint from probabilistic potential theory (due to Professor E. A. Perkins).

2. Let $X(t)$, $0 \leq t \leq 1$, be a real measurable stochastic process on some probability space $(\Omega, \mathcal{B}, P)$. For positive integer k and $0 < t_1 < t_2 < \cdots < t_k \leq 1$, let $p(t_1, \ldots, t_k; y_1, \ldots, y_k)$ denote the joint density function of $X(t_1), \ldots, X(t_k)$ at $y_1, \ldots, y_k$. Also, let $\rho(t_1, \ldots, t_k; y_1, \ldots, y_{k-1} | t_1; x)$ denote the conditional joint density function of

$X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})$ at $y_1, \ldots, y_{k-1}$; given $X(t_1) = x$. Fix a level x, we assume that

(i) There exists an open neighborhood (a, b) of x such that $p(t_1, \ldots, t_k; y_1, \ldots, y_k)$ exists for all $y_1, \ldots, y_k \in (a, b)$ and that the integral function q_k defined by

$$q_k(y_1, \ldots, y_k) = \int_0^1 \cdots \int_0^1 p(t_1, \ldots, t_k; y_1, \ldots, y_k)\, dt_1 \ldots dt_k$$

is finite and continuous in $(y_1, \ldots, y_k) \in (a, b)^k$.

(ii) There exists a nonnegative integrable function $g(t)$ on $[0, 1]$ and a sequence of positive real numbers $c_1, c_2, \ldots$, such that

$$\rho(t_1, \ldots, t_k; \underbrace{0, \ldots, 0}_{(k-1)\text{ terms}} \mid t_1; x) \leq c_k \prod_{j=1}^{k-1} g(t_{j+1} - t_j),$$

for all k and all $t_1, \ldots, t_k$.

The condition (ii) above is a "localized" version of Berman's formulation on the local nondeterminism of general stochastic processes [2, Definition 5.1]; thus, we call (ii) the conditional local g-nondeterminism (abbrev. CL_gND) at the level x. The intuitive meaning is that, once the process hits x at certain instant t_1, then the local unpredictability around x occurs henceforth.

THEOREM 2.1. *Assume that the paths $X(\cdot, \omega)$ have at most countably many discontinuities and that the conditions (i), (ii) are satisfied at x. Let $\phi(t)$ be a measure function satisfying (1.1) and, for some integer $m \geq 2$,*

$$\sum_{n=0}^{\infty} \frac{2^n}{\phi^m(2^{-n})} \left(\int_0^{2^{-n}} g(t)\, dt \right)^{m-1} < \infty.$$

Then

$$p\{\omega : \phi(Z(x, \omega)) = \infty\} \geq \frac{\left(\int_0^1 p(t; x)\, dt\right)^2}{\int_0^1 \int_0^1 p(t, t'; x, x)\, dt\, dt'}.$$

To derive Theorems 1.1 and 1.2 from Theorem 2.1 is direct. Let X be the Gaussian process in Theorem 1.1. When X is locally nondeterministic in the sense of Berman [1], standard calculations on the joint density function of $X(t_1), X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})$ show that, at each x, X is CL_gND at x, with $g(t) = b^{-1}(t)$ where $b(t)$ is defined by (1.2). Next, let X be the Markov process in Theorem 1.2. The joint density function of $X(t_1), \ldots, X(t_k)$ at $x_1, \ldots, x_k$ is $\prod_{i=1}^k p(t_i - t_{i-1}; x_{i-1}, x_i)$, $t_0 = 0$. Thus, at each x, X is CL_gND with $g(t) = p(t, x, x)$. The arguments in Berman [3,129–131] show that the condition (i) is satisfied for the X above. Finally, observe that we may assume the integer m on Theorems 1.1 and 1.2 so large that $m\varepsilon > 1$.

3. **The proof of Theorem 2.1..** For each $t, 0 \le t \le 1$, set

$$L(t,x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{[x-\varepsilon, x+\varepsilon]}(X(u))\, du.$$

Mimicking the proof of Marcus [6, Theorem 1], we can show that, under the condition (i), $L(t,x)$ exists in $L^k(\Omega)$ for all even k, hence for all k, and

$$EL^k(t,x) = \int_0^t \cdots \int_0^t p(u_1, \ldots, u_k; \underbrace{x, \ldots, x}_{k \text{ terms}})\, du_1 \ldots du_k$$

Furthermore, for a.s. ω, a measure $L(dt, x, \omega)$ is induced. From the displays (3.1) and (3.2) below, $L(dt,x)$ has no atoms a.s.. Since $X(\cdot)$ is assumed to have at most countably many discontinuities, thus $L([0,1],x) = L(A,x)$ where $A = A(\omega)$ is the set of continuities of $X(\cdot)$. Let $t \in A$ and $X(t) \neq x$. Then there exist $\varepsilon = \varepsilon(t) > 0$ and $\delta = \delta(t) > 0$ such that $|X(s) - x| > \varepsilon$ for all $s \in (t - 2\delta, t + 2\delta)$. For all rationals s, s': $t - 2\delta < s < t - \delta < t < t + \delta < s' < t + 2\delta$, $L([s,s'],x) = 0$ from the definition of L, see Marcus [6, p. 281], and hence $L((t-\delta, t+\delta), x) = 0$. Thus $L([0,1],x) = L(A,x) = L(Z(x) \cap A, x) = L(Z(x), x)$; cf. Marcus [6, p.282, line 5] where he assumed the path continuity and now we have weakened his assumption. Set

$$M_m(t,x,\omega) = \sum_{j=1}^{2^n} L^m\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right], x, \omega\right), \quad \frac{1}{2^{n+1}} < t \le \frac{1}{2^n},$$

which we regard as Berman's [3] modulator "localized" at x. It is seen that $M_m(\cdot, x, \omega)$ is nondecreasing and

$$[L(t,x,\omega) - L(s,x,\omega)]^m \le 2^m M_m(t-s, x, \omega) \tag{3.1}$$

for all rationals $s < t$ (cf. [3, Lemma 2.2]). When the process is CL_gND at x,

$$\begin{aligned} EM_m(2^{-n}, x) &= \sum_{j=1}^{2^n} \int \cdots \int_{\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]^m} p(t_1, \ldots, t_m; x, \ldots, x)\, dt_1 \ldots dt_m \\ &= \sum_{j=1}^{2^n} m! \int_{\frac{j-1}{2^n} \le t_i < t_{i+1} \le \frac{j}{2^n}} \cdots \int p(t_1, \ldots, t_m; 0, \ldots, 0 \mid t_1; x) p(t_i; x)\, dt_1 \ldots dt_m \\ &\le m! c_m 2^n \left(\int_0^1 p(t;x)\, dt\right) \left(\int_0^{2^{-n}} g(t)\, dt\right)^{m-1}. \end{aligned} \tag{3.2}$$

We also observe that the content of Berman [2, Lemma 2.3] can be expressed as follows. Let $\phi(t)$ be a measure function satisfying (1.1) and $\alpha(dt)$ be a Borel measure on $[0,1]$. If for some positive integer p,

$$\sum_{n=0}^{\infty} \frac{1}{\phi^p(2^{-n})} \sum_{j=1}^{2^n} \alpha^p\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] < \infty.$$

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Then, $\alpha(E) = 0$ for every Borel E of finite ϕ-measure. Now, all the ingredients of the arguments in the proofs of [2, Theorem 6.2] and [3, Theorem 2] have been localized to the level x. With these and the reasoning of Berman, we conclude that, under the assumptions of Theorem 2.1,

$$P\{\omega : \phi(Z(x,\omega)) = \infty\} \geq P\{\omega : L(Z(x),x,\omega) > 0\} = P\{\omega : L([0,1],x,\omega) > 0\}$$
$$\geq \frac{(EL(1,x))^2}{EL^2(1,x)} = \frac{\left(\int_0^1 p(t;x)\,dx\right)^2}{\int_0^1\int_0^1 p(t,t';x,x)\,dt\,dt'}.$$

In the above, we have used a simple distributional inequality for nonnegative random variables, see Kahane [5; p. 6].

4. Professor E. A. Perkins kindly indicates the author to note the following question. Let X be a standard Markov process and x be regular for $\{x\}$. Since it is well-known that the level set $Z(x)$ is essentially the range of a subordinator, we may apply the result in Fritedt and Pruitt [4] to obtain the exact Hausdorff measure of $Z(x)$. Then, what is the comparison between this viewpoint and Theorem 1.2? We cannot give concrete examples to compare the two aspects, but we remark as follows. On the one hand, for a general X, the structure of the subordinator associated with $Z(x)$ seems not explicitly known (it is stable with index $1/\alpha$ when X is stable with index $\alpha > 1$). On the other hand, in Theorem 1.2 the measure function ϕ and the transition density $p(t;x,x)$ are explictly related. Observe that, if for sufficiently small t

$$\phi(t) \geq \text{const.}\left(\frac{|\log t|^{1+\varepsilon}}{t}\right)^{1/p}\left(\int_0^t p(s;x,x)\,ds\right)^{(p-1)/p}$$

for some $\varepsilon > 0$ and some integer $p \geq 2$, then the summability condition in Theorem 1.2 is satisfied (cf. Berman [2, Theorem 6.2]).

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