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THREE EXAMPLES CONCERNING THE ORE CONDITION IN NOETHERIAN RINGS

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1. Introduction

A ring R is said to satisfy the right Ore condition with respect to a subset C of R if, given $a \in R$ and $c \in C$, there exist $b \in R$ and $D \in C$ such that $ad = cb$. It is well known that R has a classical right quotient ring if and only if R satisfies the right Ore condition with respect to C when C is the set of regular elements of R (a regular element of R being an element of R which is not a zero-divisor). It is also well known that not every ring has a classical right quotient ring. If we make the non-trivial assumption that R has a classical right quotient ring, it is natural to ask whether this property also holds in certain rings related to R such as the ring $M_n(R)$ of all n by n matrices over R. Some answers to this question are known when extra assumptions are made. For example, it was shown by L. W. Small in (**5**) that if R has a classical right quotient ring Q which is right Artinian then $M_n(Q)$ is the right quotient ring of $M_n(R)$ and eQe is the right quotient ring of eRe where e is an idempotent element of R. Also it was shown by C. R. Hajarnavis in (**3**) that if R is a Noetherian ring all of whose ideals satisfy the Artin-Rees property then R has a quotient ring Q and $M_n(Q)$ is the quotient ring of $M_n(R)$.

The first example we give is a right Noetherian ring R which has a right quotient ring but $M_2(R)$ does not. The second example is a Noetherian P.I. ring R which is its own quotient ring and which has an idempotent element e such that eRe has neither a right nor a left quotient ring. An ideal I of a ring R is said to satisfy the AR-property (short for Artin-Rees property) if, given a right ideal K of R, there is a positive integer n such that $K \cap I^n \subseteq KI$. The third example we give is a right Noetherian P.I. ring R which has a prime ideal P which satisfies the AR-property (in fact is nilpotent) but R does not satisfy the right Ore condition with respect to $C(P)$ where $C(P)$ denotes the set of elements of R whose images are regular elements of R/P. In (**4**) a method is indicated for constructing an example with similar properties, but that method must fail because the ring R which it produces is a principal right ideal ring whose nilpotent radical P is prime; in such a ring R the elements of $C(P)$ are precisely the right regular elements of R, and hence R satisfies the right Ore condition with respect to $C(P)$ because of the following result: If S is a right Noetherian ring with nilpotent radical N and if $a, c \in S$ with c right regular, then there exist $b \in S$ and $d \in C(N)$ such that $ad = cb$.

Conventions. $C(0)$ and $C(I)$ will denote respectively the set of regular elements of R and the set of elements of R which are regular mod (I); if there is a possibility of ambiguity we shall write $C_R(0)$ and $C_R(I)$. By "the right quotient ring of R" we shall mean the classical right quotient ring of R, i.e. the ring formed by inverting all the regular elements of R. In constructing matrix rings we shall use the following kind of notation:

$$\begin{pmatrix} S & M \\ 0 & T \end{pmatrix} = \left\{ \begin{pmatrix} s & m \\ 0 & t \end{pmatrix} : s \in S, m \in M, t \in T \right\}.$$

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2. The examples

Example 2.1. We shall show that there is a right Noetherian ring R such that R has a right quotient ring but $M_2(R)$ does not. The construction is in several stages and we start by taking T to be any right Noetherian integral domain which is not left Ore (i.e. there are non-zero left ideals A and B of T such that $A \cap B = 0$). *Let* u be an indeterminate which commutes with the elements of T and let C denote the set of all elements of the polynomial ring $T[u]$ which have non-zero constant term. We first show that $T[u]$ satisfies the right Ore condition with respect to C. We note that $T[u]$ is a right Noetherian domain so that any two non-zero right ideals of $T[u]$ have non-zero intersection. Let $a \in T[u]$ and $c \in C$. For the purposes of establishing the right Ore condition we may suppose that $a \neq 0$. We have $aT[u] \cap cT[u] \neq 0$ so that $af = cg$ for some non-zero elements f and g of $T[u]$. We can write $f = du^i$ for some $d \in C$ and non-negative-integer i. Thus $adu^i = cg$ where c has non-zero constant term. Therefore $g = bu^i$ for some $b \in T[u]$. Hence $ad = cb$ with $d \in C$.

Now let S be the partial right quotient ring of $T[u]$ with respect to C and let D be the right quotient division ring of T. We can make D into a right S-module by identifying D with S/uS, i.e. by setting $Du = 0$. Set

$$R = \begin{pmatrix} T & T \\ 0 & T[u] \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} D & D \\ 0 & S \end{pmatrix}$$

with the usual matrix operations and using the right action of S on D defined above when calculating the $(1, 2)$-entry of a product. We shall now show that Q is the right quotient ring of R. It is straightforward to check that the regular elements of R are given by

$$C_R(0) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : 0 \neq a \in T, b \in T, c \in C \right\}.$$

Let

$$r = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in C_R(0)$$

then a has an inverse a^{-1} *in* D, c has an inverse c^{-1} in S, and

$$\begin{pmatrix} a^{-1}, & -a^{-1}bc^{-1} \\ 0, & c^{-1} \end{pmatrix}$$

is an inverse for r in Q. Now let

$$q=\begin{pmatrix} f & g \\ 0 & h \end{pmatrix} \in Q;$$

we must show that $qr \in R$ for some $r \in C_R(0)$. Because D is the right quotient ring of T there is a non-zero element a of T such that $fa \in T$ and $ga \in T$. Also there exists $d \in C$ such that $hd \in T[u]$. Because both a and d are elements of C there exists $c \in C$ such that $c \in aT[u] \cap dT[u]$. Set

$$r=\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

then $r \in C_R(0)$ and $qr \in R$. Thus Q is the right quotient ring of R. In future we shall omit the details of such verifications.

Finally we shall show that $M_2(R)$ does not have a right quotient ring. Because T is not left Ore there are non-zero elements p and q of T such that $Tp \cap Tq = 0$. Thus if s and t are elements of T such that $sp = tq$ then $s = t = 0$. Let x, y, z be the following elements of R:

$$x=\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad y=\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \quad \text{and} \quad z=\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

It can easily be shown that x and y are regular elements of R and that $Rx \cap Ry = 0$ (so that $r_1 x = r_2 y$ implies $r_1 = r_2 = 0$), and z is right but not left regular (i.e. $zr = 0$ implies $r = 0$, but there is a non-zero element r of R such that $rz = 0$). Let a and c be the following elements of $M_2(R)$:

$$a=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad c=\begin{pmatrix} z & x \\ 0 & y \end{pmatrix}.$$

It is easy to show that c is a regular element of $M_2(R)$ (using $Rx \cap Ry = 0$ to prove left regularity). Suppose that there are elements b and d of $M_2(R)$ such that $ad = cb$; we shall show that d is not regular. Let $b = [b_{ij}]$ and $d = [d_{ij}]$ with b_{ij}, $d_{ij} \in R$ for $1 \leqq i$, $j \leqq 2$. Comparing entries in the second row of ad and cb gives $0 = yb_{21} = yb_{22}$ so that $b_{21} = b_{22} = 0$. Now comparing the first rows of ad and cb gives $d_{11} = zb_{11}$ and $d_{12} = zb_{12}$. But there is a non-zero element r of R such that $rz = 0$. Let w be the element of $M_2(R)$ with r in the (1, 1)-position and 0's elsewhere, then $w \neq 0$ and $wd = 0$. Therefore d is not a regular element of $M_2(R)$.

Remarks. (1) It is clear from the last paragraph that the ring of 2 by 2 upper triangular matrices over R does not have a right quotient ring, and therefore neither does the ring of 2 by 2 lower triangular matrices over R because these two rings are conjugate under an inner automorphism of $M_2(R)$.

(2) We do not know whether there is a ring R which is its own quotient ring and is such that $M_2(R)$ does not have a right quotient ring. Part of the difficulty with this sort of question is the problem of finding a convenient description for the regular elements of $M_2(R)$; for example, it is possible for a regular element of $M_2(R)$ to have entries which are all zero-divisors in R. However, the situation for triangular matrices is more manageable, and I am very grateful to H. Attarchi for communicating the following information to me.

Let R be a ring such that cR (Rc) is an essential right (left) ideal of R for each right (left) regular element c of R. Suppose that

$$w=\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

is a regular element of the ring $U_2(R)$ of all 2 by 2 upper triangular matrices over R. Clearly a is a right regular element of R and c is left regular. Let $x \in 1(a) \cap Rcb^{-1}$ where $1(a)=\{r \in R\colon ra=0\}$ and $Rcb^{-1}=\{r \in R\colon rb \in Rc\}$. There exists $y \in R$ such that $xb=-yc$. Set

$$z=\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

then $zw=0$, so that $z=0$. Thus $1(a) \cap Rcb^{-1}=0$ from which it follows that $1(a)=0$ because Rcb^{-1} is an essential left ideal of R. Thus a is a regular element of R, and by symmetry so also is c. Thus, for such a ring R, a matrix w of the form given above is regular if and only if a and c are regular elements of R. This makes it easy to prove that, for example, if R is a left and right Noetherian ring which has a right quotient ring Q then $U_2(Q)$ is the right quotient ring of $U_2(R)$.

Example 2.2. We shall show that there is a left and right Noetherian P.I. ring R which is its own quotient ring but which has an idempotent element e such that eRe does not have a right or left quotient ring. Again the construction is in several stages. Let U be the ring of integers Z localised at the prime ideal $2Z$ and set

$$T=\begin{pmatrix} U, & 2U \\ U, & U \end{pmatrix},$$

then T is a left and right Noetherian prime P.I. ring. Set

$$M=\begin{pmatrix} 2U & 2U \\ U & U \end{pmatrix} \quad \text{and} \quad A=\begin{pmatrix} 2U & 2U \\ U & 2U \end{pmatrix},$$

then M is a maximal ideal of T and A is an ideal of T (in fact A is the intersection of the two maximal ideals of T). We have

$$C_T(A)=\left\{\begin{pmatrix} a & 2b \\ c & d \end{pmatrix}\colon a, b, c, d \in U, ad \notin 2U\right\}.$$

Thus if $t \in C_T(A)$ then $\det(t) \in U$ and $\det(t) \notin 2U$. Thus $\det(t)$ is a unit of U, so that the elements of $C_T(A)$ are units of T.

Set $F=T/M$ then $F \cong Z/2Z$. Set

$$S=\begin{pmatrix} F & F & F \\ 0 & T & F \\ 0 & 0 & F \end{pmatrix},$$

then S is a left and right Noetherian P.I. ring. Let S' be the ring which is constructed in the same way as S but using Z instead of U, then S' is the ring given in (2) and S is a partial quotient ring of S'; this is because T is the partial quotient ring of the ring

$$T' = \begin{pmatrix} Z & 2Z \\ Z & Z \end{pmatrix}$$

with respect to the set of elements of T' which have odd determinant. But it was shown in (**2**) that S' has neither a left nor a right quotient ring, and hence S has neither a left nor a right quotient ring; this assertion can be justified either by using the fact that S is a partial quotient ring of S', or by modifying the argument given in (**2**) and using the fact that T does not satisfy either the right or left Ore condition with respect to $C_T(M)$.

We now aim to find suitable R and e with $eRe \cong S$. Set

$$I = \begin{pmatrix} 0 & F & F \\ 0 & A & F \\ 0 & 0 & 0 \end{pmatrix}$$

then I is an ideal of S. Set $C = C_S(0) \cap C_S(I)$ and let $s \in S$, then $s \in C$ if and only if s has 1 in the (1, 1)- and (3, 3)-positions and an element of $C_T(A)$ in the (2, 2)-position. It follows easily that the elements of C are units of S. Now set

$$R = \begin{pmatrix} S & S/I \\ 0 & S \end{pmatrix}$$

then R is a left and right Noetherian P.I. ring. Let e be the element of R which has 1 in the (1, 1)-position and 0's elsewhere, then e is idempotent and $eRe \cong S$. An element of R is regular if and only if its diagonal entries belong to $C_S(0) \cap C_S(I) = C$ (a right or left regular element of S is regular and similarly for the finite commutative ring S/I). But the elements of C are units of S, from which it follows that the regular elements of R are units modulo the nilpotent ideal consisting of all strictly upper triangular elements of R. Therefore the regular elements of R are units of R.

Example 2.3. We shall show that there is a right Noetherian P.I. ring R which has a prime ideal P such that P satisfies the AR-property but R does not satisfy the right Ore condition with respect to $C(P)$. Let F be a field and x an indeterminate. Let R be the ring of all matrices of the form

$$\begin{pmatrix} f(0) & g(x) \\ 0 & f(x) \end{pmatrix}$$

with $f(x), g(x) \in F[x]$. Let t: $R \to F[x]$ be the function which sends the matrix displayed above to $f(x)$ and set $P = \text{Ker}(t)$, then t is a surjective ring homomorphism and P consists of the strictly upper triangular elements of R. Thus P is a prime ideal of R and trivially satisfies the AR-property because it is nilpotent. Let e_{i2} denote the element of R with 1 in the (i, 2)-position and 0's elsewhere for $i = 1$ or 2. Set $a = e_{12}$ and $c = xe_{22}$ then $c \in C(P)$ and it is not possible to have $ad = cb$ with $b \in R$ and $d \in C(P)$.

Remarks. (1) The following are open questions: (a) If R is a left and right Noetherian ring which has a prime ideal P which satisfies the AR-property, does R satisfy the right Ore condition with respect to $C(P)$? This question is open even in the special case where P

is nilpotent. (b) If R is a right Noetherian prime ring which has a non-zero prime ideal P which satisfies the AR-property, does R satisfy the right Ore condition with respect to $C(P)$?

(2) It is shown in (**1**), Theorem 3.11, that if R is right Noetherian and P is a prime ideal of R which satisfies the AR-property then R satisfies the right Ore condition with respect to $C(P)$ if and only if P is weakly ideal invariant.

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