Euro. Jnl of Applied Mathematics (2017), *vol.* 28, *pp.* 208–220. © Cambridge University Press 2016 208
doi:10.1017/S0956792516000218

The mathematics of varistors

GIOVANNI CIMATTI

Department of Mathematics, University of Pisa, 56127, Pisa, Largo Bruno Pontecorvo 5, Italy
email: cimatti@dm.unipi.it

(*Received 3 December 2015; revised 4 April 2016; accepted 19 April 2016; first published online 19 May 2016*)

A three-dimensional model of the varistor device is proposed. The thermal and electric conductivity of the material are taken to depend, in addition to the electric potential, on the temperature. Two theorems of existence and uniqueness of solutions for the boundary-value problem which determine the potential and the temperature inside the device are proposed. Levy–Caccioppoli global inversion theorem is used for the proof.

Key words: varistor, nonlinear elliptic problems, existence, uniqueness of solutions, Levy–Caccioppoli global inversion theorem

1 Introduction

Voltage dependent resistors, also called varistors, are electronic components with an electrical resistivity that varies with the applied voltage [6, 7]. This highly nonlinear voltage–current characteristic is similar to that of a diode with the basic difference of being the same for both directions of transverse current. The traditional schematic symbol of the varistor, see Figure 1, reflects well the diode-like behaviour in both directions of the current flow. Varistors are made of a ceramic mass consisting of an enormous number of randomly oriented zinc oxide grains. Electrically, this is equivalent to a network of a huge number of back-to-back diode pairs. This explains the empirical current–voltage characteristic of Figure 2, which is well described by the empirical law

$$I = CV^{\alpha}, \tag{1.1}$$

where α is an odd positive integer typically greater that 31 and C a positive constant. The high value of the exponent α explains the flat region in the voltage–current characteristic which is essential for the use of varistors as circuit protectors. In this paper, we model a varistor as a three-dimensional body represented by an open and bounded subset Ω of $\mathbf{R}^3$ homeomorphic to a sphere and with a boundary Γ consisting of three regular, e.g. C^2, surfaces Γ_1, Γ_2 and Γ_3 such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ as shown in Figure 3. The surfaces Γ_1 and Γ_2 represent the electrodes of the varistor to which a potential difference is applied, whereas Γ_3 is the electrically insulated part of the device. We can easily prove that if we assume a local Ohm's law of the form

$$\mathbf{J} = -\gamma\varphi^{\beta}\nabla\varphi, \tag{1.2}$$

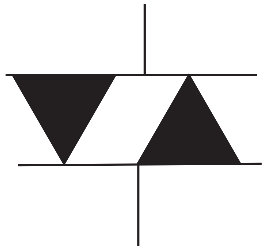

FIGURE 1. Schematic diagram of varistor.

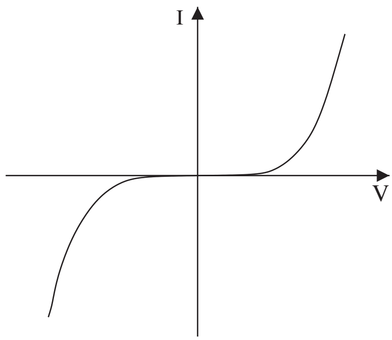

FIGURE 2. Current-Voltage characteristic of varistor.

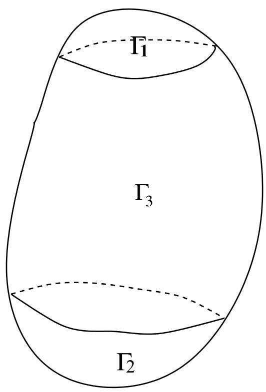

FIGURE 3. The subset Ω.

where β is an even positive integer, γ a positive constant and $\varphi(\mathbf{x})$ the potential inside Ω, we obtain, if a difference of potential V is applied between Γ_1 and Γ_2, the global relation (1.1). For, if $\mathbf{n}$ is the unit vector normal to Γ_2 pointing outward with respect to Ω, we have, for the total current crossing Ω,

$$I = -\int_{\Gamma_2} \mathbf{J} \cdot \mathbf{n} \, d\Gamma .$$

https://doi.org/10.1017/S0956792516000218 Published online by Cambridge University Press

Since $\nabla \cdot \mathbf{J} = 0$, the problem which determines $\varphi(\mathbf{x})$ in Ω is, under stationary conditions,

$$\nabla \cdot (\varphi^\beta \nabla\varphi) = 0 \ \text{ in } \Omega, \ \varphi = 0 \ \text{ on } \Gamma_1, \ \varphi = V \ \text{ on } \Gamma_2, \ \frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{1.3}$$

If we define $\psi = \frac{1}{1+\beta}\varphi^{1+\beta}$, we have $\mathbf{J} = -\gamma\nabla\psi$. Thus, problem (1.3) becomes

$$\Delta\psi = 0 \ \text{ in } \Omega, \ \psi = 0 \ \text{ on } \Gamma_1, \ \psi = \frac{1}{1+\beta}V^{1+\beta} \ \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_3.$$

If we define the purely geometric quantity

$$a = \int_{\Gamma_2} \frac{\partial\xi}{\partial n}\, d\Gamma,$$

where $\xi(\mathbf{x})$ is the solution of the problem

$$\Delta\xi = 0 \ \text{ in } \Omega, \ \xi = 0 \ \text{ on } \Gamma_1, \ \xi = 1 \ \text{ on } \Gamma_2, \ \frac{\partial\xi}{\partial n} = 0 \ \text{ on } \Gamma_3,$$

we obtain, for the total current crossing Ω,

$$I = \frac{\gamma a}{1+\beta}V^{1+\beta}. \tag{1.4}$$

Thus, we obtain (1.1) from (1.4) if we choose $\alpha = \beta + 1$ and $C = \frac{\gamma a}{1+\beta}$. In varistors, as in all semiconductors, there is a quite appreciable dependence on temperature [6]. Thus, in Section 2 we assume, instead of (1.2), the more general constitutive equation

$$\mathbf{J} = -S(u, \varphi)\nabla\varphi, \tag{1.5}$$

where u denotes the temperature. We shall make assumptions on $S(u, \varphi)$ which make (1.5) a generalisation of (1.2). We prove, using the Levy–Caccioppoli global inversion theorem [2, 3, 8] and [1], that the boundary problem which determines $\varphi(\mathbf{x})$ and $u(\mathbf{x})$, and therefore also the global current I, has at least one solution. In Section 3, we present a result of existence and uniqueness of solution which holds if we assume the special case of (1.5) given by $\mathbf{J} = -\varphi^\beta\sigma(u)\nabla\varphi$.

2 A theorem of existence of solutions

Together with (1.5), we assume for the density of heat flow

$$\mathbf{q} = -\kappa(u, \varphi)\nabla u,$$

where κ is the thermal conductivity. For the density of Joule heating, we have

$$\mathbf{E} \cdot \mathbf{J} = S(u, \varphi)|\nabla\varphi|^2.$$

Therefore, the energy equation reads

$$-\nabla \cdot (\kappa(u, \varphi)\nabla u) = S(u, \varphi)|\nabla\varphi|^2. \tag{2.1}$$

On the other hand, we have $\nabla \cdot (S(u,\varphi)\nabla\varphi) = 0$ since $\nabla \cdot \mathbf{J} = 0$. Hence, we can restate (2.1) in full divergence form as follows:

$$\nabla \cdot (\kappa(u,\varphi)\nabla u + \varphi S(u,\varphi)\nabla\varphi) = \nabla \cdot (\kappa(u,\varphi)\nabla u) + S(u,\varphi)|\nabla\varphi|^2.$$

We assume that the electrically insulated part of the device, i.e. Γ_3, is also thermally insulated[1]. Thus, for the determination of $u(\mathbf{x})$ and $\varphi(\mathbf{x})$, we have the following boundary value problem:

$$\nabla \cdot (S(u,\varphi)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\varphi = 0 \quad \text{on } \Gamma_1, \;\; \varphi = V \quad \text{on } \Gamma_2, \;\; \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma_3, \tag{2.3}$$

$$\nabla \cdot (\kappa(u,\varphi)\nabla u + \varphi S(u,\varphi)\nabla\varphi) = 0 \quad \text{in } \Omega, \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma_1, \;\; u = U \quad \text{on } \Gamma_2, \;\; \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_3, \tag{2.5}$$

where $V > 0$ and $U > 0$[2] are given constants. We assume

$$S(u,0) = 0 \tag{2.6}$$

for compatibility with (1.2). We wish to prove that problems (2.2)–(2.5) have at least one solution under suitable assumptions. To this end, we consider the following ancillary two-point problem: to find $(\mathcal{U}(\varphi),\gamma) \in C^1([0,\varphi]) \times \mathbf{R}^1$ such that

$$\kappa(\mathcal{U},\varphi)\frac{d\mathcal{U}}{d\varphi} + \varphi S(\mathcal{U},\varphi) = \gamma S(\mathcal{U},\varphi), \tag{2.7}$$

$$\mathcal{U}(0) = 0, \tag{2.8}$$

$$\mathcal{U}(V) = U. \tag{2.9}$$

The next lemma will be used to prove that the problems (2.7)–(2.9) have one and only one solution.

Lemma 2.1 *If $P(\varphi)$, $Q(\varphi)$ and $G(\varphi) \in C^0([0,V])$ and*

$$\int_0^V Q(\tau)e^{\int_0^\tau P(t)dt}d\tau \neq 0, \tag{2.10}$$

the linear two-point problem

$$\frac{dH}{d\varphi} + P(\varphi)H = \Gamma Q(\varphi) + G(\varphi), \tag{2.11}$$

$$H(0) = 0, \tag{2.12}$$

$$H(V) = 0, \tag{2.13}$$

[1] This assumption is essential for the present theory.

[2] The case $U < 0$ can be treated in a similar way. If $U = 0$, however, the theory below does not apply.

has one and only one solution.

Proof The solution of the problem (2.11), (2.12) is given by

$$H(\varphi) = e^{-\int_0^\varphi P(t)dt} \int_0^\varphi (\Gamma Q(\tau) + G(\tau)) e^{\int_0^\tau P(t)dt} d\tau.$$

Therefore, the condition (2.13) becomes

$$e^{-\int_0^V P(t)dt} \int_0^V (\Gamma Q(\tau) + G(\tau)) e^{\int_0^\tau P(t)dt} d\tau = 0. \tag{2.14}$$

By (2.10), the equation (2.14) in the unknown Γ has one and only one solution and correspondingly this is the case for problems (2.11)–(2.13). □

We use the Levy–Caccioppoli global inversion theorem [2] and Lemma 2.1 to prove

Lemma 2.2 *If in addition to (2.6), we assume*

$$a_1 \varphi^{\beta_1} \sigma(\mathcal{U}) + b \geqslant S(\mathcal{U}, \varphi) \geqslant a_0 \varphi^{\beta_0} \sigma(\mathcal{U}), \tag{2.15}$$

where $b > 0$, $a_1 > a_0 > 0$, β_1, β_0 are positive even integers with $\beta_1 > \beta_0$ and

$$\sigma_1 \geqslant \sigma(\mathcal{U}) \geqslant \sigma_0 > 0, \tag{2.16}$$

$$\kappa_1 \geqslant \kappa(\mathcal{U}, \varphi) \geqslant \kappa_0 > 0, \tag{2.17}$$

$$\text{the function } A(\mathcal{U}, \varphi) = \frac{S(\mathcal{U}, \varphi)}{\kappa(\mathcal{U}, \varphi)} \text{ is globally Lipschitz} \tag{2.18}$$

then the problems (2.7)–(2.9) have one and only one solution.

Proof Setting $\mathcal{W}(\varphi) = \mathcal{U}(\varphi) - \frac{U}{V}\varphi$, we can rewrite the problems (2.7)–(2.9) with homogeneous boundary conditions. We obtain

$$\kappa\left(\mathcal{W} + \frac{U}{V}\varphi, \varphi\right)\frac{d\mathcal{W}}{d\varphi} + \varphi S\left(\mathcal{W} + \frac{U}{V}\varphi, \varphi\right) + \frac{U}{V}\kappa\left(\mathcal{W} + \frac{U}{V}\varphi, \varphi\right) = \gamma S\left(\mathcal{W} + \frac{U}{V}\varphi, \varphi\right), \tag{2.19}$$

$$\mathcal{W}(0) = 0, \ \mathcal{W}(V) = 0. \tag{2.20}$$

Let $X = \{\mathcal{W}(\varphi) \in C^1([0, V]),\ \mathcal{W}(0) = 0,\ \mathcal{W}(V) = 0\} \times \mathbf{R}^1$ be the Banach space with norm $\|(\mathcal{W}, \gamma)\|_X = \|\mathcal{W}\|_{C^1} + |\gamma|$. Define the operator $F : X \to C^0([0, V])$ as follows:

$$F((\mathcal{W}, \gamma)) = \kappa\Big(\mathcal{W} + \frac{U}{V}\varphi, \varphi\Big)\frac{d\mathcal{W}}{d\varphi} + \varphi S\Big(\mathcal{W} + \frac{U}{V}\varphi, \varphi\Big) \tag{2.21}$$

$$+\frac{U}{V}\kappa\Big(\mathcal{W} + \frac{U}{V}\varphi, \varphi\Big) - \gamma S\Big(\mathcal{W} + \frac{U}{V}\varphi, \varphi\Big).$$

https://doi.org/10.1017/S0956792516000218 Published online by Cambridge University Press

The map $F((\mathcal{W},\gamma))$ is differentiable and it is easily seen that its differential is given by

$$\begin{aligned} dF((\mathcal{W},\gamma))(W,\Gamma) &= \kappa\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)\frac{dW}{d\varphi}+\kappa'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)W\frac{d\mathcal{W}}{d\varphi} \\ &+\frac{U}{V}\kappa'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)W+\varphi S'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)W-\Gamma S\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big) \\ &-\gamma S'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)W. \end{aligned}$$

We claim that dF, as a linear map from X to $C^0([0,V])$, is invertible for every $(\mathcal{W},\gamma)\in X$. Let $G(\varphi)\in C^0([0,V])$. The problem

$$dF((\mathcal{W},\gamma))(W,\Gamma)=G,\ \ W(0)=0,\ \ W(V)=0,$$

can be written as

$$\frac{dW}{d\varphi}+PW=\Gamma Q+\tilde{G},\ \ W(0)=0,\ \ W(V)=0, \tag{2.22}$$

where

$$P(\varphi)=\frac{\kappa'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)\frac{d\mathcal{W}}{d\varphi}+\frac{U}{V}\kappa'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)+\varphi S'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)-\gamma S'\Big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\Big)}{\kappa\big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\big)},$$

$$Q(\varphi)=\frac{S\big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\big)}{\kappa\big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\big)},\ \ \tilde{G}(\varphi)=\frac{G}{\kappa\big(\mathcal{W}+\frac{U}{V}\varphi,\varphi\big)}.$$

We can apply Lemma 2.1 to problem (2.22) since the condition (2.10) is satisfied by (2.15)–(2.17). We conclude that dF is invertible. To apply the Levy–Caccioppoli global inversion theorem it remains to prove that F is proper[3]. Let $\mathbf{K}$ be a compact subset of $C^0([0,V])$. By the theorem of Arzelá the functions in $\mathbf{K}$ are equicontinuous and equibounded. We need to prove that

$$F^{-1}(\mathbf{K})=\{(\mathcal{W}(\varphi),\gamma)\in X;\ F((\mathcal{W},\gamma))=g,\ g\in\mathbf{K}\},$$

is a compact subset of X. Define

$$\mathbf{C}_1=\{\mathcal{W}(\varphi)\in C^1([0,V]),\mathcal{W}(0)=0,\ \mathcal{W}(V)=0,\ \exists\gamma\in\mathbf{R}^1,\ (\mathcal{W},\gamma)\in F^{-1}(\mathbf{K})\},$$

$$\mathbf{C}_2=\{\gamma\in\mathbf{R}^1,\ \exists\mathcal{W}\in\mathbf{C}_1 \text{ with } (\mathcal{W},\gamma)\in F^{-1}(\mathbf{K})\}.$$

Let $g\in\mathbf{K}$ and $F((\mathcal{W},\gamma))=g$. We have, by (2.21),

$$\frac{d\mathcal{W}}{d\varphi}+\varphi\frac{S(\mathcal{W}+\frac{U}{V}\varphi,\varphi)}{\kappa(\mathcal{W}+\frac{U}{V}\varphi,\varphi)}+\frac{U}{V}-\gamma\frac{S(\mathcal{W}+\frac{U}{V}\varphi,\varphi)}{\kappa(\mathcal{W}+\frac{U}{V}\varphi,\varphi)}=\frac{g(\varphi)}{\kappa(\mathcal{W}+\frac{U}{V}\varphi,\varphi)}, \tag{2.23}$$

[3] We recall that $F:\mathbb{X}\to\mathbb{Y}$ is proper if $F^{-1}(\mathbb{K})$ is compact in $\mathbb{X}$ when $\mathbb{K}$ is compact in $\mathbb{Y}$.

$$\mathcal{W}(0) = 0,\ \mathcal{W}(V) = 0. \tag{2.24}$$

If $(\mathcal{W}, \gamma)$ is a solution of (2.23), (2.24) there exists a constant C_1 depending only on $\mathbf{K}$ such that

$$|\gamma| \leqslant C_1. \tag{2.25}$$

For, integrating (2.23) from 0 to V, we have

$$|\gamma| = \left|\int_0^V \frac{S(\mathcal{W}(t) + \frac{U}{V}t, t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt\right|^{-1} \left|\int_0^V \frac{g(t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt - U - \int_0^V \frac{tS(\mathcal{W}(t) + \frac{U}{V}t, t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt\right|.$$

By (2.17), we have

$$\left|\int_0^V \frac{g(t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt\right| \leqslant M_1,$$

where M_1 depends only on $\mathbf{K}$. On the other hand, by (2.15),

$$\int_0^V \frac{S(\mathcal{W}(t) + \frac{U}{V}t, t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt \geqslant \frac{1}{\kappa_1} \int_0^V S\left(\mathcal{W}(t) + \frac{U}{V}t, t\right) dt \geqslant \frac{1}{\kappa_1} \int_0^V t^{\beta_0} \sigma\left(\mathcal{W}(t) + \frac{U}{V}t, t\right) dt$$
$$\geqslant \frac{\sigma_0 V^{\beta_0+1}}{\kappa_1(\beta_0 + 1)},$$

and similarly

$$\int_0^V \frac{tS(\mathcal{W}(t) + \frac{U}{V}t, t)}{\kappa(\mathcal{W}(t) + \frac{U}{V}t, t)} dt \leqslant \frac{\sigma_1 V^{\beta_1+2}}{\kappa_0(\beta_1 + 2)}.$$

Thus, we obtain (2.25). From (2.23), recalling (2.25), (2.15)–(2.17), we obtain

$$\left|\frac{d\mathcal{W}}{d\varphi}\right| \leqslant M_2, \tag{2.26}$$

where M_2 depends only on $\mathbf{K}$. The functions $\mathcal{W}(\varphi)$ are therefore equicontinuous and equibounded. We claim that also the functions $\frac{d\mathcal{W}}{d\varphi}$ are equicontinuous. If $\varphi_2,\ \varphi_1 \in [0, V]$ we have from (2.23) and (2.25), by (2.16)–(2.18),

$$|\mathcal{W}'(\varphi_2) - \mathcal{W}'(\varphi_1)| \leqslant |g(\varphi_2) - g(\varphi_1)|$$
$$+ \left|\varphi_2 A\left(\mathcal{W}(\varphi_2) + \frac{U}{V}\varphi_2, \varphi_2\right) - \varphi_1 A\left(\mathcal{W}(\varphi_1) + \frac{U}{V}\varphi_1, \varphi_1\right)\right|$$
$$+ |\gamma| \left|A\left(\mathcal{W}(\varphi_2) + \frac{U}{V}\varphi_2, \varphi_2\right) - A\left(\mathcal{W}(\varphi_1) + \frac{U}{V}\varphi_1, \varphi_1\right)\right|$$
$$\leqslant |g(\varphi_2) - g(\varphi_1)| + M_3|\mathcal{W}(\varphi_2) - \mathcal{W}(\varphi_1)| + M_4|\varphi_2 - \varphi_1|.$$

Hence, $\mathbf{C}_1$ is a compact subset of $C^1([0, V])$. It remains to prove that $\mathbf{C}_2$ is closed. Let $\{\gamma_k\} \subset \mathbf{C}_2$, $\gamma_k \to \gamma^*$ and $\{g_k\} \subset \mathbf{K}$, $g_k \to g^*$ as $k \to \infty$. We claim that there exist $\mathcal{W}^* \in \mathbf{C}_1$

and $g^* \in \mathbf{K}$ such that $F(\mathcal{W}^*, \gamma^*) = g^*$. We have $F(\mathcal{W}_k, \gamma_k) = g_k$, i.e.

$$\frac{d\mathcal{W}_k}{d\varphi} + \varphi A\left(\mathcal{W}_k + \frac{U}{V}\varphi, \varphi\right) + \frac{U}{V} - \gamma_k A\left(\mathcal{W}_k + \frac{U}{V}\varphi, \varphi\right) = g_k, \tag{2.27}$$

$$\mathcal{W}_k(0) = 0, \ \mathcal{W}_k(V) = 0. \tag{2.28}$$

Since $\mathbf{C}_1$ and $\mathbf{K}$ are compact sets we can extract a subsequence from $\{\mathcal{W}_k\}$, not relabelled, such that $\mathcal{W}_k \to \mathcal{W}^*$ in $C^1([0, V])$. By continuous dependence on the data, we obtain from (2.27) and (2.28) that

$$\frac{d\mathcal{W}^*}{d\varphi} + \varphi A\Big(\mathcal{W}^* + \frac{U}{V}\varphi, \varphi\Big) + \frac{1}{V} - \gamma^* A\Big(\mathcal{W}^* + \frac{U}{V}\varphi, \varphi\Big) = g^*$$

$$\mathcal{W}^*(0) = 0, \ \mathcal{W}^*(V) = 0.$$

Thus, $\mathbf{C}_2$ is closed since $(\mathcal{W}^*, \gamma^*) \in F^{-1}(\mathbf{K})$. □

With a solution of problems (2.7)–(2.9) at our disposal, we can construct a solution of problems (2.2)–(2.5). More precisely, we have:

Theorem 2.3 *Let* $Q = \{(u, \varphi); u \geqslant 0, V \geqslant \varphi \geqslant 0\}$. *Assume*

$$a_1\varphi^{\beta_1}\sigma(u) + b \geqslant S(u, \varphi) \geqslant a_0\varphi^{\beta_0}\sigma(u), \tag{2.29}$$

where $a_1 > a_0 > 0$, *whereas* $\beta_1 > \beta_0$ *are positive even integers and*

$$\sigma_1 \geqslant \sigma(u) \geqslant \sigma_0 > 0, \tag{2.30}$$

$$\kappa_1 \geqslant \kappa(u, \varphi) \geqslant \kappa_0 > 0. \tag{2.31}$$

Suppose that the function $A(u, \varphi) = \frac{S(u,\varphi)}{\kappa(u,\varphi)}$ *satisfies a Lipschitz condition in* Q. *Then the problem for* (u, φ)

$$\nabla \cdot (S(u, \varphi)\nabla\varphi) = 0 \quad in\ \Omega, \tag{2.32}$$

$$\varphi = 0 \quad on\ \Gamma_1, \ \ \varphi = V \quad on\ \Gamma_2, \ \ \frac{\partial\varphi}{\partial n} = 0 \quad on\ \Gamma_3, \tag{2.33}$$

$$\nabla \cdot (\kappa(u, \varphi)\nabla u + \varphi S(u, \varphi)\nabla\varphi) = 0 \quad in\ \Omega, \tag{2.34}$$

$$u = 0 \quad on\ \Gamma_1, \ \ u = U \quad on\ \Gamma_2, \ \ \frac{\partial u}{\partial n} = 0 \quad on\ \Gamma_3, \tag{2.35}$$

has at least one solution.

Proof An equivalent formulation of problems (2.32)–(2.35) is obtained, in view of (2.32), if instead of (2.34) we take as energy equation

$$-\nabla \cdot (\kappa(u, \varphi)\nabla u) = S(u, \varphi)|\nabla\varphi|^2 \quad \text{in } \Omega. \tag{2.36}$$

https://doi.org/10.1017/S0956792516000218 Published online by Cambridge University Press

Applying the maximum principle to (2.36) and to (2.32) and taking into account the boundary conditions (2.33) and (2.35), we have

$$u \geqslant 0, \ \ V \geqslant \varphi \geqslant 0 \ \text{ in } \Omega. \tag{2.37}$$

Hence $(u, \varphi) \in Q$. Let $(\mathcal{U}(\varphi), \gamma)$ be the (unique) solution of (2.7)–(2.9). As a first step, we solve the problem

$$\nabla \cdot (S(\mathcal{U}(\varphi), \varphi)\nabla\varphi) = 0 \ \text{ in } \Omega, \tag{2.38}$$

$$\varphi = 0 \ \text{ on } \Gamma_1, \ \varphi = V \ \text{ on } \Gamma_2, \ \frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{2.39}$$

To this end, we define

$$\phi = \mathcal{G}(\varphi) = \int_0^{\varphi} S(\mathcal{U}(t), t)dt.$$

From (2.29) and (2.30), we have

$$\mathcal{G}'(\varphi) = S(\mathcal{U}(\varphi), \varphi) \geqslant a_0\varphi^{\beta_0}\sigma_0 > 0 \ \text{ if } \varphi > 0.$$

Hence, $\mathcal{G}(\varphi)$ maps $[0, \infty)$ one-to-one onto $[0, \infty)$. In terms of ϕ the problem (2.38), (2.39) becomes

$$\Delta\phi = 0 \ \text{ in } \Omega, \ \phi = 0 \ \text{ on } \Gamma_1, \ \phi = \mathcal{G}(V) \ \text{ on } \Gamma_2, \ \frac{\partial\phi}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{2.40}$$

If $\phi(\mathbf{x})$ is the solution of (2.40), we obtain as solution of (2.38), (2.39)

$$\tilde{\varphi}(\mathbf{x}) = \mathcal{G}^{-1}(\phi(\mathbf{x})). \tag{2.41}$$

Moreover, if $z(\mathbf{x})$ is the solution of the problem

$$\Delta z = 0 \ \text{ in } \Omega, \ z = 0 \ \text{ on } \Gamma_1, \ z = 1 \ \text{ on } \Gamma_2, \ \frac{\partial z}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{2.42}$$

(2.41) can be written

$$\tilde{\varphi}(\mathbf{x}) = \mathcal{G}^{-1}(\mathcal{G}(V)z(\mathbf{x})).$$

Define $\tilde{u}(\mathbf{x}) = \mathcal{U}(\tilde{\varphi}(\mathbf{x}))$. We have by (2.38)

$$\nabla \cdot (S(\tilde{u}, \tilde{\varphi})\nabla\tilde{\varphi}) = 0 \ \text{ in } \Omega. \tag{2.43}$$

Moreover, since $\nabla\mathcal{U}(\tilde{\varphi}) = \frac{d\mathcal{U}}{d\varphi}\nabla\tilde{\varphi}$, we have, by (2.7) and (2.43),

$$\nabla \cdot (\kappa(\tilde{u}, \tilde{\varphi})\nabla\tilde{u} + \tilde{\varphi}S(\tilde{u}, \tilde{\varphi})\nabla\tilde{\varphi}) = \nabla \cdot \left((\kappa(\mathcal{U}(\tilde{\varphi}), \tilde{\varphi})\frac{d\mathcal{U}}{d\tilde{\varphi}} + \tilde{\varphi}S(\mathcal{U}(\tilde{\varphi}), \tilde{\varphi}))\nabla\tilde{\varphi}\right)$$

$$= \gamma\nabla \cdot (S(\mathcal{U}(\tilde{\varphi}), \tilde{\varphi})\nabla\tilde{\varphi}) = 0.$$

Since $\frac{\partial\phi}{\partial n} = \mathcal{G}'(\tilde{\varphi})\frac{\partial\tilde{\varphi}}{\partial n}$ and $\frac{\partial\tilde{u}}{\partial n} = \mathcal{U}'(\varphi)\frac{\partial\tilde{\varphi}}{\partial n}$ on Γ_3, we conclude that $(\tilde{u}(\mathbf{x}), \tilde{\varphi}(\mathbf{x}))$ satisfy the boundary conditions (2.33)–(2.35) by (2.40). Hence $(\tilde{u}(\mathbf{x}), \tilde{\varphi}(\mathbf{x}))$ is a solution of (2.32)–(2.35). □

3 Uniqueness of the solution

Theorem 2.3 gives only an existence result. However, for the physically relevant case in which the current density and the density of heat flow are given by

$$\mathbf{J} = -\varphi^{\beta}\sigma(u)\nabla\varphi, \quad \mathbf{q} = -\kappa(u)\nabla u, \tag{3.1}$$

where β a positive even integer, $\sigma(u) \in C^0(\mathbf{R}^1)$ and $\kappa(u) \in C^0(\mathbf{R}^1)$ satisfy

$$\sigma_1 \geqslant \sigma(u) \geqslant \sigma_0 > 0, \tag{3.2}$$

and

$$\kappa_1 \geqslant \kappa(u) \geqslant \kappa_0 > 0, \tag{3.3}$$

an existence and uniqueness result for the corresponding problem

$$\nabla \cdot (\sigma(u)\varphi^{\beta}\nabla\varphi) = 0 \ \text{ in } \Omega, \tag{3.4}$$

$$\varphi = 0 \ \text{ on } \Gamma_1, \ \varphi = V \ \text{ on } \Gamma_2, \ \frac{\partial\varphi}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{3.5}$$

$$\nabla \cdot (\kappa(u)\nabla u + \sigma(u)\varphi^{\beta+1}\nabla\varphi) = 0 \ \text{ in } \Omega, \tag{3.6}$$

$$u = 0 \ \text{ on } \Gamma_1, \ u = U \ \text{ on } \Gamma_2, \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{3.7}$$

can be proved. As a first step, we define

$$\psi = F(\varphi) = \frac{\varphi^{\beta+1}}{\beta+1}. \tag{3.8}$$

In term of ψ, the first equation of (3.1) can be rewritten as

$$\mathbf{J} = -\sigma(u)\nabla\psi. \tag{3.9}$$

Moreover, if we express the energy equation (3.6) in terms of ψ, redefine the thermal conductivity as

$$\tilde{\kappa}(u) = (\beta+1)^{-\frac{1}{\beta+1}}\kappa(u),$$

and then write $\kappa(u)$ instead of $\tilde{\kappa}(u)$, we can restate the problems (3.4)–(3.7) in the following form,

$$\nabla \cdot (\sigma(u)\nabla\psi) = 0 \ \text{ in } \Omega, \tag{3.10}$$

$$\psi = 0 \ \text{ on } \Gamma_1, \ \psi = \frac{V^{\beta+1}}{\beta+1} \ \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{3.11}$$

$$\nabla \cdot (\kappa(u)\nabla u + \sigma(u)\psi^{\frac{1}{\beta+1}}\nabla\psi) = 0 \ \text{ in } \Omega, \tag{3.12}$$

$$u = 0 \ \text{ on } \Gamma_1, \ u = U \ \text{ on } \Gamma_2, \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{3.13}$$

We note that the differential form $\kappa(u)du + \psi^{\frac{1}{\beta+1}}\sigma(u)d\psi$ has the integrating factor $\frac{1}{\sigma(u)}$. Since the primitive of $\frac{\kappa(u)}{\sigma(u)}du + \psi^{\frac{1}{\beta+1}}d\psi$ is

$$\int_0^u \frac{\kappa(t)}{\sigma(t)}dt + \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}},$$

we are led to consider the transformation

$$\theta = F(u) + \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}} \quad \text{where} \quad F(u) = \int_0^u \frac{\kappa(t)}{\sigma(t)}dt. \tag{3.14}$$

By (3.2) and (3.3), $F(u)$ maps $[0,\infty)$ one-to-one onto $[0,\infty)$. Hence, (3.14) can be solved with respect to u, i.e.

$$u = F^{-1}\Big(\theta - \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}}\Big).$$

On the other hand,

$$\nabla\theta = \frac{\kappa(u)}{\sigma(u)}\nabla u + \psi^{\frac{1}{1+\beta}}\nabla\psi. \tag{3.15}$$

Therefore, the problems (3.10)–(3.13) can be written in terms of θ and ψ as follows:

$$\nabla\cdot\left(\sigma\left(F^{-1}\left(\theta - \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}}\right)\right)\nabla\theta\right) = 0 \ \text{ in } \Omega, \tag{3.16}$$

$$\theta = 0 \ \text{ on } \Gamma_1, \ \theta = F(U) + \frac{(1+\beta)^{-\frac{1}{1+\beta}}}{2+\beta}V^{2+\beta} \ \text{ on } \Gamma_2, \ \frac{\partial\theta}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{3.17}$$

$$\nabla\cdot\left(\sigma\left(F^{-1}\left(\theta - \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}}\right)\right)\nabla\psi\right) = 0 \ \text{ in } \Omega, \tag{3.18}$$

$$\psi = 0 \ \text{ on } \Gamma_1, \ \psi = \frac{V^{1+\beta}}{1+\beta} \ \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{3.19}$$

To the problems (3.16)–(3.19), we can apply the following:

Lemma 3.1 *If $a(\theta,\psi) \in C^0(\mathbf{R}^2)$ satisfies*

$$a_1 \geqslant a(\theta,\psi) \geqslant a_2 > 0,$$

then the problem for

$$\nabla\cdot(a(\theta,\psi)\nabla\theta) = 0 \ \ in\ \Omega, \ \theta = \theta_1 \ \ on\ \Gamma_1, \ \theta = \theta_2 \ \ on\ \Gamma_2, \ \frac{\partial\theta}{\partial n} = 0 \ \ on\ \Gamma_3,$$

$$\nabla\cdot(a(\theta,\psi)\nabla\psi) = 0 \ \ in\ \Omega, \ \psi = \psi_1 \ \ on\ \Gamma_1, \ \psi = \psi_2 \ \ on\ \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \ \ on\ \Gamma_3,$$

where θ_1, θ_2, ψ_1, ψ_2 are given constants, has one and only one solution.

We refer for the proof to [5]. We may conclude with the following,

Theorem 3.2 *If (3.1)–(3.3) hold the problems (3.4)–(3.7) have one and only one solution.*

It is interesting to note that a semi-explicit solution to problems (3.4)–(3.7) can be obtained. We make the following "Ansatz" on the solution suggested by the special structure of the system (3.16)–(3.19):

$$\theta = a\psi + b. \tag{3.20}$$

The constants a and b are immediately computed recalling (3.17) and (3.19). Thus, we find

$$b = 0, \ a = (1+\beta)V^{-(1+\beta)}F(U) + (1+\beta)^{\frac{\beta}{1+\beta}}V. \tag{3.21}$$

Hence, we arrive at the problem

$$\nabla \cdot \big(H(\psi)\nabla\psi\big) = 0 \ \text{ in } \Omega, \tag{3.22}$$

$$\psi = 0 \ \text{ on } \Gamma_1, \ \psi = \frac{V^{1+\beta}}{1+\beta} \ \text{ on } \Gamma_2, \ \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_3, \tag{3.23}$$

substituting in (3.20) the value of a and b given by (3.21), in (3.18) where

$$H(\psi) = \sigma\big(F^{-1}\big((1+\beta)V^{-(1+\beta)}F(U) + (1+\beta)^{\frac{\beta}{1+\beta}}V\big)\psi - \frac{1+\beta}{2+\beta}\psi^{\frac{2+\beta}{1+\beta}}\big). \tag{3.24}$$

The problems (3.22)–(3.24) can be solved with the transformation, invertible by (3.2),

$$\eta = \mathcal{H}(\psi) = \int_0^\psi \sigma\big(F^{-1}\big((1+\beta)V^{-(1+\beta)}F(U) + (1+\beta)^{\frac{\beta}{1+\beta}}Vt - \frac{1+\beta}{2+\beta}t^{\frac{2+\beta}{1+\beta}}\big)\big)dt. \tag{3.25}$$

In terms of η, the problems (3.22)–(3.24) becomes

$$\Delta\eta = 0 \ \text{ in } \Omega, \ \eta = 0 \ \text{ on } \Gamma_1, \ \eta = \mathcal{H}\Big(\frac{V^{1+\beta}}{1+\beta}\Big) \ \text{ on } \Gamma_2, \ \frac{\partial\eta}{\partial n} = 0 \ \text{ on } \Gamma_3. \tag{3.26}$$

If $\eta(\mathbf{x})$ is the solution of (3.26), we obtain the solution of the earlier problems (3.10)–(3.13) in the form

$$(\psi(\mathbf{x}), u(\mathbf{x})) = \Big(\mathcal{H}^{-1}(\eta(\mathbf{x})), F^{-1}\Big(\theta(\mathbf{x}) - \frac{1+\beta}{2+\beta}(\mathcal{H}^{-1}(\eta(\mathbf{x})))^{\frac{2+\beta}{1+\beta}}\Big)\Big), \tag{3.27}$$

where

$$\theta(\mathbf{x}) = \Big((1+\beta)V^{-(1+\beta)}F(U) + (1+\beta)^{\frac{\beta}{1+\beta}}V\Big)\mathcal{H}^{-1}(\eta(\mathbf{x})). \tag{3.28}$$

4 Conclusion

Certain mathematical aspects of the theory of varistors are discussed. This device has a nonlinear nonohmic current–voltage characteristic. Varistors are used as control or compensation elements in circuits to protect against excessive transient voltages. Their dependence on temperature is important in certain applications.

References

[1] Ambrosetti, A. & Arcoya, D. (2011) *An Introduction to Functional Analysis and Elliptic Problems*, Birkhäuser, Boston.

[2] Ambrosetti, A. & Prodi, G. (1973) *Analisi Non Lineare*, Quaderni S.N.S, Pisa.

[3] Caccioppoli, R. (1932) Sugli elementi uniti delle trasformazioni funzionali: Un teorema di esistenza e unicità e alcune sue applicazioni. *Rend. Sem. Mat. Univ. Padova* **3**, 123–137.

[4] Chow, S. & Hale, J. K. (1982) *Methods of Bifurcation Theory*, Springer-Verlag, New York.

[5] Cimatti, G. (2011) Remarks on the existence, uniqueness and semi-explicit solvability of systems of autonomous partial differential equations in divergence form with constant boundary conditions. *Proc. R. Soc. Edim.* **141**, 481–495.

[6] Data Sheets (1978) *Nonlinear resistors*, Philips. Amsterdam.

[7] Grondahl, L. O. & Geiger, P. H. (1927) A new electronic rectifier. *Trans. AIEE* **46**, 357–366.

[8] Levy, P. (1920) Sur les fonctions de lignes implicites. *Bull. Soc. Math. de France* **48**, 56–79.