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DITKIN CONDITIONS

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Abstract. This paper is about the connection between certain Banach-algebraic properties of a commutative Banach algebra E with unit and the associated commutative Banach algebra $C(X, E)$ of all continuous functions from a compact Hausdorff space X into E. The properties concern Ditkin's condition and bounded relative units. We show that these properties are shared by E and $C(X, E)$. We also consider the relationship between these properties in the algebras E, B and $\tilde{B}$ that appear in the so-called admissible quadruples $(X, E, B, \tilde{B})$.

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1. Introduction and preliminaries.

1.1. Introduction. Let A be a commutative Banach algebra with unit. The Gelfand transform $f \mapsto \hat{f}$ is a unital algebra homomorphism from A onto an algebra $\hat{A}$ of continuous complex-valued functions on its character space $M(A)$, the set of nonzero complex-valued multiplicative linear functionals on A, equipped with the relative weak-star topology from the dual A^*. The kernel of this homomorphism is the Jacobson radical rad(A), and so $\hat{A}$ is isomorphic to A when A is semisimple. See [**3**] for background.

For a nonempty compact Hausdorff space X and a Banach algebra E, we let $C(X, E)$ be the space of all continuous maps from X into E. We define the *uniform norm* on $C(X, E)$ by

$$\|f\|_X = \sup_{x\in X}\|f(x)\|, \quad f \in C(X, E).$$

For $f, g \in C(X, E)$ and $\lambda \in \mathbb{C}$, the pointwise operations λf, $f + g$ and fg in $C(X, E)$ are defined as usual. It is easy to see that $C(X, E)$ equipped with the norm $\|\cdot\|_X$ is a Banach algebra. If $E = \mathbb{C}$, we get the algebra $C(X, \mathbb{C}) = C(X)$ of all continuous complex-valued functions on X. Hausner [**5**] showed that if E is a commutative semisimple algebra, then $C(X, E)$ is also semisimple, with character space homeomorphic to $X \times M(E)$.

In this paper, we consider the connection between certain Banach-algebraic properties of commutative E and of $C(X, E)$. In many cases, properties of E are inherited by $C(X, E)$. The properties concerned will be detailed shortly. We also consider inheritance of properties by certain subalgebras of $C(X, E)$ called E-valued function algebras. More specifically, we consider E-valued function algebras $\tilde{B}$ that appear in what are called admissible quadruples $(X, E, B, \tilde{B})$. We now explain this concept.

1.2. E-valued function algebras. We recall definitions from our previous paper [7].

DEFINITION 1.1. By an *E-valued function algebra on X*, we mean a subalgebra $A \subset C(X, E)$, equipped with some norm that makes it complete such that (1) A has as an element, the constant function $x \mapsto 1_E$, (2) A separates points on X, i.e., given distinct points $a, b \in X$, there exists $f \in A$ such that $f(a) \neq f(b)$ and (3) the evaluation map,

$$e_x : \begin{cases} A & \to & E, \\ f & \mapsto & f(x), \end{cases}$$

is continuous, for each $x \in X$.

DEFINITION 1.2. By an *admissible quadruple,* we mean a quadruple $(X, E, B, \tilde{B})$, where

1. X is a compact Hausdorff space,
2. E is a commutative Banach algebra with unit,
3. $B \subset C(X)$ is a natural $\mathbb{C}$-valued function algebra on X,
4. $\tilde{B} \subset C(X, E)$ is an E-valued function algebra on X,
5. $B \cdot E \subset \tilde{B}$, and
6. $\{\lambda \circ f, f \in \tilde{B}, \lambda \in M(E)\} \subset B$.

One example is $(X, E, C(X), C(X, E))$. For other examples, such as the Lipschitz algebras and algebras associated with E-valued polynomials, rational functions and analytic functions, see [7], and see also Section 1.6.

Given an admissible quadruple $(X, E, B, \tilde{B})$, we define the *associated map* (also called Hausner's map)

$$\beta : \begin{cases} X \times M(E) & \to & M(\tilde{B}) \\ (x, \psi) & \mapsto & \psi \circ e_x. \end{cases}$$

The associated map is always injective.

DEFINITION 1.3. We say that an admissible quadruple $(X, E, B, \tilde{B})$ is *natural* if the associated map β is bijective.

Each quadruple of the form $(X, E, C(X), C(X, E))$ is admissible and natural. This is a more precise statement of Hausner's lemma [**5**, Lemma 2].

1.3. Properties. Let A be a commutative Banach algebra with unit.

Given an element $a \in A$, the *cozero set* of a is defined as

$$\operatorname{coz}(a) := \{\phi \in M(A) : \hat{a}(\phi) \neq 0\},$$

and the *support* $\operatorname{supp}(a)$ as the closure of $\operatorname{coz}(a)$ in $M(A)$.

To a closed set $S \subset M(A)$ are associated two ideals, the *kernel* of S,

$$I_S = I_S(A) := \{a \in A : \hat{a}(S) \subset \{0\}\},$$

and the smaller ideal

$$J_S = J_S(A) := \{a \in A : \operatorname{supp}(\hat{a}) \cap S = \emptyset\}.$$

For $\phi \in M(A)$, we abbreviate $I_\phi = I_\phi(A) := I_{\{\phi\}}$ (a maximal ideal) and $J_\phi = J_\phi(A) := J_{\{\phi\}}$.

A is said to have *bounded relative units* if, for every $\phi \in M(A)$, there exists $m_\phi > 0$ such that, for each compact subset K of $M(A) \setminus \{\phi\}$, there exists $a \in J_\phi$ with $\hat{a}(K) \subset \{1\}$ and $\|a\| \leq m_\phi$.

A satisfies *Ditkin's condition at* $\phi \in M(A)$ if $a \in \operatorname{clos}(aJ_\phi)$ for all $a \in I_\phi$.
A is a *Ditkin algebra* if A satisfies Ditkin's condition at each $\phi \in M(A)$.

A is a *strong Ditkin algebra* if I_ϕ has a bounded approximate identity contained in J_ϕ for each $\phi \in M(A)$, i.e., there exists $m_\phi > 0$ and $u_n \in J_\phi$ ($n \in \mathbb{N}$) such that $\|u_n\| \leq m_\phi$ for each n and $\|a - au_n\| \to 0$ for each $a \in I_\phi$.

1.4. Summary of results.

THEOREM 1. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then,* $C(X, E)$ *is Ditkin if and only if E is Ditkin.*

THEOREM 2. *Let* $(X, E, B, \tilde{B})$ *be a natural admissible quadruple and suppose* $\tilde{B}$ *is semisimple. Then,* $\tilde{B}$ *has bounded relative units if and only if both E and B have bounded relative units.*

COROLLARY 1.1. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then,* $C(X, E)$ *has bounded relative units if and only if E has bounded relative units*

COROLLARY 1.2. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then,* $C(X, E)$ *is a strong Ditkin algebra if and only if E is a strong Ditkin algebra.*

The 'only if' direction of Theorem 1 and hence of Corollary 1.2 extends to natural admissible quadruples: (see Propositions 2.1 and Corollary 3.3), but it appears to be unknown whether the 'if' direction does.

The results about quadruples apply to some so-called Tomiyama products, defined below. See Corollaries 2.2, 3.2 and 3.4.

We conclude the paper with an application to automatic continuity for maps $T : C(X, E) \to C(Y, F)$. See Section 4.

1.5. Properties of admissible quadruples. If $(X, E, B, \tilde{B})$ is a natural admissible quadruple, then it is easy to see that $\tilde{B}$ is semisimple if and only if E is semisimple.

Although E is not assumed semisimple in the definition, the quadruple concept really concerns semisimple E. The following is rather easily checked.

Proposition 1.3. *Let* $(X, E, B, \tilde{B})$ *satisfy conditions (1)–(5) of the definition. Define*

$$\hat{\tilde{B}} := \{x \mapsto \widehat{f(x)} : f \in \tilde{B}\}.$$

Then, $(X, E, B, \tilde{B})$ *is an admissible quadruple, if and only if* $(X, \hat{E}, B, \hat{\tilde{B}})$ *is an admissible quadruple.*

(We emphasize that, in this proposition, $\hat{E}$ denotes the Gelfand transform algebra with the quotient norm from $E/\mathrm{rad}(E)$, not the supremum norm.)

Also, for semisimple E, there is sometimes symmetry in the rôles of E and B.

Definition 1.4. We say that an admissible quadruple $(X, E, B, \tilde{B})$ is *tight* if for each $f \in \tilde{B}$ the map

$$\Phi(f) : \begin{cases} M(E) & \to & B \\ \lambda & \mapsto & \lambda \circ f \end{cases} \tag{1}$$

is continuous from $M(E)$ (with the usual relative weak-star topology from E^*) to B.

Proposition 1.4. *Suppose* $(X, E, B, \tilde{B})$ *is a tight admissible quadruple, and* E *is semisimple. Define* $\Phi(f)$ *by equation* (1)*, for each* $f \in \tilde{B}$*. Then,* Φ *is an algebra isomorphism of* $\tilde{B}$ *onto a* B*-valued function algebra on* $M(E)$*, and* $(M(E), B, E, \Phi(\tilde{B}))$ *is an admissible quadruple.*

Proof. Since the quadruple is tight, the map Φ is a well-defined linear map from the Banach space $\tilde{B}$ to the Banach space $C(M(E), B)$. An application of the closed graph theorem [**2**] shows that Φ is continuous. Thus, $\Phi(\tilde{B})$ is a B-valued function algebra on $M(E)$. The rest is clear. □

The following example shows that Proposition 1.4 would fail without the assumption of tightness.

Example 1.5. Let C denote the set of all continuous functions $f : [0, 1] \times [0, 2] \to \mathbb{C}$ such that the partial derivative $\frac{\partial f}{\partial y}$ exists at all points of the rectangle $R := [0, 1] \times [0, 2]$, is bounded on the whole rectangle, and is such that $\frac{\partial f}{\partial y}(x, y)$ is continuous on each vertical line, i.e., is continuous in y on $[0, 2]$ for each fixed $x \in [0, 1]$. With pointwise operations and the norm

$$\|f\|_C := \sup_R |f| + \sup_R \left|\frac{\partial f}{\partial y}\right|,$$

C is a natural function algebra on R.

Next, take $E = C^0([0, 1])$, $B = C^1([0, 2])$, and $X = [0, 1]$. Then, (with pointwise operations and the usual norms) E and B are semisimple separable commutative Banach algebras, with $M(E) = [0, 1]$ and $M(B) = [0, 2]$.

Let

$$C_1 := \left\{F \in E^{[0,2]} : ((x, y) \mapsto F(y)(x)) \in C\right\},$$

$$C_2 := \left\{G \in B^{[0,1]} : ((x, y) \mapsto G(x)(y)) \in C\right\}.$$

Then, C_1 is an algebra of E-valued functions on $[0, 2]$ and C_2 is an algebra of B-valued functions on [0, 1], when endowed with pointwise operations. Both algebras are algebra-isomorphic to C, via obvious isomorphisms. When they are given the norms induced by these isomorphisms, (X, E, B, C_1) is an admissible quadruple, and, in the notation of Proposition 1.4, $C_2 = \Phi(C_1)$.

We claim that $(M(E), B, E, \Phi(C_1)) = ([0, 1], B, E, C_2)$ is not an admissible quadruple, because the elements of C_2 are not all *continuous* B-valued functions. To see this, we give an example of a function $f \in C$ such that

$$\begin{cases} [0,1] & \to & C^1([0,2]), \\ x & \mapsto & (y \mapsto f(x,y)), \end{cases}$$

is not continuous.

Take

$$f(x, y) = \left\{ \begin{array}{rl} 0, & 0 \le y \le x, \\ \dfrac{(y-x)^2}{2x}, & x < y < 2x, \\ y - \dfrac{3x}{2}, & 2x \le y \le 2, \end{array} \right\}, 0 \le x \le 1.$$

Then, f is continuous on R, the partial derivative $\frac{\partial f}{\partial y}$ is continuous on each vertical line and is bounded, but it is not continuous at $(0, 0)$. Moreover, the value of $\|f(x, \cdot) - f(0, \cdot)\|_B$ exceeds 1 for all $x > 0$, so it does not tend to 0 as $x \downarrow 0$.

1.6. Tensor products. Let A and B be commutative Banach algebras with unit. A *Tomiyama product of A and B* is the completion of the algebraic tensor product $A \otimes B$ with respect to some submultiplicative cross norm not less than the injective tensor product norm. See [**8**] and [**6**, Section 2.11] for background on cross norms and tensor products of Banach algebras.

PROPOSITION 1.6. *Let C be a Tomiyama product of A and B, two commutative Banach algebras with unit. Suppose C is semisimple. Then,* $(M(B), A, B, C)$ *is a natural admissible quadruple.*

REMARK 1.1. Kaniuth shows (cf. [**6**, Corollary 2.11.3] that if C is semisimple, then so are A and B. Thus, since we are mainly interested in semisimple C, we might just as well have restricted to semisimple A and B in the definition of Tomiyama product.

Tomiyama showed [**9**, Theorem 4] that a Tomiyama product C is automatically semisimple if both A and B are semisimple, at least one of them has the Banach approximation property and the norm is either the projective or injective product norm.

Proof of Proposition. Let $X = M(B)$.

First, we have to explain how C may be regarded as an A-valued function algebra on X (condition (4) of the definition of admissible quadruple).

By the definition of Tomiyama product, we have

$$\|f\|_{A\check{\otimes}B} \leq \|f\|_C,$$

for all $f \in A \otimes B$.

Let $f \in C$. Then, there is a C-norm-Cauchy sequence $f_n \in A \otimes B$ with $\|f - f_n\|_C \to 0$. Thus, (f_n) is $A\check{\otimes}B$-norm-Cauchy as well, and so converges to an element $\Psi(f) \in A\check{\otimes}B$. We have

$$\|\Psi(f)\|_{A\check{\otimes}B} = \lim \|\Psi(f_n)\|_{A\check{\otimes}B} \leq \lim \|f_n\|_C = \|f\|_C.$$

One can check that $\Psi(f)$ does not depend on which Cauchy sequence (f_n) is chosen. So we have a well-defined continuous map $\Psi : C \to A\check{\otimes}B$, a contraction, in fact. The map Ψ is also linear, as is easily seen.

Next, we claim that Ψ is injective. Suppose $f \in C$ and $\Psi(f) = 0$. Take any sequence $f_n \in A \otimes B$ such that $f_n \to f$ in C-norm. Then, $f_n \to 0$ in $A\check{\otimes}B$-norm.

Fix any $\chi \in M(C)$. By Tomiyama's theorem, there exist $\lambda \in M(A)$ and $\gamma \in M(B)$ such that $\chi = \lambda \otimes \gamma$ when restricted to the algebraic tensor product $A \otimes B \subset C$. Moreover, there is a character χ' on $A\check{\otimes}B$ that agrees with χ on $A \otimes B$.

Fix $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\|f - f_n\|_C < \frac{\epsilon}{2} \text{ and } \|f_n\|_{A\check{\otimes}B} < \frac{\epsilon}{2}.$$

Then,

$$|\chi(f)| \leq |\chi(f - f_n)| + |\chi(f_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

because χ has norm 1 in C^*, $\chi(f_n) = \chi'(f_n)$, and χ' has norm 1 in $(A \otimes B)^*$. Thus, $|\chi(f)| < \epsilon$ for all $\epsilon > 0$.

Thus, $\chi(f) = 0$ for all $\chi \in M(C)$. Since C is semisimple, $f = 0$. Thus, Ψ is injective, as claimed.

So we have a continuous injection from C into the injective tensor product $A\check{\otimes}B$, which is a subset of $A\check{\otimes}C(X)$, and the latter is naturally identified with $C(X, A)$, as shown by Grothendieck [**8**].

Conditions (1)–(3) and (5) are straightforward, and condition (6) holds because $A \otimes B$ is dense in C.

Thus, (X, A, B, C) is an admissible quadruple. It is natural by Tomiyama's main result that $M(C)$ is homeomorphic to $M(A) \times M(B)$ [**9**, Theorem 1]. □

REMARK 1.2. The projective tensor product $A\hat{\otimes}B$ of commutative Banach algebras A and B is an example of a Tomiyama product, but it is not always semisimple if A and B are. The natural map $\Psi : A\hat{\otimes}B \to A\check{\otimes}B$ (as in the proof above) may fail to be injective. In fact [**1**, Section 9], it is injective for all B if and only if A has the Banach approximation property.

COROLLARY 1.7. *Let C be a Tomiyama product of A and B, two semisimple commutative Banach algebras with unit. Suppose C is semisimple. Then, the admissible quadruple $(M(B), A, B, C)$ is tight.*

Proof. Applying the Theorem with A and B interchanged, we conclude that $(M(A), B, A, C)$ is an admissible quadruple, so C is (isometrically isomorphic to) a B-valued function algebra on $M(A)$, i.e., $(M(B), A, B, C)$ is tight. □

Thus, we can assert that the algebra C in Example 1.5 is *not* a Tomiyama product of $C^0([0.1])$ and $C^1([0, 2])$.

2. Ditkin algebras. In this section, we will prove Theorem 1. As indicated, one direction generalises to admissible quadruples.

2.1. The 'only if' direction.

PROPOSITION 2.1. *Let* $(X, E, B, \tilde{B})$ *be an admissible quadruple. Suppose* $\tilde{B}$ *is Ditkin. Then, E and B are Ditkin.*

Proof. Suppose $\tilde{B}$ is Ditkin.

By Proposition 1.3, $(X, \hat{E}, B, \hat{\tilde{B}})$ is admissible, and since $\hat{\tilde{B}}$ inherits the Ditkin property, we may assume without loss in generality that E is semisimple.

To see that E is Ditkin, fix $\psi \in M(E)$, and $b \in E$ with $\psi(b) = 0$. We wish to show that there exist $b_n \in J_\psi(E)$ such that $\|b - b_n b\|_E \to 0$ as $n \uparrow \infty$.

Pick any $x_0 \in X$, and define $\phi = \beta(x_0, \psi)$. Then, $\phi \in M(\tilde{B})$. Define $f(x) = b$, for all $x \in X$. Then, $f \in I_\phi$, so since $\tilde{B}$ is Ditkin, we may choose $f_n \in \tilde{B}$ such that each $\hat{f}_n = 0$ near ϕ in $M(\tilde{B})$ and $\|f - f_n f\|_X \to 0$ as $n \uparrow \infty$. Take $b_n = f_n(x_0)$. Then, $\|b - b_n b\|_E \to 0$. Since β is continuous, we may choose open sets $U_n \subset X$ and $V_n \subset M(E)$ such that $x_0 \in U_n$, $\psi \in V_n$ and $\hat{f}_n = 0$ on $\beta(U_n \times V_n)$. Then, for $\chi \in V_n$, we have

$$\hat{b}_n(\chi) = \hat{f}_n(\beta(x_0, \chi)) = 0.$$

Thus, $b_n \in J_\psi$, as desired.

If $(X, E, B, \tilde{B})$ were tight, we could immediately use Proposition 1.4, to deduce that $(M(E), B, E, \Phi(\tilde{B}))$ is admissible, and the isomorphic algebra $\Phi(\tilde{B})$ is Ditkin, so B is Ditkin. However, we do not need to make this assumption.

Assume just that $(X, E, B, \tilde{B})$ is an admissible quadruple, and $\tilde{B}$ is Ditkin. The map,

$$\Psi(\lambda) : \begin{cases} \tilde{B} & \to & B \\ f & \mapsto & \lambda \circ f, \end{cases}$$

is a well-defined algebra homomorphism, for each $\lambda \in M(E)$. By using the closed graph theorem, we see that $\Psi(\lambda)$ is continuous.

Now fix $a \in X$ and $g \in B$ with $g(a) = 0$. Pick any $\lambda_0 \in M(E)$ and define $\phi = \beta(a, \lambda_0)$. Then, $\phi \in M(\tilde{B})$. Define $f(x) = g(x) \cdot 1_E$ for all $x \in X$. Then, $f \in \tilde{B}$ and $\phi(f) = 0$, so since $\tilde{B}$ is Ditkin we may choose $f_n \in \tilde{B}$ such that $\hat{f}_n = 0$ near ϕ in $M(\tilde{B})$ and $\|f - f_n f\|_{\tilde{B}} \to 0$. Let $g_n = \Psi(\lambda_0)(f_n)$. Then, $g_n \in B$ and $g_n = 0$ near a. Since $g = \Psi(\lambda_0)(f)$ and $\Psi(\lambda_0)$ is continuous, we have

$$\|g - g_n g\|_B \to 0.$$

Thus, B is Ditkin. □

Applying Proposition 1.6, we have the following corollary.

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COROLLARY 2.2. *Let A and B be semisimple commutative Banach algebras with unit, and let C be a semisimple Tomiyama product of A and B. Suppose C is Ditkin. Then, so are A and B.*

2.2. Converse direction. Turning to the other direction, we restrict to the special quadruple $(X, E, C(X), C(X, E))$.

PROPOSITION 2.3. *Let X be a compact Hausdorff space and let E be a commutative Banach algebra with unit. Suppose E is Ditkin. Then, $C(X, E)$ is Ditkin.*

Proof. Fix $\phi \in M(C(X, E))$, and $f \in I_\phi$. Let $\epsilon > 0$ be given.

Choose $\psi \in M(E)$ and $x_0 \in X$ such that $\phi = \beta(x_0, \psi)$.

Let $a = f(x_0)$. Then, $\psi(a) = 0$. Since E is Ditkin, we may choose $b \in J_\psi(E)$ such that $\|a - ba\|_E < \epsilon$. Let

$$U = \{x \in X : \|f(x)b - f(x)\|_E < \epsilon\}.$$

Then, U is an open neighbourhood of x_0. Thus, by Urysohn's lemma, we may choose $h \in J_{x_0}(C(X))$ with $h = 1$ off U and $0 \le h \le 1$ on X.

Then, for each $x \in X$, we have

$$\|(1 - h(x))(bf(x) - f(x))\|_E < \epsilon.$$

Thus, $\|(1 - h)(bf - f)\|_X < \epsilon$. Now

$$f + (1 - h)(bf - f) = f(h \cdot 1_E + b - hb) \in fJ_\phi$$

(since $h \cdot 1_E + b - hb = 0$ on $\beta^{-1}\left(h^{-1}(0) \cap b^{-1}(0)\right)$), so the distance from f to fJ_ϕ in $C(X, E)$ norm is less than ϵ.

The result follows. □

Proof of Theorem 1. Apply Proposition 2.1 (with $B = C(X)$ and $\tilde{B} = C(X, E)$) and Proposition 2.3. □

3. Bounded relative units.

3.1. Proof of Theorem 2. Note the following lemma.

LEMMA 3.1. *If A is a commutative Banach algebra with identity, then the following are equivalent:*

(1) *A has bounded relative units.*

(2) *For each $\phi \in M(A)$, there exists a constant $c_\phi > 0$ with the following property: for every closed subset K of $M(A)$ with $\phi \notin K$, there exists $x \in A$ such that $\|x\| \le c_\phi$, $\hat{x} = 0$ on K and $\hat{x} = 1$ on some neighbourhood of ϕ.*

Proof. Let e denote the identity of A.

Suppose (1) holds. Fix $\phi \in M(A)$, and let $m_\phi > 0$ be chosen as in the definition of bounded relative units. Take $c_\phi = m_\phi + \|e\|$. Let $K \subset M(A)$ be compact, with $\phi \notin K$. We may choose $a \in J_\phi$ such that $\hat{a}(K) \subset \{1\}$ and $\|a\| \le m_\phi$. Taking $x = e - a$, we have $\|x\| \le c_\phi$, $\hat{x} = 0$ on K and $\hat{x} = 1$ near ϕ. Thus, (2) holds.

The other direction is similar. □

This shows, in particular, that a unital commutative Banach algebra with bounded relative units is regular.

Proof of Theorem 2. For the 'only if' direction, suppose $(X, E, B, \tilde{B})$ is an admissible quadruple, and $\tilde{B}$ has bounded relative units. Then, we have to show that E and B have bounded relative units.

First, consider E, and fix $\psi_0 \in M(E)$. Fix any $x_0 \in X$. Since the evaluation map $f \mapsto f(x_0)$ is continuous from $\tilde{B} \to E$, there exists $\kappa > 0$ such that $\|f(x_0)\|_E \leq \kappa \|f\|_{\tilde{B}}$ for all $f \in \tilde{B}$.

Define $\phi := \beta(x_0, \psi_0) \in M(\tilde{B})$. By assumption, there exists $m > 0$ such that for each open neighbourhood W of ϕ there exists $f \in J_\phi$ such that $\hat{f} = 1$ off W and $\|f\|_{\tilde{B}} \leq m$. Let $F \subset M(E) \setminus \{\psi_0\}$ be a compact subset. Define $L := \{\beta(x_0, \chi) : \chi \in F\}$. It is clear that L is a compact subset of $M(C(X, E))\backslash\{\phi\}$. We may choose $f \in J_\phi$ such that $\hat{f}(L) \subset \{1\}$ and $\|f\|_{\tilde{B}} \leq m$. Define $b := f(x_0)$. Then, $b \in E$, $\hat{b}(F) \subset \{1\}$ and $\|b\|_E \leq \kappa \|f\|_{\tilde{B}} \leq \kappa m$.

Thus, E has bounded relative units.

Now consider B, and fix $x_0 \in X = M(B)$. Fix any $\psi_0 \in M(E)$. As noted in the proof of Proposition 2.1, the map $f \mapsto \psi_0 \circ f$ is continuous from $\tilde{B} \to B$, so there exists $\kappa > 0$ such that $\|\psi_0 \circ f\|_B \leq \kappa \|f\|_{\tilde{B}}$. So defining $\phi := \beta(x_0, \psi_0) \in M(\tilde{B})$, we may proceed in a very similar way to the above, to deduce that B has bounded relative units.

For the 'if' direction, the key observation (for which the authors would like to thank the referee) uses the classical automatic continuity theorem of Shilov [**3**, Theorem 2.3.3, p. 192] that each homomorphism from a Banach algebra into a semisimple commutative Banach algebra is necessarily continuous. We may apply this to the two homomorphisms

$$\left\{ \begin{array}{ccc} E & \to & \tilde{B} \\ a & \mapsto & 1_X \cdot a \end{array} \right\} \text{ and } \left\{ \begin{array}{ccc} B & \to & \tilde{B} \\ f & \mapsto & f \cdot e \end{array} \right\},$$

where e is the identity of E and deduce that there exist constants $\alpha > 0$ and $\gamma > 0$ such that $\|1_X \cdot a\|_{\tilde{B}} \leq \alpha \|a\|_E$ for all $a \in E$ and $\|f \cdot e\|_{\tilde{B}} \leq \gamma \|f\|_B$ for all $f \in B$.

Now every $\phi \in M(\tilde{B})$ is of the form $\psi \circ e_x$ for some $\psi \in M(E)$ and some $x \in X$. Let c_x and c_ψ be constants as guaranteed by the assumption that B and E have bounded relative units. Since $M(\tilde{B})$ is homeomorphic to $X \times M(E) = M(B) \times M(E)$, given a closed subset K of $M(\tilde{B})$ such that $\phi \notin K$, we may find closed subsets $C \subset X$ and $D \subset M(E)$ such that $K \subset (C \times M(E)) \cup (X \times D)$ and $x \notin C$ and $\psi \notin D$. Then, by hypothesis, there exist

- $f \in B$ such that $\|f\|_B \leq c_x$, $f = 0$ on C and $f = 1$ on a neighbourhood of x;
- $a \in E$ such that $\|a\|_E \leq c_\psi$, $\hat{a} = 1$ on D and $\hat{a} = 1$ on a neighbourhood of ψ.

Then, the element $f \times a$ of $\tilde{B}$ satisfies $\widehat{f \cdot a} = 0$ on $(C \times M(E)) \cup (X \times D)$ and $\widehat{f \cdot a} = 1$ in a neighbourhood of ϕ. Moreover,

$$\|f \cdot a\|_{\tilde{B}} \leq \|f \cdot e\|_{\tilde{B}} \cdot \|1_X \cdot a\|_{\tilde{B}} \leq \alpha\gamma \|a\|_E \|f\|_B \leq \alpha\gamma c_x c_\phi.$$

Thus, $\tilde{B}$ has bounded relative units. □

COROLLARY 3.2. *Let A and B be semisimple commutative Banach algebras with unit. Suppose that C is a semisimple Tomiyama product of A and B. Then, C has bounded relative units if and only if both A and B have bounded relative units.*

3.2. Proof of corollaries.

Proof of Corollary 1.1. This is immediate from Theorem 2, because E has bounded relative units if and only if $\hat{E}$ does, and $C(X, \hat{E})$ is semisimple, so the theorem applies to the quadruple
$(X, \hat{E}, C(X), C(X, \hat{E})$, and tells us that $C(X, \hat{E})$ has bounded relative units if and only if E does. But $C(X, \hat{E})$ is isometrically algebra isomorphic to $\widehat{C(X, E)}$, so $C(X, E)$ has bounded relative units if and only if E does. □

Proof of Corollary 1.2. This follows from Theorems 1 and 2, since an algebra is a strong Ditkin algebra if and only if it is Ditkin and has bounded relative units [**3**, pp. 417–8]. □

As indicated earlier, one direction of Corollary 1.2 generalises to natural admissible quadruples.

COROLLARY 3.3. *Let $(X, E, B, \tilde{B})$ be an admissible quadruple. Suppose $\tilde{B}$ is strong Ditkin. Then, E and B are strong Ditkin.*

Proof. This follows from Proposition 2.1 and Theorem 2. □

COROLLARY 3.4. *Let A and B be semisimple commutative Banach algebras with unit, and let C be a semisimple Tomiyama product of A and B. Suppose C is strong Ditkin. Then, so are A and B.*

4. Separating bijections.

DEFINITION 4.1. Let A and B be two semisimple commutative Banach algebras with identity. A linear map $T : A \to B$ is said to be *separating* or *disjointness preserving* if $\mathrm{coz}(Tf) \cap \mathrm{coz}(Tg) = \emptyset$ whenever $f, g \in A$ satisfy $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$. Moreover, T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, a map $T : A \to B$ is separating, if it is linear and $Tf \cdot Tg \equiv 0$, whenever $f, g \in A$ satisfy $f \cdot g \equiv 0$. As an application of Theorem 1, we obtain the following theorem.

THEOREM 3. *Let X, Y be two compact Hausdorff spaces and E, F be unital commutative semisimple Banach algebras that are Ditkin algebras and $T : C(X, E) \to C(Y, F)$ be a separating linear bijection, and then*
(*i*) *T is continuous,*
(*ii*) *T^{-1} is separating, and*
(*iii*) *$X \times M(E)$ and $Y \times M(F)$ are homeomorphic.*

Proof. Use [**4**, Theorem 1] and Theorem 1. □

REMARK 4.1. The results of this paper may be extended to semisimple commutative Banach algebras without identity by the device of adjoining a unit. We have confined attention to algebras with unit, to avoid clutter.

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